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Anti-Self-Dual equation
on 4-manifolds
with degenerate metric

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Juin 1997

IHES/M/97/51

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§ 1 Introduction

In this paper, we study Anti-Self-Dual equation on a product of 2 disk and a Riemann surface with degenerate metric. Our main result is a removable singularity theorem of Equation (1.1) below and a compactness theorem of its moduli space. As we will soon discuss, Equation (1.1) is a mixture of Anti-Self-Dual equation and (pseudo) holomorphic curve equation. In the case of Anti-Self-Dual equation, removable singularity theorem was established by Uhlenbeck [18] and in the case of pseudo holomorphic curve equation removable singularity theorem was established by Gromov [14] using the idea of [16]. One of our main results (Theorem 1.5) is a removable singularity theorem at a point where equation changes from Anti-Self-Dual equation to pseudo holomorphic curve equation. (Therefore, we can apply neither Uhlenbeck's nor Gromov's theorem at that point directly.) The results proved in this paper play a key role to study the moduli space of the solution of the Equation (1.1). We study the moduli space elsewhere. This moduli space will be used to define and study Floer homology of 3-manifolds with boundary, ([9], [7], [10], [12]).

The equation we study is :

$$(1.1.1) \quad \frac{\partial A}{\partial t} - d_A \Psi - *_\Sigma \left(\frac{\partial A}{\partial s} - d_A \Phi \right) = 0$$

$$(1.1.2) \quad \chi(s,t)^2 \left(\frac{\partial \Psi}{\partial s} - \frac{\partial \Phi}{\partial t} + [\Phi, \Psi] \right) + *_\Sigma F_A = 0$$

Let us explain the notation. We consider the direct product $D \times \Sigma$ where $D \subset \mathbf{C}$ is a domain and Σ is a Riemann surface. We divide D into a disjoint union $D = D_1 \cup D_2 \cup L$ where $L = \partial D_1 \cap D = \partial D_2 \cap D$ is a smooth curve and D_1, D_2 are domains. We take a smooth function $\chi : D \rightarrow [0,1]$ such that

$$(1.2.1) \quad \chi(s,t) > 0 \text{ if } (s,t) \in D_1.$$

$$(1.2.2) \quad \chi(s,t) = 0 \text{ if } (s,t) \in D_2.$$

(We specify the choice of χ later in § 4.) We consider an $SO(3)$ vector bundle $E(\Sigma)$ on Σ such that $w^1(E) = [\Sigma]$. (Such a bundle is unique.) It induces an $SO(3)$ bundle on $D \times \Sigma$ which we denote by $E(D \times \Sigma)$.

An $SO(3)$ connection $\mathcal{A}$ of $E(D \times \Sigma)$ is written as

$$(1.3) \quad \mathcal{A} = A + \Phi ds + \Psi dt.$$

Here s, t is a coordinate of D such that $s + \sqrt{-1}t$ is the standard complex coordinate of D , and $A = A(s,t)$ is a two parameter family of $SO(3)$ connections of $E(\Sigma)$. Φ, Ψ are sections of the $so(3)$ bundle induced by adjoint representation. (We discuss it a bit more detail in § 2.) Let F_A denote the curvature of A . We take and fix a Riemannian metric g_Σ on Σ , which induces a Hodge $*$ operator $*_\Sigma$. We put

$$(1.4) \quad \mathcal{E}(\mathcal{A}) = 2\pi^2 \int_{D^2 \times \Sigma} p^1(\mathcal{A}).$$

Here $p^1(\mathcal{A})$ is the first Pontryagin form. We remark that $\mathcal{E}(\mathcal{A}) \geq 0$ if $\mathcal{A}$ satisfies (1.1). (See § 2.) Our removable singularity theorem is :

Theorem 1.5 *Let $(0,0) \in L$, and let $\mathcal{A} = A + \Phi ds + \Psi dt$ be a smooth connection of $E(D \times \Sigma)$ defined on $(D - \{(0,0)\}) \times \Sigma$ such that $\mathcal{E}(\mathcal{A})$ is finite. Then there exists a gauge transformation g defined on $(D - \{(0,0)\}) \times \Sigma$ such that $g^* \mathcal{A}$ is extended to a smooth connection on $D \times \Sigma$.*

See § 2 for the definition of gauge transformation. We consider a singular Riemannian metric $\mathbf{g} = ds^2 \oplus dt^2 \oplus \chi(s,t)^2 g_\Sigma$ on $D \times \Sigma$. On $D_1 \times \Sigma$, our metric $\mathbf{g}$ is nondegenerate, and our Equation (1.1) is Anti-Self-Dual (hereafter abbreviated by ASD) equation. (See [5] and § 2-3.)

On the other hand, on $D_2 \times \Sigma$, where $\mathbf{g}$ is singular, we may regard Equation (1.1) as a pseudo holomorphic curve equation to maps from D_2 to the space of flat connections of $E(\Sigma)$, as we will explain later. Hence, we can apply either Uhlenbeck's or Sacks-Uhlenbeck-Gromov's removable singularity theorem to the solution of (1.1) except at the points on $L \times \Sigma$. And that is the case handled in Theorem 1.5.

We next state our compactness theorem.

Theorem 1.6 *Let $\mathcal{A}_i$ be a sequence of smooth connections of $E(D \times \Sigma)$ such that $\mathcal{E}(\mathcal{A}_i)$ is uniformly bounded. Then there exists a finite subset P_1 of $D_1 \times \Sigma$, a finite subset P_2 of $L \cup D_2$ and a sequence of gauge transformations g_i defined on $((D_1 \times \Sigma) - P_1) \cup ((L \cup D_2) - P_2) \times \Sigma$ such that a subsequence of $g_i^* \mathcal{A}_i$ converges in C^∞ topology on any compact subset of $((D_1 \times \Sigma) - P_1) \cup ((L \cup D_2) - P_2) \times \Sigma$.*

We remark that Theorem 1.6 is again Uhlenbeck's and Gromov's results plus some more work on the singular line $L \times \Sigma$.

We will prove one more result which implies that there is actually some fixed amount of mass lost at $(\hat{s}, \hat{t}) \times \Sigma$ for each $(\hat{s}, \hat{t}) \in P_2 \cap L$. We take a neighborhood U of $(\hat{s}, \hat{t})$ so that $g_i^* \mathcal{A}_i$ converges on any compact subset of $(U - \{(\hat{s}, \hat{t})\}) \times \Sigma$ in C^∞ topology. Let $\mathcal{A}_\infty$ be the limit.

Theorem 1.7 *If $\mathcal{E}(F_{\mathcal{A}_\infty}|_{U \times \Sigma}) > \liminf \mathcal{E}(F_{\mathcal{A}_i}|_{U \times \Sigma}) - 4\pi^2$, then we can take a sequence of gauge transformations g'_i on $U \times \Sigma$ such that a subsequence of $g_i'^* \mathcal{A}_i$ converges in C^∞ topology on $U \times \Sigma$.*

Outline of the proof of the above results are as follows.

One important observation (which the author learned from T. Yoshida's paper [19]) is that Equation (1.1.1) is invariant of the complex gauge transformation. (We review complex gauge transformation and prove this fact in § 2). The proof of Theorem 1.5 then proceeds as follows.

Let $R(\Sigma; E)$ be the space of gauge equivalence classes of flat connections of $E(\Sigma)$. It is a symplectic manifold (Goldman [13]) and the metric g_Σ determines a complex structure on $R(\Sigma; E)$ such that the symplectic form is the Kähler form. (We review these facts in § 2.) Equation (1.1.2) implies that for $(s, t) \in D$ the connection $A(s, t)$ is a flat connection. Hence it defines a map $\varphi : D_2 \rightarrow R(\Sigma; E)$. Equation (1.1.1) then means that φ is a holomorphic map.

We recall that Narasimhan-Seshadri's theorem [15], [4] implies that if the curvature of $A(s, t)$ is small then it is complex gauge equivalent to a flat connection. (In § 2 we prove

an easier version of Narashimhan-Seshadri's theorem which is enough for our purpose.)

We will find (using finiteness of $\mathcal{E}(\mathcal{A})$) that there exists a neighborhood U of L such that the curvature of $A(s, t)$ is small for $(s, t) \in U \cap D_1$. Hence φ is extended to a smooth map $\varphi : U \cup D_2 - \{(0, 0)\} \rightarrow R(\Sigma; E)$. Then, the complex gauge invariance of (1.1.1) implies that this extended map is again holomorphic.

We use finiteness of $\mathcal{E}(\mathcal{A})$ to show that $\int_{U \cup D_2} \varphi^* \omega$ is finite. This fact enables us to apply removable singularity theorem of holomorphic (or harmonic) maps and to show that φ is extended to a holomorphic map $\varphi : U \cup D_2 \rightarrow R(\Sigma; E)$.

It means that there exists a smooth connection $\mathcal{A}' = A' \oplus \Phi' ds \oplus \Psi' dt$ solving (1.1) for $\chi \equiv 0$ and $\mathcal{A}'$ is complex gauge equivalent to $\mathcal{A}$.

Hence there exists a complex gauge transformation g' on $((U \cup D_2) - \{(0, 0)\}) \times \Sigma$ such that $\mathcal{A} = g'^* \mathcal{A}'$. We decompose $g' = g'' g$, where g is real (that is $\bar{g} = g^{-1}$) and g'' is hermitian, (that is $\bar{g}'' = g''$). To prove Theorem 1.5, it suffices to show that $g''^* \mathcal{A}'$ is extended to a smooth connection on $(U \cup D_2) \times \Sigma$. For this purpose we prove

$$(1.8) \quad \|g'' - 1\|_{C^k} < C_k \chi(s, t)^2.$$

The above argument is discussed in more detail in § 4. In § 3, we prove a lemma which says that if a curvature of an ASD connection is small then it is gauge equivalent to a connection which is C^∞ close to a flat connection. Inequality (1.8) is proved in § 6.

To prove Theorem 1.6, we first apply Uhlenbeck's compactness theorem to show that restriction of $\mathcal{A}_i$ to $\Sigma \times D_1$ minus finitely many points converges in C^∞ topology on any compact set (after taking a subsequence and gauge transformations).

We recall that the restriction of $\mathcal{A}_i$ to $\Sigma \times D_2$ defines a holomorphic map $\varphi_i : D_2 \rightarrow R(\Sigma; E)$.

We next prove the following: There exists a finite subset P_3 on L and a neighborhood U of L such that, for any neighborhood W of P_3 , the curvature of $\mathcal{A}_i(s, t)$ is small for $i > I(W)$, $(s, t) \in U - W$.

It implies that φ_i is extended to a map $\varphi_i : D_2 \cup U - W \rightarrow R(\Sigma; E)$. Using uniformly boundedness of $\mathcal{E}(\mathcal{A}_i)$, Sacks-Uhlenbeck-Gromov compactness of holomorphic curve implies that there exists a finite set P_4 on $D_2 \cup U - W$ such that φ_i converges in C^∞ topology on any compact subsets of $D_2 \cup U - W - P_4$. We put $P_2 = P_3 \cup P_4$.

Since W is arbitrary small, we have $\varphi_\infty : D_2 \cup U - P_2 \rightarrow R(\Sigma; E)$.

It means that there exists a smooth connection $\mathcal{A}'_i$ on $(D_2 \cup U - P_2) \times \Sigma$ converging in C^∞ topology on compact sets such that $\mathcal{A}'_i$ solves (1.1) for $\chi \equiv 0$ and that $\mathcal{A}'_i$ is complex gauge equivalent to $\mathcal{A}_i$ on $(D_2 \cup U - P_2) \times \Sigma$.

Let g'_i be the complex gauge transformation such that $g'^*_i \mathcal{A}'_i = \mathcal{A}_i$. We take the polar decomposition and obtain $g'_i = g''_i g_i$ where g_i is real and g''_i is hermitian. We have $g''_i \equiv 1$ on $D_2 \times \Sigma$.

Since $\mathcal{A}_i$ converges on $(D_1 \cap U) \times \Sigma$ finitely many points, it is easy to see that g_i converges there. Also g_i converges on $(D_2 - P_2) \times \Sigma$. Hence all we need is to show the estimate

$$(1.9) \quad \|g''_i - 1\|_{C^k} < C_k \chi(s, t)^2.$$

Here C_k is independent of i . (The argument explained above is discussed in more detail in

§ 5.) Estimate (1.9) is proved in §6. We prove Theorem 1.7 in § 5 also.

The author would like to thank Mikio Furuta for helpful discussion. He also would like to thank the referee of [8] (an earlier version of this paper) for his patience and valuable suggestions. This paper is completed during the author's stay in University of Chicago and Institute des Hautes Etudes Scientifique, to which the author would like to thank for hospitality.

§ 2 Background materials

§ 2-1 Real Gauge transformation

Let Σ be a Riemann surface and $E(\Sigma) \rightarrow \Sigma$ be an $SO(3)$ bundle on Σ such that its restriction to each connected component is nontrivial. In other words, we assume the Stiefel Whitney class $w^2(E(\Sigma))$ is the fundamental class. Such a bundle is unique.

Let $P_{U(2)}(\Sigma) \rightarrow \Sigma$ be the principal $U(2)$ bundle such that $c^1(P_{U(2)}(\Sigma)) = [\Sigma]$. (Here c^1 denotes the 1st Chern class. This condition determines the bundle.) We recall that $U(2)/U(1) = SO(3)$. Hence we have a homomorphism $U(2) \rightarrow SO(3)$. Since $c^1(P_{U(2)}(\Sigma)) \equiv w^2(E(\Sigma)) \pmod{2}$ we have $P_{U(2)}(\Sigma) \times \mathbf{R}^3 = E(\Sigma)$. Let $\text{Det } P_{U(2)}(\Sigma)$ be the determinant line bundle of $P_{U(2)}(\Sigma)$. The first Chern class of $\text{Det } P_{U(2)}(\Sigma)$ is the fundamental class.

We identify an $so(3)$ connection on $E(\Sigma)$ to a unitary connection $P_{U(2)}(\Sigma)$ as follows. We fix a connection ∇^{det} of $\text{Det } P_{U(2)}(\Sigma)$. We may choose so that the curvature $F_{\nabla^{\text{det}}}$ coincides to the volume form Ω_{Σ} of our Riemann surface. (We fix a Riemannian metric g_{Σ} of Σ .) We remark that $u(2) \cong so(3) \oplus u(1)$. Hence, for each connection A of $E(\Sigma)$, the direct sum $\hat{A} = A \oplus \nabla^{\text{det}}$ is a connection on $P_{U(2)}(\Sigma)$. It is easy to see that $\hat{A}$ induces a connection ∇^{det} on $\text{Det } P_{U(2)}(\Sigma)$. We put

$$\mathcal{A}(\Sigma, E) = \{A \mid (\text{smooth}) so(3) \text{ connections on } E\}$$

$$\begin{aligned} \mathcal{A}(\Sigma, E) = \{A \mid (\text{smooth}) U(2) \text{ connections on } P_{U(2)} \text{ which} \\ \text{induces } \nabla^{\text{det}} \text{ on the determinant bundle}\} \end{aligned}$$

We have a bijection $\mathcal{A}(\Sigma, E) = \mathcal{A}'(\Sigma, E)$. We consider the adjoint action of $SU(2)$ to $SU(2)$. It induces an action of $U(2)$ to $SU(2)$ and to $SO(3) = SU(2)/\{\pm 1\}$. We put

$$Ad_0 E(\Sigma) = P_{U(2)}(\Sigma) \times SU(2), \quad Ad E(\Sigma) = P_{U(2)}(\Sigma) \times SO(3).$$

A section of the bundle $Ad_0 E(\Sigma)$ induces an automorphism of $P_{U(2)}(\Sigma)$. Hence it acts on the set of unitary connections of $P_{U(2)}(\Sigma)$. It is easy to see that $\mathcal{A}(\Sigma, E)$ is invariant of this action. Namely $\Gamma(\Sigma, Ad_0 E(\Sigma))$ acts on $\mathcal{A}(\Sigma, E)$.

Our convention of gauge transformation is $g^* A = A + g^{-1} d_A g$. Hence $(g_1 g_2)^* = g_2^* g_1^*$.

We can describe this action by using the bijection $\mathcal{A}(\Sigma, E) = \mathcal{A}'(\Sigma, E)$ as follows. We have a map $Ad_0 E(\Sigma) \rightarrow Ad E(\Sigma)$. Hence an element of $\Gamma(\Sigma, Ad_0 E(\Sigma))$ can be regarded as an $SO(3)$ gauge transformation of $E(\Sigma)$ and acts on $\mathcal{A}(\Sigma, E)$. Clearly this action coincides with the action of $\Gamma(\Sigma, Ad_0 E(\Sigma))$ on $\mathcal{A}(\Sigma, E)$.

We remark that $\Gamma(\Sigma, Ad_0 E(\Sigma))$ is a subgroup of $\Gamma(\Sigma, Ad E(\Sigma))$ of finite index.

§ 2-2 Moduli space of flat connections

Let Ω_{Σ} be the volume form of Σ . We regard it as a $u(1)$ valued 2 form on Σ . By $u(2) \cong so(3) \oplus u(1)$, we regard Ω_{Σ} as a section of $P_{U(2)}(\Sigma) \times_{ad} u(2)$. In other words, we

write Ω_Σ in place of $\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix} \Omega_\Sigma \in \Gamma(\Sigma, P_{U(2)}(\Sigma) \times_{ad} u(2))$. We put

$$\hat{R}(\Sigma, E) = \{A \in \mathcal{A}(\Sigma, E) \mid F_A = \Omega_\Sigma\}.$$

Using the identification $\mathcal{A}(\Sigma, E) = \mathcal{A}'(\Sigma, E)$, the set $\hat{R}(\Sigma, E)$ can be identified to the set of flat $SO(3)$ connections of E . The action of $\Gamma(\Sigma, Ad_0 E(\Sigma))$ on $\mathcal{A}(\Sigma, E)$ preserves the set $\hat{R}(\Sigma, E)$. We put

$$R(\Sigma, E) = \frac{\hat{R}(\Sigma, E)}{\Gamma(\Sigma, Ad_0 E(\Sigma))}.$$

We recall that $R(\Sigma, E)$ is a smooth manifolds (see [2]). We put

$$R'(\Sigma, E) = \frac{\hat{R}'(\Sigma, E)}{\Gamma(\Sigma, AdE(\Sigma))}.$$

$R'(\Sigma, E)$ is a smooth orbifold and $R(\Sigma, E)$ is a (branched) covering of it.

We next recall the symplectic and complex structures of $R(\Sigma, E)$. Let $A \in \hat{R}(\Sigma, E)$. It induces a flat connection on $P_{U(2)}(\Sigma) \times_{ad} su(2)$, since the $su(2)$ component of the curvature is zero. We put

$$\mathcal{H}^1(\Sigma; A) = \text{Ker}(\Delta_A : \Gamma(\Sigma, \Lambda^1 \otimes P_{U(2)} \times_{ad} su(2)) \rightarrow \Gamma(\Sigma, \Lambda^1 \otimes P_{U(2)} \times_{ad} su(2))).$$

Here Δ_A is the flat Laplacian. Since $su(2)$ is isomorphic to $so(3)$, this group is identified to the cohomology $H^1(\Sigma; so(3)_A)$. Here $so(3)_A$ is the local system defined by the flat connection on $E(\Sigma)$ induced by A . It is well know that $T_{[A]} R'(\Sigma, E) \cong H^1(\Sigma; so(3)_A)$. Hence we have $T_{[A]} R(\Sigma, E) \cong \mathcal{H}^1(\Sigma; A)$.

Let $u, v \in \mathcal{H}^1(\Sigma; A)$ and $*_\Sigma$ be the Hodge $*$ operator of the surface Σ . We put

$$g_{R(\Sigma, E)}(u, v) = \int_\Sigma -\text{Tr}(u \wedge *_\Sigma v) \quad , \quad \omega(u, v) = \int_\Sigma -\text{Tr}(u \wedge v) \quad , \quad J_{R(\Sigma, E)}(u) = *_\Sigma u.$$

(We recall $-\text{Tr}$ is the casimir operator of $su(2)$.) We recall the following famous result.

Lemma 2.1 $R(\Sigma, E)$ is a Kähler manifold with complex structure $J_{R(\Sigma, E)}$, Kähler form $\omega_{R(\Sigma, E)}$ and the Kähler metric $g_{R(\Sigma, E)}$.

§ 2-3 Energy functional

We review the relation of $\mathcal{E}(\mathcal{A})$ to Yang-Mills functional and symplectic volume. Let $P_{U(2)}(D \times \Sigma)$, $E(D \times \Sigma)$ be pull backs of $P_{U(2)}(\Sigma)$ and $E(\Sigma)$ respectively. $\mathcal{A}(D \times \Sigma, E)$ and $\mathcal{A}'(D \times \Sigma, E)$ are defined in a similar way. There is a natural bijection between them. As we mentioned in introduction, we can decompose an element $\mathcal{A}$ of $\mathcal{A}(D \times \Sigma, E)$ as $\mathcal{A} = A + \Phi ds + \Psi dt$. Here $A = A(s, t) \in \mathcal{A}(\Sigma, E)$ for each (s, t) and $\Phi, \Psi \in \Gamma(D \times \Sigma, P_{U(2)} \times_{ad} su(2))$.

Let us consider a singular metric $\mathbf{g} = ds^2 \oplus dt^2 \oplus \chi(s,t)^2 g_\Sigma$ as in introduction. We will prove that the functional $\mathcal{E}(\mathcal{A})$ is equal to a sum of two parts. One part is Yang-Mills functional on D_1 . We recall that $D_1 \times \Sigma$ is the part of $D \times \Sigma$ where the metric $\mathbf{g}$ is nondegenerate. We put :

$$(2.2) \quad \mathcal{E}_1(\mathcal{A}) = \int_{D_1} \|F_{\mathcal{A}}\|_{\mathbf{g}}^2 \Omega_{\mathbf{g}} .$$

The other part of $\mathcal{E}(\mathcal{A})$ is the energy of the map $(s,t) \mapsto [A(s,t)]$. We assume that our connection $\mathcal{A} = A + \Phi ds + \Psi dt$ satisfies Equation (1.1). Then (1.1.2) implies that $F_{A(s,t)} = 0$ for $(s,t) \in D_2$. Therefore we may regards $A(s,t) \in \hat{R}(\Sigma, E)$. We let $\varphi(s,t)$ be the equivalence class of $A(s,t)$ in $R(\Sigma, E)$. We have

$$(2.3) \quad d_A \left(\frac{\partial A(s,t)}{\partial s} \right) = d_A \left(\frac{\partial A(s,t)}{\partial t} \right) = 0$$

since $A(s,t)$ are flat. Hence by (1.1.1)

$$(2.4) \quad \frac{\partial A}{\partial s} - d_A \Psi = *_\Sigma \left(\frac{\partial A}{\partial t} - d_A \Phi \right) \in \mathcal{H}^1(\Sigma, A).$$

Using the identification $T_{[A]} R'(\Sigma, E) \cong \mathcal{H}^1(\Sigma, A)$ and (2.4), it is easy to see

$$(2.5) \quad \frac{\partial A}{\partial s} - d_A \Psi = \frac{\partial \varphi}{\partial s}, \quad \frac{\partial A}{\partial t} - d_A \Phi = \frac{\partial \varphi}{\partial t}.$$

We now put

$$(2.6) \quad \mathcal{E}_2(\mathcal{A}) = \frac{1}{2} \int_{D_2} \left(\left\| \frac{\partial \varphi}{\partial s} \right\|_{g_{R(\Sigma, E)}}^2 + \left\| \frac{\partial \varphi}{\partial t} \right\|_{g_{R(\Sigma, E)}}^2 \right) ds dt .$$

(2.5) implies that

$$(2.7) \quad \mathcal{E}_2(\mathcal{A}) = \frac{1}{2} \int_{D_2 \times \Sigma} \left(\left\| \frac{\partial A}{\partial s} - d_A \Phi \right\|_{g_\Sigma}^2 + \left\| \frac{\partial A}{\partial t} - d_A \Psi \right\|_{g_\Sigma}^2 \right) \Omega_{\mathbf{g}_{prod}} .$$

Here $\Omega_{\mathbf{g}_{prod}}$ is the volume form of the product metric $\mathbf{g}_{prod} = ds^2 \oplus dt^2 \oplus g_\Sigma$. We remark that $\mathcal{E}_2(\mathcal{A}) = \int_{D_2} \varphi^* \omega$. We recall also that $c^1(R(\Sigma, E)) = 2\omega$. Hence $2\mathcal{E}_2(\mathcal{A}) = \int_{D_2} \varphi^* c^1$.

We have the following equality in case $\mathcal{A}$ satisfies Equation (1.1).

$$(2.8) \quad \begin{aligned} \mathcal{E}_1(\mathcal{A}) = & \int_{D_1 \times \Sigma} \left(\left\| \frac{\partial A}{\partial s} - d_A \Phi \right\|_{g_\Sigma}^2 + \left\| \frac{\partial A}{\partial t} - d_A \Psi \right\|_{g_\Sigma}^2 \right) \Omega_{\mathbf{g}_{prod}} \\ & + \int_{D_2 \times \Sigma} \left(\chi(s,t)^2 \left\| \frac{\partial \Psi}{\partial s} - \frac{\partial \Phi}{\partial t} + [\Phi, \Psi] \right\|_{g_\Sigma}^2 + \chi(s,t)^{-2} \|F_{\mathcal{A}}\|_{g_\Sigma}^2 \right) \Omega_{\mathbf{g}_{prod}} \end{aligned}$$

Hence $2\mathcal{E}_2(\mathcal{A})$ coincides with $\mathcal{E}_1(\mathcal{A})$ formally by putting $\chi \equiv 0$. To see (2.8), it suffices

to remark the following formula (2.9), where u is a k form on Σ and $c > 0$.

$$(2.9) \quad \int_{\Sigma} |u|_{g_{\Sigma}}^2 \Omega_{g_{\Sigma}} = c^{2(1-k)} \int_{\Sigma} |u|_{g_{\Sigma}}^2 \Omega_{g_{\Sigma}}.$$

Lemma 2.10 $\mathcal{E}(\mathcal{A}) = \mathcal{E}_1(\mathcal{A}) + 2\mathcal{E}_2(\mathcal{A})$.

Proof: It is well known that $\mathcal{E}_1(\mathcal{A}) = 2\pi^2 \int_{D_1 \times \Sigma} p^1(\mathcal{A})$ (see for example [6]). Using the fact that $F_{A(s,t)} = 0$ for $(s,t) \in D_2$, we find that

$$F_{\mathcal{A}} \wedge F_{\mathcal{A}} = -2 \left(\frac{\partial A}{\partial s} - d_A \Phi \right) \wedge \left(\frac{\partial A}{\partial t} - d_A \Psi \right) \wedge ds \wedge dt.$$

Using equation (1.1.1) and $2\pi^2 p^1(\mathcal{A}) = -\text{Tr}(F_{\mathcal{A}} \wedge F_{\mathcal{A}})$, we find $\mathcal{E}_2(\mathcal{A}) = \pi^2 \int_{D_2 \times \Sigma} p^1(\mathcal{A})$. Lemma 2.10 follows.

§ 2-4 Complex Gauge transformation on Riemann surface

We next define complex gauge transformations. We embed $SU(2) \subset SL(2; \mathbf{C})$ and put

$$Ad_{SL(2; \mathbf{C})} P(\Sigma) = P_{U(2)}(\Sigma) \times_{Ad} SL(2; \mathbf{C}).$$

In this paper, we call $\Gamma(\Sigma, Ad_0 E)$ the *real gauge transformation group* and $\Gamma(\Sigma, Ad_{SL(2; \mathbf{C})} P(\Sigma))$ the *complex gauge transformation group*. Let us define its action to the set of $so(3)$ connections of $E(\Sigma) \rightarrow \Sigma$.

We fixed a Riemannian metric on Σ . We take the complex structure of Σ such that it is Kähler.

