Matching the Heavy Vector Meson Theory

J. Bijnens\textsuperscript{a}, P. Gosdzinsky\textsuperscript{b} and P. Talavera\textsuperscript{c}

\textsuperscript{a}Department of Theoretical Physics, University of Lund

\textsuperscript{b}NORDITA, Blegdamsvej 17, DK-2100 Copenhagen Ø, Denmark.

\textsuperscript{c}Departament de Física i Enginyeria Nuclear
Universitat Politècnica de Catalunya, E-08034 Barcelona, Spain.


We show how to obtain a “heavy” meson effective lagrangian for the case where the number of heavy particles is not conserved. We apply the method in a simple example at tree level and perform then the reduction for the case of vector mesons in Chiral Perturbation Theory in a manifestly chiral invariant fashion. Some examples showing that “heavy” meson effective theory also works at the one–loop level are included.
I. INTRODUCTION

In a well-known paper [1] Isgur and Wise derived extra consequences of the heavy mass of a particle in a restricted class of processes. Their method was then quickly generalized [2] and extended for use in various sectors of Chiral Perturbation Theory [3,4]. The general class of processes dealt with in the latter references is with one “heavy” particle going in and moving through the whole process and going out again. Everything else has low momenta compared to the heavy particle. The common point is that the number of “heavy” particles remains constant during the process due to a conserved quantum number. This is not the case in all processes where we expect this type of expansion to be useful. A simple but naive example would be the process in which intermediate state contains particles of even higher masses. Those can be integrated out first and present no theoretical problem. We could, however, study the process \( W^+ b \rightarrow W^+ b \) in a theory with a light \( W \)-boson. Then we would have to consider the charm quark intermediate state. The charm quark here has a large momentum and the process should be expressible as a power series in \( 1/m_b \) in an effective lagrangian where the \( b \) quark always takes the large momentum.

A similar problem arises when we try to do Chiral Perturbation Theory for vector mesons. In reference [5] we conjectured that a similar formalism should exist there. (See [6] for other applications of this formalism). The main obstacle of dealing with this situation is that the methods used in [2]—or e.g. [7] for the extension to other spins—do not take all terms correctly into account in performing the reduction from the full theory to the “heavy” effective one.

In order to estimate the parameters at higher order, in our previous calculation [8] we performed the matching of the relativistic models with the “heavy meson” formulation in a diagrammatic fashion. The diagram by diagram formulation becomes difficult if we want to do the determination of all terms resulting from this reduction. In particular, chiral symmetry relates processes with different number of pions and it would therefore be useful to have a procedure that fully generates the correct terms immediately. For interactions among pions only when integrating out vector mesons this can be done using the equations of motion for the vector meson field [9,10]. The objective of this paper is to show how this can be done for the “heavy meson” effective theory. The main complication is that we have to correctly treat the diagrams with only light intermediate states, one of which then has to carry high momentum, and at the same time the contributions from the heavy state to subprocesses involving light momenta only. For the “light” and “heavy” states in our theory we thus have to keep two possible momenta regimes. One around the mass-shell of the “light” mode and the other around the mass-shell of the “heavy” mode. In the usual cases only the momentum regime where all particles were close to their mass-shell is kept.∗

Keeping track of several momentum regions of the particles can in principle be done by introducing several components in the field, each of which has only a low momentum:

\[
\phi_{\text{full}} = e^{-iMv\cdot x} \tilde{\phi}_v + \phi_0 + e^{iMv\cdot x} \tilde{\phi}_0^\dagger.
\]

Notice that effective fields are traditionally normalized differently. It corresponds to \( \tilde{\phi}_v \rightarrow \tilde{\phi}_v/\sqrt{2M} \). This factor should be understood in what follows. The component with the large (small) momentum we will refer to as the high (low) component in order not to confuse with light (heavy) for the mass.

We then integrate out for the heavy field the component \( \tilde{\phi}_0 \) and for the light field the component \( \tilde{\phi}_v \). This procedure works at tree level since there is always only one line carrying the heavy momentum through the whole diagram. We therefore also only need to keep at most two powers of the high components. The end result is then an effective theory formulated in terms of the high component of the heavy field and the low component of the light field. At any time, terms in the lagrangian which cannot possibly conserve momentum do not contribute since they vanish after the integration over all space.

The approach outlined above will obviously work in simple models. In the chiral models with vertices with any number of fields it becomes rather cumbersome. We will therefore choose a different method. By introducing a “hidden” symmetry we can introduce an extra spurious degree of freedom. The “gauge fixing” of this symmetry consists in choosing an equation for the extra degree of freedom in such a way that the spurious degree of freedom takes care of the far off-shell components of the fields.

We will present the main idea in three stages. First we show it in a simple model where this approach is identical to the one in Eq. (1) but shows the procedure in a simple way. Then we show the procedure in a model with an external scalar field coupling to pions in a chirally invariant way. Finally we present the case for vector mesons interacting with pions. Here we rederive and generalize the results of [8].

