Semiconductor Lasers and Kolmogorov Spectra.

Yuri V. Lvov\textsuperscript{1,2} and Alan C. Newell\textsuperscript{1,3}

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\textsuperscript{1} Department of Mathematics, The University of Arizona, Tucson AZ 85721 USA
\textsuperscript{2} Department of Physics, The University of Arizona, Tucson AZ 85721 USA
\textsuperscript{3} Mathematical Institute, The University of Warwick, Coventry CV47AL UK

Abstract

In this letter, we make a prima facie case that there could be distinct advantages to exploiting a new class of finite flux equilibrium solutions of the Quantum Boltzmann equation in semiconductor lasers.

1 Introduction

At first sight, it may very well seem that the two subjects linked in the title have little in common. What do semiconductor lasers have to do with behavior normally associated with fully developed hydrodynamic turbulence? In order to make the connection, we begin by reviewing the salient features of semiconductor lasers. In many ways, they are like two level lasers in that the coherent light output is associated with the in phase transitions of an electron from a higher to lower energy state. In semiconductors, the lower energy state is the valence band from which sea electrons are removed leaving behind positively charged holes. The higher energy state is the conduction band. The quantum of energy released corresponds to an excited electron in the conduction band combining with a hole in the lower band below the bandgap. Bandgaps, or forbidden energy zones are features of the energy spectrum of an electron in periodic potentials introduced in this case by the periodic nature of the semiconductor lattice.

However, there are two important ways in which the semiconductor laser differs from and is more complicated than the traditional two-level laser model. First, there is a continuum of bandgaps parameterized by the electron momentum $k$ and the
laser output is a weighted sum of contributions from polarizations corresponding to electron-hole pairs at each momentum value. In this feature, the semiconductor laser resembles an inhomogeneously broadened two level laser. Second, electrons and holes interact with each other via Coulomb forces. Although this interaction is screened by the presence of many electrons and holes, it is nonetheless sufficiently strong to lead to a nonlinear coupling between electrons and holes at different momenta. The net effect of these collisions is a redistribution of carriers (the common name for both electrons and holes) across the momentum spectrum. In fact it is the fastest ($\approx 100$ fs.) process (for electric field pulses of duration greater than picoseconds) and because of this, the gas of carriers essentially relaxes to a distribution corresponding to an equilibrium of this collision process. This equilibrium state is commonly taken to be that of thermodynamic equilibrium for fermion gases, the Fermi-Dirac distribution characterized by two parameters, the chemical potential $\mu$ and temperature $T$, slightly modified by the presence of broadband pumping and damping.

But the Fermi-Dirac distribution is not the only equilibrium of the collision process. There are other stationary solutions, called finite flux equilibria, for which there is a finite and constant flux of carriers and energy across a given spectral window. The Fermi-Dirac solution has zero flux of both quantities. It is the aim of this letter to suggest that these finite flux equilibria are more relevant to situations in which energy and carriers are added in one region of the spectrum, redistributed via collision processes to another region where they are absorbed. Moreover, it may be advantageous to pump the laser in this way because such a strategy may partially overcome the deleterious effects of Pauli blocking. The Pauli exclusion principle means that two electrons with the same energy and spin cannot occupy the same state at a given momentum. This leads to inefficiency because the pumping is effectively multiplied by a factor $(1 - n_s(k)), s = e, h$ for electrons and holes respectively, denoting the probability of not finding electron (hole) in a certain $k$ (used to denote both momentum and spin) state. But, near the momentum value corresponding to the lasing frequency $\omega_L$, $n_s(k)$ is large ($n_e(k) + n_h(k)$ must exceed unity) and Pauli blocking significant. Therefore, pumping the laser in a window about $\omega_0 > \omega_L$ in such a way that one balances the savings gained by lessening the Pauli blocking (because the carriers density $n_s(k)$ decreases with $k = |k|$) with the extra input energy required (because $k$ is larger), and then using the finite flux solution to transport carriers (and energy) back to lasing frequency, seems an option worth considering. The aim of this letter is to demonstrate, using the simplest possible model, that this alternative is viable. More detailed results using more sophisticated (but far more complicated) models will be given later.