We recall that the set of unitary ($u(2)$) connections on $P_{U(2)}(\Sigma) \times \mathbf{C}^2$ is identified to the set of holomorphic structures on $P_{U(2)}(\Sigma) \times \mathbf{C}^2$. Therefore we can identify $\mathcal{A}(\Sigma, E)$ to the set of holomorphic structures on $P_{U(2)}(\Sigma) \times \mathbf{C}^2$ which induces a given holomorphic connection on the determinant line bundle $\text{Det } P_{U(2)}$.

Let us describe this identification more explicitly. We take a unitary frame of our bundle $P_{U(2)}(\Sigma) \times \mathbf{C}^2$ locally and write an element of $\mathcal{A}(\Sigma, E)$ as $d + A$, where A is a 2×2 matrix valued 2 form such that ${}^t \bar{A} = -A$, $\text{Tr} A = \omega$. Here $d + \omega = \nabla^{\text{Det}}$. ($d\omega$ is the $\sqrt{-1}$ times the volume form.) We use decomposition $\Lambda^1 \Sigma \otimes \mathbf{C} = \Lambda^{0,1} \Sigma \oplus \Lambda^{1,0} \Sigma$ to write

$$(2.11) \quad A = A^{1,0} + A^{0,1}.$$

Then, $\bar{\partial}_A = \bar{\partial} + A^{0,1}$ gives the holomorphic structure of $P_{U(2)} \times \mathbf{C}^2$ corresponding to our connection $d + A$.

On the contrary, suppose we have a holomorphic structure $\bar{\partial}_A = \bar{\partial} + A^{0,1}$, such that it induces given complex structure on determinant line bundle. We put

$$(2.12) \quad A = -\sqrt{-1} \bar{A}^{0,1} + A^{0,1}.$$

It then satisfies ${}^t \bar{A} = -A$. The other relation $\text{Tr} A = \omega$ follows from the fact that $\bar{\partial}_A$ induces given holomorphic structure on the determinant line bundle. (In other word, from

the fact that $(0,1)$ part of ω is equal to $\text{Tr}A^{0,1}$ and $\bar{\omega} = -\omega$.)

Now we define an action of $\Gamma(\Sigma, \text{Ad}_{\text{SL}(2, \mathbb{C})}P)$ on $\mathcal{A}(\Sigma, E)$. Let $\bar{\partial}_A$ be a holomorphic structure on $P_{U(2)} \times \mathbb{C}^2$ and $g \in \Gamma(\Sigma, \text{Ad}_{\text{SL}(2, \mathbb{C})}P)$. Then $g^{-1} \circ \bar{\partial}_A \circ g$ defines another holomorphic structure on $P_{U(2)} \times \mathbb{C}^2$. Since $\det g = 1$, $\bar{\partial}_A$ and $g^{-1} \circ \bar{\partial}_A \circ g$ induces the same holomorphic structure on the determinant bundle. Hence, using the identification of $\mathcal{A}(\Sigma, E)$ to a set of holomorphic structure, we have an action of $\Gamma(\Sigma, \text{Ad}_{\text{SL}(2, \mathbb{C})}P)$ to $\mathcal{A}(\Sigma, E)$. In view of (2.11) and (2.12), the action is given explicitly as

$$(2.13) \quad A = A^{0,1} + A^{1,0} \mapsto (g^{-1} \bar{\partial} g + g^{-1} A^{0,1} g) - \overline{(g^{-1} \bar{\partial} g + g^{-1} A^{0,1} g)} \\ = A + g^{-1} \bar{\partial}_A g + {}^t \bar{g} \partial_A {}^t \bar{g}^{-1}$$

We remark that if $g \in \Gamma(\Sigma, \text{Ad}_0 E)$ namely if g is a real gauge transformation, then ${}^t \bar{g} = g^{-1}$ implies

$$g^{-1} \bar{\partial}_A g - {}^t \bar{g}^{-1} \partial_A {}^t \bar{g}^{-1} = g^{-1} \bar{\partial}_A g + g^{-1} \partial_A g = g^{-1} d_A g.$$

Therefore (2.13) coincides with the usual action of real gauge transformation group.

We next recall the relation between complex gauge transformation and flat connection.

[15], [4] proved that all connections in $\mathcal{A}(\Sigma, E)$ which correspond to the stable holomorphic structure on $P_{U(2)} \times \mathbb{C}^2$ is transformed by a complex gauge transformation to an element of $\hat{R}(\Sigma, E)$. We need a local version of it which we prove here. (The proof is much easier than global one.)

We take a sufficiently large k_0 and fix it. Let $\mathcal{A}_{L_{k_0}^2}(\Sigma, E)$ be the $L_{k_0}^2$ completion of $\mathcal{A}(\Sigma, E)$. $\mathcal{G}_{L_{k_0}^2, \mathbb{C}}(\Sigma, E)$ and $\mathcal{G}_{L_{k_0}^2, \mathbb{R}}(\Sigma, E)$ be the $L_{k_0}^2$ completions of $\Gamma(\Sigma, \text{Ad}_{\text{SL}(2, \mathbb{C})}P)$ and $\Gamma(\Sigma, \text{Ad}_0 E)$ respectively. $\mathcal{G}_{L_{k_0+1}^2, \mathbb{C}}(\Sigma, E)$ and $\mathcal{G}_{L_{k_0+1}^2, \mathbb{R}}(\Sigma, E)$ are Banach Lie groups and they act on $\mathcal{A}_{L_{k_0}^2}(\Sigma, E)$ by Formula (2.13).

Let $A_0 \in \hat{R}(\Sigma, E)$. We consider a representative of a neighborhood of $[A_0]$ in the quotient space $R(\Sigma, E)$ by Coulomb gauge. Namely we put

$$\mathcal{D}^+(A_0) = \left\{ A \in \mathcal{A}_{L_{k_0}^2}(\Sigma, E) \mid d_{A_0}^*(A - A_0) = 0, F_A = \Omega_\Sigma \right\}.$$

Lemma 2.14 *There exists an open neighborhood of identity $\mathcal{G}_0(A_0) \subseteq \mathcal{G}_{L_{k_0+1}^2, \mathbb{C}}(\Sigma, E)$ and an open neighborhood $\mathcal{D}(A_0)$ of A_0 in $\mathcal{D}^+(A_0)$, such that the map $\Pi : \mathcal{G}_0(A_0) \times \mathcal{D}(A_0) \rightarrow \mathcal{A}_{L_{k_0}^2}(\Sigma, E)$, $(g, A) \mapsto g^* A$ is a diffeomorphism onto an open neighborhood of A_0 in $\mathcal{A}_{L_{k_0}^2}(\Sigma, E)$.*

Proof: We first remark that there is no reducible flat connection on $E(\Sigma)$. It follows that the action of $\mathcal{G}_{L_{k_0+1}^2, \mathbb{C}}(\Sigma, E) / \{\pm 1\}$ is free at a neighborhood of A_0 in $\mathcal{A}_{L_{k_0}^2}(\Sigma, E)$.

We next remark that, by an implicit function theorem, a neighborhood of A_0 in $\mathcal{D}^+(A_0)$ is a smooth finite dimensional submanifold of $\mathcal{A}_{L_{k_0}^2}(\Sigma, E)$.

Therefore, in view of implicit function theorem, it suffices to show that the derivative of

Π at $(1, A_0)$ is invertible. We are going to verify it.

First we find

$$T_{A_0} \mathcal{D}^+(A_0) = \left\{ B \in L_{k_0}^2(\Sigma; \Lambda^1 \otimes P_{U(2)} \times_{ad} so(3)) \mid d_{A_0}^* B = 0, \quad d_{A_0} B = 0 \right\}.$$

The Lie algebra of $\mathcal{G}_{L_{k_0+1}, \mathbf{C}}(\Sigma, E)$ is

$$T_1 \mathcal{G}_{L_{k_0+1}, \mathbf{C}}(\Sigma, E) = L_{k_0+1}^2(\Sigma; P_{U(2)} \times_{ad} sl(2; \mathbf{C})).$$

By differentiating Formula (2.13), the derivative of the map $\Pi: \mathcal{G}_0(A_0) \times \mathcal{D}(A_0) \rightarrow \mathcal{A}_{L_k^2}(\Sigma, E)$ is given by

$$(2.15) \quad (u, B) \mapsto \bar{\partial}_{A_0} u - \partial_{A_0} {}^t \bar{u} + B.$$

We use the decomposition $sl(2; \mathbf{C}) = su(2) \oplus \sqrt{-1}su(2)$. Then, by putting $u = u_1 + \sqrt{-1}u_2$, where ${}^t \bar{u}_i = -u_i$, we find that (2.15) is

$$(2.16) \quad (u_1 + \sqrt{-1}u_2, B) \mapsto d_{A_0} u_1 + \sqrt{-1}(\bar{\partial}_{A_0} - \partial_{A_0})u_2 + B.$$

Since Hodge $*$ is equal to $\sqrt{-1}$ for $(1,0)$ form and is $-\sqrt{-1}$ for $(0,1)$ form, it follows that

$$(2.17) \quad d_{A_0} u_1 + \sqrt{-1}(\bar{\partial}_{A_0} - \partial_{A_0})u_2 + B = d_{A_0} u_1 - *_\Sigma d_{A_0} u_2 + B.$$

Thus the derivative of the map $\Pi: \mathcal{G}_0(A_0) \times \mathcal{D}(A_0) \rightarrow \mathcal{A}_{L_{k_0}^2}(E)$ is identified to a map

$$L_{k_0+1}^2(\Sigma; P_{U(2)} \times_{ad} su(2)) \oplus L_{k_0+1}^2(\Sigma; P_{U(2)} \times_{ad} su(2)) \oplus \mathcal{H}^1(\Sigma, A_0) \rightarrow L_k^2(\Sigma; P_{U(2)} \times_{ad} su(2) \otimes \Lambda^1)$$

and is

$$(u_1, u_2, B) \mapsto d_{A_0} u_1 - *_\Sigma d_{A_0} u_2 + B.$$

Since A_0 is irreducible this map is invertible. The proof of Lemma 2.14 is now complete.

§ 2-5 Complex Gauge transformation on $D \times \Sigma$

We next study Gauge transformation on $D \times \Sigma$. Let $D \subseteq \mathbf{C}$ be an open set. We consider $D \times \Sigma$ with metric $ds^2 \oplus dt^2 \oplus \alpha(s, t)^2 g_\Sigma$. Here g_Σ is a fixed metric on Σ and $\alpha(s, t)$ is a positive valued function of (s, t) . Let $\mathcal{A} = A + \Phi ds + \Psi dt$ be an element of $\mathcal{A}(D \times \Sigma, E)$.

Complex structures of D and Σ induce a complex structure on $D \times \Sigma$. Using it, we decompose $\mathcal{A} = \mathcal{A}^{1,0} + \mathcal{A}^{0,1}$. We remark that $\bar{\partial}_{\mathcal{A}} = \bar{\partial} + \mathcal{A}^{0,1}$ does not necessary define a holomorphic structure of the bundle $P_{U(2)} \times \mathbf{C}^2$.

In the case of Riemann surface, it always define a holomorphic structure. In the case of $D \times \Sigma$, the $(0,2)$ part of the curvature, which is equal to $\bar{\partial}_{\mathcal{A}}^2$, may not vanish. This condition, in turn, is a sufficient condition. Namely we have the following well known:

Lemma 2.18 $\bar{\partial}_{\mathcal{A}} = \bar{\partial} + \mathcal{A}^{0,1}$ defines a holomorphic structure on $P_{U(2)} \times \mathbf{C}^2$ if and only if $\bar{\partial}_{\mathcal{A}}^2 = 0$.

We now consider the complex gauge transformation group $\Gamma(D \times \Sigma, Ad_{SL(2; \mathbb{C})} P)$ and its action to the set of connections. An element of $\Gamma(D \times \Sigma, Ad_{SL(2; \mathbb{C})} P)$ may be regarded also as a two parameter family of gauge transformations on Σ . So we write elements of $\Gamma(D \times \Sigma, Ad_{SL(2; \mathbb{C})} P)$ as $g(s, t)$, where $g(s, t) \in \Gamma(\Sigma, Ad_{SL(2; \mathbb{C})} P)$ for each $(s, t) \in D$.

Let $\mathcal{A}(D \times \Sigma; E)$ be as before. We define an action of $\Gamma(D \times \Sigma, Ad_{SL(2; \mathbb{C})} P)$ in the same way as the case of Riemann surface. Namely by

$$(2.19) \quad \mathcal{A} = \mathcal{A}^{0,1} + \mathcal{A}^{1,0} \mapsto g^* \mathcal{A} = \mathcal{A} + g^{-1} \bar{\partial}_{\mathcal{A}} g + {}^t \bar{g} \partial_{\mathcal{A}} {}^t \bar{g}^{-1}.$$

The following lemma is immediate from Lemma 2.18.

Lemma 2.20 *If $\bar{\partial}_{\mathcal{A}}^2 = 0$, then $\bar{\partial}_{g^* \mathcal{A}}^2 = 0$.*

Let us now make some explicit calculation. We take our complex structure on D so that $s + \sqrt{-1}t$ is the complex coordinate. We first calculate (2.20) and find that :

Lemma 2.21 *$\bar{\partial}_{\mathcal{A}}^2 = 0$ is equivalent to the equation (1.1.1)*

Proof: We recall that the vector bundle $\Lambda_+^2(M)$ of self dual 2 forms on Kähler surface M decompose as

$$(2.22) \quad \Lambda_+^2(M) = (\Lambda^{0,2}(M) \oplus \Lambda^{2,0}(M))_{\mathbb{R}} \oplus \mathbb{R}\omega,$$

where ω is the Kähler form. On the other hand, the operator $\bar{\partial}_{\mathcal{A}}^2$ is equal (0,2) part of the curvature of $\mathcal{A}$.

In case of $D \times \Sigma$, the local frame of $(\Lambda^{0,2}(M) \oplus \Lambda^{2,0}(M))_{\mathbb{R}}$ and $\mathbb{R}\omega$ are $ds \wedge e^1 - dt \wedge e^2$, $ds \wedge e^2 + dt \wedge e^1$ and $ds \wedge dt + \chi^2 e^1 \wedge e^2$ respectively. Here e^1, e^2 is an orthonormal basis of $\Lambda^1 \Sigma$ such that $J_{\Sigma} e^1 = e^2$.

Therefore, according to the decomposition (2.22), the $\mathbb{R}\omega$ component of self dual part of $F_{\mathcal{A}}^+$ the curvature of $\mathcal{A}$ is

$$\frac{1}{2} \left(\chi(s, t)^{-2} *_{\Sigma} F_{\mathcal{A}(s, t)} + \left(\frac{\partial \Psi}{\partial s} - \frac{\partial \Phi}{\partial t} + [\Phi, \Psi] \right) (ds \wedge dt + \chi(s, t)^2 \Omega_{\Sigma}) \right)$$

(This vanishes if and only if (1.1.2) is satisfied.) The component of $F_{\mathcal{A}}^+$ orthogonal to it is :

$$(2.23) \quad \begin{aligned} & \frac{1}{2} \left(\left(-\frac{\partial A}{\partial s} + d_A \Phi \right) \wedge ds + \left(-\frac{\partial A}{\partial t} + d_A \Psi \right) \wedge dt \right) \\ & + *_{D \times \Sigma} \frac{1}{2} \left(\left(-\frac{\partial A}{\partial s} + d_A \Phi \right) \wedge ds + \left(-\frac{\partial A}{\partial t} + d_A \Psi \right) \wedge dt \right) \\ & = \frac{1}{2} \left(\left(-\frac{\partial A}{\partial s} + d_A \Phi \right) + *_{\Sigma} \left(-\frac{\partial A}{\partial t} + d_A \Psi \right) \right) \wedge ds \\ & + \frac{1}{2} \left(\left(-\frac{\partial A}{\partial t} + d_A \Psi \right) - *_{\Sigma} \left(-\frac{\partial A}{\partial s} + d_A \Phi \right) \right) \wedge dt. \end{aligned}$$

(2.23) vanishes if and only if (1.1.1) is satisfied. The proof of Lemma 2.22 is now complete.

Corollary 2.24 *Equation (1.1.1) is invariant of complex gauge transformations.*

Proof: This is an immediate consequence of the fact that $\bar{\partial}_{g^*,A}^2 = g^{-1} \circ \bar{\partial}_A^2 \circ g$ and Lemma 2.23.

We finally remark that the action can be extend to connections and gauge transformations in an appropriate Sobolev classes.

§ 3 Almost flat ASD connection

In [18] [17], Uhlenbeck proved that, if an L^2 -norm of an ASD connection on 4 disk is small, then one can find a gauge so that the connection itself is small. The purpose of this section is to prove Proposition 3.1 which is a generalization of it to general 4 manifolds. Of course, there exist nontrivial flat connections. So the conclusion is that, after gauge transformation, the connection is close to a flat connection. It seems that such a generalization is in principle known or at least expected to experts of gauge theory for a long time. (It was written in [11].) However since the author did not find an appropriate reference written in western language, we give a full detail of the proof. We apply it in this paper only in the case when 4 manifold is $[-2, 2]^2 \times \Sigma$. However we prove it in general since it might be useful to have a reference of this result.

Proposition 3.1 *Let Y be a compact Riemannian 4 manifold with (or without) boundary, $P_G \rightarrow Y$ be a principal G bundle on it and $K \subseteq Y$ be a compact subset such that $K \cap \partial Y = \emptyset$. Then there exist positive numbers $\varepsilon_1(K, Y, P_G)$, $C_{1,k}(K, Y, P_G)$ and $\lambda \in (0, 1]$ with the following properties.*

Let A be an ASD connection on Y such that $\|F_A\|_{L^2(Y)} \leq \varepsilon_1(K, Y, P_G)$. Then there exists a flat connection A_0 and a gauge transformation g on a neighborhood of K such that

$$\|g^*A - A_0\|_{C^k(K)} \leq C_{1,k} \|F_A\|_{L^2(Y)}^\lambda$$

The proposition holds for an arbitrary (compact) structure group G .

Remark 3.2 If the space of flat connections $R(K)$ on a sufficiently small neighborhood of K is smooth, then we may choose $\lambda = 1$ in Proposition 3.1. (See the remark after Lemma 3.30.)

The constants $C_{1,k}$ in Proposition 3.1 depends on the metric on Y . However when we consider a family of metrics which are of bounded geometry in the sense we define below, we can choose them independent of the metric in the family.

Definition 3.3 Let B_k be a sequence of positive numbers and $\delta > 0$. We say that a compact Riemannian manifold (Y, g) and its compact subset $K \subseteq Y$ is of $B_{-1}, B_0, B_1, \dots, \delta$ bounded geometry if the following is satisfied.

$$(3.4.1) \quad \begin{cases} \text{injectivity radius of } g > \delta \\ \sup |\nabla^k R_g| < B_k \end{cases}$$

$$(3.4.2) \quad \text{disk}(K, \partial Y) > \delta$$

$$(3.4.3) \quad \text{Diameter of } K \leq B_{-1}.$$

Definition 3.5 Let A_0 be a flat connection of a principal G bundle $P_G \rightarrow Y$ on a Riemannian manifold Y . Let A be another connection of P_G . We define the C^k norm

$\|A - A_0\|_{C^k(Y), A_0}$ as follows.

For each $p \in Y$, let $\varphi : B(0, \varepsilon) \rightarrow T_p Y \rightarrow Y$ be the composition of linear isometric embedding and exponential map. Here $B(0, \varepsilon)$ denote the metric ball of radius ε centered at 0. We take a trivialization of $\varphi^* P$ such that $\varphi^* A_0 = 0$. We take the C^k norm of $\varphi^* A$ at 0 with respect to this trivialization. Taking supremum over $p \in Y$, we define $\|A - A_0\|_{C^k(Y), A_0}$.

We define the C^k norm $\|\cdot\|_{C^k(Y), A_0}$ of gauge transformations and sections of induced bundles in a similar way.

Proposition 3.6 *If (Y, g) and $K \subseteq Y$ is of $\delta, B_\bullet$ bounded geometry. Then there exists $\varepsilon_2(\delta, B_\bullet)$ and $C_{2,k}(\delta, B_\bullet)$ with the following properties. If A is an ASD connection on Y satisfying $\|F_A\|_{L^2(Y)} \leq \varepsilon_2$, then there exists a flat connection A_0 and a gauge transformation g on a neighborhood of K such that*

$$\|g^* A - A_0\|_{C^k(K), A_0} \leq C_{2,k} \|F_A\|_{L^2(Y)}^\lambda.$$

We use Propositions 3.1 and 3.6 in the case when $Y = [-2, 2]^2 \times \Sigma$, $K = [-1, 1]^2 \times \Sigma$ and the metric $g' = ds^2 \oplus dt^2 \oplus \alpha(s, t)^2 g_\Sigma$, such that

$$(3.7.1) \quad 0 < c_0 < \alpha(s, t) < c_1,$$

$$(3.7.2) \quad |\alpha|_{C^1} < B_k.$$

It is easy to see that such a pair is of bounded geometry in the sense of Definition 3.3. Therefore Proposition 3.6 implies the following Corollary 3.8. We consider $E([-2, 2]^2 \times \Sigma)$ as in § 1, 2 and real gauge transformations as we defined in § 2.

Corollary 3.8 *There exists $\varepsilon_3(c_0, c_1, B_\bullet)$ and $C_{3,k}(c_0, c_1, B_\bullet)$ with the following properties. If $\mathcal{A}$ is an ASD connection of $E([-2, 2]^2 \times \Sigma)$ with $\|F_{\mathcal{A}}\|_{L^2([-2, 2]^2 \times \Sigma)} \leq \varepsilon_3(c_0, c_1, B_\bullet)$ then there exists a gauge transformation $g \in \Gamma([-1, 1]^2 \times \Sigma, Ad_0 E)$ and a flat connection $\mathcal{A}_0$ such that*

$$\|g^* \mathcal{A} - \mathcal{A}_0\|_{C^k([-1, 1]^2 \times \Sigma, g'), \mathcal{A}_0} \leq C_{3,k} \|F_{\mathcal{A}}\|_{L^2([-2, 2]^2 \times \Sigma, g')}.$$

Remark 3.9 There is one point where Corollary 3.8 is not so immediate from Proposition 3.6. Namely in Corollary 3.8, we consider the real gauge transformation in the sense of § 2, that is a section of $\Gamma([-1, 1]^2 \times \Sigma, Ad_0 E)$. On the other hand, the gauge transformation in Proposition 3.6 is a section of $\Gamma([-1, 1]^2 \times \Sigma, AdE)$ in the situation of Corollary 3.8.

However, since $\Gamma([-1, 1]^2 \times \Sigma, Ad_0 E) / \{\pm 1\}$ is a finite index subgroup of $\Gamma([-1, 1]^2 \times \Sigma, AdE)$, we find a finite subset $\{g_i\}$ of $\Gamma([-1, 1]^2 \times \Sigma, AdE)$ such that $\bigcup g_i \cdot \Gamma([-1, 1]^2 \times \Sigma, Ad_0 E) / \{\pm 1\} = \Gamma([-1, 1]^2 \times \Sigma, AdE)$. Then, since the set $\{g_i\}$ is finite, we have

$$\|g_i^* g^* \mathcal{A} - g_i^* \mathcal{A}_0\|_{C^k([-1, 1]^2 \times \Sigma, g_i^* \mathcal{A}_0)} \leq C'_k \|g^* \mathcal{A} - \mathcal{A}_0\|_{C^k([-1, 1]^2 \times \Sigma, \mathcal{A}_0)} \leq C'_k C_{2,k} \|F_{\mathcal{A}}\|_{L^2([-1, 1]^2 \times \Sigma)}.$$

Thus Corollary 3.8 follows from Proposition 3.6.

Remark 3.10 We remark that in the situation of Corollary 3.8, there is another natural way to define C^k norm as follows. We fix chart, coordinate and local trivialization of $E([-2, 2]^2 \times \Sigma)$ and define C^k norm using them in the usual way. We write this norm $\|\cdot\|_{C^k}$. This is the norm we use in §§ 2,4,5,6. However this norm is equivalent to one used in Corollary 3.8 by the following :

Addendum 3.11 *In the situation of Corollary 3.8, there exist $C_{4,k}(c_0, c_1, B_*)$ such that*

$$(3.12) \quad \|A_0\|_{C^k([-1,1]^2 \times \Sigma, g')} \leq C_{4,k}.$$

Proof of Proposition 3.1: We take a sufficiently fine simplicial decomposition of a neighborhood of K , (which we denote by K^+ hereafter). Then the star neighborhoods of its double barycentric subdivision gives an open covering $\bigcup_{j=0}^4 \bigcup_{i=1}^{N_j} U_i^j \supseteq K^+$ with the following properties.

$$(3.13.1) \quad U_i^j \text{ is diffeomorphic to } D^4.$$

$$(3.13.2) \quad U_i^j \cap U_{i'}^{j'} = \emptyset \text{ for } i \neq i'.$$

$$(3.13.3) \quad \text{We put } U_{(j_0)} = \bigcup_{j=0}^{j_0} \bigcup_{i=1}^{N_j} U_i^j. \text{ Then } U_{(j_0-1)} \cap U_{i_0}^{j_0} \text{ is diffeomorphic to } S^{j_0-1} \times D^{4-j_0+1}. (S^0 \text{ is two points and } S^{-1} \text{ is empty set.})$$

The proof of (3.13) is a part of the standard result, which says that a simplicial decomposition of a manifold induces a handle decomposition. See also Figures 3.19 below.

We choose $U_{i,m}^j$, $m = 0, \dots, 4$ such that

$$(3.14.1) \quad U_{i,m}^j \supset U_{i,m+1}^j, \quad U_{i,0}^j = U_i^j.$$

$$(3.14.2) \quad \text{dist}(U_{i,m+1}^j, Y - U_{i,m}^j) \geq \mu_1 > 0, \text{ where } \text{dist} \text{ is the Riemannian distance and } \mu_1 \text{ is independent of } A.$$

$$(3.14.3) \quad \bigcup_{j=0}^4 \bigcup_{i=1}^{N_j} U_{i,m}^j \supseteq K^+ \text{ holds for each } m.$$

$$(3.14.4) \quad \text{For each } m, \{U_{i,m}^j \mid j = 0, 1, 2, 3, 4, i = 1, \dots, N_j\} \text{ satisfies (3.13.1), (3.13.2), (3.13.3).}$$

We take a trivialization of our bundle at each U_i^j and let $\hat{B}_{i,j}$ be the trivial connection on U_i^j which is 0 with respect to this trivialization. We may choose trivializations so that

$$(3.15) \quad \|\hat{B}_{i',j'} - \hat{B}_{i,j}\|_{C^k(U_i^j \cap U_{i'}^{j'}, \hat{B}_{i,j})} \leq C_k$$

where C_k is independent of A .

By applying Uhlenbeck's theorem (Proposition 8.3 of [6]) to $U_{i,k}^j$, there exists a gauge transformation $h_{i,j}$ on U_i^j such that

$$(3.16) \quad \|h_{i,j}^* A - \hat{B}_{i,j}\|_{C^k(U_i^j, \hat{B}_{i,j})} \leq C_k \|F_A\|_{L^2(Y)}.$$

We put $W_m = \bigcup_{j=0}^m \bigcup_{i=1}^{N_j} U_{i,m}^j$ for $m = 0, \dots, 4$. We are going to construct a flat connections B_m and a gauge transformations g_m on W_m such that

$$(3.17) \quad \|g_m^* A - B_m\|_{C^k(W), B_m} \leq C_k \|F_A\|_{L^2(Y)},$$

by induction on m . For $m = 0$, we put

$$\begin{cases} B_0 = \hat{B}_{i,0} & \text{on } U_{i,0}^0 \\ g_0 = h_{i,0} & \text{on } U_{i,0}^0 \end{cases}.$$

They are smooth since $U_{i,0}^0$ are disjoint to each other. (3.17) is immediate from (3.16).

We next extend them to W_1 . We put $W_m^- = \bigcup_{j=0}^m \bigcup_{i=1}^{N_j} U_{i,m+1}^j$. We remark that

(3.18.1) $W_0 \cap U_{i,0}^1$ is a diffeomorphic to a disjoint union of two D^+ 's.