∗For the pions close to their mass-shell here means low-momentum.
We also show in a few examples that the “heavy” meson method works at the one–loop level and present a short discussion of the relevance of the width.

II. A SIMPLE MODEL

Consider the following lagrangian with a heavy \( \phi \) field and a light \( \pi \) field:

\[
L_0 = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi + \frac{1}{2} \partial_\mu \pi \partial^\mu \pi - \frac{1}{2} M^2 \phi^2 - \lambda M \phi \pi \pi
\]  

(2)

We will be interested in the limit, in which the \( \phi \) mesons gets very heavy, that is, \( M \to \infty \). We will consider that \( \lambda \) in this limit is a non-vanishing finite constant.

The “heavy” meson theory (HMT) now looks at processes of the type

\[
\phi + n \pi \rightarrow \phi + k \pi, 
\]

(3)

with the \( \phi \) on-shell and the momenta of the pions small compared to the \( \phi \)-mass \( M \). \( n \pi \rightarrow k \pi \) processes are relevant as well, they occur as subprocesses in (3). An important point is to notice that processes such as a decay of a \( \phi \) into two \( \pi \)'s, allowed by (2), do not lie within the heavy meson theory. The vertex \( \lambda M \phi \pi \pi \) can, however, generate Green functions that lie within the HMT. In figure 1c, for example, we have combined four of these vertices to construct a six–point function that is described within the HMT. This Green function cannot be generated in the same way in the heavy meson theory, and we will see how the HMT-vertices that accounts for the Green function can be generated.

The naive limit to the heavy meson theory would be to just write

\[
\phi = e^{-iMv \cdot x} \tilde{\phi} + e^{iMv \cdot x} \tilde{\phi}^\dagger
\]

(4)

and restrict \( \tilde{\phi} \) and \( \pi \) to momenta much smaller than \( Mv \). Here \( v \) is a four-velocity as introduced in [1], \( v^2 = 1 \). For \( L_0 \) this naive procedure would lead to a lagrangian without interaction terms \( \dagger \), an obviously wrong conclusion. The method where we also keep the particular far off-shell regions relevant for the processes of Eq. (3) as in Eq. (1) can be used here and gives the same results as a diagram by diagram matching.

A different way to achieve the same result is introducing first two extra symmetries \( R_1 \times R_2 \) with the extra fields \( \sigma \) and \( \psi \). The symmetry transformations are with \( \alpha_1 \in R_1 \) and \( \alpha_2 \in R_2 \):

\[
\phi \rightarrow \phi + \alpha_1, \quad \psi \rightarrow \psi - \alpha_1, \quad \pi \rightarrow \pi + \alpha_2, \quad \text{and} \quad \sigma \rightarrow \sigma - \alpha_2 .
\]

(5)

The lagrangian,

\[
L_1 = \frac{1}{2} \partial_\mu \Phi \partial^\mu \Phi + \frac{1}{2} \partial_\mu \Pi \partial^\mu \Pi - \frac{1}{2} M^2 \Phi^2 - M \lambda \Phi \Pi \Pi
\]

\[
\Phi = \phi + \psi, 
\]

\[
\Pi = \pi + \sigma,
\]

(6)

is equivalent to \( L_0 \) and has the extra symmetry. We can simply return to \( L_0 \) by choosing \( \psi = \sigma = 0 \). We use the freedom of gauge to have \( \psi \) take care of the low part of \( \phi \) and \( \sigma \) of the high part of \( \pi \). We choose:

\[
0 = -\partial^2 \psi - M^2 \psi - \lambda M \pi \pi - \lambda M \sigma 
\]

\[
0 = -\partial^2 \sigma - 2 \lambda M \phi \pi - 2 \lambda M \psi \sigma .
\]

(7)

Here it should be understood that inside the \( \sigma \sigma \) term only the “low momentum part” is taken, (i.e. the term behaving like: \( 2\tilde{\sigma} \tilde{\sigma}^\dagger \), see below). This choice now allows us to make the consistent set of approximations:

\[
\pi = \tilde{\pi} 
\]

\[
\psi = \tilde{\psi} 
\]

\[
\phi = e^{-iMv \cdot x} \tilde{\phi} + e^{iMv \cdot x} \tilde{\phi}^\dagger 
\]

\[
\sigma = e^{-iMv \cdot \tilde{\sigma}} + e^{iMv \cdot \tilde{\sigma}^\dagger} .
\]

(8)

\[^{\dagger}\text{The interaction term always vanishes in this approximation because of momentum conservation.}\]
This is consistent since the equations of motion are now such that all terms in the equations that contribute to the processes (3) are correctly taken care of. The problem before was that, if we used Eq. (4) only, there were “driving” terms in the equations that had the wrong momentum\(^1\) that had nowhere to go. The choice of gauge for the spurious fields of (7) solves this problem. All the tilded fields in (8) are “low momentum fields”, that is, their Fourier transforms are only non-vanishing for momenta \(p \sim 0\). It is this fact that will allow us to perform a consistent and well defined expansion in \(1/M\).

