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SMOOTH GROUP ACTIONS ON 4-MANIFOLDS AND
THE SEIBERG-WITTEN INVARIANTS : II

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Abstract

In this paper we study the Seiberg-Witten invariants of 4-manifold with a finite group(or a compact Lie group) acting on. Among other things, we will prove the following result:

Let X be a smooth closed 4-manifold. Suppose $H_1(X, \mathbb{R}) = 0$ and $b_2^+ \geq 2$, where b_2^+ is the rank of $H_+^2(X)$. Let $\mathcal{C}$ be a $Spin^c$ -structure on X . Assume that $\mathcal{C}$ is equivariant with respect to an $\mathbb{Z}_p$ action on X , where p is a prime. If $\mathbb{Z}_p$ acts on the space $H_+^2(X, \mathbb{R})$ of harmonic self dual 2-forms trivially. Then the Seiberg-Witten invariant

$$SW(\mathcal{C}) = 0(mod p)$$

if $k_i \leq \frac{1}{2}(b_2^+ - 1)$ for $i = 0, 1, \dots, p-1$. Here $k_i = m_i - n_i$, m_i and n_i are the dimensions of the ω^i -eigenspaces of the linear $\mathbb{Z}_p$ actions on $ker D_A$ and $coker D_A$ respectively, $\omega = e^{\frac{2\pi i}{p}}$ is the p -th unit root. $D_A : \Gamma(W^+) \rightarrow \Gamma(W^-)$ is the Dirac operator corresponding to $\mathcal{C}$ and an equivariant connection A on $\det W^+$.

§1 Introduction

In this paper we study the Seiberg-Witten invariant of smooth 4-manifolds with the symmetry of some finite group or compact Lie group. This is a continuation of [3] where we have used the Seiberg-Witten moduli space of a Spin manifold in the presence of a finite group action. We will prove some mod p vanishing theorems for the Seiberg-Witten invariants. The strategy of this work is to use the “finite dimensional approximation” technique due to Furuta to interpret the Seiberg-Witten invariant as certain equivariant degree. In this sequel equivariant K-theory especially the Adams ψ -operation plays a key role. In his celebrated paper [12], E.Witten proved that the Seiberg-Witten invariant for a Kähler manifold is ± 1 for the standard $Spin^c$ -structure. C.Taubes [11] has generalized this result to symplectic 4-manifolds. It is may be interesting to compare these with our vanishing theorem, because this would exclude the existence of certain group actions on symplectic 4-manifold.

To state our main results, we need to give some necessary preliminaries. Let X be a closed Riemannian $Spin^c$ four manifold. Let $\mathcal{C}$ be a $Spin^c$ structure. A $Spin^c$ structure

consists of a principal $Spin^c$ -bundle $P_{Spin^c}(X)$ over X with a $Spin^c$ -equivariant bundle map

$$P_{Spin^c}(X) \rightarrow P_{SO}(X) \times P_{U(1)}(X)$$

The first Chern class of the bundle $P_{U(1)}(X)$ is called the *canonical class* of the $Spin^c$ -structure. Let us write W^+ and W^- to denote the associated complex spinor bundles. Let $L = \det W^+$, the determinant line bundle on X . Note that L is the associated line bundle of $P_{U(1)}(X)$.

Let G be a finite group or a compact Lie group acting on X preserving the orientation. We can assume that G preserves the isometries without loss of generality. Hence G acts on the frame bundle $P_{SO}(X)$. We say that the $Spin^c$ -structure $\mathcal{C}$ is *preserved* by the G -action if $P_{U(1)}(X)$ is a principal G - $U(1)$ bundle, and the product action of G on $P_{SO}(X) \times P_{U(1)}(X)$ lifts to the bundle $P_{Spin^c}(X)$. We say that $\mathcal{C}$ is *G -equivariant* if the induced G -action on $P_{SO}(X) \times P_{U(1)}(X)$ lifts to a G -action on $P_{Spin^c}(X)$. For such a $Spin^c$ -structure, if A is a G -connection on L , the Dirac operator D_A is G -equivariant. Therefore the index $ind_G D_A = \ker D_A - \operatorname{coker} D_A \in R(G)$.