These finite flux equilibria are the analogies of the Kolmogorov spectra associ-
ated with fully developed, high Reynolds number hydrodynamic turbulence and the
wave turbulence of surface gravity waves on the sea. In the former context, energy
is essentially added at large scales (by stirring or some instability mechanism), is
dissipated at small (Kolmogorov and smaller) scales of the order of less than the
inverse three quarter power of the Reynolds number. It cascades via nonlinear inter-
actions from the large scales to the small scales through a window of transparency
(the inertial range in which neither forcing nor damping is important) by the con-
tant energy flux Kolmogorov solution. Indeed, for hydrodynamic turbulence, the
analogue to the Fermi-Dirac distribution, the Rayleigh-Jeans spectrum of equipati-
tions, is irrelevant altogether. The weak turbulence of surface gravity waves is the
classical analogue of the case of weakly interacting fermions. The mechanism for
energy and carrier density (particle number) transfer is ”energy” and ”momentum”
conserving binary collisions satisfying the ”four wave resonance” conditions
\[ k + k_1 = k_2 + k_3, \quad \omega(k) + \omega(k_1) = \omega(k_2) + \omega(k_3). \] (1)

In the semiconductor context, \( \hbar \omega(k) = \hbar \omega_{\text{gap}} + \epsilon_e(k) + \epsilon_h(k) \) (which can be well
approximated by \( \alpha + \beta k^2 \)) where \( \hbar \omega_{\text{gap}} = \epsilon_{\text{gap}} \) corresponds to the minimum bandgap
and \( \epsilon_e(k), \epsilon_h(k) \) are electron and hole energies. In each case, there is also a simple
relation \( E(k) = \omega n(k) \) between the spectral energy density \( E(k) \) and carrier (par-
ticle number) density \( n(k) \). As a consequence of conservation of both energy and
carriers, it can be argued (schematically shown in Figure 1 and described in its cap-
tion), that the flux energy (and some carriers) from intermediate momentum scales
(around \( k_0 \) say) at which it is injected, to higher momenta (where it is converted
into heat) must be accompanied by the flux of carriers and some energy from \( k_0 \) to
lower momenta at which it will be absorbed by the laser. It is the latter solution
that we plan to exploit.

2 Model

We present the results of a numerical simulation of a greatly simplified model of
semiconductor lasing in which we use parameter values which are realistic but make
fairly severe approximations in which we (a) assume that the densities of electrons
and holes are the same (even though their masses differ considerably) (b) ignore
carrier recombination losses and (c) model the collision integral by a differential
approximation [1], [2], [3] in which the principal contributions to wavevector quartets
satisfying (1) are assumed to come from nearby neighbors. Despite the brutality
of the approximations, the results we obtain are qualitatively similar to what we
obtain using more sophisticated and complicated descriptions.
The semiconductor Maxwell-Bloch equations are [4],[5],

\[
\begin{align*}
\frac{\partial e}{\partial t} &= i \frac{\Omega}{2\epsilon_0} \int \mu_k p_k dk - \gamma_E e, \quad (2) \\
\frac{\partial p_k}{\partial t} &= (i\Omega - i\omega_k - \gamma_P)p_k - \frac{i\mu_k}{2\hbar}(2n_k - 1)e, \quad (3) \\
\frac{\partial n_k}{\partial t} &= \Lambda(1 - n_k) - \gamma_k n_k + \left( \frac{\partial n_k}{\partial t} \right)_{\text{collision}} - \frac{i}{2\hbar} (\mu_k p_k^* - \mu_k p_k) e. \quad (4)
\end{align*}
\]