The “heavy” lagrangian should only depend on the fields close to their mass–shell. We therefore integrate out \(\tilde{\psi}, \tilde{\sigma}\) and \(\tilde{\sigma}^\dagger\). At tree level, this can be done by choosing the gauge and replacing the solution in the lagrangian. We have solved (7) by performing an expansion in \(1/M\). Up to order \(O(1/M^3)\), we find:

\[
\tilde{\psi} = -\frac{\lambda}{M} \pi^2 + \frac{\lambda}{M^3} \Box \pi^2 - 8 \frac{\lambda^3}{M^3} \pi^2 \tilde{\phi} \tilde{\phi}^\dagger \tag{9}
\]

\[
\tilde{\sigma} = 2 \frac{\lambda}{M} \pi \tilde{\phi} - 4i \frac{\lambda}{M^2} \nu \cdot \partial (\pi \tilde{\phi}) + \frac{\lambda}{M^3} \left( 2 \Box (\pi \tilde{\phi}) - 8 (\nu \cdot \partial)^2 (\pi \tilde{\phi}) - 4 \lambda^2 \pi^3 \right) \tag{10}
\]

To go to the nonrelativistic limit, we introduce the correct normalization,

\[
\tilde{\phi} = \frac{1}{\sqrt{2M}} \tilde{\phi}, \quad \tilde{\phi}^\dagger = \frac{1}{\sqrt{2M}} \tilde{\phi}^\dagger. \tag{11}
\]

Replacing (8), (9) and (10) in (6), and expanding up to order \(O(1/M^2)\), we find:

\[
\mathcal{L}_1^E = \frac{1}{2} \partial_\mu \tilde{\pi}^\mu + \frac{1}{2M} \partial_\mu \tilde{\phi} \partial^\mu \tilde{\phi} + i \frac{1}{2} (\tilde{\phi}^\dagger \nu \cdot \partial \tilde{\phi} - \tilde{\phi} v \cdot \partial \tilde{\phi}^\dagger) + \frac{\lambda^2}{2} \tilde{\pi}^2 - \frac{\lambda^2}{2M} \pi^2 \tilde{\phi} \tilde{\phi}^\dagger + 2i \frac{\lambda^2}{M^2} \left( \tilde{\phi}^\dagger \nu \cdot \partial (\tilde{\pi} \tilde{\phi}) - \tilde{\phi} \nu \cdot \partial (\tilde{\pi} \tilde{\phi}^\dagger) \right) + \frac{\lambda^2}{M^3} \left( \frac{1}{2} \tilde{\pi}^2 \pi^2 + \frac{1}{M} \left( 4 \lambda^2 \pi^4 \tilde{\phi} \tilde{\phi}^\dagger \right) \right) \tag{11}
\]

In (11) we have kept terms which apparently are of order \(O(1/M^3)\). The reason is that for Green functions involving \(\tilde{\phi} \tilde{\phi}^\dagger\), an extra factor \(2M\) has to be introduced, (10). Clearly, we have generated HMT vertices for the Green functions discussed at the beginning of this section. Two four–point vertices are not suppressed in the limit \(M \to \infty\). In fact, they are the only non-suppressed Green functions in this limit. The four pion function has been obtained by integrating out the low component, \(\psi\), of the heavy particle, \(\phi\). This is the same attitude followed for example in [9], where the vector mesons where integrated out to obtain their contributions to the pseudo-scalar interactions at low-energy. The other four–point Green function, \(\tilde{\phi} \tilde{\phi}^\dagger \pi \pi\), has been obtained by integrating out the high components, \(\tilde{\sigma}\) and \(\tilde{\sigma}^\dagger\) of the light particle, the \(\pi\). To our knowledge, this procedure was used for the first time in [8], there we used a diagrammatic approach.

We also find a six point function. It is suppressed by a factor \(1/M^2\). We have checked that the four and six point functions generated by (11) reproduce exactly, up to order \(O(1/M^2)\) the corresponding Green functions of the full relativistic theory, (2).

\(^1\)The other terms can be neglected since at tree level we can never have intermediate lines with momenta far away from either 0 or \(Mv\).
Let us have a closer look at the situation of the six point functions in general, see Fig. 1. The first three diagrams, \(a\), \(b\) and \(c\) are diagrams in the full theory, (2), while diagram \(d\), \(e\) and \(f\) live in the effective theory.

Cutting the internal heavy meson of diagram \(a\), we obtain two four point functions that lie within the HMT. The HMT can describe these two four–point functions, (11). The HMT counterpart for diagram \(a\) is diagram \(d\). The situation is very similar for diagram \(b\): here, we again obtain two four point functions that are described within the HMT by cutting one of the internal pion line. The fact that one of these is a four pion function shows why at the beginning of the section we claimed the processes \(n\pi \rightarrow k\pi\) also have to be taken into account. The HMT counterpart of \(b\) is \(e\). The situation is however different for diagram \(c\). Here no HMT diagram can be generated by cutting any of the internal lines. In the HMT, we can not construct its counterpart from “smaller” diagrams. A new vertex, the six point vertex in (11) has been generated to account for it. The HMT counterpart for \(c\) is \(f\).