(3.18.2) $W_0^- \cap U_{i,1}^1$ is a diffeomorphic to a disjoint union of two D^+ 's.

(3.18.3) $\text{dist}(W_0^- \cap U_{i,0}^1, Y - (W_0 \cap U_{i,0}^1)) \geq \mu_2$. Here μ_2 is independent of A .

Figure 3.19

We next choose a cut off function $\chi_1 : W_0 \cup \bigcup U_{i,1}^1 \rightarrow [0,1]$ such that

$$\chi_1 = \begin{cases} 0 & \text{on } W_0^- - \bigcup_i (W_0 \cap U_{i,0}^1) \\ 1 & \text{on } U_{i,1}^1 - \bigcup_i (W_0 \cap U_{i,0}^1) \end{cases}.$$

Let $g_{i,1}$ be a gauge transformation defined at $W_0 \cap U_{i,0}^1$ such that

$$(3.20) \quad \hat{B}_{i,1} = g_{i,1}^* B_0.$$

By (3.15), we have

$$(3.21) \quad \|g_{i,1}\|_{C^k(W_0 \cap U_{i,0}^1), B_0} < C_k.$$

We take $h_{i,1}$ is as in (3.16) and put

$$(3.22) \quad g'_{i,1} = g_0^{-1} h_{i,1} g_{i,1}^{-1}$$

Lemma 3.23

$$\|g_{i,1}^* B_0 - B_0\|_{C^k(W_0 \cap U_{i,0}^1), B_0} \leq C_k \|F_A\|_{L^2(Y)}$$

Proof: By (3.16) and (3.20), (3.21), we have

$$\|B_0 - g_{i,1}^{-1*} h_{i,1}^* A\|_{C^k(W_0 \cap U_{i,0}^1), B_0} \leq C_k \|F_A\|_{L^2(Y)}.$$

Therefore, using $g_{i,1}^* g_0^* = g_{i,1}^{-1*} h_{i,1}^*$ (note $(g_1 g_2)^* = g_2^* g_1^*$), we have

$$(3.24) \quad \|g_{i,1}'^{-1} d_{B_0} g'_{i,1} + g_{i,1}'^{-1} (g_0^* A - B_0) g_{i,1}'\|_{C^k(W_0 \cap U_{i,0}^1), B_0} \leq C_k \|F_A\|_{L^2(Y)}.$$

Using (3.17) (for $m = 0$) and (3.24), we can prove

$$\|g'_{i,1}{}^{-1} d_{B_0} g'_{i,1}\|_{C^k(W_0 \cap U_{i,0}^1, B_0)} \leq C_k \|F_A\|_{L^2(Y)}.$$

inductively on k . Lemma 3.23 follows.

By Lemma 3.23, we can find a gauge transformation $g''_{i,1}$ on $W_0 \cap U_{i,0}^1$ such that

$$(3.25.1) \quad g'_{i,1} = \exp(X_{i,1}) g''_{i,1} \text{ on } W_0 \cap U_{i,0}^1,$$

$$(3.25.2) \quad g''_{i,1}{}^* B_0 = B_0. \text{ (In other words } d_{B_0} g''_{i,1} = 0.)$$

and that $X_{i,1} \in C^\infty(W_0 \cap U_{i,0}^1, \mathcal{G})$ (where $\mathcal{G}$ is the Lie algebra of the structure group G) satisfies

$$(3.25.3) \quad \|X_{i,1}\|_{C^k(W_0 \cap U_{i,0}^1, B_0)} \leq C_k \|F_A\|_{L^2(Y)}.$$

We remark that (3.25.2) means that $g''_{i,1}$ is a *locally* constant gauge transformation if we take the trivialization of the bundle so that $B_0 = 0$.

Lemma 3.26 *There exists a gauge transformation $g'''_{i,1}$ on $U_{i,0}^1$ such that*

$$(3.27.1) \quad g'''_{i,1} = g''_{i,1} g_{i,1} \text{ on } U_{i,1}^1.$$

$$(3.27.2) \quad \|g'''_{i,1}\|_{C^k(U_{i,0}^1, \hat{B}_{i,1})} \leq C_k.$$

Proof: Using the trivialization $\hat{B}_{i,1}$ we regard $g''_{i,1} g_{i,1}$ as a map $W_0 \cap U_{i,0}^1 \rightarrow G$. (3.20), (3.21) and (3.25.2) imply $\|g''_{i,1} g_{i,1}\|_{C^k(U_{i,0}^1 \cap W_0, \hat{B}_{i,1})} \leq C_k$. This inequality means that $g''_{i,1} g_{i,1}$ as a map $: W_0 \cap U_{i,0}^1 \rightarrow G$ has uniformly bounded C^k norm. Hence Lemma 3.26 follows from the lemma below.

Lemma 3.28 *Let $M \supset \bar{N}^+ \supset N^+ \supset \bar{N} \supset N$ be relatively compact open subsets of Riemannian manifolds. For each D there exists $C_k(M, N^+, N, G, D)$ with the following properties. Let $f : N^+ \rightarrow G$ be a map such that $\|f\|_{C^k(B^+)} < D$ and that f is homotopic to 0. Then there exists $\tilde{f} : M \rightarrow G$ such that $\tilde{f} = f$ on N and that $\|\tilde{f}\|_{C^k(M)} < C_k$.*

Proof: We take $\chi : M \rightarrow [0, 1]$ such that $\chi = 1$ on N and $\chi = 0$ outside a neighborhood of $\bar{M} - N^+$.

For each $f : N^+ \rightarrow G$ satisfying the assumption, we take $\hat{f} : M \rightarrow G$ such that $\hat{f} = f$ on N . If $f' : B^+ \rightarrow G$ and $\sup \text{dist}(f(p), f'(p)) < \varepsilon(G)$ then there exists $X : N^+ \rightarrow \mathcal{G}$ such that $f'(p) = f(p) \exp(X(p))$ with $\|X\|_{C^k(B^+)} < C_k(f)$. We put

$$U(f) = \{f' : N^+ \rightarrow G \mid \|f\|_{C^k(B^+)} < D, \sup \text{dist}(f(p), f'(p)) < \varepsilon(G)\}.$$

By Ascoli-Alzera, there exists a finitely many f_i such that if f satisfies the assumption of Lemma 3.28 then $f \in U(f_i)$ for some i . We put $\tilde{f} = \hat{f}_i \exp(\chi X)$ where $f = f_i \exp(X)$. It

is easy to see that the C^k norm of $\tilde{f}$ is estimated by $C_k(f_i)$ and C^k norm of $\hat{f}_i$, and hence is uniformly bounded. The proof of Lemma 3.28 is complete.

We now use Lemma 3.26 to define a flat connection B_1 on W_1 as follows.

$$B_1(x) = \begin{cases} B_0(x) & x \in W_0^- \\ (g_{i,1}''')^{-1*} \hat{B}_{i,1}(x) & x \in U_{i,1}^1 \end{cases}$$

(3.20), (3.25.2), (3.27.1) imply that B_1 is well defined and flat. We next define a gauge transformation on W_1 by

$$g_1(x) = \begin{cases} g_0(x) & x \in W_0^- - \bigcup_i U_{i,0}^1 \\ h_{i,1}(g_{i,1}''')^{-1} & x \in U_{i,1}^1 - W_0 \\ g_0(x) \exp(\chi_1(x) X_{i,1}(x)) & x \in U_{i,0}^1 \cap W_0 \end{cases}$$

By (3.2.1) and (3.2.2), we have $g_0 \exp(X_{i,1}) = g_0 g_{i,1}' g_{i,1}''^{-1} = h_{i,1}(g_{i,1}'' g_{i,1}')^{-1}$. It follows from (3.27.1) that g_1 is smooth. Then (3.16), induction hypothesis (3.17), (3.25.3) and (3.27.2) imply

$$(3.29) \quad \|g_1^* A - B_1\|_{C^k(W_1), B_1} \leq C_k \|F_A\|_{L^2(Y)}.$$

Next we are going to extend our flat connection B_1 and gauge transformation g_1 to W_2 . For this purpose, let us recall some well known facts on fundamental group and its representation space. We first remark that, by Van-Kampen's theorem, $\Lambda = \pi_1(W_1)$ is a free group with $N_1 - N_0$ generators γ_i and each of 2 simplex e_i^2 corresponds to the relation among generators γ_i . The simplexes of dimension ≥ 3 are not related to the fundamental group. Hence the group $\Gamma = \pi_1(K^+)$ has a presentation

$$\Gamma = \langle \gamma_1, \dots, \gamma_{N_1-N_0} \mid \omega_1, \dots, \omega_{N_2} \rangle$$

here $\omega_i(\gamma_1, \dots, \gamma_{N_1-N_0})$ are elements of Λ . We put

$$\hat{R}(\Gamma) = \left\{ (h_1, \dots, h_{N_1-N_0}) \in G^{N_1-N_0} \mid \omega_i(h_1, \dots, h_{N_1-N_0}) = 1, i = 1, \dots, N_2 \right\}$$

The space of flat connections on W_1 is a quotient of $\hat{R}(\Gamma)$ by the adjoint action of G . More precisely the quotient of $\hat{R}(\Gamma)$ by the adjoint action of G coincides with the union of the spaces of the flat connections on G bundles (of various topological types). We next need the following lemma.

Lemma 3.30 *There exists a positive numbers $\varepsilon(G)$, $C(G)$ and $\lambda(G) \in (0,1]$ such that the following holds. Suppose $(h_1, \dots, h_{N_1-N_0}) \in G^{N_1-N_0}$ and*

$$|\omega_i(h_1, \dots, h_{N_1-N_0}) - 1| \leq \varepsilon(G), \quad i = 1, \dots, N_2$$

Then there exists $(h'_1, \dots, h'_{N_1-N_0}) \in \hat{R}(\Gamma)$ such that

$$|h'_i - h_i| \leq C(G) \left(\max_{i=1, \dots, N_2} |\omega_i(h_1, \dots, h_{N_1-N_0}) - 1| \right)^{\lambda(G)}.$$

Lemma 3.30 is an immediate consequence of the fact that $\hat{R}(\Gamma)$ is a compact real algebraic set. We remark that, by implicit function theorem, we may take $\lambda = 1$ if $\hat{R}(\Gamma)$ is smooth. Here we say that $\hat{R}(\Gamma)$ is smooth if the map $G^{N_1-N_0} \rightarrow G^{N_2} : (h_1, \dots, h_{N_1-N_0}) \mapsto (\omega_i(h_1, \dots, h_{N_1-N_0}))$ is of maximal rank on $\hat{R}(\Gamma)$. This is the case of Corollary 3.8 which we use in later sections.

We now go back to the proof of Proposition 3.1. We have constructed a flat connection B_1 on W_1 . By choosing a base point, we obtain a (holonomy) representation of $\Lambda = \pi_1(W_1)$ to G associated to B_1 . Namely an element $(h_1, \dots, h_{N_1-N_0}) \in G^{N_1-N_0}$.

Lemma 3.31 $|\omega_i(h_1, \dots, h_{N_1-N_0}) - 1| \leq C \|F_A\|_{L^2(Y)}, \quad i = 1, \dots, N_2.$

Proof. By (3.13.3) the intersection $U_{k,1}^2 \cap W_1$ is diffeomorphic to $S^1 \times D^3$. The holonomy of B_1 along $S^1 \subseteq S^1 \times D^3$ is $\omega_i(h_1, \dots, h_{N_1-N_0})$. On the other hand, we have by (3.16) and (3.29)

$$\left\| (h_{i,2})^* (g_1^{-1})^* B_1 - \hat{B}_{i,2} \right\|_{C^0(U_{i,1}^2 \cap W_1, \hat{B}_{i,2})} \leq C \|F_A\|_{L^2(Y)}.$$

The lemma follows immediately.

Now using Lemmata 3.30 and 3.31, we obtain $(h'_1, \dots, h'_{N_1-N_0}) \in \hat{R}(\Gamma)$ such that $|h'_i - h_i| \leq C \left(\max_{i=1, \dots, N_2} |\omega_i(h_1, \dots, h_{N_1-N_0}) - 1| \right)^{\lambda}$. Therefore there exists a flat connection B'_1 on W_1 such that

$$(3.32.1) \quad \|B'_1 - B_1\|_{C^k(W_1), B'_1} \leq C_k \|F_A\|_{L^2(Y)}^{\lambda}$$

$$(3.32.2) \quad \text{The holonomy of } B'_1 \text{ is } (h'_1, \dots, h'_{N_1-N_0}) \in \hat{R}(\Gamma) \subseteq \hat{R}(\Lambda) = \text{Hom}(\pi_1(W_1), G).$$

(3.32.1) and (3.29) implies

$$(3.33) \quad \|g_1^* A - B'_1\|_{C^k(W), B'_1} \leq C_k \|F_A\|_{L^2(Y)}^{\lambda}.$$

We next remark that

$$(3.34.1) \quad W_1 \cap U_{i,1}^2 \text{ is diffeomorphic to } S^1 \times D^3.$$

$$(3.34.2) \quad W_1^- \cap U_{i,2}^2 \text{ is diffeomorphic to } S^1 \times D^3.$$

$$(3.34.3) \quad \text{dist}(W_1^- \cap U_{i,2}^2, Y - W_1 \cap U_{i,1}^2) \geq \mu_3.$$

Since the holonomy of B'_1 on $W_1 \cap U_{i,1}^2 \cong S^1 \times D^3$ is trivial by (3.32.2), there exists a gauge transformation $g_{i,2}$ on $W_1 \cap U_{i,1}^2$ such that

$$(3.35) \quad \hat{B}_{i,2} = g_{i,2}^* B'_1.$$

($\hat{B}_{i,2}$ is a trivial connection on U_i^2 .) By (3.15), (3.27.2), Definition of B_1 , (3.32.1), we have :

$$(3.36) \quad \|g_{i,2}\|_{C^k(W_1 \cap U_{i,1}^2), B_1} < C_k.$$

We put

$$(3.37) \quad g'_{i,2} = g_1^{-1} h_{i,2} g_{i,2}^{-1}.$$

Here $h_{i,2}$ is as in (3.16). In the same way as Lemma 3.23, we can prove :

$$(3.38) \quad \|g_{i,2}^{**} B_1 - B_1\|_{C^k(W_1 \cap U_{i,1}^2), B_1} \leq C_k \|F_A\|_{L^2(Y)}^2.$$

Hence we can find a gauge transformation $g''_{i,2}$ on $W_1 \cap U_{k,1}^2$ such that

$$(3.39.1) \quad g'_{i,2} = \exp(X_{i,2}) g''_{i,2} \text{ on } W_1 \cap U_{k,1}^2,$$

$$(3.39.2) \quad g_{i,2}^{**} B_1 = B_1.$$

$$(3.39.3) \quad \|X_{i,2}\|_{C^k(W_1 \cap U_{k,1}^2), B_1} \leq C_k \|F_A\|_{L^2(Y)}^2.$$

We next want to extend $g''_{i,2} g_{i,2}$ to $U_{i,2}^2$. A trouble here is that $W_1 \cap U_{k,1}^2$ is not contractible. So in the case when $\pi_1(G) \neq 1$, we need some more argument to extend $g''_{i,2} g_{i,2}$.

We use obstruction theory for this purpose. We remark that each of $i=1, \dots, N_2$ corresponds to a 2 simplex Δ_i^2 of K_+ . We define a two cochain $s \in C^2(K_+, \pi_1(G))$ as follows.

We choosed a trivialization $\hat{B}_{i,2}$ of our bundle on $U_{i,1}^2$. Then $g''_{i,2} g_{i,2}$ is regarded as a map $W_1 \cap U_{i,1}^2 \cong S^1 \times D^3 \rightarrow G$. We define $s(\Delta_i^2) \in \pi_1(G)$ to be the homotopy class of this map.

Lemma 3.40 $s = -\delta s'$ for some 1 cochain $s' \in C^1(K_+, \pi_1(G))$.

Proof: It is a standard fact of obstruction theory that $\delta s = 0$. We then have $[s] \in H^2(K_+, \pi_1(G))$.

In fact $[s] = 0$ is also a standard fact of obstruction theory. To make it clear, let us generalize the statement. Let $P_1, P_2 \rightarrow K_+$ be G principal bundles and $\varphi : P_1|_{K_{+,1}} \rightarrow P_2|_{K_{+,1}}$ be a bundle isomorphism on 1 skeleton $K_{+,1}$. We define $s(\varphi)$ as follows. On each 2 simplex Δ_i^2 , we trivialize P_j . Then $\varphi|_{\partial \Delta_i^2}$ is regarded as a map $S^1 = \partial \Delta_i^2 \rightarrow G$. Let $s(\varphi)(\Delta_i^2)$ be the homotopy class of this map. It is a standard obstruction theory that $\delta s(\varphi) = 0$ and that $[s(\varphi)] \in H^2(K_+, \pi_1(G))$ is zero if and only if the restriction of φ to 0 skeleton is extended to an isomorphism between P_1 and P_2 on 2 skeleton. In our case, $P_1 = P_2$. Hence $[s] = 0$.

Lemma 3.41 *There exists a finite subset Z of $\pi_1(G)$ independent of A , such that*

$s(\Delta_i^2) \in Z$ for each i .

Proof: By (3.36) and (3.39.2) $\|g_{i,2}'' g_{i,2}\|_{C^k(W_1 \cap U_{k,1}^2), \hat{B}_{i,2}}$ is uniformly bounded. The lemma then follows from Ascoli-Alzera.

Lemmata 3.41 and 3.40 imply that there exists a finite set Z' of $\pi_1(G)$ independent of A such that we can take s' , satisfying $\delta s' = -s$ and $s'(\Delta_i^1) \in Z'$ for each i . (This is because there is only finitely many cochains satisfying $s(\Delta_i^2) \in Z$.)

Now we modify $g_{i,1}'''$ in Lemma 3.26 using this chain s' as follows. For each $\gamma \in Z'$ we take a map $h_\gamma : U_{k,0}^1 \rightarrow G$ such that

$$(3.42.1) \quad h_\gamma = 1 \text{ on } U_{k,0}^1 \cap W_0.$$

(3.42.2) Let $p_1, p_2 \in U_{k,0}^1 \cap W_0$ belongs to the different component of $U_{k,0}^1 \cap W_0 = D^4 \amalg D^4$. Let ℓ be a path joining p_1 and p_2 . By (3.42.1), $h_\gamma \circ \ell$ determines an element of $\pi_1(G)$. We require that this element is γ .

We now replace $g_{i,1}'''$ by $h_{s'(\Delta_i^1)} \circ g_{i,1}'''$. By (3.42.1), the condition (3.27.1) is still satisfied. Since Z' is a finite set, the C^k norm of h_γ is uniformly bounded. Hence (3.27.2) is also satisfied. Using this modified $g_{i,1}'''$, we construct B_1, g_1 .

Then it is a consequence of $\delta s' = -s$ and a standard argument of obstruction theory that $s = 0$ as a cochain after modification. In other words, $g_{i,2}'' g_{i,2} : W_1 \cap U_{k,1}^2 \rightarrow G$ is 0 homotopic for each i . We now can apply Lemma 3.28 and obtain a gauge transformation $g_{i,2}'''$ on $U_{k,1}^2$ such that

$$(3.43.1) \quad g_{i,2}''' = g_{i,2}'' g_{i,2} \text{ on } W \cap U_{i,2}^2.$$

$$(3.43.2) \quad \|g_{i,2}'''\|_{C^k(U_{i,1}^2), \hat{B}_{2,i}} \leq C_k.$$

We put

$$B_2(x) = \begin{cases} B_1(x) & x \in W_1^- \\ (g_{i,2}''')^{-1*} \hat{B}_{i,2}(x) & x \in U_{i,2}^2 \end{cases}$$

$$g_2(x) = \begin{cases} g_1(x) & x \in W_1^- - \bigcup_i U_{i,1}^2 \\ h_{i,2}(g_{i,2}''')^{-1} & x \in U_{i,2}^2 - W_1 \\ g_1(x) \exp(\chi_2(x) X_{i,2}(x)) & x \in U_{i,2}^2 \cap W_1 \end{cases}$$

where χ_2 is a cut off function similar to χ_1 . We have

$$\|g_2^* A - B_2\|_{C^k(W_1), B_2} \leq C_k \|F_A\|_{L^2(Y)}.$$

Thus we constructed required flat connection and gauge transformation on W_2 . We can proceed in the same way on W_3, W_4 and prove Proposition 3.1.

We next prove Corollary 3.8 and Addendum 3.11. We first remark that, in the situation of Corollary 3.8, we can take covering $U_{i,k}^j$ independent of the metric $\mathbf{g}'$ we use. We remark also that constant C_k in (3.16) is independent of the metric $\mathbf{g}'$ satisfying (3.7). This follows immediately from Uhlenbeck's proof. Other constants are obviously independent of the metric. Corollary 3.8 follows. Addendum 3.11 is also obvious from construction. We only need to remark that we can take $\hat{B}_{i,j}$ such that $\|\hat{B}_{i,j}\|_{C^k} < C_k$. Then $\|B_j\|_{C^k} < C_k$ by (3.17.1), (3.27.2) etc.

To prove Proposition 3.6, we need to recall the proof of Cheeger's finiteness theorem [3]. We find that we need only a finite number of possibilities of the simplicial decomposition of K_+ and also we may choose U_i^j to be contained in a geodesic coordinate. Moreover the coordinate change of the geodesic coordinate has a uniformly bounded C^k norm. It implies that the constants μ_j in (3.14.2), (3.18.3), (3.34.3) are also uniformly bounded away from 0.

We also remark that the constant C_k in Formula (3.16) is uniformly bounded by Uhlenbeck's proof and the fact that the C^k norm of the metric tensor with respect to the geodesic coordinate is uniformly bounded.

We next remark that the cut off functions χ_j we use have uniformly bounded C^k norm. The existence of such cut off functions can be proved easily in our case by using uniformly C^k boundedness of distance function and $\mu_j > c > 0$.

Then the uniformly boundedness of the constant $C_{2,k}$ is an immediate consequence of the construction. The proof of Proposition 3.6 is now complete.

§ 4 Removable Singularity

In this section, we prove Theorem 1.5 assuming Mainlemma 4.42. The proof of Mainlemma 4.42 will be given in § 6. We first take an open covering of $D_1 \times \Sigma$ as follows. (We use this open covering also in the proof of compactness theorem in § 5.)

For each $(s_0, t_0) \in D_1$, $\mu \in [0, \frac{1}{2}]$, we put

$$Y_\mu(s_0, t_0) = [s_0 - 10\chi(s_0, t_0)^{1-\mu}, s_0 + 10\chi(s_0, t_0)^{1-\mu}] \times [t_0 - 10\chi(s_0, t_0)^{1-\mu}, t_0 + 10\chi(s_0, t_0)^{1-\mu}].$$

We study $Y_\mu(s_0, t_0) \times \Sigma$ together with scaled metric $\mathbf{g}' = \mathbf{g}'_{s_0, t_0} = \mathbf{g}/\chi(s_0, t_0)^2$. We change the coordinate by putting

$$(4.1) \quad \begin{aligned} s &= \chi(s_0, t_0)S + s_0 \\ t &= \chi(s_0, t_0)T + t_0 \end{aligned}$$

In this coordinate, the metric $\mathbf{g}'_{s_0, t_0}$ is

$$(4.2) \quad \mathbf{g}'_{s_0, t_0} = dS^2 \oplus dT^2 \oplus \alpha_{s_0, t_0}(S, T)^2 g_\Sigma.$$

Here

$$(4.3) \quad \alpha_{s_0, t_0}(S, T) = \frac{\chi(s, t)}{\chi(s_0, t_0)}, \quad \text{and}$$

$$(4.4) \quad K_\mu(s_0, t_0) = [s_0 - \chi(s_0, t_0)^{1-\mu}, s_0 + \chi(s_0, t_0)^{1-\mu}] \times [t_0 - \chi(s_0, t_0)^{1-\mu}, t_0 + \chi(s_0, t_0)^{1-\mu}].$$

We write $Y(s_0, t_0)$, $K(s_0, t_0)$ in place of $Y_0(s_0, t_0)$, $K_0(s_0, t_0)$, respectively. Let $\mathcal{K} \subseteq D$ be an arbitrary compact set, which we fix them for the rest of this section. We find an (infinite) set $\{(s_v, t_v) \in D_1 \mid v = 1, 2, \dots\}$ with the following properties

$$(4.5.1) \quad Y(s_v, t_v) \subseteq D_1 \times \Sigma,$$

$$(4.5.2) \quad \text{There exists an open neighborhood } U_1 \text{ of } L \cap \mathcal{K} \text{ such that } U_1 \cap D_1 \subseteq \bigcup_v \text{Int } K(s_v, t_v).$$

$$(4.5.3) \quad \text{The covering } \{Y(s_v, t_v) \mid v\} \text{ is locally finite. Moreover, for each } (s, t) \in D_1, \text{ the order of the set } \{v \mid (s, t) \in Y(s_v, t_v)\} \text{ is smaller than a number } N_0 \text{ which is independent of } (s, t).$$

We consider the covering

$$(4.6) \quad (U_1 \cap D_1) \times \Sigma \subseteq \bigcup_v \text{Int } K(s_v, t_v) \times \Sigma.$$

The member of our covering (4.6) becomes finer and finer as it approaches to $L = \partial D_1$. For each $\mathcal{K}$, we choose a covering (4.6) and fix it in § 4.5. On $\text{Int } K(s_v, t_v)$, we use metric (4.2) together with coordinate (4.1) (with (s_0, t_0) replaced by (s_v, t_v) .) They are of bounded geometry. In fact we can prove more.

Lemma 4.7 *We can choose our cut off function χ such that, for each $\lambda > 0$, $\mu \in [0, \frac{1}{2}]$, there exists $C_{5,k}(\lambda, \mu)$, $C_{6,k}$ satisfying :*

$$(4.8.1) \quad \left| \frac{\partial^k \alpha_{s_0, t_0}}{\partial S^l \partial T^{k-l}} \right| (S, T) \leq C_{5,k} \chi(s_0, t_0)^{k-\lambda} \leq C_{6,k},$$

$$(4.8.2) \quad \frac{1}{2} < \alpha_{s_0, t_0}(S, T) < 2.$$

for $-2\chi(s_0, t_0)^{-\mu} \leq S, T \leq 2\chi(s_0, t_0)^{-\mu}$, if $Y_\mu(s_0, t_0) \subseteq D \times \Sigma$. We remark that $C_{5,k}$ is independent of (s_0, t_0) .

Remark 4.9 In order $\mathbf{g}'_{(s_0, t_0)}$ to have a bonded geometry, the existence of $C_{6,k}$ is enough. We use stronger inequality (4.8.1) in the proof of Mainlemma 4.42 in § 6.

Proof: Let $f: D \rightarrow [-1, 1]$ be a smooth function such that $L = f^{-1}(0)$, $D_1 = f^{-1}([-1, 0])$, and that $df \neq 0$ on L . We put

$$\chi(s, t) = \begin{cases} e^{-\frac{1}{f(s, t)}} & (s, t) \in D_1 \\ 0 & (s, t) \in L \cup D_2 \end{cases}.$$

We remark that $\frac{1}{2} < \frac{\chi(s', t')}{\chi(s, t)} < 2$ if $\max\{|s - s'|, |t - t'|\} < \chi(s, t)^\mu$ and if $\chi(s, t)$ is small. (4.8.2) follows. (4.8.1) then follows by an easy calculation.

We thus have a family of metrics $\mathbf{g}'_{(s_0, t_0)}$ of bounded geometry on $[-2, 2]^2 \times \Sigma$. We then apply Corollary 3.8 and Addendum 3.11 to obtain constants $\varepsilon_4 = \varepsilon_3(\frac{1}{2}, 2, C_{6,\bullet})$, $C_{7,k} = C_{3,k}(\frac{1}{2}, 2, C_{6,\bullet})$, $C_{8,k} = C_{4,k}(\frac{1}{2}, 2, C_{6,\bullet})$. Here ε_3 and $C_{3,k}$ are as in Corollary 3.8 and $C_{4,k}$ is as in Addendum 3.11.