In particular, if $G = \mathbb{Z}_p$ is a cyclic group and $\mathcal{C}$ is $\mathbb{Z}_p$ -equivariant. Then the eigenvalues of the $\mathbb{Z}_p$ -actions on $\ker D_A$ and $\operatorname{coker} D_A$ are the p -th unit roots of 1. Let $\omega_p = e^{2\pi\sqrt{-1}/p}$ and let $k_i = m_i - n_i$, where m_i and n_i are the dimensions of the ω_p^i -eigenspace of $\ker D_A$ and $\operatorname{coker} D_A$ respectively. Obviously $k_0 + k_1 + \cdots + k_{p-1} = ind D_A$. For convenience we use b_i to denote the i -th Betti number of X and b_2^+ for the rank of $H_+^2(X, \mathbb{R})$.

Our main results are as follows

Theorem 1 *Let X be a smooth closed $Spin^c$ four manifold with $b_1 = 0$ and $b_2^+ \geq 2$. Let $\mathcal{C}$ be a $Spin^c$ structure equivariant with respect to a smooth $\mathbb{Z}_p$ action on X , where p is a prime. If $b_2^+(X/\mathbb{Z}_p) = b_2^+$, then the Seiberg-Witten invariant*

$$SW(\mathcal{C}) = 0 \pmod{p}$$

when $k_i \leq \frac{1}{2}(b_2^+ - 1)$ for $i = 0, 1, \dots, p-1$.

Recall that for an oriented 4-manifold X with $H_1(X, \mathbb{Z}_2) = 0$, a $Spin^c$ structure on X is completely determined by its canonical class. In this case, $\mathcal{C}$ is equivariant with respect to an $\mathbb{Z}_p$ action on X for an odd p if and only if L is a $\mathbb{Z}_p$ - $U(1)$ -bundle. On the other hand, by [6] L is an $\mathbb{Z}_p$ - $U(1)$ -bundle if and only if $c_1(L) \in H^2(X, \mathbb{Z})$ is $\mathbb{Z}_p$ -invariant.

However, for p even, sometimes the lifted action to $P_{Spin^c}(X)$ must have higher order. For example, if $p = 2$ and X is spin, let $\mathcal{C}$ be the $Spin^c$ structure with trivial canonical class, then $\mathcal{C}$ is $\mathbb{Z}_2$ -equivariant if and only if this $\mathbb{Z}_2$ action on X has only isolated fixed points [1].

Let $\tau : X \rightarrow X$ be an involution preserving the $Spin^c$ -structure $\mathcal{C}$. We say that τ is of *even type* if $\mathcal{C}$ is $\mathbb{Z}_2$ -equivariant, where $\mathbb{Z}_2$ is generated by τ . Otherwise, we say that τ is of *odd type*. In the latter case, it is easy to see that

$$\tau : P_{SO}(X) \times P_{U(1)}(X) \rightarrow P_{SO}(X) \times P_{U(1)}(X)$$

can be lifted to an fiberwise diffeomorphism $\hat{\tau} : P_{Spin^c}(X) \rightarrow P_{Spin^c}(X)$ such that $\hat{\tau}^2 = -1$. Therefore, the eigenvalues of $\hat{\tau}$ on the spinor bundles W^+ and W^- are $\pm\sqrt{-1}$. If A is a $\mathbb{Z}_2$ -connection on L , the eigenvalues of $\hat{\tau}$ on $\ker D_A$ and $\operatorname{coker} D_A$ are $\pm\sqrt{-1}$. Define $k_+ := m_+ - n_+$ and $k_- := m_- - n_-$, where $m_{\pm}$ and $n_{\pm}$ denote the dimensions of $\sqrt{-1}$ and $-\sqrt{-1}$ eigenspaces of $\hat{\tau}$ on $\ker D_A$ and $\operatorname{coker} D_A$ respectively. We have the following

Theorem 2 *Let X be a smooth closed $Spin^c$ four manifold. Suppose $b_1 = 0$ and $b_2^+ \geq 2$. Let $\mathcal{C}$ be the $Spin^c$ -structure on X . Suppose that $\tau : X \rightarrow X$ is a smooth odd type involution with respect to $\mathcal{C}$ such that $b_2^+(X/\tau) = b_2^+$. Then the Seiberg-Witten invariant*

$$SW(\mathcal{C}) = 0 \pmod{2}$$

if $k_{\pm} \leq \frac{1}{2}(b_2^+ - 1)$.