This can be generalized, and if we continue our expansions of \(\tilde{\psi}, \tilde{\sigma}\), (9) and \(\mathcal{L}_1^E\), (11) to higher orders in \(O(1/M)\), we would generate new eight, ten, and higher order vertices.

III. ONE LOOP MATCHING IN THE SIMPLE MODEL

We will now illustrate with a few examples that the effective lagrangian we have constructed in the previous section, which does reproduce correctly the tree level Green functions of the full relativistic theory, also reproduces correctly the non-analytic parts of the Green functions at the one loop level. The first non-trivial results in that model show up in four–point functions. We add a pion mass\(^\S\) and a coupling of pions to an external scalar field \(S(x)\) to the model. Then we have nontrivial behaviour already in two– and three–point functions. The Lagrangian becomes

\[
\mathcal{L} = \mathcal{L}_0 + \frac{1}{2} m^2 \pi^2 - S(x) \pi^2.
\]  

(12)

The first example will be the one–loop self-energy of the \(\phi\), at leading order in \(1/M\). In the effective theory (HMT), its contribution is given by

\[
\Pi^E_\phi = -4i\lambda^2 \frac{m^2}{16\pi^2} \left( -\frac{1}{\epsilon} + \gamma_e - \log 4\pi - \log \frac{m^2}{\mu^2} \right) + O(1/M^2)
\]

(13)

Where we have followed the standard notation and used dimensional regularization with \(D = 4 - 2\epsilon\) and \(\gamma_e\) is Euler’s constant.

\(^\S\)For the quantities and to the order considered here the HMT Lagrangian is identical to the one of Sect. II.
It is also easy to obtain the result in the full relativistic theory:

\[
\Pi_\phi^F = 2\lambda^2 M^2 \frac{i}{16\pi^2} \left( \frac{1}{\epsilon} + \log 4\pi - \gamma_e - \log \frac{M^2}{\mu^2} \right) \\
+ 4\lambda^2 m^2 \frac{i}{16\pi^2} \left( 1 - \log \frac{m^2}{\mu^2} + \log \frac{M^2}{\mu^2} \right) + O(1/M^2)
\]

(14)

Let us make a few observations here:

1. The nonanalytic terms we are interested in are the logarithms of \( m^2 \). It follows from (13) and (14) that the effective theory does reproduce this. This is the main point.

2. The analytic dependence on \( m^2 \) gets a correction at one–loop compared to the tree–level one. This correction is even infinite in the \( M \to \infty \) limit. This dependence cannot be obtained from the HMT. We have to leave a term proportional to \( m^2 \) for the \( \phi \) mass in the HMT.

3. It is interesting to notice that in the full theory, \( \Pi_\phi^F \) has an imaginary part, which is absent in the effective theory. This is due to the fact that in the full theory the \( \phi \) has a finite width due to the decay \( \phi \to 2\pi \). Since this decay cannot be described within the heavy meson theory, no imaginary part is present in \( \Pi_\phi^E \). A small discussion about this fact can be found in Sect. VII.

As a second example we consider the scalar form–factor of the \( \phi \), with the scalar source \( S(x) \) defined above. To leading order in \( 1/M \), the only diagram that contributes in the effective theory is the one of Fig. 2b. It contributes

\[
8i\lambda^2 \frac{1}{(4\pi)^2} \left[ \frac{1}{\epsilon} - \gamma_e + \log(4\pi) - \int_0^1 \log \left( \frac{m^2 - q^2 x(1-x) - i\epsilon}{\mu^2} \right) dx \right]
\]

where \( q \) is the momentum that flows through the scalar source.

In the full theory, we have to consider only the diagram of Fig. 2a at this order.

As in the previous case, \( q \) is the momentum that enters through the source \( S(x) \), and \( Q^2 = (Q-q)^2 = M^2 \). We are only interested in the leading order in \( 1/M \). The contributions away from \( y \approx 0 \) and \( y \approx 1 \) can always be expanded in \( 1/M^2 \), while we expand in \( (1-y) \) and \( y \) near 1 and 0 respectively.

\[
I \approx 8i\lambda^2 M^2 \frac{1}{(4\pi)^2} \int_0^1 dx \left\{ \int_0^{1-\delta} \frac{dy}{-m^2 + (1-y)Q^2 + x(1-x)q^2 + i\epsilon} + \int_{1-\delta}^{1} \frac{dy}{-m^2 + yQ^2 + i\epsilon} \right\}
\]

As in the previous case, \( q \) is the momentum that enters through the source \( S(x) \), and \( Q \) is the momentum of the heavy field, \( \phi \), \( Q^2 = (Q-q)^2 = M^2 \). We are only interested in the leading order in \( 1/M \). The contributions away from \( y \approx 0 \) and \( y \approx 1 \) can always be expanded in \( 1/M^2 \), while we expand in \( (1-y) \) and \( y \) near 1 and 0 respectively.