The next lemma is obvious.

Lemma 4.10 *There exists a compact (with respect to the C^∞ topology) set $\mathcal{R} \subset \hat{\mathcal{R}}(\Sigma, E)$ and $C_{9,k}(C_{8,\bullet})$ with the following properties. If $\mathcal{A}_0$ is a flat connection on $[-1, 1]^2 \times \Sigma$ with $\|\mathcal{A}_0\|_{C^k([-1, 1]^2 \times \Sigma, \mathbf{g}')} \leq C_{8,k}$, then there exists a gauge transformation g_1 and $A_0 \in \mathcal{R}$ such that*

$$(4.11.1) \quad g_1^* \mathcal{A}_0 = A_0 + 0dS + 0dT.$$

$$(4.11.2) \quad \|g_1\|_{C^k([-1, 1]^2 \times \Sigma, \mathbf{g}')} \leq C_{9,k}.$$

Let $\mathcal{A}$ be as in Corollary 3.8. By Corollary 3.8, Addendum 3.11, we have $\mathcal{A}_0$ and g_0 such that $\|\mathcal{A}_0\|_{C^k([-1, 1]^2 \times \Sigma, \mathbf{g}')} \leq C_{8,k}$ and $\|g_0^* \mathcal{A} - \mathcal{A}_0\|_{C^k([-1, 1]^2 \times \Sigma, \mathbf{g}')} \leq C_{7,k} \|F_{\mathcal{A}}\|_{L^2([-2, 2]^2 \times \Sigma, \mathbf{g}')}.$ By Lemma 4.10, we obtain g_1 and A_0 . We put $g = g_0 g_1$. We then have

$$(4.12.1) \quad \|g^* \mathcal{A} - A_0\|_{C^k([-1, 1]^2 \times \Sigma, \mathbf{g}')} \leq C_{10,k} \|F_{\mathcal{A}}\|_{L^2([-2, 2]^2 \times \Sigma, \mathbf{g}')}.$$

$$(4.12.2) \quad A_0 \in \mathcal{R}.$$

In other words, we may take $\mathcal{A}_0$ in Corollary 3.8 so that it has no dT , dS component. We next apply Lemma 2.14 to each element A_0 of $\mathcal{R} \subseteq \hat{R}(\Sigma, E)$ and obtain $\mathcal{G}_0(A_0)$ and $\mathcal{D}_0(A_0)$. Let $\Pi: \mathcal{G}_0(A_0) \times \mathcal{D}_0(A_0) \rightarrow \mathcal{A}_{L^2_{k_0}}(\Sigma, E)$ be the map in Lemma 2.14. We put

$$(4.13) \quad \Pi(\mathcal{G}_0(A_0) \times \mathcal{D}_0(A_0)) = \mathcal{U}(A_0).$$

Lemma 2.14 implies that $\mathcal{U}(A_0)$ is an open neighborhood of A_0 (with respect to the $L^2_{k_0}$ -topology.)

Lemma 4.14 *Let $\mathcal{A}$ be a connection satisfying the assumption of Theorem 1.5. Then, the following holds for every but finitely many v .*

There exists $A_{0,v} \in \mathcal{R} \subseteq R(\Sigma, E)$ and a gauge transformation g_v on $[-1, 1]^2 \times \Sigma \cong K(s_v, t_v)$ with the following properties.

We regard the restriction of $\mathcal{A}$ to $K(s_v, t_v)$ as a connection on $[-1, 1]^2 \times \Sigma$, and put $g_v^ \mathcal{A} = A'(S, T) + \Phi' dS + \Psi' dT$. Then $A'(S, T) \in \mathcal{U}(A_{0,v})$ for every $(S, T) \in [-1, 1]^2$.*

Proof: Let $\varepsilon_5 \leq \varepsilon_4$ be a positive number we determine later. By (4.5.3) and $\mathcal{E}(\mathcal{A}) < \infty$, we find that

$$(4.15) \quad \|F_{\mathcal{A}}\|_{L^2(Y(s_v, t_v) \times \Sigma, \mathbf{g})} < \varepsilon_5$$

holds for every but finitely many v . (In fact, the number of v such that (4.15) does not hold is smaller than $N_0 \mathcal{E}(\mathcal{A}) / \varepsilon_5^2$.)

We remark $\|F_{\mathcal{A}}\|_{L^2(Y(s_v, t_v) \times \Sigma, \mathbf{g})} = \|F_{\mathcal{A}}\|_{L^2(Y(s_v, t_v) \times \Sigma, \mathbf{g}'_{s_v, t_v})}$ since L^2 norm of the curvature is invariant of scaling. It follows from Corollary 3.8, Addendum 3.11 ((4.12.1)), (4.15) and $\varepsilon_5 \leq \varepsilon_4$ that there exists $A_{0,v} \in \mathcal{R} \subseteq R(\Sigma, E)$ and a gauge transformation g_v on $[-1, 1]^2 \times \Sigma \cong K(s_v, t_v) \times \Sigma$ such that

$$(4.16) \quad \|g_v^* \mathcal{A} - A_{0,v}\|_{C^k([-1, 1]^2 \times \Sigma, \mathbf{g}')} < C_{10,k} \varepsilon_5$$

Therefore (using (4.8.2) also), we find

$$(4.17) \quad \|A'(S, T) - A_{0,v}\|_{C^k(\Sigma, \mathbf{g}_{\Sigma})} < 2C_{10,k} \varepsilon_5$$

where $g_v^* \mathcal{A} = A'(S, T) + \Phi' dS + \Psi' dT$. We may choose ε_5 so that (4.17) implies $A'(S, T) \in \mathcal{U}(A_{0,v})$, since $\mathcal{U}(A_{0,v})$ is open in C^k topology. The proof of Lemma 4.14 is now complete.

By the same proof as above, we can prove the following lemma, which we use in the next section.

Lemma 4.18 *Let $\mathcal{A}$ be an ASD connection on $(D_1 \times \Sigma, \mathbf{g})$. We assume $\mathcal{E}_1(\mathcal{A}) < E_0$. Then, the number of v which does not satisfy the following $(\star)$ is smaller than a number $N(E_0)$ which depends only on E_0 (and the choice of $\mathcal{G}_0(A_0)$ and $\mathcal{D}_0(A_0)$.)*

$(\star)$ *There exists $A_{0,v} \in \mathcal{R} \subseteq R(\Sigma, E)$ and a gauge transformation g_v on $[-1, 1]^2 \times \Sigma \cong K(s_v, t_v) \times \Sigma$ with the following properties.*

We regards the restriction of $\mathcal{A}$ to $K(s_v, t_v)$ as a connection on $[-1, 1]^2 \times \Sigma$, and put $g_v^* \mathcal{A} = A'(S, T) + \Phi' dS + \Psi' dT$. Then $A'(S, T) \in \mathcal{U}(A_{0,v})$ for every $(S, T) \in [-1, 1]^2$.

Now let v satisfy the conclusion of Lemma 4.14 or 4.18 for $\mathcal{A}$ and g_v , $g_v^* \mathcal{A} = A'(S, T) + \Phi' dS + \Psi' dT$ as in the lemma. Since $\Pi : \mathcal{G}_0(A_0) \times \mathcal{D}_0(A_0) \rightarrow \mathcal{U}(A_{0,v})$ is a diffeomorphism, we have

Lemma 4.19 *Let $\mathcal{A}$ be a connection satisfying the assumption of Theorem 1.5. Then there exists a neighborhood U_2 of $\mathcal{K} \cap L$ such that $A(s, t)$ is complex gauge equivalent to a flat connection for $(s, t) \in U_2 \cap D_1$.*

We next recall the following fact. See [1] § 8 for the proof.

Lemma 4.20 *Let A_0, A_1 be flat connections on $E(\Sigma)$. Suppose that $g^* A_0 = A_1$ for a complex gauge transformation g . Then g is a real gauge transformation.*

By Lemmata 4.19 and 4.20, we obtain a map $\varphi : U_2 \cap D_1 \rightarrow R(\Sigma, E)$ such that $\varphi(s, t)$ is in the real gauge equivalence class of $g^* A(s, t)$ where g is a complex gauge transformation and $g^* A(s, t)$ is flat.

On the other hand, since $A(s, t)$ is flat for $(s, t) \in D_2 \cup L - \{(0, 0)\}$, we already have a map $\varphi : D_2 \cup L - \{(0, 0)\} \rightarrow R(\Sigma, E)$.

Thus we have $\varphi : D_2 \cup L \cup U_2 - \{(0, 0)\} \rightarrow R(\Sigma, E)$. This map is smooth. In fact we have the following easy :

Lemma 4.21 *Let $A(x)$ be a smooth family of elements of $\mathcal{A}(\Sigma, E)$ parametrized by $x \in Z$ for any manifold Z . Suppose that $A(x)$ is complex gauge equivalent to a flat connection for each x . Then the map $\varphi : Z \rightarrow R(\Sigma, E)$ defined by $\varphi(x) = [A(x)]$ is smooth.*

Proof: The problem is local, so we may work in a neighborhood V of $x_0 \in Z$. Let g_0 be a complex gauge transformation such that $g_0^* A(x_0)$ is flat. We put $g_0^* A(x_0) = A_0$. If x is close to x_0 , then $g_0^* A(x)$ is C^{k_0} close to A_0 . Therefore $g_0^* A(x) \in \mathcal{U}(A_0)$. Therefore the map $\bar{\varphi} : V \rightarrow \mathcal{D}_0(A_0)$ defined by $x \mapsto pr_2 \circ \Pi^{-1}(g_0^* A(x))$ is smooth. (Here $\Pi : \mathcal{G}_0(A_0) \times \mathcal{D}_0(A_0) \rightarrow \mathcal{U}(A_0)$ is as in Lemma 2.14 and $pr_2 : \mathcal{G}_0(A_0) \times \mathcal{D}_0(A_0) \rightarrow \mathcal{D}_0(A_0)$ is the projection to the second factor.) Since $\mathcal{D}_0(A_0)$ is diffeomorphic to an open subset of $R(\Sigma, E)$, Lemma 4.21 follows.

We thus obtained a smooth map $\varphi : D_2 \cup L \cup U_2 - \{(0, 0)\} \rightarrow R(\Sigma, E)$.

Lemma 4.22 $\varphi : (D_2 \cup L \cup U) \cap \mathcal{K} - \{(0, 0)\} \rightarrow R(\Sigma, E)$ is holomorphic.

Proof: For any open set W , we may find a lift $\hat{\varphi} : W \rightarrow \hat{R}(\Sigma, E)$ of φ . Then, Equation (1.1.1) together with Corollary 2.24 implies

$$\frac{\partial \hat{\varphi}}{\partial t} - *_\Sigma \frac{\partial \hat{\varphi}}{\partial s} \in \text{Im}(d_A) + *_\Sigma \text{Im}(d_A).$$

The lemma then follows from the fact that $*_\Sigma$ defines the complex structure on $R(\Sigma, E)$.

Lemma 4.23 $\int_{D_2 \cup L \cup U_2 - \{(0,0)\}} \varphi^* \omega < \infty.$

Proof: Lemma 4.23 is a consequence of the following Lemma 4.24.

Lemma 4.24 For each metric g' on $[-2, 2]^2 \times \Sigma$ satisfying (3.7.1), (3.7.2), there exists $\varepsilon_6(c_0, c_1, B_{k_0})$ depending only on c_0, c_1, B_{k_0} in (3.7.1), (3.7.2) with the following properties. Let $\mathcal{A} = A + \Phi dS + \Psi dT$ be an ASD connection on $[-2, 2]^2 \times \Sigma$ with respect to the metric g' . Assume $\|F_{\mathcal{A}}\|_{L^2([-2, 2]^2 \times \Sigma, g')} < \varepsilon_6$. Then

(4.25.1) $A(S, T)$ is complex gauge equivalent to a flat connection for any $(S, T) \in [-1, 1]^2$.

(4.25.2) We define $\varphi : [-1, 1]^2 \rightarrow R(E; \Sigma)$ by sending (s, t) to the complex gauge class of $A(s, t)$. (We remark that by (4.25.1) and Lemma 4.20, the complex gauge class of $A(s, t)$ determines an element of $R(E; \Sigma)$.) The map φ is smooth because of Lemma 4.21. Then we have

$$\int_{[-1, 1]^2} \varphi^* \omega \leq 10 \int_{[-1, 1]^2 \times \Sigma} \|F_{\mathcal{A}}\|_{g'}^2 \Omega_g.$$

Proof of Lemma 4.24 $\Rightarrow$ Lemma 4.23: Let us consider the covering (4.6). By (4.5.3), we find that $\|F_{\mathcal{A}}\|_{L^2(Y(s_v, t_v), g)} < \varepsilon_6$ except finitely many v . Let $v(1), \dots, v(N)$ be all v 's such that $\|F_{\mathcal{A}}\|_{L^2(Y(s_v, t_v), g)} \geq \varepsilon_6$. Then by Lemma 4.24 and conformal invariance of the L^2 norm of the curvature on 4 manifolds, we have

$$\int_{K(s_v, t_v)} \varphi^* \omega \leq 10 \int_{K(s_v, t_v) \times \Sigma} \left(\left\| \frac{\partial A}{\partial s} - d_A \Phi \right\|_{g_{\Sigma}}^2 + \left\| \frac{\partial A}{\partial t} - d_A \Psi \right\|_{g_{\Sigma}}^2 \right) \Omega_{g_{\Sigma}}$$

for any $v \neq v(i)$. Therefore, using (4.5.3), we have

$$\int_{U_2 \cap D_1} \varphi^* \omega < 10 N_0 \mathcal{E}_1(\mathcal{A}) + \sum_i \int_{K(s_{v(i)}, t_{v(i)})} \varphi^* \omega.$$

The right hand side is finite. Lemma 4.23 follows.

Proof of Lemma 4.24: We first use Corollary 3.8 to find a real gauge transformations $g_0(S, T)$ such that $g_0(S, T)^* A(S, T) \in \Pi(\mathcal{G}_0(A_0) \times \mathcal{D}_0(A_0))$ for some $A_0 \in \mathcal{R}$. (4.25.1) then follows. We are going to prove (4.25.2). Since both sides of the inequality are independent of real gauge transformation, we may assume that $A(S, T) \in \Pi(\mathcal{G}_0(A_0) \times \mathcal{D}_0(A_0))$.

We put $A(S, T) = \Pi(g(S, T), A'(S, T))$. Here $(g(S, T), A'(S, T)) \in \mathcal{G}_0(A_0) \times \mathcal{D}_0(A_0)$. (We remark that $g(S, T)$ is a complex gauge transformation.) We put

$$(4.26) \quad \begin{cases} \frac{\partial A}{\partial S} = B_1(S, T) + d_{A'(S, T)} u_1(S, T) + *_\Sigma d_{A'(S, T)} v_1(S, T) \\ \frac{\partial A}{\partial T} = B_2(S, T) + d_{A'(S, T)} u_2(S, T) + *_\Sigma d_{A'(S, T)} v_2(S, T) \end{cases}$$

$$(4.27) \quad \begin{cases} \frac{\partial A'}{\partial S} = B'_1(S, T) + d_{A'(S, T)} u'_1(S, T) + *_\Sigma d_{A'(S, T)} v'_1(S, T) \\ \frac{\partial A'}{\partial T} = B'_2(S, T) + d_{A'(S, T)} u'_2(S, T) + *_\Sigma d_{A'(S, T)} v'_2(S, T) \end{cases}$$

where $B_i(S, T) \in \text{Ker} d_{A(S, T)} \cap \text{Ker} d_{A(S, T)}^*$, $B'_i(S, T) \in \text{Ker} d_{A'(S, T)} \cap \text{Ker} d_{A'(S, T)}^*$. We remark that $\text{Ker} d_{A(S, T)} \cap \text{Ker} d_{A(S, T)}^*$ is orthogonal to $\text{Im} d_{A(S, T)} + \text{Im} *_\Sigma d_{A(S, T)}$.

Note however $\text{Im} d_{A(S, T)}$ is not orthogonal to $\text{Im} *_\Sigma d_{A(S, T)}$ in general. (The author thanks the referee of [8] who pointed out this fact.) (For the flat connection $A'(S, T)$, it is easy to see that $\text{Ker} d_{A'(S, T)}$ is orthogonal to $\text{Im} *_\Sigma d_{A'(S, T)}$.) We put $\mathcal{H}^1(\Sigma, A) \in \text{Ker} d_A \cap \text{Ker} d_A^*$.

A complex gauge transformation $g \in \Gamma(\Sigma; P_{U(2)} \times_{\mathcal{M}} SL(2; \mathbb{C}))$ acts on $\Gamma(\Sigma; P_{U(2)} \times_{\mathcal{M}} su(2))$ by

$$(4.28) \quad g^* B = g^{-1} B^{0,1} g + {}^t \bar{g} B^{1,0} {}^t \bar{g}^{-1}.$$

(4.28) is the linear part of the affine map (2.13) (which is defined on an infinite dimensional affine space $\mathcal{AE}(\Sigma)$.) Since $\text{Im} d_A + \text{Im} *_\Sigma d_A$ is the tangent space of the orbit of complex gauge transformation (see the proof of Lemma 2.14), it follows that complex gauge transformation preserves it. Namely we have

$$(4.29) \quad \begin{aligned} g(S, T)^* (\text{Im} d_{A'(S, T)} + \text{Im} *_\Sigma d_{A'(S, T)}) \\ = \text{Im} d_{g(S, T)^* A'(S, T)} + \text{Im} *_\Sigma d_{g(S, T)^* A'(S, T)} \end{aligned}$$

However, since complex gauge transformation is not necessary unitary, (4.29) does not imply that $\mathcal{H}^1(\Sigma, A)$ is preserved by complex gauge transformation. Namely

$$(4.30) \quad g^* \mathcal{H}^1(\Sigma, A) \not\subseteq \mathcal{H}^1(\Sigma, g^* A)$$

in general. So we define $g^{**} : \mathcal{H}^1(\Sigma, A) \rightarrow \mathcal{H}^1(\Sigma, g^* A)$ as follows. Let $B \in \mathcal{H}^1(\Sigma, A)$. Then

$$(4.31) \quad g^*(B) = g^{**} B + d_{g^* B} u + *_\Sigma d_{g^* B} v.$$

(4.31) implies

$$(4.32) \quad g^{**} g'^{**} = (g' g)^{**}.$$

Sublemma 4.33 $g(S, T)^{**} B'_i = B_i.$

Proof: We take the derivative of the formula $A(S, T) = g(S, T)^* A'(S, T)$, that is

$$A(S, T) = A'(S, T) + g(S, T)^{-1} \bar{\partial}_{A'(S, T)} g(S, T) + {}^t \overline{g(S, T)} \partial_{A'(S, T)} {}^t \overline{g(S, T)}^{-1}.$$

We then obtain

$$\begin{aligned}
\frac{\partial A}{\partial S} &= g(S, T)^{-1} \frac{\partial A'}{\partial S} g(S, T) + \overline{g(S, T)} \frac{\partial A'}{\partial S} \overline{g(S, T)}^{-1} \\
&\quad + \frac{\partial(g(S, T)^{-1})}{\partial S} \bar{\partial}_{A'(S, T)} g(S, T) + g(S, T)^{-1} \bar{\partial}_{A'(S, T)} \frac{\partial g(S, T)}{\partial S} \\
&\quad + \frac{\overline{\partial g(S, T)}}{\partial S} \partial_{A'(S, T)} \overline{g(S, T)}^{-1} + \overline{g(S, T)} \partial_{A'(S, T)} \frac{\overline{\partial(g(S, T)^{-1})}}{\partial S}.
\end{aligned}$$

In view of (4.27) and (4.29), we find that

$$g(S, T)^{-1} \frac{\partial A'}{\partial S} g(S, T) + \overline{g(S, T)} \frac{\partial A'}{\partial S} \overline{g(S, T)}^{-1} = g^{**} B_1' + d_A u + *_\Sigma d_A v.$$

On the other hand, by putting

$$\frac{\partial g(S, T)}{\partial S} = g(S, T) e(S, T)$$

and

$$\begin{aligned}
e(S, T) &= e_1(S, T) + \sqrt{-1} e_2(S, T) \\
\bar{e}_1(S, T) &= -e_1(S, T), \quad \bar{e}_2(S, T) = -e_2(S, T),
\end{aligned}$$

we have

$$\begin{aligned}
&\frac{\partial g(S, T)^{-1}}{\partial S} \bar{\partial}_{A'(S, T)} g(S, T) + g(S, T)^{-1} \bar{\partial}_{A'(S, T)} \frac{\partial g(S, T)}{\partial S} \\
&\quad + \frac{\overline{\partial g(S, T)}}{\partial S} \partial_{A'(S, T)} \overline{g(S, T)}^{-1} + \overline{g(S, T)} \partial_{A'(S, T)} \frac{\overline{\partial(g(S, T)^{-1})}}{\partial S} \\
&= d_{A(S, T)} e_1 - *_\Sigma d_{A(S, T)} e_2
\end{aligned}$$

Sublemma 4.33 follows.

Sublemma 4.34 *There exists a neighborhood of identity $\mathcal{G}_1(A_0)$ in $\mathcal{G}_0(A_0)$ and an neighborhood $\mathcal{D}_1(A_0)$ of A_0 in $\mathcal{D}_0(A_0)$ with the following property. For each $A_1 \in \mathcal{D}_1(A_0)$, $g \in \mathcal{G}_1(A_0)$ and $B \in \mathcal{H}^1(\Sigma, A_1)$, we have*

$$\frac{1}{2} \|B\|_{L^2} < \|g^{**}(B)\|_{L^2} < 2 \|B\|_{L^2}.$$

The sublemma follows easily from the continuity of g^{**} with respect to the parameters g and A_1 .

Let us go back to the proof of Lemma 4.24. We may choose ε_6 small enough so that Assumption $\|F_A\|_{L^2([-2, 2]^2 \times \Sigma, g)} \leq \varepsilon_6$ implies $(g(S, T), A'(S, T)) \in \mathcal{G}_1(A_0) \times \mathcal{D}_1(A_0)$, $A(S, T) = \Pi(g(S, T), A'(S, T))$.

We remark that $2\|\varphi^* \omega(\partial/\partial S, \partial/\partial T)\| = \|B_1'\|_{L^2(\Sigma)}^2 + \|B_2'\|_{L^2(\Sigma)}^2$, since φ is holomorphic. So we have

$$\int_{[-1,1]^2} \varphi^* \omega \leq \frac{1}{2} \int_{[-1,1]^2} \left(\|B'_1(S, T)\|_{L^2(\Sigma)}^2 + \|B'_2(S, T)\|_{L^2(\Sigma)}^2 \right) dS dT.$$

Hence by Sublemmata 4.33 and 4.34 we have

$$(4.35) \quad \int_{[-1,1]^2} \varphi^* \omega \leq \int_{[-1,1]^2} \left(\|B_1(S, T)\|_{L^2(\Sigma)}^2 + \|B_2(S, T)\|_{L^2(\Sigma)}^2 \right) dS dT.$$

We are going to estimate the right hand side by L^2 norm of the curvature.

We remark that since $\mathcal{A}$ is ASD we have

$$(4.36) \quad B_2 = *_\Sigma B_1.$$

We here recall a decomposition of $\Lambda^2(D \times \Sigma)$. Namely we have

$$\Lambda^2(D \times \Sigma) = \left(\Lambda^1(D) \otimes \Lambda^1(\Sigma) \right) \oplus \left(\Lambda^2(D) \otimes \Lambda^0(\Sigma) \right) \oplus \left(\Lambda^0(D) \otimes \Lambda^2(\Sigma) \right).$$

We can also decompose $\left(\Lambda^1(D) \otimes \Lambda^1(\Sigma) \right)$ to self dual and anti self dual part. Let $F_{\mathcal{A}}^{11+}$ and $F_{\mathcal{A}}^{11-}$ be the components of the curvature of $\mathcal{A}$ belonging to self dual and anti self dual part of $\Lambda^1(D) \otimes \Lambda^1(\Sigma) \otimes \left(P_{\ell(2)} \times_{\mathcal{M}} su(2) \right)$, respectively. Then we have

$$(4.37) \quad \|F_{\mathcal{A}}^{11-}\|^2 \leq \|F_{\mathcal{A}}\|^2,$$

pointwise. On the other hand, we can calculate $F_{\mathcal{A}}^{11-}$ in a similar way as (2.24) and obtain

$$\begin{aligned} F_{\mathcal{A}}^{11-} = & \frac{1}{2} \left(\left(-\frac{\partial A}{\partial S} + d_A \Phi \right) - *_\Sigma \left(-\frac{\partial A}{\partial T} + d_A \Psi \right) \right) \wedge dS \\ & - \frac{1}{2} \left(*_\Sigma \left(-\frac{\partial A}{\partial S} + d_A \Phi \right) + \left(-\frac{\partial A}{\partial T} + d_A \Psi \right) \right) \wedge dT. \end{aligned}$$

It follows from (1.1.1) that

$$(4.38) \quad \|F_{\mathcal{A}}^{11-}\|^2 = \frac{1}{4} \left(\left\| \left(-\frac{\partial A}{\partial S} + d_A \Phi \right) - *_\Sigma \left(-\frac{\partial A}{\partial T} + d_A \Psi \right) \right\|^2 \right).$$

pointwise. Then using (4.26) we have

$$(4.39) \quad \begin{aligned} & \left(-\frac{\partial A}{\partial S} + d_A \Phi \right) - *_\Sigma \left(-\frac{\partial A}{\partial T} + d_A \Psi \right) \\ & = -B_1 + *_\Sigma B_2 + d_{A(S, T)} u''(S, T) + *_\Sigma d_{A(S, T)} v''(S, T) \end{aligned},$$

for some u'', v'' . The orthogonality $\mathcal{H}^1(\Sigma, A) \perp (\text{Im } d_A + \text{Im } *_\Sigma d_A)$ and (4.36), (4.38), (4.39) imply

$$(4.40) \quad \|F_{\mathcal{A}}^{11-}\|_{L^2(\{(S, T)\} \times \Sigma; \alpha^2(S, T) g_\Sigma)}^2 \geq \|B_1(S, T)\|_{L^2(\Sigma; \alpha(S, T) g_\Sigma)}^2 = \|B_2(S, T)\|_{L^2(\Sigma; \alpha^2(S, T) g_\Sigma)}^2.$$

(4.35), (4.37) and (4.40) together with $c_1 < \alpha(s, t) < c_2$ imply Lemma 4.24.

To complete the proof of Theorem 1.5, we need Mainlemma 4.42. To state it we need some notation.

Let $k \geq k_0$. For each $[A_0] \in R(\Sigma, E)$, we take its representative $A_0 \in \hat{R}(\Sigma, E)$. We apply Lemma 2.14 to it, (with k_0 replaced by k) and obtain $\mathcal{G}_{0,k}(A_0) \subset \mathcal{G}_{\hat{E}_{k+1},c}(\Sigma, E)$, $\mathcal{D}_{0,k}(A_0) \subset \mathcal{D}_{L_k^2}(A_0)$ where

$$\mathcal{D}_{L_k^2}(A_0) = \left\{ A \in \mathcal{A}_{L_k^2}(\Sigma; E) \mid d_A^*(A - A_0) = 0, F_A = \Omega_A \right\},$$

such that $\Pi: \mathcal{G}_{0,k}(A_0) \times \mathcal{D}_{0,k}(A_0) \rightarrow \mathcal{A}_{0,k}(\Sigma; E)$, $(g, A) \mapsto g^* A$ is a diffeomorphism to a neighborhood of A_0 in $\mathcal{A}_{0,k}(\Sigma; E)$. In particular, $\mathcal{D}_{0,k}(A_0) \rightarrow \hat{R}(\Sigma; E) \rightarrow R(\Sigma; E)$ is a diffeomorphism to a neighborhood $R_k(A_0)$ of $[A_0] \in R(\Sigma, E)$. Choose finitely many $[A_\alpha]$ such that

$$(4.41) \quad \bigcup_{\alpha} R_k(A_\alpha) = R(\Sigma, E).$$

We put $\mathcal{R}_k(A_\alpha) = \mathcal{D}_{0,k}(A_0)$. (We remark our finite set $\{A_\alpha\}$ depends on k .) We put $\lambda = \frac{1}{100k}$.