Here \(e(t)\) and \(p_k(t)\) are the electric field and polarization at momentum \(k\) envelopes of the carrier wave \(\exp(-i\Omega t + iKz)\) where \(\Omega\) is the cavity frequency (we assume single mode operation only) and \(n(k)\) is the carrier density for electrons and holes. The constants \(\gamma_E, \gamma_P\) model electric field and homogeneous broadening losses, \(\epsilon_0\) is dielectric constant, \(\mu_k\) is the weighting accorded to different \(k\) momentum (modeled by \(\mu_k = \mu_{k=0}/(1 + \epsilon_k/\epsilon_{\text{gap}})\)), \(\Lambda_k\) and \(\gamma_k\) represent carrier pumping and damping. In (4), the collision term is all important and is given by

\[
\frac{\partial}{\partial t} n_k = 4\pi \int \left| T_{kk_1} k_{k_2} k_{k_3} \right|^2 \left( n_{k_2} n_{k_3} (1 - n_{k_1} - n_k) + n_k n_{k_1} (n_{k_2} + n_{k_3} - 1) \right) \delta(k + k_1 - k_2 - k_3) \delta(\omega_k + \omega_{k_1} - \omega_{k_2} - \omega_{k_3}) dk_1 dk_2 dk_3, \quad (5)
\]

where \(T_{kk_1} k_{k_2} k_{k_3}\) is the coupling coefficient measuring mutual electron and hole interactions. We make the weak assumption that all fields are isotropic and make a convenient transformation from \(k (= |k|)\) to \(\omega\) via the dispersion relation \(\omega = \omega(k)\) defining the carrier density \(N_\omega\) by \(\int N_\omega d\omega = \int n(k)dk\) or \(N_\omega = 4\pi k^2 dk/(d\omega) n(k)\).

Then, in the differential approximation, (5) can be written as both,

\[
\begin{align*}
\frac{\partial N_\omega}{\partial t} &= \frac{\partial^2 K}{\partial \omega^2} \quad \text{and} \quad \frac{\partial \omega N_\omega}{\partial t} = -\frac{\partial}{\partial \omega} \left( K - \omega \frac{\partial K}{\partial \omega} \right), \quad (6)
\end{align*}
\]

with

\[
K = -I^s \left( n_\omega^2 \left( n_\omega^{-1} \right)^{'''} + n_\omega^2 (\ln n_\omega)^{''} \right), \quad \left( \right)'' = \frac{\partial}{\partial \omega}, \quad n_\omega = n(k(\omega)),
\]

where \(s\) is the number computed from the dispersion relation, the dependence of \(T_{kk_1} k_{k_2} k_{k_3}\) on \(k\) and dimensions (\(s\) is of the order of 7 for semiconductors.) The conservation forms of the equations for \(N_\omega\) and \(E_\omega = \omega N_\omega\) allow us identify \(Q = \frac{\partial K}{\partial \omega}\) (positive if carriers flow from high to low momenta) and \(P = K - \omega \frac{\partial K}{\partial \omega}\) (positive if energy flows from low to high momenta) as the fluxes of carriers and energy respectively. Moreover, the equilibrium solutions are now all transparent. The general stationary solution of (6) is the integral of \(K = Q_\omega + P\) which contains four
parameters, two (chemical potential and temperature) associated with the fact that 
$K$ is a second derivative, and two constant fluxes $Q$ and $P$ of carriers and energy.
The Fermi-Dirac solution $n_\omega = (\exp(A\omega + B) + 1)^{-1}$, the solution of $K = 0$, has 
zero flux. We will now solve (2), (3) and (4) after angle averaging (4) and replacing 
$4\pi k^2 \frac{\partial k}{\partial \omega} \left( \frac{\partial n_k}{\partial \omega} \right)_{\text{collision}}$ by $\frac{\partial^2 K}{\partial \omega^2}$. The value of the constant $I$ is chosen to ensure that 
solutions of (6) relax in a time of 100 fs.

3 Results

We show the results in figures 2, 3, and 4. First, to test accuracy, we show, in Figure 
2, the relaxation of (6) to a pure Fermi-Dirac spectrum in the window $\omega_L = 1 < \omega < 
\omega_0 = 2$. The boundary conditions correspond to $P = Q = 0$ at both ends. We then 
modify boundary conditions to read $Q = Q_0 > 0$ and $P = 0$ at both ends. Next, in 
Figure 3 and 4, we show the results of two experiments in which we compare the 
efficiencies of two experiments in which we arrange to (i) pump broadly so that the 
effective carrier distribution equilibrium has zero flux and (ii) pump carriers and 
energy into a narrow band of frequencies about $\omega_0$ and simulate this by specifying 
carrier and energy flux rates $Q_L$ and $P_L = -\omega_L Q_L$ ($P_L$ chosen so that the energy 
absorbed by the laser is consistent with the number of carriers absorbed there) at 
the boundary $\omega = \omega_0$. $\omega = \omega_L$ is the frequency at which the system lases.