\[
I \approx 8i\lambda^2 M^2 \frac{1}{(4\pi)^2} \int_0^1 dx \left\{ \int_0^{1-\delta} \frac{dy}{-m^2 + (1-y)Q^2 + x(1-x)q^2 + i\epsilon} + \int_{1-\delta}^{1} \frac{dy}{-m^2 + yQ^2 + i\epsilon} \right\}
\]

As in the previous case, \( q \) is the momentum that enters through the source \( S(x) \), and \( Q \) is the momentum of the heavy field, \( \phi \), \( Q^2 = (Q-q)^2 = M^2 \). We are only interested in the leading order in \( 1/M \). The contributions away from \( y \approx 0 \) and \( y \approx 1 \) can always be expanded in \( 1/M^2 \), while we expand in \( (1-y) \) and \( y \) near 1 and 0 respectively.

\[
I \approx 8i\lambda^2 M^2 \frac{1}{(4\pi)^2} \int_0^1 dx \left\{ \int_0^{1-\delta} \frac{dy}{-m^2 + (1-y)Q^2 + x(1-x)q^2 + i\epsilon} + \int_{1-\delta}^{1} \frac{dy}{-m^2 + yQ^2 + i\epsilon} \right\}
\]

As in the previous case, \( q \) is the momentum that enters through the source \( S(x) \), and \( Q \) is the momentum of the heavy field, \( \phi \), \( Q^2 = (Q-q)^2 = M^2 \). We are only interested in the leading order in \( 1/M \). The contributions away from \( y \approx 0 \) and \( y \approx 1 \) can always be expanded in \( 1/M^2 \), while we expand in \( (1-y) \) and \( y \) near 1 and 0 respectively.

\[
I \approx 8i\lambda^2 M^2 \frac{1}{(4\pi)^2} \int_0^1 dx \left\{ \int_0^{1-\delta} \frac{dy}{-m^2 + (1-y)Q^2 + x(1-x)q^2 + i\epsilon} + \int_{1-\delta}^{1} \frac{dy}{-m^2 + yQ^2 + i\epsilon} \right\}
\]

As in the previous case, \( q \) is the momentum that enters through the source \( S(x) \), and \( Q \) is the momentum of the heavy field, \( \phi \), \( Q^2 = (Q-q)^2 = M^2 \). We are only interested in the leading order in \( 1/M \). The contributions away from \( y \approx 0 \) and \( y \approx 1 \) can always be expanded in \( 1/M^2 \), while we expand in \( (1-y) \) and \( y \) near 1 and 0 respectively.
where $\delta$ and $\alpha$ are small numbers, but much larger than $\max(m^2, |q^2|)/M^2$. After integrating over $y$ we can again make the same three remarks made above:

1. The nonanalytic term in $m$ and $q$ are exactly the same as in the integral of (15).
2. The argument of the logarithm is $M^2$. To obtain $\mu^2$ we have to add a term proportional to $\log(M^2/\mu^2)$. This term requires us to introduce a direct $S(x)\tilde{\phi}\tilde{\phi}$ coupling in the HMT to obtain matching.
3. The imaginary part only coincides provided the part corresponding to the $\phi$ width can be neglected. It is equal to $-\lambda^2 M^2/(2\pi) \left\{ \theta(1 - 4m^2/q^2) \sqrt{1 - 4m^2/q^2} - 1 \right\}$. The second term is the imaginary part due to the $\phi$-width and the first part is the long-distance part present in the effective theory (15).

IV. A CHIRALLY SYMMETRIC MODEL

The QCD lagrangian has an approximate chiral symmetry $SU(N_f)_{L} \times SU(N_f)_{R}$ for $N_f$ light flavours. This implies the existence of light pseudo-goldstone Bosons, which we will refer to as pions. The method of dealing with these at low energy is Chiral Perturbation Theory [11]. In (2) there were only vertices with a limited number of legs. Effective lagrangians with pions tend to have vertices with any number of legs. The method illustrated on the simple example becomes much more powerful here. In this section we will introduce a pion lagrangian coupling to an external scalar source $S(x)$ and we will show how to obtain an effective lagrangian with terms bilinear in $S(x)$ to reproduce all Green functions of the type $S(\tau - p)S(p + k)\pi^n$ with $p$ a large momentum and $k$ and the $\pi$-momenta small. Again we will restrict our discussion to tree level.