Mainlemma 4.42 *There exists $\varepsilon_7(k, C_{5,0}(\mu, \lambda), \dots, C_{5,k+100}(\mu, \lambda), D_1, \dots, D_{k+100}) > 0$, $C_{11,k}(C_{5,0}(\mu, \lambda), \dots, C_{5,k+100}(\mu, \lambda), D_1, \dots, D_{k+100})$ with the following properties. ($C_{5,k}(\mu, \lambda)$ is as in Lemma 4.7.)*

Let $\mathcal{A}' = A' + \Phi' dS + \Psi' dT$ be a connection on $Y_\mu(s_0, t_0)$ which satisfies (1.1) with $\chi \equiv 0$. Assume

$$(4.43.1) \quad A'(S, T) \in \mathcal{R}_{k+100}(A_\alpha) \text{ for some } \alpha \text{ independent of } (S, T),$$

$$(4.43.2) \quad \left| \frac{\partial^m A'}{\partial S^\ell \partial T^{m-\ell}} \right|_{C^{k+100}(\Sigma, g_\Sigma)} \leq D_m \chi(s_0, t_0)^m \text{ for } m \leq k+100,$$

$$(4.43.3) \quad \left| \frac{\partial^m \Phi'}{\partial S^\ell \partial T^{m-\ell}} \right|_{C^{k+100}(\Sigma, g_\Sigma)} \leq D_m \chi(s_0, t_0)^{m+1} \text{ for } m \leq k+99,$$

$$(4.43.4) \quad \left| \frac{\partial^m \Psi'}{\partial S^\ell \partial T^{m-\ell}} \right|_{C^{k+100}(\Sigma, g_\Sigma)} \leq D_m \chi(s_0, t_0)^{m+1} \text{ for } m \leq k+99.$$

Let g'' be a complex gauge transformation group on $Y_\mu(s_0, t_0)$ with $\overline{g''} = g''$. We assume that $g''^ \mathcal{A}'$ is an ASD connection with respect to the metric $g'_{s_0, t_0} = dS^2 \oplus dT^2 \oplus \alpha_{s_0, t_0}(S, T)^2 g_\Sigma$ satisfying (4.8). We also assume that*

$$(4.44.1) \quad \|F_{g''^* \mathcal{A}'}\|_{L^2([S_0-1, S_0+1] \times [T_0-1, T_0+1] \times \Sigma, g')} < \varepsilon_7,$$

$$(4.44.2) \quad \chi(s_0, t_0) \leq \varepsilon_7.$$

for any $(S_0, T_0) \in [-\chi(s_0, t_0)^{-\mu} + 1, \chi(s_0, t_0)^{-\mu} - 1]^2$. (Here we use the metric $g' = g'_{s_0, t_0}$ in

(4.2).) Then we have

$$(4.45.1) \quad \|g''(S_0, T_0) - 1\|_{C^k(\Sigma, g_\Sigma)} \leq C_{11,k} \left(e^{\frac{-\chi(s_0, t_0)}{C_{11,k}}} + \chi(s_0, t_0)^2 \right),$$

$$(4.45.2) \quad \left\| \frac{\partial^m g''}{\partial S^i \partial T^{m-i}}(S_0, T_0) \right\|_{C^k(\Sigma, g_\Sigma)} \leq C_{11,k} \left(e^{\frac{-\chi(s_0, t_0)}{C_{11,k}}} + \chi(s_0, t_0)^{m+2} \right),$$

for $m \leq k$ and $(S_0, T_0) \in [-1, 1]^2$.

Postponing the proof of Mainlemma 4.42 to § 6, we are now in the position to complete the proof of Theorem 1.5. We use removable singularity theorem of holomorphic map together with Lemma 4.23, to show that $\varphi : D_2 \cup L \cup U_2 - \{(0,0)\} \rightarrow R(\Sigma, E)$ is extended to $\varphi : D_2 \cup L \cup U_2 \rightarrow R(\Sigma, E)$. Since $D_2 \cup L \cup U_2$ is contractible, we can lift φ to a smooth map $\hat{\varphi} : D_2 \cup L \cup U_2 \rightarrow \hat{R}(\Sigma, E)$. We put $\hat{\varphi}(s, t) = \hat{A}'(s, t)$. By construction, there is a family of complex gauge transformations $g'(s, t)$ parametrized by $(s, t) \in D_2 \cup L \cup U_2 - \{(0,0)\}$ such that $A(s, t) = g'(s, t)^* \hat{A}'(s, t)$. We here use the following easy

Lemma 4.46 *Let $A(s, t) = g'(s, t)^* \hat{A}'(s, t)$. Here $A(s, t)$, $\hat{A}'(s, t)$ are smooth families of smooth connections on $E(\Sigma)$ and $g'(s, t)$ be a family of elements of $\Gamma(\Sigma; \text{Ad}_{SL(2, \mathbb{C})} P(\Sigma))$. Let $[g'(s, t)]$ be the equivalence class in $\Gamma(\Sigma; \text{Ad}_{SL(2, \mathbb{C})} P(\Sigma)) / \{\pm 1\}$. Then $(s, t) \mapsto [g'(s, t)]$ is smooth.*

The proof of Lemma 4.46 is similar to the proof of Lemma 4.21 and is omitted.

By Lemma 4.46, we may assume that $g'(s, t)$ is smooth with respect to (s, t) , replacing $g'(s, t)$ by $-g'(s, t)$ for some (s, t) . We next take the polar decomposition. Namely we put $g'(s, t) = \hat{g}''(s, t)g(s, t)^{-1}$ where $\overline{g(s, t)} = g(s, t)^{-1}$ and $\overline{\hat{g}''(s, t)} = \hat{g}''(s, t)$. $g(s, t), \hat{g}''(s, t)$ are both smooth. We regards $g(s, t)$ as a real gauge transformation g on $(D_2 \cup L \cup U_2) \times \Sigma$. By changing it at $(D_2 \cup L \cup U_2) - \mathcal{K}$, we can extend the restriction of g to $((D_2 \cup L \cup U_2) \cap \mathcal{K} - \{(0,0)\}) \times \Sigma$ to a complex gauge transformation on $(D - \{(0,0)\}) \times \Sigma$. We are going to verify that $g^* \mathcal{A} = g''^* \hat{\mathcal{A}}'$ is extended to a smooth connection on $D \times \Sigma$. (We recall $\mathcal{A} = g' \hat{\mathcal{A}}'$.)

We regards $\hat{g}''(s, t)$ as a complex gauge transformation $\hat{g}''$ on $(D - \{(0,0)\}) \times \Sigma$. We remark that $\hat{g}'' = 1$ on $D_2 \times \Sigma$, by Lemma 4.20. We extend $\hat{g}''$ to $D \times \Sigma$ by putting $\hat{g}'' = 1$ on $\{(0,0)\} \times \Sigma$. It suffices to show that this extended $\hat{g}''$ is smooth at $\{(0,0)\} \times \Sigma$. For this purpose it suffices to show that

$$(4.47) \quad \left\| \frac{\partial^k \hat{g}''}{\partial s^i \partial t^{k-i}}(s_0, t_0) - 1 \right\|_{C^k(\Sigma, g_\Sigma)} < C_k \chi(s_0, t_0)^2.$$

We are going to prove (4.47) by using Mainlemma 4.42.

We take an arbitrary k and fix it. There exists a neighborhood W_k of $(0,0)$ such that $\varphi(W_k) \subseteq R_{k+100}(A_\alpha)$ for some α . Then, for each $(s, t) \in W_k$, there exists a unique $A'(s, t) \in \mathcal{R}_{k+100}(A_\alpha)$ such that

$$(4.48) \quad \varphi(s, t) = [A'(s, t)].$$

There exists a real gauge transformation $h(s, t)$ such that $\hat{A}'(s, t) = h(s, t)^* A'(s, t)$. We put

$$(4.49) \quad g''(s, t) = h(s, t) \hat{g}''(s, t) h(s, t)^{-1}.$$

Then $\overline{g}''(s, t) = g''(s, t)$. We regard $g''(s, t)$ as a complex gauge transformation g'' on $(W_k - \{(0, 0)\}) \times \Sigma$.

We next use the change of coordinate (4.1). Since φ is a holomorphic function, we have Φ', Ψ' such that

$$(4.50) \quad \frac{\partial A'}{\partial S} + *_\Sigma \frac{\partial A'}{\partial T} = d_{A'} \Psi' + *_\Sigma d_{A'} \Phi'.$$

Since A' is a flat connection, it follows that $d_{A'}$ and $*_\Sigma d_{A'}$ are invertible. Therefore such Φ', Ψ' are unique. We put $\mathcal{A}' = A' + \Phi' dS + \Psi' dT$. We have

$$\begin{aligned} g''(S, T)^* A'(S, T) &= h(S, T)^{-1*} \hat{g}''(S, T)^* \hat{A}'(S, T) \\ &= h(S, T)^{-1*} g'(S, T)^* \hat{A}'(S, T) = h(S, T)^{-1*} g(S, T)^* A(S, T). \end{aligned}$$

On the other hand, $g''^{-1*} h^* g^* \mathcal{A}$ satisfies (4.50) by Corollary 2.25. Hence $h^{-1*} g^* \mathcal{A} = g''^* \mathcal{A}'$. It follows that $g''^* \mathcal{A}'$ is ASD. (Note gh^{-1} is a real gauge transformation and $\mathcal{A}$ is ASD.)

We use the fact that $A'(s, t)$ is smooth, to show that there exists $C_{12, k}$ such that

$$(4.51) \quad \left| \frac{\partial^k A'}{\partial S^i \partial T^{k-i}} \right| (S, T) \leq C_{12, k} \chi(s_0, t_0)^k.$$

Namely (4.43.2) holds. (4.50), (4.51) and invertibility of $d_{A'}$ and $*_\Sigma d_{A'}$ implies that Φ', Ψ' satisfies Assumption (4.43.3), (4.43.4), where D_k depends only on $C_{12, m}$, $m < k + 10$.

We may choose our neighborhood W_k sufficiently small such that $\|F_{\mathcal{A}}\|_{L^2((W_k \cap D_1) \times \Sigma_g)} < \varepsilon_7$. Hence

$$(4.52) \quad \|F_{g''^* \mathcal{A}'}\|_{L^2((W_k \cap D_1) \times \Sigma_g)} < \varepsilon_7.$$

We take (s_0, t_0) sufficiently close to $(0, 0)$ such that

$$Y_\mu(s_0, t_0) \subset W_k \cap D_1.$$

for some fixed positive μ , say $\mu = \frac{1}{2}$. It then follows from (4.52) that Assumption (4.44.1) is satisfied.

By shrinking the neighborhood W_k if necessary, we may assume (4.44.2) and :

$$(4.53) \quad \exp\left(-\frac{\chi(s_0, t_0)^{-\mu}}{C_{11, k}}\right) + \chi(s_0, t_0)^{2+k} < 2\chi(s_0, t_0)^{2+k}$$

if $(s_0, t_0) \in W_k$. Now Mainlemma 4.42 applied to $A'(S, T)$ and g'' together with (4.53) implies that

$$(4.54.1) \quad \|g''(0,0) - id\|_{C^k(\Sigma, g_\Sigma)} \leq 2C_{11,k} \chi(s_0, t_0)^2,$$

$$(4.54.2) \quad \left| \frac{\partial^m g''}{\partial S^l \partial T^{m-l}}(0,0) \right|_{C^l(\Sigma, g_\Sigma)} \leq 2C_{11,k} \chi(s_0, t_0)^{m+2}$$

for $m \leq k$.

(4.54) and uniformly boundedness of C^{k+10} norm of h implies (4.47) and hence Theorem 1.5.

§ 5 Compactness

In this section, we prove Theorems 1.6 and 1.7 assuming Mainlemma 4.42. Let $\mathcal{A}_i = A_i + \Phi_i ds + \Psi_i dt$ be as in Theorems 1.6. Let $\mathcal{K} \subseteq D$ be an arbitrary compact set. We use open covering (4.6). We put

$$(5.1) \quad \mathcal{M}_v(\mathcal{A}_i) = \int_{Y(s_v, t_v) \times \Sigma} \|F_{\mathcal{A}_i}\|^2 \Omega_g.$$

Assumption $\mathcal{E}(\mathcal{A}_i) \leq C$ and (4.5.3) imply that

$$(5.2) \quad \sum_v \mathcal{M}_v(\mathcal{A}_i) \leq C_{13},$$

for some positive number C_{13} independent of i . We use this facts to show the following :

Lemma 5.3 *For any $\varepsilon > 0$, there exists a subsequence of $\mathcal{A}_i$ (still denoted by $\mathcal{A}_i$), a finitely many $v_1, \dots, v_{N_1(\varepsilon)}$ independent of i , and a finite subset $P(\varepsilon)$ of L independent of i , with the following properties.*

For any neighborhood W of $P(\varepsilon)$, there exists $N(W, \varepsilon)$ such that for any $i > N(W, \varepsilon)$ and v with $Y(s_v, t_v) \cap \mathcal{K} \neq \emptyset$, one of the following holds.

$$(5.4.1) \quad Y(s_v, t_v) \subseteq W.$$

$$(5.4.2) \quad v = v_j \text{ for some } j.$$

$$(5.4.3) \quad \mathcal{M}_v(\mathcal{A}_i) < \varepsilon.$$

Proof: (5.2) implies that, for each i , the number of v such that $\mathcal{M}_v(\mathcal{A}_i) \geq \varepsilon$ is smaller than C_{13}/ε . We take points $p_{i,1} \in Y(s_{v_{i,1}}, t_{v_{i,1}})$, $\dots$, $p_{N(i),i} \in Y(s_{v_{i,N(i)}}, t_{v_{i,N(i)}})$. By increasing the number of points if necessary, we may assume that $N(i)$ is independent of i . By taking a subsequence if necessary, we may assume that each of $p_{i,j}$ converges to $p_{\infty,j}$. We remark $p_{\infty,j} \in \mathcal{K} \subseteq U_2$. By changing the index, we may assume that $p_{\infty,j} \in D_1$ for $j=1, \dots, N_1(\varepsilon)$ and $p_{\infty,j} \in L$ for $j > N_1(\varepsilon)$.

It is easy to see that, we may assume, by taking a subsequence, that $v_{i,j}$ is independent of j for $j=1, \dots, N_1(\varepsilon)$. We put $v_j = v_{i,j}$ for $j=1, \dots, N_1(\varepsilon)$. We put $N(W, \varepsilon) = N(\varepsilon) - N_1(\varepsilon)$, and $P(\varepsilon) = \{p_{\infty,j+N_1(\varepsilon)} \mid j=1, \dots, N_2(\varepsilon)\}$. It is easy to see that the conclusion of the lemma is satisfied.

Lemma 5.5 *There exist finite subsets $P'_3 \subseteq L$ and $P_4 \subseteq D_2$ and a neighborhood U_2 of $L \cap \mathcal{K}$ and a subsequence of $\mathcal{A}_i$ (still denoted by $\mathcal{A}_i$) with the following properties.*

For each neighborhood W of P'_3 there exists $N(W)$ such that if $i > N(W)$ then $A_i(s, t)$ is complex gauge equivalent to a flat connection for each $(s, t) \in U - W$.

Moreover defining $\varphi_i : U \cup D_2 - W \rightarrow R(\Sigma, E)$ by $\varphi_i(s, t) = [A_i(s, t)]$ we have

$$(5.6) \quad \int_{U_2 \cup D_2 - W} \varphi_i^* \omega < 10C_{14}.$$

Proof: The proof is similar to one of Lemma 4.23. Let $\varepsilon_n = \varepsilon_5(\frac{1}{2}, 2, C_{k_0+100})$ be as in Lemma 4.24. (Here $C_{n,k}$ is as in (4.8).) We apply Lemma 5.3 to $\varepsilon = \varepsilon_n$ to get $P'_3 = P(\varepsilon_n)$ and a subsequence of $\mathcal{A}_i$.

We may take U so that if $(s, t) \in U$ then $(s, t) \in K(s_v, t_v)$ with $v \neq v_j$, $j = 1, \dots, N_1(\varepsilon_n)$.

Now let $(s, t) \in U - W$ and $i > N(W, \varepsilon_n)$. Take v such that $(s, t) \in K(s_v, t_v)$. By the above choice of U , (5.4.2) does not occur. Since $(s, t) \in U - W$, (5.4.1) does not occur either. Therefore Lemma 5.3 implies $\mathcal{M}_v(\mathcal{A}_i) < \varepsilon_n$. It follows from Lemma 4.24, (4.28.1) that $A_i(s, t)$ is complex gauge equivalent to a flat connection.

(5.6) also follows from Lemma 4.24, (4.25.2) and (5.2).

Lemma 5.5 and Sacks-Uhlenbeck-Gromov compactness implies the following. We use the same notation as Lemma 5.5.

Lemma 5.7 *There exist a subsequence of $\mathcal{A}_i$ (still denoted by $\mathcal{A}_i$) and a finite subset $P_4 \subseteq D_2$ such that $\varphi_i : U_2 \cup D_2 - W - P_4 \rightarrow R(\Sigma, E)$ converges in C^∞ topology on any compact subset of $U_2 \cup D_2 - W - P_4$.*

Since W is an arbitrary neighborhood of P'_3 , it follows from diagonal argument that we have a subsequence of φ_i (denoted by the same symbol) converging to φ_∞ on any compact subset of $U_2 \cup D_2 - P'_3 - P_4$.

We fix a sufficiently large positive integer k_0 and first show C^{k_0} convergence in place of C^∞ convergence, by a technical reason. We also fix $\mu \in (0, \frac{1}{2}]$.

Let $\mathcal{K}_1 = \mathcal{K} \cap U_2 - W_1$ where W_1 is a neighborhood of $P'_3 \cup (U \cap P_4)$. φ_i is defined on a neighborhood of $\mathcal{K}_1$ for large i and converges to φ_∞ in C^∞ topology on a neighborhood of $\mathcal{K}_1$.

We consider the open covering (4.41) $R(\Sigma; E) = \bigcup_{\alpha} R_{k_0+100}(A_{\alpha})$. We choose $\overline{R'_{k_0+100}(A_{\alpha})} \subseteq R_{k_0+100}(A_{\alpha})$ such that $R_{k_0+100}(\Sigma; E) = \bigcup_{\alpha} R'_{k_0+100}(A_{\alpha})$.

By shrinking U_2 if necessary we may assume that

$$(5.8) \quad \varphi_\infty(Y_\mu(s_v, t_v)) \subseteq R'_{k_0+100}(A_{\alpha_v}),$$

for each $Y_\mu(s_v, t_v)$ such that $Y_\mu(s_v, t_v) \cap \mathcal{K}_1 \neq \emptyset$. (This is because the diameter of $Y_\mu(s_v, t_v)$ (with respect to the Euclidean metric on $D \subseteq \mathbf{C}$) converges to 0 as $(s_v, t_v) \rightarrow L$.) Therefore

$$(5.9) \quad \varphi_i(Y_\mu(s_v, t_v)) \subseteq R_{k_0+100}(A_{\alpha_v}),$$

for large i . Therefore we can lift $\varphi_i : Y_\mu(s_v, t_v) \rightarrow R_{k_0+100}(A_{\alpha_v})$ to $\hat{\varphi}_i : Y_\mu(s_v, t_v) \rightarrow R_{k_0+100}(A_{\alpha_v})$. We put

$$(5.10) \quad \hat{\varphi}_i(s, t) = A'_i(v : s, t),$$

for $(s, t) \in Y_\mu(s_v, t_v)$. We then change the coordinate by (4.1), with (s_0, t_0) replaced by (s_v, t_v) . Since φ_i is holomorphic, we have $\Phi'_i(v : S, T)$ and $\Psi'_i(v : S, T)$ such that

$$(5.11) \quad \frac{\partial A'_i(v: S, T)}{\partial S} + *_\Sigma \frac{\partial A'_i(v: S, T)}{\partial T} = d_A \Psi'_i(v: S, T) + *_\Sigma d_A \Phi'_i(v: S, T).$$

On the other hand, we have :

$$(5.12) \quad \left\| \frac{\partial^m \hat{A}'_i(v: S, T)}{\partial S^\ell \partial T^{m-\ell}}(S_0, T_0) \right\|_{C^{k_0+100}(\Sigma, g_\Sigma)} \leq C_k \chi(s_v, t_v)^m.$$

for $m \leq k_0 + 100$, $(S_0, T_0) \in Y_\mu(s_v, t_v)$. Therefore, in a way similar to the argument at the end of the last section, we have

$$(5.13.1) \quad \left\| \frac{\partial^m \Phi'_i(v: S, T)}{\partial S^\ell \partial T^{m-\ell}}(S_0, T_0) \right\|_{C^{k_0+100}(\Sigma, g_\Sigma)} \leq C_{15} \chi(s_v, t_v)^{m+1} \text{ for } m \leq k_0 + 99,$$

$$(5.13.2) \quad \left\| \frac{\partial^m \Psi'_i(v: S, T)}{\partial S^\ell \partial T^{m-\ell}}(S_0, T_0) \right\|_{C^{k_0+100}(\Sigma, g_\Sigma)} \leq C_{15} \chi(s_v, t_v)^{m+1} \text{ for } m \leq k_0 + 99.$$

We remark that C_{15} are independent of v and i . We put

$$(5.14) \quad \varepsilon_8 = \varepsilon_7(k_0, C_{5,1}(\mu, \lambda), \dots, C_{5,k_0+100}(\mu, \lambda); C_{15,1}, \dots, C_{15,k_0+100})$$

where ε_7 is as in MainLemma 4.42. We apply Lemma 5.3 again for $\varepsilon = \min\{\varepsilon_8, \varepsilon_6\}$. (Here $\varepsilon_6 = \varepsilon_6(\frac{1}{2}, 2, C_{6,k_0})$ is as in Lemma 4.24.) We then obtain $P'_3 = P(\min\{\varepsilon_8, \varepsilon_6\})$. Let $\mathcal{K}_2 = \mathcal{K} \cap U_2 - W_2$ where W_2 is a neighborhood of $P'_3 \cup (U_2 \cap P_4)$. ($W_2 \supseteq W_1$.)

Lemma 5.3 then implies that

$$(5.15) \quad \gamma \mathcal{M}_v(\mathcal{A}_i) < \varepsilon_8$$

if $Y_\mu(s_v, t_v) \cap \mathcal{K}_2 \neq \emptyset$ and i large.

We are going to show that $\mathcal{A}_i$ is gauge equivalent to a sequence of connections which converges in C^{k_0-1} topology on $\mathcal{K}_2 \times \Sigma$.

Since φ_i converges to φ_∞ in C^∞ topology on $\mathcal{K}_2 \times \Sigma$, it follows that we have $\hat{\varphi}'_i: \mathcal{K}_2 \rightarrow \hat{R}(\Sigma; E)$, $\hat{\varphi}'_\infty: \mathcal{K}_2 \rightarrow \hat{R}(\Sigma; E)$ such that $\hat{\varphi}'_i$ converges to $\hat{\varphi}'_\infty$ in C^∞ topology. In other words, if we put $\hat{\varphi}'_i(s, t) = \hat{A}'_i(s, t)$, $\hat{\varphi}'_\infty(s, t) = \hat{A}'_\infty(s, t)$ then $\hat{A}'_i(s, t)$ converges to $\hat{A}'_\infty(s, t)$ on $\mathcal{K}_2 \times \Sigma$ in C^∞ topology. By construction there exists a family of complex gauge transformations $g'_i(s, t)$ such that

$$(5.16) \quad A_i(s, t) = g'_i(s, t)^* \hat{A}'_i(s, t),$$

We use polar decomposition and obtain $g'_i(s, t) = \hat{g}'_i(s, t) g_{1,i}(s, t)^{-1}$ where $g_{1,i}(s, t)$ is real and $\hat{g}'_i(s, t)$ is hermitian. We are going to show the following inequality.

Lemma 5.17 *For large i , the following inequality holds for each $(s, t) \in \mathcal{K}_2$*

$$(5.18.1) \quad \|\hat{g}'_i(s, t) - 1\|_{C^{k_0}(\Sigma, g_\Sigma)} \leq C_{16,0} \chi(s, t)^2,$$

$$(5.18.2) \quad \left\| \frac{\partial^m \hat{g}''_i}{\partial s^\ell \partial t^{m-\ell}}(s, t) \right\|_{C^{k_0}(\Sigma, g_\Sigma)} \leq C_{16,m} \chi(s, t)^{m+2} \text{ for } m \leq k_0.$$

Proof: The proof is similar to the proof of (4.47) in § 4. Let $Y_\mu(s_v, t_v) \cap \mathcal{K}_2 \neq \emptyset$. There exist a real gauge transformation $h_i(v; s, t)$ on $Y_\mu(s_v, t_v) \times \Sigma$ such that

$$(5.19) \quad h_i(v; s, t)^* A'_i(v; s, t) = \hat{A}'_i(s, t),$$

C^{k_0+100} norm of $h_i(v; s, t)$ is bounded by a constant independent of i, v , since $A'_i(v; s, t) \in \bigcup \mathcal{R}_{k_0+100}(A_\alpha)$ and $\mathcal{R}_{k_0+100}(A_\alpha)$ is compact, and since $\hat{A}'_i$ converges in C^∞ -topology. We put

$$(5.20) \quad g_i''(v; s, t) = h_i(v; s, t) \hat{g}_i'(s, t) h_i(v; s, t)^{-1},$$

We change the variable by (4.1) with (s_0, t_0) replaced by (s_v, t_v) . We want to apply Mainlemma 4.42 to $g_i''(v; S, T)$, $A'_i(v; S, T)$. We can check easily that $g_i''^* A'_i$ is ASD. The Condition (4.43.1) holds by construction. (4.43.2), (4.43.3) and (4.43.4) hold by (5.12), (5.13.1) and (5.13.2). To check (4.44.1) we remark that by (4.1)

$$[S_0 - 1, S_0 + 1] \times [T_0 - 1, T_0 + 1] \cong [s_0 - \chi(s_v, t_v), s_0 + \chi(s_v, t_v)] \times [t_0 - \chi(s_v, t_v), t_0 + \chi(s_v, t_v)]$$

is contained in some $Y(s_v, t_v)$. (Here we put $s_0 = s_v + S_0 \chi(s_v, t_v)$, $t_0 = t_v + T_0 \chi(s_v, t_v)$.) Therefore (4.44.1) follows from (5.14) and (5.15). We may assume that (s_0, t_0) is close enough to L so that (4.44.2) holds.

We thus can apply Mainlemma 4.42 and obtain

$$(5.21.1) \quad \|g_i''(v; S_0, T_0) - 1\|_{C^{k_0}(\Sigma, g_2)} \leq 2C_{11, k_0} \chi(s_v, t_v)^2,$$

$$(5.21.2) \quad \left\| \frac{\partial^m g_i''(v, S, T)}{\partial S^\ell \partial T^{m-\ell}}(S_0, T_0) \right\|_{C^{k_0}(\Sigma, g_2)} \leq 2C_{11, k_0} \chi(s_v, t_v)^{m+2}$$

for $m \leq k_0$ $(S_0, T_0) \in [0, 1]^2$.

(5.21.2) implies

$$(5.22) \quad \|g_i''(v, s, t) - 1\|_{C^{k_0}(K(s_v, t_v) \times \Sigma, g_{m, s, t})} \leq 2C_{11, k_0} \chi(s_v, t_v)^2, \text{ for } m \leq k_0.$$

Since $\bigcup K(s_v, t_v) \supseteq \mathcal{K}_2$ and since C^{k_0+100} norm of $h_i(v; s, t)$ is bounded, (5.21.1), (5.22) implies Lemma 5.17.

We next use Uhlenbeck compactness theorem (see [6]) on $(D_1 \cap \mathcal{K}) \times \Sigma$ to obtain the following.