In both cases, the rate of addition of carriers and energy is (approximately) 
the same. The results support the idea that it is worth exploring the exploitation 
of the finite flux equilibrium. The carrier density of the equilibrium solutions at 
$\omega_0$ is small thus making pumping more efficient there. The output of the laser is 
greater by a factor of 10. While we do not claim that, when all effects are taken 
account of, this advantage will necessary remain, we do suggest that the strategy of 
using finite flux equilibrium solutions of the Quantum Boltzmann equation is worth 
further exploration.

4 Acknowledgments

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5 Figure Captions.

• Figure 1
Carriers and energy are added at $\omega_0$ at rates $Q_0$ and $\omega_0 Q_0$. Energy and some carriers are dissipated at $\omega_R > \omega_0$ (an idealization) and carriers and some energy are absorbed by the laser at $\omega_L$. (The carriers number will build until the laser switches on.) A little calculation shows $Q_L = Q_0(\omega_R - \omega_0)/(\omega_R - \omega_L)$, $Q_R = Q_0(\omega_L - \omega_0)/(\omega_R - \omega_L)$, $P_R = Q_0 \omega R(\omega_0 - \omega_L)/(\omega_R - \omega_L)$, $P_L = \omega L Q_0(\omega_0 - \omega_R)/(\omega_R - \omega_L)$. Finite flux stationary solutions are realized in the windows $(\omega_L, \omega_0)$ and $(\omega_0, \omega_R)$ although in practice there will be some losses through both these regions.

• Figure 2.
To test accuracy we take some initial distribution function (thin line) and plot its time evolution as described by (6) with boundary conditions $P = Q = 0$ at both ends. The distribution function relaxes to Fermi-Dirac state (thick line). Several intermediate states is shown by long-dashed and short-dashed lines (Figure 2a). We then modify boundary conditions to $P = 0, Q = Q_0 > 0$ at both ends. Then initial distribution function (thick line) relaxes to finite-$Q$-equilibria as shown by long-dashed line. We then change boundary conditions to $P = 0, Q = Q_1 > Q_0$ at both ends, so that distribution function is shown by short-dashed line. Increasing $Q$ at boundaries even further, so that $P = 0, Q = Q_2 > Q_1$ at both ends, the distribution function is given by dotted line (Figure 2b).

• Figure 3.
We now solve (2-4) with the collision term given by (6). We pump broadly, so that the effective carrier distribution has zero flux. The initial distribution function (thin line) builds up because of a global pumping (dashed lines), until the laser switches on. The final (steady) distribution function is shown by thick solid line (Figure 3a). The output power (in arbitrary units) as a function of time (measured in relaxation times $\approx 100 \text{fs}$) is also shown (Figure 3b).

• Figure 4.
We pump in the narrow region around $\omega_0 \approx 200 \text{meV}$ and we model this by specifying carrier and energy flux rates $Q_L$ and $P_L = -\omega_L Q_L$. The initial distribution function (thin line) builds up because of influx of particles and energy from right boundary (dashed lines), until the laser switches on. The final (steady) distribution function is shown by thick solid line and corresponds to a flux of particles and energy from right boundary (where we add particles and energy) to the left boundary, where the system lases (Figure 4a). The
output power as a function of time is also shown (Figure 4b).
Figure 1

\[ \omega_0, \omega_0 Q_0 \]

\[ P_L \quad Q_L \quad P_R \quad Q_R \]

\[ \omega_L \quad \omega_0 \quad \omega_R \]

IN

OUT

\[ Q_0, \omega_0 Q_0 \]
References


Figure 2(a)
Relaxation to Fermi-Dirac distribution
Distribution for Different Fluxes

Figure 2 (b)
Figure 3(a)
Distribution Function
Conventional Pumping
Figure 3(b)
OutPut Power, AU;

``Conventional'' Pumping
Figure 4(a)
Distribution Function
Figure 4(b)

OutPut Power, AU; FluxPumping