Consider the following lagrangian:

$$L_2 = \frac{F^2}{4} \left\langle u_\mu u^\mu + S(x)\chi_+ \right\rangle. \tag{18}$$

Here we used the following notation

$$\chi_{\pm} = u_R^\dagger \chi u_L \pm u_L^\dagger \chi u_R$$
$$u_\mu = i \left( u_R^\dagger (\partial_\mu - ir_\mu) u_R - u_L^\dagger (\partial_\mu - il_\mu) u_L \right)$$
$$\Gamma_\mu = \frac{1}{2} \left( u_R^\dagger (\partial_\mu - ir_\mu) u_R + u_L^\dagger (\partial_\mu - il_\mu) u_L \right) \tag{19}$$

and $(C)$ denotes the trace of $C$. The external fields $\chi, l_\mu, r_\mu$ are defined in the usual way, see [9]. $\chi$ contains the quark masses. All of these are $N_f$ by $N_f$ matrices in flavour space. The matrices $u_R$ and $u_L$ are given in terms of the pions as

$$u_R = u_L^\dagger = u = \exp \left( \frac{i}{F} \pi^a \chi^a \right). \tag{20}$$

The transformations under a chiral transformation $g_L \times g_R \in SU(N_f)_{L} \times SU(N_f)_{R}$ are given by

$$u_R \to g_R u_R h_\mu^l \quad u_L \to \frac{g_L u_L h_\mu^r}{h_\mu^l} \quad u_\mu \to h_\mu^s u_\mu h_\mu^l$$
and $S(x) \to h_\ell S(x)h_\ell^\dagger$. \tag{21}

This together with (20) determines $h_\ell$ in terms of the pion fields $\pi^a$ and $g_L, g_R$.

We now set

$$S(x) = e^{-iMv \cdot \tilde{S}(x)} + e^{iMv \cdot \tilde{S}(x)} \tag{22}$$

where $\tilde{S}(x)$ only has momenta small compared to $Mv$ and look at Green functions with one insertion of $\tilde{S}(x)$ and one of $\tilde{S}(x)^\dagger$. These have contributions with pions of momenta of order $Mv$. We treat these pions in the same way as in the previous section by introducing a spurious field that takes its role. The difficulty here is to do it in a manifestly chiral invariant fashion.

This can be done by introducing a “hidden” $SU(N_f)_3$ symmetry that is rather nonlinearly realized. We first introduce three $N_f$-by-$N_f$ special unitary matrices, $u_R, u_L$ and $W = w^2$ transforming under $(g_L \times g_R \times h_1 \times h_2) \in SU(N_f)_L \times SU(N_f)_R \times SU(N_f)_1 \times SU(N_f)_2$ as:
and we define \( h_3 \) via

\[
\mathcal{L}_2 = \frac{F^2}{4} \left[ (u_w^t u_w^\mu) + \frac{1}{8M^2} \left( \langle \{ \bar{S}(x), \chi_+ \} \{ \bar{S}(x)^\dagger, \chi_- \} - \frac{1}{N_f} \langle \{ \bar{S}(x), \chi_+ \} \{ \bar{S}(x)^\dagger, \chi_- \} \rangle \right) \right]
\]
We will now apply our method to the vector meson case. Here the situation is a little bit more involved because of the structure of the heavy meson propagator in the nonrelativistic limit, and we will construct the effective theory in two steps: First, we integrate out the non-relevant degrees of freedom, that is, the low component of the vector mesons and the high components of the pseudoscalars. Then, project the theory on the “orthogonal subspace” (see section IV of [8]), extract the large momentum proportional to $M_V$ from the vectors and construct an effective lagrangian that allows to perform a consistent expansion in $1/M_V$ We again restrict ourselves to the processes of the type:

$$\text{Vector} + n\pi \longrightarrow \text{Vector} + k\pi.$$  (32)

There exist several models that describe the interaction of vector mesons with the pseudoscalar Goldstone bosons, see [12] and references therein. We will present our result in terms of model III of that reference (see below for a short discussion about other models).

Then the lagrangian reads:

$$\mathcal{L}_3 = \frac{F^2}{4} (u_\mu u^\mu) - \frac{1}{4} (\nabla_\mu \nabla'^{\mu}) + \frac{M^2}{2} (\nabla_\mu - \frac{i}{g} \Gamma_\mu)^2.$$  (33)

With $\nabla_{\mu\nu} = \partial_\mu \nabla - \partial_\nu \nabla - ig \left[ \nabla_\mu, \nabla_\nu \right]$. We have disregarded the terms containing quark masses for the present discussion, because they are $O(1/M)$ suppressed. The pseudoscalar fields transform as in (20), and the vector fields transform as:

$$\nabla_\mu \rightarrow h_c \nabla_\mu h^\dagger_c + \frac{i}{g} h_c \partial_\mu h^\dagger_c.$$  (34)

where $h_c$ is defined in (21).

We now enlarge the symmetry for the pseudoscalar sector in the same way as we did in the previous section to obtain an extra degree of freedom, $u_R = u_L^\dagger = u w$, and a $\mathbb{R}^4$ hidden symmetry to introduce a spurious vector degree of freedom (corresponding to the $\psi$ field in Sect. II). We have as a total set of transformations:

$$\alpha_\mu \in \mathbb{R}^4 : W_\mu \rightarrow W_\mu + \alpha_\mu$$

$$X_\mu \rightarrow X_\mu - \alpha_\mu$$

$$h_3 : W_\mu \rightarrow h_3 W_\mu h^\dagger_3$$

$$X_\mu \rightarrow h_3 X_\mu h^\dagger_3 + \frac{i}{g} h_3 \partial_\mu h^\dagger_3.$$  (36)

We now use $\nabla_\mu = W_\mu + X_\mu$ together with (19) and (25) in (33), with $h_3$ defined as in Sect. IV. The model is then exactly equivalent to the original model (33) setting $X_\mu = 0$ and $\xi = 0$.