Let $\mathcal{M}_{\mathcal{C}}$ denote the Seiberg-Witten moduli space. Define $2d$ to be the dimension of $\mathcal{M}_{\mathcal{C}}$. It is well known [12] that $2d = \frac{1}{4}(c_1(L)^2 - 2\chi(X) - 3\sigma(X))$. By the definition of the Seiberg-Witten invariant, if $d \notin \mathbb{Z}$ or negative, $SW(\mathcal{C}) = 0$. It is conjectured that $SW(\mathcal{C}) = 0$ if $d \geq 1$, X is simply connected or $b_1(X) = 0$. Furuta announced an interesting partial result towards this conjecture

Theorem 3 (Furuta) *Let X be a $Spin^c$ 4-manifold. Assume that $b_1 = 0$, $b_2^+ \geq 2$. Let $\mathcal{C}$ denote the $Spin^c$ structure. Let $2d$ denote the dimension of the moduli space $\mathcal{M}_{\mathcal{C}}$. Then the Seiberg-Witten invariant $a_i SW(\mathcal{C}) \in \mathbb{Z}$ for $1 \leq i \leq d$, where a_i is given by the following power series $\{\frac{\log(1+x)}{x}\}^l = 1 + a_1x + a_2x^2 + \dots$. Here $l = \frac{b_2^+ - 1}{2}$.*

So far we have not yet seen the details for the proof. For convenience we will include some details of the proof.

If $d = 0$, by the dimension formula it is easy to check that $\operatorname{ind} D_A = l + 1$, where $l = \frac{1}{2}(b_2^+ - 1)$ as above. On the other hand, $\operatorname{ind} D_A \in R(G)$ can be calculated in terms of the fixed point data using the Atiyah-Singer index theorem. Thus $k_0, k_1, \dots, k_{p-1}$ and $k_{\pm}$ in theorems 1 and 2 are determined by the local data about the fixed point set.

Corollary 4 *Let (X, ω) be a closed simply connected symplectic 4-manifold with $\Lambda^{0,2}T^*(X)$ trivial. Let τ be an involution on X preserving the symplectic structure. If $b_2^+(X) \geq 2$ and τ has no isolated fixed point, then $b_2^+(X/\tau) < b_2^+(X)$.*

Proof: By [11] there is a canonical $Spin^c$ -structure on X say $\mathcal{C}$, with determinant line bundle $L = \Lambda^{0,2}T^*(X)$. By assumption, we conclude that $\mathcal{C}$ is $\mathbb{Z}_2$ -equivariant and X is a spin manifold.

Suppose $b_2^+(X/\tau) = b_2^+(X)$, in other words τ acts trivially on $H_+^2(X, \mathbb{R})$. By the Atiyah-Singer G -index theorem it is easy to verify that $\operatorname{ind}_{\tau} D_A = 0$, because that τ has no isolated fixed point. Thus $k_+ = k_- = \frac{1}{2}\operatorname{ind} D_A$. Since the moduli space $\mathcal{M}_{\mathcal{C}}$ is zero dimensional, $\operatorname{ind} D_A = l + 1$. By Theorem 2 the Seiberg-Witten invariant $SW(\mathcal{C}) = 0 \pmod{2}$. This contradicts with Taubes' result [11] that $SW(\mathcal{C}) = \pm 1$. Thus we conclude that $b_2^+(X/\tau) < b_2^+(X)$, q.e.d.