Lemma 5.23 *There exist a finite subset $P_1 \subseteq (D_1 \cap \mathcal{K}) \times \Sigma$, a sequence of (real) gauge transformations $g_{2, i}$ on $(D_1 \cap \mathcal{K}) \times \Sigma - P_1$ and a subsequence of $\mathcal{A}_i$ (still denoted by the same symbol) such that $g_{2, i}^* \mathcal{A}_i$ converges in C^∞ topology on any compact subset of $(D_1 \cap \mathcal{K}) \times \Sigma - P_1$.*

We put

$$(5.24) \quad \lim_{i \rightarrow \infty} g_{2, i}^* \mathcal{A}_i = \mathcal{A}_\infty.$$

$$(5.25) \quad \mathcal{A}_\infty = A_\infty + \Phi_\infty ds + \Psi_\infty dt.$$

We may assume that $\mathcal{K}_2 \cap P_1 = \emptyset$. Since $A_i(s, t)$ is complex gauge equivalent to $\hat{A}'_i(s, t)$ and since $\hat{A}'_i(s, t)$ converges to $\hat{A}'_\infty(s, t)$, it follows that $A_\infty(s, t)$ is complex gauge equivalent to $\hat{A}'_\infty(s, t)$ for $(s, t) \in \mathcal{K}_2$. We put

$$(5.26) \quad A_\infty(s, t) = g'_\infty(s, t)^* \hat{A}'_\infty(s, t).$$

We apply polar decomposition to $g'_\infty(s, t)$ and obtain $g'_\infty(s, t) = \hat{g}''_\infty(s, t)g_\infty(s, t)^{-1}$ where $g_\infty(s, t)$ is real and $\hat{g}''_\infty(s, t)$ is hermitian. In exactly the same way as the proof of Lemma 5.17, we have :

Lemma 5.27 *The following inequality holds for each $(s, t) \in \mathcal{K}_2$*

$$(5.28.1) \quad \|\hat{g}''_\infty(s, t) - 1\|_{C^{k_0}(\Sigma, g_\Sigma)} \leq C_{17,0} \chi(s, t)^2,$$

$$(5.28.2) \quad \left\| \frac{\partial^m \hat{g}''_\infty}{\partial s^\ell \partial t^{m-\ell}}(s, t) \right\|_{C^{k_0}(\Sigma, g_\Sigma)} \leq C_{17,m} \chi(s, t)^2 \text{ for } m \leq k_0.$$

We now show

Lemma 5.29 $g_{1,i}^{-1*} \mathcal{A}_i$ converges in C^{k_0-1} topology on $\mathcal{K}_2 \times \Sigma$.

Proof: (5.16), (5.24), (5.26) and $\lim_{i \rightarrow \infty} \hat{A}'_i(s, t) = \hat{A}'_\infty(s, t)$ implies that $g'_i g_{2,i}$ converges to g'_∞ in C^∞ topology on any compact set of $(D_1 \cap \mathcal{K}_2) \times \Sigma$.

On the other hand, $g'_i g_{2,i} = \hat{g}''_i(g_{1,i} g_{2,i})$, $g'_\infty = \hat{g}''_\infty g_\infty$ are polar decompositions. Therefore $\hat{g}''_i$ converges to $\hat{g}''_\infty$ in C^∞ topology on any compact set of $(D_1 \cap \mathcal{K}_2) \times \Sigma$.

Now let $\varepsilon > 0$ be any positive number. By Lemmata 5.17 and 5.27, there exists a neighborhood $V(\varepsilon)$ of L such that

$$(5.30) \quad \|\hat{g}''_i - 1\|_{C^{k_0}((V(\varepsilon) \cap \mathcal{K}_2) \times \Sigma)} < \varepsilon, \quad \|\hat{g}''_\infty - 1\|_{C^{k_0}((V(\varepsilon) \cap \mathcal{K}_2) \times \Sigma)} < \varepsilon$$

for any i . Since $\hat{g}''_i$ converges to $\hat{g}''_\infty$ in C^∞ topology on $(D_1 \cap \mathcal{K}_2 - V(\varepsilon)) \times \Sigma$, there exists $N(\varepsilon)$ such that

$$(5.31) \quad \|\hat{g}''_i - \hat{g}''_\infty\|_{C^{k_0}((D_1 \cap \mathcal{K}_2 - V(\varepsilon)) \times \Sigma)} < \varepsilon, \text{ for } i > N(\varepsilon).$$

On $D_2 \times \Sigma$, $\hat{g}''_i = \hat{g}''_\infty = 1$. Therefore (5.30) and (5.31) imply

$$(5.32) \quad \|\hat{g}''_i - \hat{g}''_\infty\|_{C^{k_0}(\mathcal{K}_2 \times \Sigma)} < \varepsilon, \text{ for } i > N(\varepsilon).$$

Since $\lim_{i \rightarrow \infty} \hat{A}'_i = \hat{A}'_\infty$ in C^∞ topology, and $g_{1,i}^{-1*} \mathcal{A}_i = \hat{g}''_i^* \hat{A}'_i$, $g_{1,\infty}^{-1*} \mathcal{A}_\infty = \hat{g}''_\infty^* \hat{A}'_\infty$, Lemma 5.29 follows from (5.32).

We next are going to improve Lemma 5.29 and show that $g_{1,i}^{-1*} \mathcal{A}_i$ converges in C^∞ topology on $\mathcal{K}_2 \times \Sigma$. For this purpose we prove the following analogue of Lemmata 5.17, 5.27.

Lemma 5.33 For $i > N(\varepsilon, k)$, the following inequality holds for each $(s, t) \in \mathcal{K}_2$

$$(5.34.1) \quad \|\hat{g}_i'(s, t) - 1\|_{C^1(\Sigma, g_\Sigma)} \leq C_{18k} \chi(s, t)^2,$$

$$(5.34.2) \quad \left\| \frac{\partial^k \hat{g}_i''}{\partial s^i \partial t^{k-i}}(s, t) \right\|_{C^1(\Sigma, g_\Sigma)} \leq C_{18k} \chi(s, t)^2,$$

$$(5.34.3) \quad \|\hat{g}_\infty''(s, t) - 1\|_{C^1(\Sigma, g_\Sigma)} \leq C_{18k} \chi(s, t)^2,$$

$$(5.34.4) \quad \left\| \frac{\partial^k \hat{g}_\infty''}{\partial s^i \partial t^{k-i}}(s, t) \right\|_{C^1(\Sigma, g_\Sigma)} \leq C_{18k} \chi(s, t)^2.$$

Proof: The proof is similar to Lemma 5.17. (But we do not use Lemma 5.5.) Let k be an arbitrary positive number. We take the covering (4.41) for $k+100$. Take $\overline{R'_{k+100}(A_\alpha)} \subseteq R_{k+100}(A_\alpha)$ such that $\bigcup R'_{k+100}(A_\alpha) = R(\Sigma; E)$. We take a finite number of open subsets V_β such that

$$(5.35.1) \quad \bigcup V_\beta \supset L \cap \mathcal{K}_2$$

$$(5.35.2) \quad \varphi_\infty(V_\beta) \subset R'_{k+100}(A_{\alpha(\beta)}).$$

It follows that

$$(5.35.3) \quad \varphi_i(V_\beta) \subset R_{k+100}(A_{\alpha(\beta)}), \text{ for } i > N(\varepsilon, k).$$

We lift $\varphi_i|_{V_\beta}$ to $\hat{\varphi}_{i,\beta}: V_\beta \rightarrow \mathcal{R}_{k+100}(A_{\alpha(\beta)})$ and put $\hat{\varphi}_{i,\beta}(s, t) = A_i'(\beta; s, t)$.

Suppose $Y_\mu(s_v, t_v) \subseteq V_\beta$. We change the coordinate by (4.11) with (s_0, t_0) replaced by (s_v, t_v) . Then we can find $\Phi_i'(v; S, T)$ and $\Psi_i'(v; S, T)$ such that

$$(5.36) \quad \frac{\partial A_i'(\beta; S, T)}{\partial S} + *_\Sigma \frac{\partial A_i'(\beta; S, T)}{\partial T} = d_{A_i'} \Psi_i'(v; S, T) + *_\Sigma d_{A_i'} \Phi_i'(v; S, T).$$

In a way similar to (5.13) we can verify (4.43.2), (4.43.3) and (4.43.4) for D_k independent of i . We put

$$(5.37) \quad \varepsilon_9 = \varepsilon_7(k; C_{5,1}, \dots, C_{5,k+100}; D_1, \dots, D_{k+100}).$$

Let $\mathcal{A}_\infty$ be the C^{k_0-1} limit of $g_i^{-1*} \mathcal{A}_i$. Since $p^1(\mathcal{A}_\infty)$ is continuous we can choose V'_β such that

$$(5.38.1) \quad 2\pi^2 \int_{V'_\beta} p^1(\mathcal{A}_\infty) < \frac{\varepsilon_9}{2}$$

$$(5.38.2) \quad V_\beta \cap L = V'_\beta \cap L.$$

We then use Lemma 5.29 (the C^2 convergence) to find

$$(5.38.3) \quad 2\pi^2 \int_{V'_\beta} p^1(\mathcal{A}_i) < \varepsilon_9, \text{ for } i > N_2(\varepsilon, k).$$

(This is the point where the proof of Lemma 5.33 is different from the proof of Lemma 5.17.) Now suppose $Y_\mu(s_v, t_v) \subseteq V'_\beta$. We put $h_i(\beta; s, t) \star A_i'(\beta; s, t) = \hat{A}_i'(s, t)$,

$$g_i''(\beta; s, t) = h_i(\beta; s, t) \hat{g}_i''(s, t) h_i(\beta; s, t)^{-1}.$$

We remark that the set

$$\left(\bigcup_{\beta} \bigcup_{v: Y_n(s_v, t_v) \subseteq V_\beta'} Y(s_v, t_v) \right) \cup (L \cap \mathcal{K}_2) \cup (L \cap D_2)$$

is a neighborhood of $L \cap \mathcal{K}_2$ because of (5.38.2).

We can apply Mainlemma 4.42 to $g_i''(\beta; S, T)$ and $A_i'(\beta; s, t)$ because of (5.38.3). Therefore we obtain (5.34.1) and (5.34.2) in the same way as the proof of Lemma 5.17. The proofs of (5.34.3) and (5.34.4) are similar. The proof of Lemma 5.33 is now complete.

Using Lemma 5.33 in the same way as Lemmata 5.17 and 5.27, we have the following :

Lemma 5.39 $g_{1,i}^{-1*} \mathcal{A}_i$ converges in C^∞ topology on $\mathcal{K}_2 \times \Sigma$.

The rest of the proof of Theorem 1.6 is now routine. Let $V \times \Sigma$ be a small compact neighborhood of $(D_1 \cap \partial U_2 \cap \mathcal{K}) \times \Sigma$. We find that both $g_{1,i}^{-1*} \mathcal{A}_i$ and $g_{2,i}^* \mathcal{A}_i$ converges in C^∞ topology on $(V \cap U) \times \Sigma$. Hence $g_{2,i} g_{1,i}$ converges on $(V \cap U) \times \Sigma$. We find a gauge transformation $g_{3,i}$ on $((D_1 \cap \mathcal{K} - U) \cup V) \times \Sigma$ such that $g_{3,i} = g_{2,i} g_{1,i}$ on a neighborhood of $(\partial V \cap U) \times \Sigma$ and that $g_{3,i}$ converges in C^∞ topology. We put

Figure 5.40

$$g_i = \begin{cases} g_{1,i}^{-1} & \text{on } (\mathcal{K}_2 \cup D_2 - V - P_4) \times \Sigma \\ g_{3,i}^{-1} g_{2,i} & \text{on } ((D_1 \cap \mathcal{K} - U) \cup V) \times \Sigma - P_1 \end{cases}$$

Then $g_i^* \mathcal{A}_i$ converges on $\mathcal{K}_2 \times \Sigma$ in C^∞ topology by Lemma 5.39. $g_i^* \mathcal{A}_i$ converges in C^∞ topology on any compact subset on $D_2 - P_4$ since $g_i^* \mathcal{A}_i = \mathcal{A}_i'$ there. $g_i^* \mathcal{A}_i$ converges in C^∞ topology on any compact subset on $((D_1 \cap \mathcal{K} - U) \cup V - P_1) \times \Sigma$ because $g_{3,i}$ converges in C^∞ topology and because of Lemma 5.23. We thus have proved the following.

Lemma 5.41 For any compact subset $\mathcal{K} \subseteq D$, there exist finite subsets $P_1(\mathcal{K}) \subseteq (\mathcal{K} \cap D_1) \times \Sigma$, $P_2(\mathcal{K}) \subseteq \mathcal{K} \cap (D_2 \cup L)$ with the following properties.

If $\mathcal{K}' \subseteq ((\mathcal{K} \cap D_1) \times \Sigma - P_1(\mathcal{K})) \cup ((\mathcal{K} \cap (D_2 \cup L) - P_2(\mathcal{K})) \times \Sigma)$ is a compact subset then there exists a subsequence of $\mathcal{A}_i$ (still denoted by the same symbol) and a sequence of gauge transformations $g_i(\mathcal{K}')$ on $\mathcal{K}'$ such that $g_i(\mathcal{K}')^* \mathcal{A}_i$ converges in C^∞ topology on $\mathcal{K}'$.

We remark that we can find an upper bound of the order of $P_1(\mathcal{K})$ and $P_2(\mathcal{K})$ independent of $\mathcal{K}$. This is a consequence of Theorem 1.7 which we will prove later in this section. Hence we may assume that $P_1 = P_1(\mathcal{K})$ and $P_2 = P_2(\mathcal{K})$ are independent of $\mathcal{K}$.

Let $\mathcal{K}'_n \subseteq (D_1 \times \Sigma - P_1) \cup ((D_2 \cup L - P_2) \times \Sigma)$ be a compact subsets such that $\mathcal{K}'_n \uparrow (D_1 \times \Sigma - P_1) \cup ((D_2 \cup L - P_2) \times \Sigma)$. By a diagonal argument we may assume that the subsequence in Lemma 5.41 is independent of n .

Furthermore by a gluing argument similar to one we gave before, we may assume that $g_i(\mathcal{K}'_n) = g_i(\mathcal{K}'_{n+1})$ on $\mathcal{K}'_n$. Hence we have a gauge transformation g_i on

$(D_1 \times \Sigma - P_1) \cup ((D_2 \cup L - P_2) \times \Sigma)$ such that $g_i^* \mathcal{A}_i$ converges on any compact set. The proof of Theorem 1.6 is complete modulo Mainlemma 4.42.

We turn to the proof of Theorem 1.7. We use the notation of Theorem 1.7. We first show :

Lemma 5.42 *For any open set V with $\bar{V} \subseteq U$, we have*

$$(5.43) \quad \lim_{i \rightarrow \infty} \int_{V \times \Sigma} p^1(\mathcal{A}_i) = \int_{V \times \Sigma} p^1(\mathcal{A}_\infty) .$$

Proof: We remark that $\mathcal{A}_\infty$ can be extended to a smooth connection on $U \times \Sigma$ by Theorem 1.5. Let us embed $U \subseteq S^2$. We can find a bundle $\hat{E} \rightarrow S^2 \times \Sigma$, a connection $\hat{\mathcal{A}}_\infty$ on it and an isomorphism $\psi : \hat{E}|_{U \times \Sigma} \cong E(U \times \Sigma)$ such that $\psi^* \mathcal{A}_\infty = \hat{\mathcal{A}}_\infty$ on U .

We may assume that $g_i^* \mathcal{A}_i$ converges to $\mathcal{A}_\infty$ on a compact neighborhood $W \times \Sigma$ of $\partial V \times \Sigma$. We can find a connection $\hat{\mathcal{A}}_i$ on $\hat{E}|_{(S^2 - V) \cup W \times \Sigma}$ such that

$$(5.44.1) \quad \hat{\mathcal{A}}_i = \hat{\mathcal{A}}_\infty \text{ on } (S^2 - V) \times \Sigma .$$

$$(5.44.2) \quad \psi^* g_i^* \mathcal{A}_i = \hat{\mathcal{A}}_i \text{ in a neighborhood of } (\partial W \cap U) \times \Sigma .$$

$$(5.44.3) \quad \|\hat{\mathcal{A}}_\infty - \hat{\mathcal{A}}_i\|_{C^2(W \times \Sigma)} \rightarrow 0 .$$

We glue $\hat{E}|_{(S^2 - V) \cup W \times \Sigma}$ and $E(V \times \Sigma)$ by g_i to obtain $\hat{E}_i \rightarrow S^2 \times \Sigma$. By (5.44.2) the connection $\hat{\mathcal{A}}_i$ is extended to a connection on $\hat{E}_i$. We denote it by the same symbol. By (5.44.1) and (5.44.3) we have

$$(5.45) \quad \limsup_{i \rightarrow \infty} \left| \int_{V \times \Sigma} p^1(\mathcal{A}_i) - \int_{V \times \Sigma} p^1(\mathcal{A}_\infty) \right| = \limsup_{i \rightarrow \infty} \left| \int_{S^2 \times \Sigma} p^1(\hat{E}_i) - \int_{S^2 \times \Sigma} p^1(\hat{E}) \right| .$$

Since the intersection form on $H^2(S^2 \times \Sigma)$ is an even form, it follows from $(w^2)^2 \equiv p^1 \pmod{4}$ that the right hand side of (5.45) is an even integer. (This is related to the fact that the minimal Chern number of $R(\Sigma; E)$ is 2.) On the other hand, the left hand side is strictly smaller than 2 by assumption. Lemma 5.42 follows.

Hereafter V_i denotes neighborhoods of $(\hat{s}, \hat{t})$. Let ε_6 be as in Lemma 4.24. We choose $V_1 \subseteq U$ such that $\int_{V_1 \times \Sigma} p^1(\mathcal{A}_\infty) < \frac{\varepsilon_6}{2}$. Then by Lemma 5.42, we have

$\int_{V_1 \times \Sigma} c^2 p^1(\mathcal{A}_i) < \varepsilon_6$ for large i . Hence by Lemma 4.24, the connection $\mathcal{A}_i(s, t)$ is complex gauge equivalent to a flat connection for $(s, t) \in V_1$. We put $\varphi_i(s, t) = [\mathcal{A}_i(s, t)]$. By Lemma 4.24, we have

$$(5.46) \quad \int_{V_2} \varphi_i^* \omega \leq 20\pi^2 \int_{V_2 \times \Sigma} p^1(\mathcal{A}_i) \text{ for } V_2 \subseteq V_1 \text{ and large } i .$$

Therefore, we may assume that φ_i converges in C^∞ topology to φ_∞ on V_2 . (We do

not need to remove finitely many points since we may assume $\int_{V_2} \varphi_i^* \omega \leq \varepsilon$ for small ε .

We can then find flat connections $\hat{A}'_i(s, t)$ and $\hat{A}'_\infty(s, t)$ and complex gauge transformations $g'_i(s, t)$, $g'_\infty(s, t)$ such that $\lim_{i \rightarrow \infty} \hat{A}'_i(s, t) = \hat{A}'_\infty(s, t)$ in C^∞ topology on $V_2 \times \Sigma$ and $g'_i(s, t)^* \hat{A}'_i(s, t) = A_i(s, t)$, $g'_\infty(s, t)^* \hat{A}'_\infty(s, t) = A_\infty(s, t)$. We put $g'_i(s, t) = \hat{g}'_i(s, t) g_{1,i}(s, t)^{-1}$, $g'_\infty(s, t) = \hat{g}''_\infty(s, t) g_{1,\infty}(s, t)^{-1}$ where $g_{1,i}(s, t), g_{1,\infty}(s, t)$ are real and $\hat{g}'_i(s, t), \hat{g}''_\infty(s, t)$ are hermitian.

Lemma 5.47 *For each k there exists V_k and $C_{19,k}$ such that if $(s, t) \in V_k$ then*

$$(5.48.1) \quad \|\hat{g}'_i(s, t) - 1\|_{C^k(\Sigma, g_\Sigma)} \leq C_{19,k} \chi(s, t)^2, \text{ for } i > N_k,$$

$$(5.48.2) \quad \left\| \frac{\partial^k \hat{g}''_i}{\partial s^i \partial t^{k-i}}(s, t) \right\|_{C^k(\Sigma, g_\Sigma)} \leq C_{19,k} \chi(s, t)^2, \text{ for } i > N_k,$$

$$(5.48.3) \quad \|\hat{g}''_\infty(s, t) - 1\|_{C^k(\Sigma, g_\Sigma)} \leq C_{19,k} \chi(s, t)^2,$$

$$(5.48.4) \quad \left\| \frac{\partial^k \hat{g}''_\infty}{\partial s^i \partial t^{k-i}}(s, t) \right\|_{C^k(\Sigma, g_\Sigma)} \leq C_{19,k} \chi(s, t)^2.$$

Proof: We first take $V_{3,k}$ such that $\varphi_i(V_{3,k}) \subseteq R_{k+100}(A_\beta)$, $\varphi_\infty(V_{3,k}) \subseteq R_{k+100}(A_\beta)$. We lift to $\hat{\varphi}_i : V_{3,k} \rightarrow \mathcal{R}_{k+100}(A_\beta)$, $\hat{\varphi}_\infty : V_{3,k} \rightarrow \mathcal{R}_{k+100}(A_\beta)$. Then, for each v with $Y_\mu(s_v, t_v) \subseteq V_{3,k}$, we find $\Phi'_i(v : S, T)$, $\Psi'_i(v : S, T)$, $\hat{\varphi}_i(S, T) = A'_i(v : S, T)$ such that (5.36), (4.43.2), (4.43.3), (4.43.4) are satisfied for D_k independent of i . We define ε_9 by (5.37). Using Lemma 5.42, we then can choose V_k such that

$$(5.49) \quad 2\pi^2 \int_{V_k \times \Sigma} p^1(\mathcal{A}_i) < \varepsilon_9 \text{ for } i > N_k.$$

Suppose $Y_\mu(s_v, t_v) \subseteq V_k$. Then by (5.49), we can apply Mainlemma 4.42 in the same way as the proof of Lemma 5.33 and obtain (5.48).

We can now complete the proof of Theorem 1.7. Since $g_i(s, t)^* g'_i(s, t)^* \hat{A}'_i(s, t) = g_i(s, t)^* A_i(s, t)$ and $\hat{A}'_i(s, t)$ both converges in C^∞ topology on any compact subset on $(U - (\hat{s}, \hat{t})) \times \Sigma$, it follows that $g'_i(s, t) g_i(s, t)$ converges in C^∞ topology on any compact subset on $(U - (\hat{s}, \hat{t})) \times \Sigma$.

Then, since $g'_i(s, t) g_i(s, t) = \hat{g}'_i(s, t) (g_{1,i}(s, t)^{-1} g_i(s, t))$ is the polar decomposition, it follows that $\hat{g}'_i(s, t)$ converges in C^∞ topology on any compact subset on $(U - (\hat{s}, \hat{t})) \times \Sigma$.

Therefore Lemma 5.47 implies that $\hat{g}'_i(s, t)$ converges in C^∞ topology on $U \times \Sigma$. Hence $g_{1,i}^{-1*} \mathcal{A}_i = \hat{g}'_i(s, t)^* \mathcal{A}'_i$ converges in C^∞ topology on $U \times \Sigma$. The proof of Theorem 1.7 is now complete.

§ 6 Mainlemma

In this section, we prove Mainlemma 4.42. We use the same notation as Mainlemma 4.42. We put $\mathcal{A} = g''^* \mathcal{A}' = A(S, T) + \Phi dS + \Psi dT$ and $\delta = \chi(s_0, t_0)$. We take k as in Mainlemma 4.42 and fix it in this section.

We start with preliminary lemma proving that $|g'' - 1| < C\varepsilon_7$ under the assumption of Mainlemma 4.42. We remark that our goal is to show something like $|g'' - 1| < C\delta^2$.

Lemma 6.1 *There exists $C_{20}(C_{3,*}, D_*)$ such that (4.43.1), (4.44.1) imply*

$$(6.2.1) \quad \|g''(S_1, T_1) - 1\|_{C^\ell(\Sigma, g_\Sigma)} \leq C_{20}\varepsilon_7,$$

$$(6.2.2) \quad \left\| \frac{\partial' g''}{\partial S^m \partial T^{\ell-m}}(S_1, T_1) \right\|_{C^\ell(\Sigma, g_\Sigma)} \leq C_{20}\varepsilon_7$$

for $m \leq k + 90$ and $(S_1, T_1) \in [-\delta^{-\mu} + 2, \delta^{-\mu} - 2]^2$.

Proof: We apply Corollary 3.8, Addendum 3.11 and (4.12) to $[S_1 - 1, S_1 + 1] \times [T_1 - 1, T_1 + 1] \cong [-1, 1]^2$ and obtain $A_0 \in \mathcal{R}$ and a real gauge transformation g_1 on $[S_1 - 1, S_1 + 1] \times [T_1 - 1, T_1 + 1] \times \Sigma$ such that

$$(6.3) \quad \|g_1^* \mathcal{A} - A_0\|_{C^\ell([S_1 - 1, S_1 + 1] \times [T_1 - 1, T_1 + 1] \times \Sigma, g')} < C_\ell \varepsilon_7$$

for any ℓ . Here we used the scaled metric g' , (4.2). We may assume that the element $[A_0] \in R(\Sigma, E)$ is as close to $[A'(S_1, T_1)]$ as we want, by choosing ε_7 (which may depend on k) small. Hence we may assume that $[A_0] \in \mathcal{R}_{k+95}(A_\alpha)$. Namely there exists a gauge transformation g_2 on Σ such that $g_2^* A_0 \in \mathcal{R}_{k+95}(A_\alpha)$. Since $\mathcal{R}, \mathcal{R}_{k+95}(A_\alpha)$ are both compact in C^∞ topology, it follows that C^ℓ norm of g_2 is uniformly bounded for any ℓ . Therefore, by (6.3), we may assume

$$g_2^* g_1(S, T)^* A(S, T) \in \Pi(\mathcal{G}_{k+96,0}(A_0) \times \mathcal{D}_{k+95,0}(A_0))$$

for $(S, T) \in [S_1 - 1, S_1 + 1] \times [T_1 - 1, T_1 + 1]$, by taking ε_7 small enough. We put

$$g_2^* g_1(S, T)^* A(S, T) = g_3(S, T)^* A_1(S, T)$$

where $g_3(S, T) \in \mathcal{G}_{k+96,0}(A_0)$, $A_1(S, T) \in \mathcal{D}_{k+95,0}(A_0)$. We remark that $A_1(S, T)$ is complex gauge equivalent to $A'(S, T)$. Therefore, since $\mathcal{D}_{k+95,0}(A_0) \rightarrow R(\Sigma, E)$ is injective, we have $A_1(S, T) = A'(S, T)$. Namely

$$(6.4) \quad g_2^* g_1(S, T)^* A(S, T) = g_3(S, T)^* A'(S, T).$$

Since $A(S, T) = g''^* A'(S, T)$ we have

$$(6.5) \quad g''(S, T) g_1(S, T) g_2 = \pm g_3(S, T).$$

We decompose $g_3(S, T) = g_4(S, T) g_5(S, T)$ where g_4 is hermitian and g_5 is real. Then,

since g'' is hermitian and g_1 and g_2 are real, it follows from (6.5) that $g'' = \pm g_4$. Therefore, to prove the lemma, it suffices to estimate $g_3 - 1$.