Notice that the non-linear term in (36) has been chosen inside $X_\mu$, the reason is that later we will choose the $\mathbb{R}^4$ gauge such that this field becomes the “low” momentum component of the vector. It therefore has to take the low momentum nonlinear term. In principle, for the vectors we would have to use the general parametrization (1), and we would have to integrate its low component out.

The equation of motion for the pseudoscalar fields reads:

$$\nabla_\mu u^\mu - \frac{i M^2}{2 g F^2} \left[ \nabla_\mu - \frac{i}{g} \Gamma_\mu, u^\mu \right] = 0.$$  (37)

and the vector equation of motion:

$$\partial_\mu \nabla_{\mu\nu} - ig [\nabla', \nabla_{\mu\nu}] + M^2 (\nabla_\nu - \frac{i}{g} \Gamma_\nu) = 0.$$  (38)

We now expand these equations up to second order in the $W$ and $\xi$ fields**. Choosing the gauge fixing conditions such that for the processes (32) $X_\mu$ only has “low” momentum and $\xi$ only “high” momentum:

**This is all we need for the processes (32) at tree level, when later $W$ and $\xi$ become the “high” momentum components.
\[-(\nabla^l \mu)^2 \xi - \frac{1}{4} [u^l \mu, u^l \nu] \]
\[-\frac{iM^2}{2g^2} \left( [W_\mu + \frac{i}{4g} [\xi, u^l \mu], u^l \mu] - \left[ X_\mu - \frac{i}{g} \Gamma^l \mu \nabla^l \mu \xi \right] \right) + O((\xi, W)^3) = 0, \tag{39} \]

for the pseudoscalars and

\[
\nabla^X X_{\mu \nu} - ig \nabla^X [W_\mu, W_\nu] - ig [W_\mu, \nabla^X \nu W_\nu - \nabla^X W^\nu] + M^2 \left( X_\mu - \frac{i}{g} \Gamma^l \mu - \frac{i}{8g} [\xi, \nabla^l \xi] \right) + O((\xi, W)^4) = 0, \tag{40} \]

for the vectors. The superscripts $l$ denotes quantities with $w = 1$ as in Sect. IV and we have defined the following covariant derivative:

\[
\nabla^X \mu A = \partial_\mu A - ig [X_\mu, A]. \tag{41} \]

This choice now allows us to make the *consistent* set of approximations

\[
\sqrt{2M} W_\mu = e^{-iM \nu x} \tilde{W}_\mu + e^{iM \nu x} \tilde{W}_\mu^\dagger \quad \text{and} \quad X_\mu = \tilde{X}_\mu \tag{42} \]

together with those of (29). Again, tilded symbols means that they are restricted to low momenta.

Solving iteratively the gauge conditions we find the following solutions for the spurious fields, using $g = M/(2F)$ [12] to count $g$ as $O(M)$:

\[
\xi = \frac{i}{2gF^2} [W_\mu, u^l \mu] + \ldots \\
\tilde{X}_\mu = \frac{i}{g} \Gamma^l \mu + \ldots. \tag{43} \]

Here we only keep the leading terms contributing to (32), and set the external sources $l_\mu$ and $r_\mu$ to zero.

Putting these solutions inside the lagrangian (33), after some algebra, we obtain

\[
\mathcal{L}_3^E = -\frac{1}{2} (\nabla^l \mu W_\nu \nabla^l \mu W^\nu) + \frac{1}{2} (\nabla^l \mu W^\mu \nabla_\nu W^\nu) + \frac{F^2}{4} (u^l \mu u^l \nu) + \frac{1}{4} (W_\mu, u^l \mu) \]
\[-\frac{1}{4} ([u^l \mu, u^l \nu] [W^\mu, W^\nu]) + \frac{M^2}{2} (W_\mu W^\mu) + \ldots. \tag{44} \]

The dots in (44) denote terms with zero or more than two vectors, as well as terms suppressed in the $1/M$ counting. The term $(1/4)(W_\mu u^l \mu)^2$ is generated by the high component of the pions, $\xi$, while the the rest of the terms (excluding the mass term) are generated by the low component of the vectors.