To conclude this section, let us give a remark. It is quite obvious that our results can be applied to study $\mathbb{Z}_p$ actions on 4-manifolds with nontrivial Seiberg-Witten invariant. For example, it may be applied to study group actions on K_3 -surface. This is related to a question, namely whether there is homologically trivial $\mathbb{Z}_p$ action on K_3 surface. For $p = 2$, it is already shown [10] that no such nontrivial involution even topologically. However, it is not hard to construct such a homologically trivial topological action for odd p .

The rest of this paper is organized as follows. In §2 we briefly recall some necessary details about the Seiberg-Witten theory and the finite dimensional approximation method. A key result for the rest, Theorem 2.4, will be proved. In §3 we give a proof of Theorem 3 following Furuta. In §4 we study the Seiberg-Witten invariant of 4-manifolds with an involution action on. The proof theorem 1 for $p = 2$ and theorem 2 will be given there. In §5 we carry out the analogue of §4 for odd order group action and prove the rest of theorem 1.

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§2 Seiberg-Witten Theory

In this section we first give a brief review on Seiberg-Witten theory and Furuta's "finite dimensional approximation" method. Using this method, we will interpret the Seiberg-Witten invariant of a 4-manifold with a group action as certain equivariant degree.

Throughout the rest of this paper let X be a smooth, closed, connected, oriented Riemannian 4-manifold satisfying $b_1(X) = 0$ and $b_2^+(X) \geq 2$. Recall that there always exist $Spin^c$ structures on X , which are one to one correspondent to classes in $H^2(X, \mathbb{Z})$. Let $\mathcal{C}$ be a $Spin^c$ structure on X . As above we use W^+ and W^- to denote the associated complex spinor bundles. Let $L = \det W^+$, the determinant line bundle on X . Recall that when $H^1(X, \mathbb{Z}_2) = 0$, $\mathcal{C}$ is entirely determined by the topology of L .

Let $i = \sqrt{-1}$. Using the Clifford multiplication, we will identify $i\Omega^+$ with the Lie algebra bundle $isu(W^+)$.

The Seiberg-Witten's monopole equations are a pair of equations for a unitary connection A on L and a section $\phi \in \Gamma(W^+)$:

$$\begin{aligned} D_A \phi &= 0 \\ F_A^+ &= q(\phi) \end{aligned}$$

Here $q(\phi) := \phi \otimes \phi^* - \frac{|\phi|^2}{2} Id$, $D_A : \Gamma(W^+) \rightarrow \Gamma(W^-)$ is the twisted Dirac operator. F_A^+ is the self dual part of the curvature F_A .

The gauge group $\mathcal{G}_L = Map(X, U(1))$ acts on the set of solutions. Let $\mathcal{M}_c$ denote the moduli space, the quotient of the set of solutions by the gauge group.

Let $\mathcal{A} := \{\text{unitary connections on } L\}$. $\Omega^+ := \{\text{self dual part of harmonic 2-forms}\}$.

The above equations give a $\mathcal{G}_L$ -equivariant map $\mathcal{F}$

$$\begin{aligned}\mathcal{A} \times \Gamma(W^+) &\rightarrow i\Omega^+ \times \Gamma(W^-) \\ (A, \phi) &\rightarrow (F_A^+ - q(\phi), D_A\phi)\end{aligned}$$

The moduli space $\mathcal{M}_c = \mathcal{F}^{-1}(0)/\mathcal{G}_L$.

To study the moduli space and its property, one has to work in the completed Banach space with suitable Sobolev norm. By [4], one needs to complete $\Gamma(W^+)$ and Ω^1 with the L_4^2 -norm and complete $\Gamma(W^-)$ and Ω^2 with the L_3^2 -norm. For the sake of simplicity we use the same notations for these completed spaces.

Instead of viewing the moduli space as the orbits space of the gauge group $\mathcal{G}_L$, we consider the restriction of the map $\mathcal{F}$ to a slice at a base point $A_0 \in \mathcal{A}$. Note that $A = A_0 + i\Omega^1$. The slice is given by $T_{A_0} := \{A_0 + ia | d_{A_0}^* a = 0\} \subset \mathcal{A}$. Let Ω_c^1 denote the space of co-closed 1-forms, i.e, $\Omega_c^1 = \ker\{d^* : \Omega^1 \rightarrow \Omega^0\}$. Observe that T_{A_0} is an affine space diffeomorphic to $i\Omega_c^1$. The stablizer of the gauge group $\mathcal{G}_L$ action at A_0 is S^1 since $b_1(X) = 0$.