On the other hand, by (6.3) and uniformly boundedness of C^ℓ norm of g_2 , we have

$$(6.6) \quad \left\| g_2^* g_1^*(S, T) A(S, T) - g_2^* A_0 \right\|_{C^\ell(\{S_0-1, S_0+1\} \times [T_0-1, T_0+1] \times \Sigma, g'_{S_0, T_0})} < C_\ell \varepsilon_7$$

We recall $g_2^* A_0 \in \mathcal{R}_{k+95}(A_0) \subseteq \mathcal{D}_{k+95,0}(A_0)$. Therefore, since $\Pi: \mathcal{G}_{k+96,0}(A_0) \times \mathcal{D}_{k+95,0}(A_0) \rightarrow \mathcal{A}_{k+95,0}(\Sigma; E)$ is a diffeomorphism to its image, we have

$$\left\| pr_1(\Pi^{-1}(g_2^* g_1^*(S, T) A(S, T))) - 1 \right\|_{L^2_{k+96}(\Sigma, g_\Sigma)} \leq C \varepsilon_7.$$

Since $pr_1(\Pi^{-1}(g_2^* g_1^*(S, T) A(S, T))) = g_3(S, T)$, we have

$$\left\| g''(S, T) - 1 \right\|_{C^{k+90}(\Sigma, g_\Sigma)} \left(= \left\| g_4(S, T) - 1 \right\|_{C^{k+90}(\Sigma, g_\Sigma)} \right) \leq C \varepsilon_7.$$

We have proved (6.2.1). To prove (6.2.2), we remark that A_0 is independent of (S, T) . Hence (6.3) implies that

$$\left\| \frac{\partial'}{\partial S^m \partial T^{\ell-m}} (g_1(S, T)^* A(S, T)) \right\|_{C^{k+90}(\Sigma, g_\Sigma)} \leq C \varepsilon_7.$$

Therefore

$$\left\| \frac{\partial'}{\partial S^m \partial T^{\ell-m}} (pr_1(\Pi^{-1}(g_2^* g_1^*(S, T) A(S, T)))) \right\|_{C^{k+90}(\Sigma, g_\Sigma)} \leq C \varepsilon_7.$$

(6.2.2) then follows from $pr_1(\Pi^{-1}(g_2^* g_1^*(S, T) A(S, T))) = g_3(S, T)$, $g_3(S, T) = g_4(S, T) g_5(S, T)$. The proof of Lemma 6.1 is complete.

We put

$$(6.7) \quad g''(S, T) = \exp(\sqrt{-1} \varepsilon(S, T)),$$

where $\varepsilon(S, T)$ is a section of $P_{\ell(2)}(\Sigma) \times_{ad} su(2)$. We put

$$e_{a,b} = \frac{\partial^{a+b} e}{\partial S^a \partial T^b}, \quad f_{a,b}(S, T) = \|e_{a,b}(S, T)\|_{L^2(\Sigma, g_\Sigma)}^2, \quad f = f_{0,0}, \quad h_m^2 = \sum_{a+b=m} f_{a,b}.$$

We use Lemma 6.1 to show the following regularity estimate. We remark that regularity estimate of Yang Mills connection requires gauge fixing since Yang Mills equation is elliptic only modulo gauge transformation. Usually Coulomb gauge is used for this purpose. Our choice of gauge is a bit different from Coulomb gauge. Namely we require g'' to be hermitian.

Lemma 6.8 *There exists C_{21} such that, for each $(S, T) \in [-\delta^{-\mu} + 2, \delta^{-\mu} - 2]^2$, we have:*

$$(6.9.1) \quad \|e_{a,b}(S,T)\|_{C^\ell(\Sigma, g_\Sigma)} \leq C_{21} \|e_{\alpha,\beta}(S,T)\|_{L^2(\Sigma, g_\Sigma)} \text{ if } a+b \leq k+89, \ell < k+89.$$

$$(6.9.2) \quad \|e_{a,b}(S_1, T_1)\|_{C^\ell(\Sigma, g_\Sigma)}^2 \leq C_{21} \int_{S_1-1}^{S_1+1} \int_{T_1-1}^{T_1+1} \|e_{a',b'}(S,T)\|_{L^2(\Sigma, g_\Sigma)}^2 dSdT \text{ if } a' \leq a, b' \leq b, \text{ and } a+b \leq k+89, \ell < k+89.$$

Proof: Lemma 6.8 follows immediately from Lemma 6.1 and the following elementary Lemma 6.10.

Lemma 6.10 There exists a constant C_ℓ such that for any function $f: B(2) \rightarrow \mathbf{R}$ of $C^{\ell+1}$ class on a disk of radius 2 we have:

$$\|f\|_{C^\ell(B(1))} \leq C_\ell (1 + \|f\|_{C^{\ell+1}(B(2))}) \|f\|_{L^2(B(2))}.$$

Proof: Replacing f by $\frac{f}{\|f\|_{C^{\ell+1}(B(2))}}$, we may assume that $\|f\|_{C^{\ell+1}(B(2))} = 1$. Then, if Lemma 6.10 is false, there exists f_i such that $\|f_i\|_{C^{\ell+1}(B(2))} = 1$ and $\|f_i\|_{C^\ell(B(1))} / \|f_i\|_{L^2(B(2))} \rightarrow \infty$. Since $\|f_i\|_{C^{\ell+1}(B(2))} = 1$, it follows that $\|f_i\|_{L^2(B(2))} > c > 0$. Therefore $\|f_i\|_{C^\ell(B(1))} \rightarrow \infty$. On the other hand, since $\|f_i\|_{C^{\ell+1}(B(2))} = 1$, Ascoli-Alzera implies that f_i has a C^ℓ convergent subsequence. This is a contradiction.

Lemma 6.8 implies

$$(6.11.1) \quad \|e_{a,b}(S,T)\|_{C^m(\Sigma, g_\Sigma)}^2 \leq C_{21} f_{a,b}(S,T) \text{ if } a+b \leq k+89, m < k+89.$$

$$(6.11.2) \quad f_{a,b}(S_1, T_1) \leq C_{21} \int_{S_1-1}^{S_1+1} \int_{T_1-1}^{T_1+1} f_{a',b'}(S,T) dSdT \text{ if } a' \leq a, b' \leq b, \text{ and } a+b \leq k+89.$$

We now start the main part of the proof. We remark that Mainlemma 4.42 is an a priori estimate of Equation (1.1). However we do not need to use (1.1.1), since it is automatically satisfied for $\mathcal{A} = g''^* \mathcal{A}'$. (This is because (1.1.1) is invariant of complex gauge transformation.) Hence we only need to study (1.1.2). Our next task is to write (1.1.2) for $\mathcal{A} = g''^* \mathcal{A}'$ in terms of the hermitian gauge transformation g'' . We are going to calculate the first order term with respect to e . ($g'' = \exp \sqrt{-1}e$.) Lemma 6.12 below gives the second term of (1.1.2) and Lemma 6.21 will be used the calculation of first term of (1.1.2). Roughly speaking, we are going to prove that (1.1.2) for $\mathcal{A} = g''^* \mathcal{A}'$ is:

$$\left(-\frac{\partial^2 e}{\partial S^2} - \frac{\partial^2 e}{\partial T^2} \right) - \alpha(S,T)^{-2} *_\Sigma d_{\mathcal{A}'(S,T)} *_\Sigma d_{\mathcal{A}'(S,T)} e = (\text{higher order term}).$$

Lemma 6.12 There exist $\varepsilon_{10}(C_{6,\bullet}, D_\bullet)$ and $C_{22}(C_{6,\bullet}, D_\bullet)$ such that (4.43) and

$$(6.13.1) \quad \|e(S_1, T_1)\|_{C^{k+80}(\Sigma, g_\Sigma)} \leq \varepsilon_{10}, \text{ for any } (S_1, T_1) \in [-\delta^{-\mu} + 3, \delta^{-\mu} - 3]^2,$$

$$(6.13.2) \quad \|e_{a,b}(S_1, T_1)\|_{C^{k+80}(\Sigma, g_\Sigma)} \leq \varepsilon_{10} \text{ for any } a+b \leq k+80, (S_1, T_1) \in [-\delta^{-\mu} + 3, \delta^{-\mu} - 3]^2,$$

imply the following inequality for $(S_1, T_1) \in [-\delta^{-\mu} + 4, \delta^{-\mu} - 4]^2$, $a+b \leq k+70$, $\ell \leq k+70$:

$$(6.14.1) \quad \|F_{\mathcal{A}(S_1, T_1)} + d_{\mathcal{A}'(S_1, T_1)} *_\Sigma d_{\mathcal{A}'(S_1, T_1)}(e(S_1, T_1))\|_{C^\ell(\Sigma, g_\Sigma)} \leq C_{22} h_0(S_1, T_1)^2,$$

$$\begin{aligned}
(6.14.2) \quad & \left\| \frac{\partial^{a+b} F_A}{\partial S^a \partial T^b}(S_1, T_1) + d_{A'(S_1, T_1)} *_{\Sigma} d_{A'(S_1, T_1)}(e_{a,b}(S_1, T_1)) \right\|_{C^{\ell}(\Sigma, g_{\Sigma})} \\
& \leq C_{22} \left(\sum_{j>0} \delta^j h_{a+b-j}(S_1, T_1) + \sum_{j \geq 0} \sum_{n \geq 2} \sum_{m_1 + \dots + m_n + j = a+b} \delta^j h_{m_1}(S_1, T_1) \dots h_{m_n}(S_1, T_1) \right).
\end{aligned}$$

Proof: By (2.13) and $\overline{g''(S, T)} = g''(S, T)$, we have

$$\begin{aligned}
(6.15) \quad & A(S, T) = g''(S, T)^* A'(S, T) \\
& = A'(S, T) + g''(S, T)^{-1} \bar{\partial}_{A'(S, T)} g''(S, T) - (\partial_{A'(S, T)} g''(S, T)) g''(S, T)^{-1}.
\end{aligned}$$

Since $g''(S, T) = \exp(\sqrt{-1}e(S, T))$, we can calculate in the same way as (2.16), (2.17) and obtain :

$$(6.16) \quad \|A(S, T) - A'(S, T) + *_{\Sigma} d_{A'(S, T)} e(S, T)\|_{C^{\ell}(\Sigma, g_{\Sigma})} \leq C \|e(S, T)\|_{C^{\ell-1}(\Sigma, g_{\Sigma})}^2.$$

Since $A'(S, T)$ is flat, it follows from (6.16) that

$$(6.17) \quad \|F_{A(S, T)} + d_{A'(S, T)} *_{\Sigma} d_{A'(S, T)} e(S, T)\|_{C^{\ell}(\Sigma, g_{\Sigma})} \leq C \|e(S, T)\|_{C^{\ell+2}(\Sigma, g_{\Sigma})}^2.$$

Using (6.11.1), the right hand side of (6.17) is estimated by the right hand side of (6.14.1). We thus proved (6.4.1).

To prove (6.4.2) we put

$$\mathcal{F}(S, T, e) = F_{\exp(\sqrt{-1}e)^* A'(S, T)}.$$

Here

$$\begin{aligned}
\mathcal{F}^{(\ell)} : [-\delta^{-\mu} + 2, \delta^{-\mu} - 2]^2 \times C^{\ell}(\Sigma, P_{U(2)}(\Sigma) \times_{ad} su(2)) \\
\rightarrow C^{\ell-2}(\Sigma, \Lambda^1(\Sigma) \otimes P_{U(2)}(\Sigma) \times_{ad} su(2))
\end{aligned}$$

is a C^{∞} map between Banach spaces. Let $D\mathcal{F}_{(S, T, e^{(0)})}^{(\ell)}$ be the derivative of $\mathcal{F}^{(\ell)}$ with respect to e at $(S, T, e^{(0)})$. Then as we calculated above

$$(6.18) \quad D\mathcal{F}_{(S, T, e^{(0)})}^{(\ell)} = -d_{A'(S, T)} *_{\Sigma} d_{A'(S, T)}.$$

We denote by $D_n \mathcal{F}_{(S, T, e^{(0)})}^{(\ell)}$ the n -th derivative of $\mathcal{F}^{(\ell)}$ with respect to e at $(S, T, e^{(0)})$. The map :

$$\begin{aligned}
D_n \mathcal{F}^{(\ell)} : [-\delta^{-\mu} + 2, \delta^{-\mu} - 2]^2 \times C^{\ell}(\Sigma, P_{U(2)}(\Sigma) \times_{ad} su(2)) \\
\times C^{\ell}(\Sigma, P_{U(2)}(\Sigma) \times_{ad} su(2))^n \rightarrow C^{\ell-2}(\Sigma, \Lambda^1(\Sigma) \otimes P_{U(2)}(\Sigma) \times_{ad} su(2))
\end{aligned}$$

is of C^{∞} class and is multilinear on $C^{\ell}(\Sigma, P_{U(2)}(\Sigma) \times_{ad} su(2))^n$ factor. Therefore, using Assumption (4.43.2), we have

$$(6.19) \quad \left\| \left(\frac{\partial^{a'+b'}}{\partial S^{a'} \partial T^{b'}} (D_n \mathcal{F}^{(\ell)}) \right)_{(S, T, e^{(0)})} (e^{(1)}, \dots, e^{(n)}) \right\|_{C^{\ell-2}(\Sigma, g_\Sigma)} \leq C(K) \delta^{a'+b'} \|e^{(1)}\|_{C^\ell(\Sigma, g_\Sigma)} \cdots \|e^{(n)}\|_{C^\ell(\Sigma, g_\Sigma)}.$$

for $a_1 + b_1 \leq 75$, $\ell < 75$, and $e^{(0)}$ in a compact set K of $C^\ell(\Sigma, P_{U(2)}(\Sigma) \times_{ad} su(2))$. We consider $e^{(0)} = e(S_1, T_1)$ satisfying (6.13.1). Hence it is in a compact set of $C^\ell(\Sigma, P_{U(2)}(\Sigma) \times_{ad} su(2))$ for $\ell \leq k + 70$.

We are to calculate $\frac{\partial^{a+b}}{\partial S^a \partial T^b} \mathcal{F}(S, T, e(S, T))$. It is easy to see that $\frac{\partial^{a+b}}{\partial S^a \partial T^b} \mathcal{F}(S, T, e(S, T))$ is a sum of the terms

$$(6.20) \quad \left(\frac{\partial^{a'+b'}}{\partial S^{a'} \partial T^{b'}} (D_n \mathcal{F}^{(\ell)}) \right)_{(S_1, T_1, e(S_1, T_1))} (e_{c_1, d_1}(S_1, T_1), \dots, e_{c_n, d_n}(S_1, T_1)),$$

where $a' + c_1 + \dots + c_n = a$, $b' + d_1 + \dots + d_n = b$. In fact, $\frac{\partial}{\partial S_1}$ of (6.20) is

$$\begin{aligned} & \left(\frac{\partial^{a'+b'+1}}{\partial S^{a'+1} \partial T^{b'}} (D_n \mathcal{F}^{(\ell)}) \right)_{(S_1, T_1, e(S_1, T_1))} (e_{c_1, d_1}(S_1, T_1), \dots, e_{c_n, d_n}(S_1, T_1)) \\ & + \left(\frac{\partial^{a'+b'}}{\partial S^{a'} \partial T^{b'}} (D_{n+1} \mathcal{F}^{(\ell)}) \right)_{(S_1, T_1, e(S_1, T_1))} (e_{1,0}(S_1, T_1), e_{c_1, d_1}(S_1, T_1), \dots, e_{c_n, d_n}(S_1, T_1)) \\ & + \sum_{m=1}^n \left(\frac{\partial^{a'+b'}}{\partial S^{a'} \partial T^{b'}} (D_n \mathcal{F}^{(\ell)}) \right)_{(S_1, T_1, e(S_1, T_1))} (e_{c_1, d_1}(S_1, T_1), \dots, e_{c_{m+1}, d_{m+1}}(S_1, T_1), \dots, e_{c_n, d_n}(S_1, T_1)) \end{aligned}$$

Using (6.19) and (6.11.1), we can estimate $C^{\ell-2}$ norm of (6.20) by $C \delta^{a'+b'} h_{c_1+d_1} \cdots h_{c_n+d_n}$. This number $C \delta^{a'+b'} h_{c_1+d_1} \cdots h_{c_n+d_n}$ is estimated by the right hand side of (6.14.2) unless $a' + b' = 0$, $n = 1$ or $a = a', b = b'$.

We remark that $\mathcal{F}(S, T, 0) = 0$ for any (S, T) . Hence, (6.20) in case $a = a', b = b'$ is estimated as

$$\begin{aligned} & \left\| \left(\frac{\partial^{a+b}}{\partial S^a \partial T^b} \mathcal{F}^{(\ell)} \right)_{(S_1, T_1, e(S_1, T_1))} \right\|_{C^\ell(\Sigma, g_\Sigma)} \\ & = \left\| \left(\frac{\partial^{a+b}}{\partial S^a \partial T^b} \mathcal{F}^{(\ell)} \right)_{(S_1, T_1, e(S_1, T_1))} - \left(\frac{\partial^{a+b}}{\partial S^a \partial T^b} \mathcal{F}^{(\ell)} \right)_{(S_1, T_1, 0)} \right\|_{C^\ell(\Sigma, g_\Sigma)} \\ & \leq \int_0^1 \left\| \left(\frac{\partial^{a+b}}{\partial S^a \partial T^b} (D_1 \mathcal{F}^{(\ell)}) \right)_{(S_1, T_1, \theta e(S_1, T_1))} (e(S_1, T_1)) \right\|_{C^\ell(\Sigma, g_\Sigma)} d\theta \\ & \leq C \delta^{a+b} h_0(S_1, T_1) \end{aligned}$$

and is estimated by right hand side of (6.14.2).

Finally (6.20) in case $a' + b' = 0$, $n = 1$ is equal to $(D_1 \mathcal{F}^{(\ell)})_{(S_1, T_1, e(S_1, T_1))}(e_{a, b})$. Therefore, by (6.18) and (6.19), we have

$$\begin{aligned} & \left\| (D_1 \mathcal{F}^{(\ell)})_{(S_1, T_1, e(S_1, T_1))}(e_{a, b}) + d_{A'(S, T)} *_{\Sigma} d_{A'(S, T)}(e_{a, b}) \right\|_{C^{\ell}(\Sigma, g_{\Sigma})} \\ &= \left\| (D_1 \mathcal{F}^{(\ell)})_{(S_1, T_1, e(S_1, T_1))}(e_{a, b}) - (D_1 \mathcal{F}^{(\ell)})_{(S_1, T_1, 0)}(e_{a, b}) \right\|_{C^{\ell}(\Sigma, g_{\Sigma})} \\ &\leq \int_0^1 \left\| (D_2 \mathcal{F}^{(\ell)})_{(S_1, T_1, \theta e(S_1, T_1))}(e(S_1, T_1), e_{a, b}) \right\|_{C^{\ell}(\Sigma, g_{\Sigma})} d\theta \\ &\leq Ch_0(S_1, T_1) h_{a+b}(S_1, T_1) \end{aligned}$$

This is estimated also by the right hand side of (6.14.2). The proof of Lemma 6.12 is now complete.

Lemma 6.21 *There exists $\varepsilon_{10}(C_{6, \bullet}, D_{\bullet})$ and $C_{23}(C_{6, \bullet}, D_{\bullet})$ such that (4.43) and (6.13) imply the following inequality for $a + b \leq k + 70$, $\ell \leq k + 70$, $(S_1, T_1) \in [-\delta^{-\mu} + 4, \delta^{-\mu} - 4]^2$.*

$$(6.22.1) \quad \left\| \frac{\partial^{a+b}}{\partial S^a \partial T^b} \left(\Phi - \left(\Phi' - \frac{\partial \Phi}{\partial T} \right) \right) (S_1, T_1) \right\|_{C^{\ell}(\Sigma, g_{\Sigma})} \leq C_{23} \sum \delta^j h_{m_1}(S_1, T_1) \cdots h_{m_n}(S_1, T_1),$$

$$(6.22.2) \quad \left\| \frac{\partial^{a+b}}{\partial S^a \partial T^b} \left(\Psi - \left(\Psi' + \frac{\partial \Psi}{\partial S} \right) \right) (S_1, T_1) \right\|_{C^{\ell}(\Sigma, g_{\Sigma})} \leq C_{23} \sum \delta^j h_{n_1}(S_1, T_1) \cdots h_{n_m}(S_1, T_1).$$

Here the sum in the right hand sides of (6.22.1), (6.22.2), is taken over nonnegative integers $j, m_1, \dots, m_n$ such that

$$(6.23) \quad a + b + 1 = j + m_1 + \cdots + m_n, \quad j + n \geq 2, \quad n \geq 1.$$

Proof: We first calculate Φ, Ψ in terms of $g'', \mathcal{A}'$. We put $Z = S + \sqrt{-1}T$. Note

$$\Phi' dS + \Psi' dT = \frac{1}{2}(\Phi' - \sqrt{-1}\Psi') dZ + \frac{1}{2}(\Phi' + \sqrt{-1}\Psi') d\bar{Z}.$$

We put

$$\begin{aligned} \partial^{\mathcal{A}'} g &= \frac{\partial g}{\partial Z} dZ + \frac{1}{2}[\Phi' - \sqrt{-1}\Psi', g] dZ \\ \bar{\partial}^{\mathcal{A}'} g &= \frac{\partial g}{\partial \bar{Z}} d\bar{Z} + \frac{1}{2}[\Phi' + \sqrt{-1}\Psi', g] d\bar{Z}. \end{aligned}$$

Then, we have, by (2.13) and $\overline{g''} = g''$, that

$$\begin{aligned}
(\Phi - \Phi')dS + (\Psi - \Psi')dT &= g''^{-1}\bar{\partial}^3 g'' - (\partial^3 g'')g''^{-1} \\
&= g''^{-1}\frac{\partial g''}{\partial \bar{Z}}d\bar{Z} - \frac{\partial g''}{\partial Z}g''^{-1}dZ \\
&\quad + \frac{1}{2}g''^{-1}[\Phi' - \sqrt{-1}\Psi', g'']d\bar{Z} - \frac{1}{2}[\Phi' + \sqrt{-1}\Psi', g'']g''^{-1}dZ
\end{aligned}$$

Therefore

$$\begin{aligned}
(6.24) \quad \Phi &= \frac{1}{2}\left(g''^{-1}\frac{\partial g''}{\partial S} - \frac{\partial g''}{\partial S}g''^{-1}\right) + \frac{\sqrt{-1}}{2}\left(g''^{-1}\frac{\partial g''}{\partial T} + \frac{\partial g''}{\partial T}g''^{-1}\right) \\
&\quad + \Phi' + \frac{1}{2}\left(g''^{-1}[\Phi', g''] - [\Phi', g'']g''^{-1}\right) - \frac{\sqrt{-1}}{2}\left(g''^{-1}[\Psi', g''] + [\Psi', g'']g''^{-1}\right)
\end{aligned}$$

Let $M(2)$ be the set of all complex valued 2×2 matrix. $\exp : M(2) \rightarrow M(2)$ is a C^∞ map. We define $u : M(2) \times M(2) \rightarrow M(2)$ by

$$u(U, V) = \frac{\partial \exp U}{\partial U} \Big|_U (V) = V + \frac{1}{2}(UV + VU) + \dots$$

We put $g'' = \exp(\sqrt{-1}e)$ and define

$$(6.25.1) \quad \mathcal{H}_1(e, f) = \frac{\sqrt{-1}}{2}(g''^{-1}u(g'', f) - u(g'', f)g''^{-1}),$$

$$(6.25.2) \quad \mathcal{H}_2(e, f) = -\frac{1}{2}(g''^{-1}u(g'', f) + u(g'', f)g''^{-1})$$

$$\begin{aligned}
(6.25.3) \quad \mathcal{H}_3(S, T, e) &= \Phi'(S, T) + \frac{1}{2}(g''^{-1}[\Phi'(S, T), g''] - [\Phi'(S, T), g'']g''^{-1}) \\
&\quad - \frac{\sqrt{-1}}{2}(g''^{-1}[\Psi'(S, T), g''] + [\Psi'(S, T), g'']g''^{-1})
\end{aligned}$$

Here

$$\begin{aligned}
\mathcal{H}_1^{(\ell)} : C^\ell(\Sigma, P_{U(2)}(\Sigma) \times_{ad} su(2))^2 &\rightarrow C^\ell(\Sigma, P_{U(2)}(\Sigma) \times_{ad} su(2)) \\
\mathcal{H}_2^{(\ell)} : C^\ell(\Sigma, P_{U(2)}(\Sigma) \times_{ad} su(2))^2 &\rightarrow C^\ell(\Sigma, P_{U(2)}(\Sigma) \times_{ad} su(2)) \\
\mathcal{H}_3^{(\ell)} : [-\delta^{-\mu} + 2, \delta^{-\mu} - 2]^2 \times C^\ell(\Sigma, P_{U(2)}(\Sigma) \times_{ad} su(2)) &\rightarrow C^\ell(\Sigma, P_{U(2)}(\Sigma) \times_{ad} su(2))
\end{aligned}$$

are C^∞ maps between Banach spaces. By (6.24) we have

$$(6.26) \quad \Phi(S, T) = \mathcal{H}_1(e(S, T), e_{1,0}(S, T)) + \mathcal{H}_2(e(S, T), e_{0,1}(S, T)) + \mathcal{H}_3(S, T, e(S, T)).$$

We remark that $\mathcal{H}_1(e, f)$ and $\mathcal{H}_2(e, f)$ are linear with respect to f . We denote by $D_n \mathcal{H}_i^{(\ell)}$ be the n -th derivative with respect to e direction. By a simple calculation, we have

$$(6.27.1) \quad \mathcal{H}_1^{(\ell)}(0, f) = 0, \quad \mathcal{H}_2^{(\ell)}(0, f) = -f.$$

$$(6.27.2) \quad \mathcal{H}_3^{(\ell)}(S, T, 0) = \Phi', \quad (D_1 \mathcal{H}_3^{(\ell)})_{(S, T, 0)}(e) = [\Psi', e].$$

Now, by Assumption (4.43.3), (4.43.4), we have

$$(6.28.1) \quad \left\| \left(D_n \mathcal{H}_i^{(\ell)} \right)_{(e^{(0)}, f)}(e^{(1)}, \dots, e^{(n)}) \right\|_{C^\ell(\Sigma, g_\Sigma)} \leq C \|f\|_{C^\ell(\Sigma, g_\Sigma)} \|e^{(1)}\|_{C^\ell(\Sigma, g_\Sigma)} \cdots \|e^{(n)}\|_{C^\ell(\Sigma, g_\Sigma)} \quad i = 1, 2.$$

$$(6.28.2) \quad \left\| \left(\frac{\partial^{a'+b'}}{\partial S^a \partial T^b} (D_n \mathcal{H}_3^{(\ell)}) \right)_{(S, T, e^{(0)})}(e^{(1)}, \dots, e^{(n)}) \right\|_{C^\ell(\Sigma, g_\Sigma)} \leq C \delta^{a'+b'+1} \|e^{(1)}\|_{C^\ell(\Sigma, g_\Sigma)} \cdots \|e^{(n)}\|_{C^\ell(\Sigma, g_\Sigma)}$$

We can take the constant C independent of $e^{(0)} = e(S, T)$ because of (6.13).

We first study $\frac{\partial^{a+b}}{\partial S^a \partial T^b} \mathcal{H}_3(S, T, e(S, T))$. It is a sum of the terms

$$(6.29) \quad \left(\frac{\partial^{a'+b'}}{\partial S^a \partial T^b} (D_n \mathcal{H}_3^{(\ell)}) \right)_{(S_1, T_1, e(S_1, T_1))}(e_{c_1, d_1}(S_1, T_1), \dots, e_{c_n, d_n}(S_1, T_1)),$$

where $a' + c_1 + \dots + c_n = a$, $b' + d_1 + \dots + d_n = b$. By (6.28.2) and (6.11.1), the Formula (6.29) is estimated by $C \delta^{a'+b'+1} h_{c_1+d_1} \cdots h_{c_n+d_n}$. This number $C \delta^{a'+b'+1} h_{c_1+d_1} \cdots h_{c_n+d_n}$ is estimated by the right hand side of (6.22.1) unless $(a' + b', n) = (a + b, 0)$.