This last step to be taken is to go from (44) to an effective theory which allows us to perform a $1/M$ expansion. The presence of terms involving $\partial_\mu W^\mu$ in (44) makes this last step a little bit more subtle, and we cannot simply replace (42) in (44). We will follow the method of section 4 of [8] and introduce a parallel component, $\tilde{W}_\mu^\parallel = \nu^\mu (\nu \cdot \tilde{W})$, and a perpendicular component $\tilde{W}_\mu^\perp = \tilde{W}_\mu - \nu^\mu (\nu \cdot \tilde{W})$, and integrate the parallel component out, see [8] for details. To leading order in $1/M$, and in the notation of [8] (Eq.(13) of that paper), this leads to

\[
a_2 = -\frac{1}{2M}, \quad a_3 = 0, \quad a_7 = -\frac{1}{4M}, \quad a_8 = 0, \quad a_9 = \frac{1}{4M}. \tag{45} \]

We have checked that these terms can be reproduced *exactly* diagrammatically. The fact that $a_3$ vanishes in (45) might seem surprising, since in (44) the term

\[
\frac{1}{2} (\nabla^l \mu W^\mu \nabla_\nu W^\nu) \tag{46} \]

is present. To understand $a_3 = 0$, we first notice that (46) does not contribute to pion-vector scattering, since for on-shell vectors, $\partial_\mu W^\mu = 0$. For processes such as $2\pi V \rightarrow 2\pi V$, the contribution of (46) to the diagram of Fig. 1f does not vanish. To leading order in the $1/M$ expansion, this is cancelled by the contribution of (46) to the diagram of Fig. 1d. Contributions where only one of the vertices of Fig. 1d is generated by (46) are suppressed in $1/M$. We
VI. VECTORS AT ONE LOOP

Contributions to the vector masses in the relativistic and the heavy meson formulation are given by the diagrams in Fig. 2 with the crosses removed. We introduce here a pion mass term, $m = m_\pi$, to have relevant nonanalytic contributions and only consider the pion contributions. In the effective formulation we obtain a contribution to the $\rho$ mass-shift

$$\delta M = -\frac{m^4}{32\pi^2 M^2} \log \left( \frac{m^2}{\mu^2} \right) .$$

In the relativistic formulation we obtain

$$\delta(M^2) = \frac{M^4}{2g^2 F^4} B_{20}(M^2, m^2) .$$

with

$$\int \frac{d^4 p}{i(2\pi)^4} \frac{p_\mu p_\nu}{(p^2 - m^2)((p + Q)^2 - m^2)} = g_{\mu\nu} B_{20}(Q^2, m^2) + Q_\mu Q_\nu B_{22}(Q^2, m^2) .$$

We can now check by expanding $B_{20}(M^2, m^2)$ that the first nonanalytic dependence on $m^2$ only appears at order $m^4/M^2 \log(m^2/M^2)$ and that the coefficients agree if we use the value of $g$ used above. So here we have an indication that the procedure of the heavy meson theory also works in this case at one loop.

VII. WIDTH

The width of the “heavy” particle due to its decays is not included in the HMT. Here we discuss shortly under what circumstances we expect the HMT to be useful given that it cannot easily describe the width. In the tree level processes in the relativistic theory we can describe the width of the heavy particle by using as propagator instead

$$\frac{1}{p^2 - M^2 + i\Gamma} \approx \frac{1}{M} \frac{1}{2v \cdot k + i\Gamma} \approx \frac{1}{2M} \frac{1}{v \cdot k} \left\{ 1 - \frac{i\Gamma}{2v \cdot k} + \cdots \right\} .$$

Here we see that if the typical off-shellness of the “heavy” particle is large compared to its width the latter can be neglected. For vector mesons at tree–level this will always be the case except for the $\rho$. But even there we are helped by the extra factor of two in the expansion in (50).

In the loop diagrams a similar argument will hold if the contributions of the integrals very near to the mass-shell is small compared to the others. Again for vector mesons we expect this to be the case except possibly for the $\rho$. Even for the $\rho$, the dependence on the strange quark mass and similar effects come from intermediate states that are far off-shell so those should be reliably estimated in the HMT.

VIII. CONCLUSIONS

The problem of reducing a relativistic theory to a “heavy” effective formulation restricted to a particular type of processes has been solved in the case where the number of “heavy” particles is not conserved. The problem of “high” components of light particles and “low” components of heavy particles has been treated in a natural way. We have enlarged the symmetry by a “hidden” symmetry. The spurious degrees of freedom thus introduced, can be chosen by
a particular choice of “gauge” for the extra symmetry to play the role of the “high” components of the light particle and of the “low” components of the heavy particle.

The choice of gauge allows then for a simple reduction to the “heavy” effective theory. We explicitly matched all relevant Green functions at tree level, first in a toy model and afterwards in two Chiral models, showing that matching already at this level is rather subtle. In the last two examples our method allows to have chiral symmetry explicit during all stages of the calculation. The calculation for the model for the vector mesons agrees with our previous matching procedure [8] which was done by matching specific Green functions.

We have shown that with this procedure the non-analytical parts at one-loop level are also recovered in a few examples. We included both a two– and a three–point function in the simple model and a two–point function in the vector chiral case. These examples provide support that the “heavy” meson theory will also be correct at the quantum level.

We discussed shortly the relevance of the nonzero width of the “heavy” particle.

ACKNOWLEDGMENTS

PG acknowledges a grant form the Spanish Ministry for Education and Culture.
PG and PT thank the hospitality of the Theoretical Physics Department of Lund University, where part of this work has been carried out.

   Phys. B.