In the case $(a' + b', n) = (a + b, 0)$, Formula (6.29) is $\left(\frac{\partial^{a+b}}{\partial S^a \partial T^b} \mathcal{H}_3^{(\ell)} \right)_{(S_1, T_1, e(S_1, T_1))}$. We can use (6.27.2) and (6.28.2) in the same way as the proof of Lemma 6.12 to show

$$\left\| \left(\frac{\partial^{a+b}}{\partial S^a \partial T^b} \mathcal{H}_3^{(\ell)} \right)_{(S_1, T_1, e(S_1, T_1))} - \frac{\partial^{a+b} \Phi'}{\partial S^a \partial T^b}(S_1, T_1) \right\|_{C^\ell(\Sigma, g_\Sigma)} \leq C \delta^{a+b+1} h_0.$$

The right hand side is estimated by the right hand side of (6.22.1). Summing up, we find that

$$(6.30) \quad \left\| \left(\frac{\partial^{a+b}}{\partial S^a \partial T^b} \mathcal{H}_3(S, T, e(S, T)) \right)_{(S_1, T_1)} - \frac{\partial^{a+b} \Phi'}{\partial S^a \partial T^b}(S_1, T_1) \right\|_{C^\ell(\Sigma, g_\Sigma)} \leq (\text{right hand side of (6.22.1)}).$$

We next study $\frac{\partial^{a+b}}{\partial S^a \partial T^b} \mathcal{H}_1(e(S, T), e_{1,0}(S, T))$ and $\frac{\partial^{a+b}}{\partial S^a \partial T^b} \mathcal{H}_2(e(S, T), e_{0,1}(S, T))$. Since $\mathcal{H}_i^{(\ell)}(e, f)$ ($i = 1, 2$) is independent of (S, T) and is linear on f , it follows that they are sums of the terms

$$(6.31.1) \quad \left(D_n \mathcal{H}_1^{(\ell)} \right)_{(e(S_1, T_1), e_{a'+1, b'})}(e_{c_1, d_1}(S_1, T_1), \dots, e_{c_n, d_n}(S_1, T_1)), \text{ and}$$

$$(6.31.2) \quad \left(D_n \mathcal{H}_2^{(\ell)} \right)_{(e(S_1, T_1), e_{a', b'+1})}(e_{c_1, d_1}(S_1, T_1), \dots, e_{c_n, d_n}(S_1, T_1)) \text{ respectively,}$$

where $a' + c_1 + \dots + c_n = a$, $b' + d_1 + \dots + d_n = b$. By (6.28.1) and (6.11.1), the Formula (6.31.1), (6.31.2) is estimated by $C h_{a'+b'+1} h_{c_1+d_1} \cdots h_{c_n+d_n}$. This number $C h_{a'+b'+1} h_{c_1+d_1} \cdots h_{c_n+d_n}$ is estimated by the right hand side of (6.22.1) unless $n = 0$, that is $a = a', b = b'$. In the case $a = a', b = b'$, (6.31) is $\mathcal{H}_1^{(\ell)}(e(S_1, T_1), e_{a+1, b})$ and $\mathcal{H}_2^{(\ell)}(e(S_1, T_1), e_{a, b+1})$. Using (6.28.1) in the same way as the proof of Lemma 6.12, we have

$$\begin{aligned} \|\mathcal{H}_1^{(\ell)}(e(S_1, T_1), e_{a+1, b}) - \mathcal{H}_1^{(\ell)}(0, e_{a+1, b})\|_{C^\ell(\Sigma, g_\Sigma)} &\leq Ch_0 h_{a+b+1}, \\ \|\mathcal{H}_2^{(\ell)}(e(S_1, T_1), e_{a, b+1}) - \mathcal{H}_2^{(\ell)}(0, e_{a, b+1})\|_{C^\ell(\Sigma, g_\Sigma)} &\leq Ch_0 h_{a+b+1}. \end{aligned}$$

The right hand side of (6.31) is estimated by the right hand side of (6.22.1). On the other hand, by (6.27.1), we have $\mathcal{H}_1^{(\ell)}(0, e_{a+1, b}) = 0$, and $\mathcal{H}_2^{(\ell)}(0, e_{a, b+1}) = -e_{a, b+1}$. Thus we have

$$(6.32.1) \quad \|\mathcal{H}_1^{(\ell)}(e(S_1, T_1), e_{a+1, b})\|_{C^\ell(\Sigma, g_\Sigma)} \leq Ch_0 h_{a+b+1}$$

$$(6.32.2) \quad \|\mathcal{H}_2^{(\ell)}(e(S_1, T_1), e_{a, b+1}) + e_{a, b+1}\|_{C^\ell(\Sigma, g_\Sigma)} \leq Ch_0 h_{a+b+1}.$$

(6.22.1) follows from (6.26), (6.30), (6.32). The proof of (6.22.2) is similar.

Now, we put

$$(6.33.1) \quad \begin{aligned} Q(e: S, T) &= \Delta_{S, T} e + \alpha(S, T)^{-2} \Delta_{A'(S, T)} e \\ &\quad - \left(\frac{\partial \Psi}{\partial S} - \frac{\partial \Phi}{\partial T} + [\Phi, \Psi] \right) - \alpha(S, T)^{-2} *_\Sigma F_A, \end{aligned}$$

$$(6.33.2) \quad \begin{aligned} Q(e: a, b: S, T) &= \Delta_{S, T} e_{a, b} + \alpha(S, T)^{-2} \Delta_{A'(S, T)} e_{a, b} \\ &\quad - \frac{\partial^{a+b}}{\partial S^a \partial T^b} \left(\left(\frac{\partial \Psi}{\partial S} - \frac{\partial \Phi}{\partial T} + [\Phi, \Psi] \right) + \alpha(S, T)^{-2} *_\Sigma F_A \right) \end{aligned}$$

where

$$\Delta_{S, T} = -\frac{\partial^2}{\partial S^2} - \frac{\partial^2}{\partial T^2}, \quad \Delta_{A'(S, T)} = -*_\Sigma d_{A'(S, T)} *_\Sigma d_{A'(S, T)}.$$

We remark

$$\begin{aligned} \Delta_{S, T} e + \alpha(S, T)^{-2} \Delta_{A'(S, T)} e &= Q(e: S, T), \\ \Delta_{S, T} e_{a, b} + \alpha(S, T)^{-2} \Delta_{A'(S, T)} e_{a, b} &= Q(e: a, b: S, T). \end{aligned}$$

if $\mathcal{A} = g''^* \mathcal{A}'$ satisfies (1.1.2). Hence Lemma 6.34 below is an estimate of the nonlinear part of Equation (1.1.2). We recall $\lambda = \frac{1}{100k}$.

Lemma 6.34 *There exist $C_{24}(C_{5*}(\mu, \lambda), D_*)$ and $\varepsilon_{11}(C_{5*}(\mu, \lambda), D_*)$ such that (4.43) and $\|F_{g''^* \mathcal{A}}\|_{L^2([S_0-1, S_0+1] \times [T_0-1, T_0+1] \times \Sigma, g')}$ $< \varepsilon_{11}$ (for every $(S_0, T_0) \in [-\delta^{-\mu} + 1, \delta^{-\mu} - 1]^2$) imply*

$$(6.35.1) \quad \|Q(e: S, T)\|_{C^\ell(\Sigma, g_\Sigma)} \leq C_{24}(\delta^2 + h_0(S, T)^2 + h_1(S, T)^2 + h_2(S, T)^2),$$

$$(6.35.2) \quad \begin{aligned} &\|Q(e: a, b: S, T)\|_{C^\ell(\Sigma, g_\Sigma)} \\ &\leq C_{24} \delta^{2+a+b} + C_{24} \sum \delta^{\max\{0, j-\lambda\}} h_{m_1}(S, T) \cdots h_{m_n}(S, T), \end{aligned}$$

for $\ell \leq k+60$. Here the sum in the right hand sides of (6.35.2) is taken over nonpositive integers $j, m_1, \dots, m_n$ such that

$$(6.36) \quad a+b \leq j + m_1 + \cdots + m_n \leq a+b+2, \quad j+n \geq 2, \quad n \geq 1.$$

Proof: $\Delta_{A'(S,T)}e - *_\Sigma F_A$ is estimated by right hand side of (6.35.1) by virtue of Lemma 6.12. We calculate

$$(6.37) \quad \begin{aligned} & \frac{\partial}{\partial S} \left(\Psi' - \frac{\partial e}{\partial S} \right) - \frac{\partial}{\partial T} \left(\Phi' + \frac{\partial e}{\partial T} \right) + \left[\Psi' - \frac{\partial e}{\partial S}, \Phi' + \frac{\partial e}{\partial T} \right] \\ &= \Delta_{S,T}e + \frac{\partial \Psi'}{\partial S} - \frac{\partial \Phi'}{\partial T} + \left[\Psi' - \frac{\partial e}{\partial S}, \Phi' + \frac{\partial e}{\partial T} \right] \end{aligned}$$

We remark that, by (4.43),

$$(6.38) \quad \begin{aligned} & \left\| \frac{\partial \Psi'}{\partial S} - \frac{\partial \Phi'}{\partial T} + \left[\Psi' - \frac{\partial e}{\partial S}, \Phi' + \frac{\partial e}{\partial T} \right] \right\|_{C^t(\Sigma, g_\Sigma)} \\ & \leq C \left(\delta^2 + \delta \left(\|e_{1,0}\|_{C^t(\Sigma, g_\Sigma)} + \|e_{0,1}\|_{C^t(\Sigma, g_\Sigma)} \right) + \|e_{1,0}\|_{C^t(\Sigma, g_\Sigma)} \|e_{0,1}\|_{C^t(\Sigma, g_\Sigma)} \right) \end{aligned}$$

The right hand side of (6.38) is estimated by the right hand side of (6.35.1). On the other hand, Lemma 6.21 and (4.43) imply that C^t norm of

$$(6.39) \quad \frac{\partial \Psi}{\partial S} - \frac{\partial \Phi}{\partial T} + [\Phi, \Psi] - \left(\frac{\partial}{\partial S} \left(\Psi' - \frac{\partial e}{\partial S} \right) - \frac{\partial}{\partial T} \left(\Phi' + \frac{\partial e}{\partial T} \right) + \left[\Psi' - \frac{\partial e}{\partial S}, \Phi' + \frac{\partial e}{\partial T} \right] \right)$$

is estimated by the right hand side of (6.35.1). We remark also $\alpha \leq 2$. (6.35.1) follows.

Next we prove (6.35.2). First we remark that

$$(6.40) \quad \left\| \frac{\partial^{a+b}}{\partial S^a \partial T^b} (\Delta_{A'(S,T)}e - *_\Sigma F_A) \right\|_{C^t(\Sigma, g_\Sigma)} \leq C \sum \delta^j h_{m_1}(S_1, T_1) \cdots h_{m_n}(S_1, T_1)$$

by virtue of Lemma 6.12. Here $j, m_1, \dots, m_n$ is as in (6.36). Next (4.43) and (6.11.1) imply that

$$\left\| \frac{\partial^{a+b}}{\partial S^a \partial T^b} \left(\frac{\partial \Psi'}{\partial S} - \frac{\partial \Phi'}{\partial T} + \left[\Psi' - \frac{\partial e}{\partial S}, \Phi' + \frac{\partial e}{\partial T} \right] \right) \right\|_{C^t(\Sigma, g_\Sigma)}$$

is estimated by

$$C\delta^{2+a+b} + C \sum \delta^{j+1} h_{1+a+b-j} + \sum h_{1+j} h_{1+a+b-j}.$$

On the other hand, using Lemma 6.21 and (4.43), the C^t norm of $\frac{\partial^{a+b}}{\partial S^a \partial T^b}$ of Formula (6.39) is estimated by $C \sum \delta^j h_{m_1} \cdots h_{m_n}$ where $j, n_1, \dots, n_\ell$ in the first term is as in (6.36). Thus we have

$$(6.41) \quad \left\| \frac{\partial^{a+b}}{\partial S^a \partial T^b} \left(\frac{\partial \Psi}{\partial S} - \frac{\partial \Phi}{\partial T} + [\Phi, \Psi] - \Delta_{S,T}e \right) \right\|_{C^t(\Sigma, g_\Sigma)} \leq C\delta^{2+a+b} + C \sum \delta^j h_{m_1} \cdots h_{m_n}$$

Finally we use Lemma 4.7 (4.8.1) to find that

$$(6.42) \quad \left| \frac{\partial^{a+b} \alpha}{\partial S^a \partial T^b} \right| < C \delta^{a+b-\lambda}.$$

(6.35.2) follows from (6.40), (6.41), (6.42). (We remark that the term $C \delta^{a+b+2}$ is only in (6.41) and not in (6.40), and α^{-2} is multiplied only to (6.40).)

We denote by $\langle \rangle$ the L^2 inner product of (Σ, g_Σ) .

Lemma 6.43 *There exists $\tau > 0$ such that*

$$\langle \Delta_{A'(S,T)} e, e \rangle \geq \tau \langle e, e \rangle$$

for any (S, T) and $e \in \Gamma(\Sigma, P_{U(2)}(\Sigma) \times_{ad} su(2))$.

Proof: Since $A'(S, T)$ is flat and hence is irreducible by our choice of bundles, it follows that there exists such τ for each (S, T) . Independence of τ of $A'(S, T)$ is a consequence of the compactness of $\mathcal{R}_{k+100}(A_\alpha)$.

Remark 6.44 This is one of the main points where the argument of this paper does not work for the trivial bundle on Σ and works only for nontrivial $SO(3)$ bundle.

Lemma 6.45 *There exist $\varepsilon_{12}(C_{5,\bullet}, D_\bullet)$ and $C_{25}(C_{5,\bullet}, D_\bullet)$ such that (4.43),*

$$\left| F_{g', \mathcal{A}} \right|_{L^2([S_0-1, S_0+1] \times [T_0-1, T_0+1] \times \Sigma, g')} < \varepsilon_{12} \text{ (for all } (S_0, T_0) \in [-\delta^{-\mu} + 1, \delta^{-\mu} - 1]) \text{ imply}$$

$$(6.46) \quad \Delta_{S,T} f(S_1, T_1) \leq -\tau f(S_1, T_1) + C_{25} \delta^2 \sqrt{f(S_1, T_1)} + C_{24} \sqrt{f(S_1, T_1)} \int_{S_1-1}^{S_1+1} \int_{T_1-1}^{T_1+1} f dS dT.$$

Proof: By our Equation (1.1.2), Lemmata 6.34, 6.43, and (6.11.2) (and Hölder inequality) we have

$$\begin{aligned} (\Delta_{S,T} f)(S_1, T_1) &= 2 \langle \Delta_{S,T} e, e \rangle - 2 \left\langle \frac{\partial e}{\partial S}, \frac{\partial e}{\partial S} \right\rangle - 2 \left\langle \frac{\partial e}{\partial T}, \frac{\partial e}{\partial T} \right\rangle \\ &\leq -2 \langle \Delta_{A'(S_1, T_1)} e, e \rangle + \langle Q(e : S_1, T_1), f \rangle \\ &\leq -2 \tau f(S_1, T_1) + \|Q(e : S_1, T_1)\|_{L^2(\Sigma, g_\Sigma)} \sqrt{f(S_1, T_1)} \\ &\leq -2 \tau f + C \delta^2 \sqrt{f} + C \sqrt{f} \int_{S_1-1}^{S_1+1} \int_{T_1-1}^{T_1+1} f dS dT \end{aligned}$$

The proof of Lemma 6.45 is complete.

Therefore, the following Lemma 6.46, together with Lemma 6.45 and (6.11.1) implies (4.45.1).

Lemma 6.46 *For any $\tau > 0$ there exists $\varepsilon_{13}(\tau)$ and $C_{26}(\tau)$ (independent of Λ) such that if $f : [-\Lambda, \Lambda] \times [-\Lambda, \Lambda] = V \rightarrow [0, \infty)$ is a smooth function such that*

$$(6.47) \quad \Delta_{S,T} f(S_1, T_1) \leq -\tau f(S_1, T_1) + \kappa \sqrt{f(S_1, T_1)} + \varepsilon_{13} \int_{S_1-1}^{S_1+1} \int_{T_1-1}^{T_1+1} f dS dT$$

then

$$f(S_1, T_1) \leq C_{26} \kappa^2 + C_{26} \exp\left(-\frac{\text{dist}((S_1, T_1), \partial V)}{C_{26}}\right) \sup f.$$

Proof: We first prove the following sublemma.

Sublemma 6.48 *There exists $C_{27}(\tau)$ with the following property. Suppose f satisfies (6.47). Then, for each $(S_0, T_0) \in V$ with $\text{dist}((S_0, T_0), \partial V) > 3$, one of the following (6.49) or (6.50) holds.*

$$(6.49) \quad f(S_0, T_0) \leq C_{27} \kappa^2$$

$$(6.50) \quad \text{There exists } (S_1, T_1) \text{ such that}$$

$$(6.50.1) \quad \text{dist}((S_0, T_0), (S_1, T_1)) \leq 3,$$

$$(6.50.2) \quad f(S_1, T_1) \geq \left(1 + \frac{1}{C_{27}}\right) f(S_0, T_0).$$

Proof: Let $\theta > 0$ be a number depending only on τ and determined later. We put

$$g(S, T) = \exp(-\theta((S - S_0)^2 + (T - T_0)^2)) f(S, T)$$

and $A = \sup\{g(S, T) \mid (S, T) \in B((S_0, T_0), 1)\}$. Here $B((S_0, T_0), 1)$ is the metric ball of radius 1 centered at (S_0, T_0) . Let us take $(S_2, T_2) \in B((S_0, T_0), 1)$ such that $g(S_2, T_2) = A$. If $(S_2, T_2) \in \partial B((S_0, T_0), 1)$ we have

$$f(S_0, T_0) = g(S_0, T_0) \leq g(S_2, T_2) = e^{-\theta} f(S_2, T_2).$$

Hence (6.50) is satisfied by putting $(S_1, T_1) = (S_2, T_2)$. If $(S_2, T_2) \notin \partial B((S_0, T_0), 1)$ we have

$$\frac{\partial g}{\partial S}(S_2, T_2) = \frac{\partial g}{\partial T}(S_2, T_2) = 0.$$

Therefore, by a simple calculation, we have

$$(6.51) \quad \begin{cases} \frac{\partial f}{\partial S}(S_2, T_2) = 2\theta(S_1 - S_0)f(S_2, T_2) \\ \frac{\partial f}{\partial T}(S_2, T_2) = 2\theta(T_1 - T_0)f(S_2, T_2) \end{cases}.$$

Using (6.47) and (6.51), we obtain

$$\begin{aligned} 0 \leq \Delta_{S,T} g(S_2, T_2) &\leq -2\tau g(S_2, T_2) + \kappa \exp(-\theta((S_2 - S_0)^2 + (T_2 - T_0)^2)) \sqrt{f(S_2, T_2)} \\ &\quad + \varepsilon_{12} \exp(-\theta((S_2 - S_0)^2 + (T_2 - T_0)^2)) \int_{S_2-1}^{S_2+1} \int_{T_2-1}^{T_2+1} f dS dT \\ &\quad + (2\theta + 4\theta^2((S_2 - S_0)^2 + (T_2 - T_0)^2)) g(S_2, T_2) \end{aligned}$$

We choose θ so that $2\theta + 4\theta^2 < \tau$. Then, since $(S_2, T_2) \in B((S_0, T_0), 1)$, we have

$$(6.52) \quad \mathcal{F}(S_2, T_2) \leq \kappa \sqrt{f(S_2, T_2)} + \varepsilon_{13} \int_{S-1}^{S+1} \int_{T-1}^{T+1} f dS dT.$$

Suppose $\kappa \sqrt{f(S_2, T_2)} \geq \varepsilon_{13} \int_{S-1}^{S+1} \int_{T-1}^{T+1} f dS dT$. Then (6.52) implies $\mathcal{F}(S_2, T_2) \leq \frac{1}{2} \kappa \sqrt{f(S_2, T_2)}$. Therefore

$$f(S_0, T_0) = g(S_0, T_0) \leq g(S_2, T_2) \leq f(S_2, T_2) \leq \left(\frac{1}{2\tau}\right)^2 \kappa^2.$$

Thus (6.49) holds.

Suppose $\kappa \sqrt{f(S_2, T_2)} \leq \varepsilon_{13} \int_{S-1}^{S+1} \int_{T-1}^{T+1} f dS dT$. Then (6.52) implies

$$\mathcal{F}(S_2, T_2) \leq \frac{1}{2} \varepsilon_{13} \int_{S-1}^{S+1} \int_{T-1}^{T+1} f dS dT.$$

It follows that, there exists $(S_1, T_1) \in [S_2 - 1, S_2 + 1] \times [T_2 - 1, T_2 + 1]$ such that

$$(6.53) \quad f(S_0, T_0) \leq f(S_2, T_2) \leq \frac{\varepsilon_{13}}{2\tau} \int_{S-1}^{S+1} \int_{T-1}^{T+1} f dS dT \leq \frac{2\varepsilon_{13}}{\tau} f(S_1, T_1).$$

We may assume $2\varepsilon_{13}/\tau < 1/2$ by choosing ε_{13} small. Then, (6.53) implies $f(S_1, T_1) \geq 2f(S_0, T_0)$. Hence (S_1, T_1) satisfies (6.50). The proof of Sublemma 6.48 is now complete.

Proof of Lemma 6.46: If $f(S_0, T_0) \leq C_{27}\kappa^2$, there is nothing to prove. Suppose $f(S_0, T_0) > C_{27}\kappa^2$. Then by Sublemma 6.48 we obtain (S_1, T_1) satisfying (6.50). Since $f(S_1, T_1) > f(S_0, T_0) > C_{27}\kappa^2$, we find (S_2, T_2) such that

$$(6.54.1) \quad \text{dist}((S_1, T_1), (S_2, T_2)) \leq 3,$$

$$(6.54.2) \quad f(S_2, T_2) \geq \left(1 + \frac{1}{C_{27}}\right) f(S_1, T_1).$$

We can repeat until $\text{dist}((S_N, T_N), \partial V) \leq 3$. (6.54.1) implies that

$$(6.55) \quad N > \frac{\text{dist}((S_0, T_0), \partial V)}{3} - 10.$$

Therefore by (6.54.2) we have

$$(6.56) \quad f(S_0, T_0) \leq \left(1 + \frac{1}{C_{27}}\right)^{-N} f(S_N, T_N) \leq \left(1 + \frac{1}{C_{27}}\right)^{-N} \sup f.$$

(6.55) and (6.56) imply

$$f(S_0, T_0) \leq C \exp\left(-\frac{\text{dist}((S_0, T_0), \partial V)}{C}\right) \sup f.$$

Hence Lemma 6.48 holds in this case also.

We thus completed the proof of (4.45.1).

We next prove (4.45.2). We are going to show the following by induction on j ($\leq k$).

We put

$$V_j = \left[\frac{-(9k-j)\delta^{-\mu}}{10k} + 10, \frac{(9k-j)\delta^{-\mu}}{10k} - 10 \right]^2.$$

Lemma 6.57 *There exists $\varepsilon_{14,j}$ and $C_{27,j}$ such that if the Assumptions (4.43) and (4.44) are satisfied for $\varepsilon_7 \leq \varepsilon_{14,j}$, then*

$$h_j(S_0, T_0) \leq C_{27,j} \delta^{2+j} + C_{27,j} \exp\left(-\frac{\delta^{-\mu}}{C_{27,j}}\right),$$

for $(S_0, T_0) \in V_j$. Here $\varepsilon_{14,j}$ depends only on $\varepsilon_{14,\ell}$, $C_{27,\ell}$, $\ell < j$ and $C_{5,\bullet}(\mu, \lambda)$, $D_\bullet$ ($\bullet \leq k+100$.) And $C_{27,j}$ depends only on $C_{27,\ell}$, $\ell < j$ and $C_{5,\bullet}(\mu, \lambda)$, $D_\bullet$ ($\bullet \leq k+100$.)

Proof: We already proved the case $j=0$. We assume that the lemma is true for $\leq j-1$. We may choose $\varepsilon_{14,j}$ so that

$$C_{27,\ell} \exp\left(-\frac{\delta^{-\mu}}{C_{27,\ell}}\right) \leq \delta^{10k}$$

for $\delta < \varepsilon_{14,j}$ and $\ell < j$. We remark that $\delta < \varepsilon_{14,j}$ is assumed by (4.44.2). Therefore, by induction hypothesis, we have

$$(6.58) \quad h_\ell(S_0, T_0) \leq 2C_{27,\ell} \delta^{2+\ell}$$

for $\ell < j$, $(S_0, T_0) \in V_{j-1}$. Let $a+b=j$. We use (6.58) and Lemma 6.34 and find C_{28} such that

$$(6.59) \quad \begin{aligned} \|Q(e; a, b; S_1, T_1)\|_{C^{k+70}(\Sigma, g_\Sigma)} &\leq C_{28} \delta^{2+j} + C_{28} \delta^{1-\lambda} (h_j + h_{j+1}) \\ &\quad + C_{28} (h_0 + \cdots + h_j) (h_j + h_{j+1} + h_{j+2}) \end{aligned}$$

for $(S_1, T_1) \in V_{j-1}$. We remark

$$(h_j + h_{j+1})^2 < C \int_{S_1-1}^{S_1+1} \int_{T_1-1}^{T_1+1} h_{j-1}^2 dS dT < C \delta^{2(1+j)}$$

by (6.11.2). Also we have

$$(h_0 + \cdots + h_j) (h_j + h_{j+1} + h_{j+2}) \leq C \delta \sqrt{\int_{S_1-1}^{S_1+1} \int_{T_1-1}^{T_1+1} h_j^2 dS dT},$$

by (6.11.2). Therefore, (6.59) implies :

$$\begin{aligned}
 (\Delta_{S,T} f_{a,b})(S_1, T_1) &= 2\langle \Delta_{S,T} e_{a,b}, e_{a,b} \rangle - 2\left\langle \frac{\partial e_{a,b}}{\partial S}, \frac{\partial e_{a,b}}{\partial S} \right\rangle - 2\left\langle \frac{\partial e_{a,b}}{\partial T}, \frac{\partial e_{a,b}}{\partial T} \right\rangle \\
 &\leq -2\langle \Delta_{A'(S,T)} e_{a,b}, e_{a,b} \rangle + 2\langle Q(e: a, b: S, T), e_{a,b} \rangle \\
 &\leq -2\mathfrak{f}_{a,b} + C_{29}\delta^{j+2}\sqrt{f_{a,b}} + C_{29}\delta^{1-\lambda}\sqrt{f_{a,b}}\sqrt{\int_{S_1-1}^{S_1+1}\int_{T_1-1}^{T_1+1}h_j^2dSdT}
 \end{aligned}$$

Take the sum over all a, b with $a + b = j$ and using $f_{a,b} \leq C_{30}\int_{S_1-1}^{S_1+1}\int_{T_1-1}^{T_1+1}h^2dSdT$, we have

$$(\Delta_{S,T} h_j^2)(S_1, T_1) \leq -2\mathfrak{h}_j^2 + C_{29}\delta^{j+2}h_m + j\sqrt{C_{30}}C_{29}\delta^{1-\lambda}\int_{S_1-1}^{S_1+1}\int_{T_1-1}^{T_1+1}h_j^2dSdT.$$

We may choose $\varepsilon_{14,j}$ enough small so that we $j\sqrt{C_{30}}C_{29}\delta^{1-\lambda} < \varepsilon_{13}(\tau)$. (Here ε_{13} is as in Lemma 6.46.) Hence we can apply Lemma 6.46 to h_j^2 and obtain

$$(6.60) \quad h_m^2(S, T) \leq C_{31}\delta^{2(j+2)} + C_{31}\exp\left(-\frac{\text{dist}((S, T), \partial V_{j-1})}{C_{31}}\right).$$

If $(S, T) \in V_j$, then $\text{dist}((S, T), \partial V_{j-1}) > \delta^{-\mu}/10k$. Hence (6.60) implies Lemma 6.57 for j . The proof of Lemma 6.57 is now complete.

We put $j = k$ in Lemma 6.57. Hence using (6.11.1) we have

$$\left\| \frac{\partial^k e}{\partial S^m \partial T^{k-m}} \right\|_{C^k(\Sigma, g_\Sigma)} < C\delta^{k+2} + C\exp\left(-\delta^{-\mu}/C\right).$$

The proof of Mainlemma 4.42 is now complete.

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Figure 3.19

Figure 5.40