Let $A - A_0 = ia$. Note that $D_A\phi = D_{A_0}\phi + ia \cdot \phi$ where $i\Omega^1$ acts on $\Gamma(W^+)$ by the Clifford multiplication. Define a map

$$\begin{aligned}\mathcal{F}_0 : i\Omega_c^1 \times \Gamma(W^+) &\rightarrow i\Omega^+ \times \Gamma(W^-) \\ (ia, \phi) &\mapsto (d^+(ia) - q(\phi), D_{A_0}\phi + ia \cdot \phi)\end{aligned}$$

In view of this point, the moduli space $\mathcal{M}_c$ can be naturally identified with $\mathcal{F}_0^{-1}(\mu_0)/S^1$, where $\mu_0 = -F_{A_0}^+ \in i\Omega^+$, S^1 acts on $\Gamma(W^+)$ by the complex multiplication and acts trivially on Ω^1 .

Observe that $\mathcal{F}_0 = \mathcal{D} + \mathcal{Q}$, where

$$\begin{aligned}\mathcal{D} &= (d^+, D_{A_0}) : i\Omega_c^1 \times \Gamma(W^+) \rightarrow i\Omega^+ \times \Gamma(W^-) \\ \mathcal{Q} : & (ia, \phi) \mapsto (-q(\phi), ia \cdot \phi)\end{aligned}$$

For $\nu \in i\Omega^+$, we define $\mathcal{M}_c(\nu) = \mathcal{F}_0^{-1}(\mu_0 + \nu)/S^1$ to be the perturbed moduli space. Under the assumption of $b_2^+(X) \geq 1$, by [7] there is a dense set of ν such that $\mathcal{M}_c(\nu)$ is a closed smooth manifold of dimension $2d = \frac{1}{4}(c_1(L)^2 - (2\chi(X) + 3\sigma(X)))$. Moreover, this moduli space does not contain reducible point and nonsmooth point. Consequently S^1 acts freely on $\mathcal{F}_0^{-1}(\mu_0 + \nu)$. By [12] the moduli space $\mathcal{M}_c(\nu)$ has a natural orientation.

If $d = 0$, by Witten's definition, the Seiberg-Witten invariant $SW(c)$ is the algebraic sum of points in $\mathcal{M}_c(\nu)$ with signs. If $d \geq 0$, $SW(c) = \chi^d[\mathcal{M}_c(\nu)]$. Here χ is the Euler class of the free S^1 action.

When $b_2^+(X) \geq 2$, it is well known (c.f:[9]) that the Seiberg-Witten invariant is independent of the pertubation and the Riemannian metric on X .

Now let us review Furuta's finite dimensional approximation method. Let $U = \Gamma(W^+)$ and $U' = \Gamma(W^-)$. For each positive real number $\lambda \in \mathbb{R}$, we use U_λ and U'_λ to denote the vector spaces spanned by the eigenvectors of the operator $D_{A_0}^* D_{A_0}$ and $D_{A_0} D_{A_0}^*$ with

eigenvalues less than or equal to λ , respectively. Similarly, we define V_λ and V'_λ to be the vector spaces spanned by the eigenvectors of the operator $d_+^* d_+$ and $d_+ d_+^*$ with eigenvalues less than or equal to λ , where $d^+ : i\Omega_c^1 \rightarrow i\Omega^+$. Recall that both d^+ and D_{A_0} are elliptic operators. U_λ , U'_λ , V_λ and V'_λ are finite dimensional spaces.

We let $p_\lambda : U' \times V' \rightarrow U'_\lambda \oplus V'_\lambda$ denote the orthogonal projection. The composition of p_λ and the restriction of $\mathcal{F}_0$ to $U_\lambda \oplus V_\lambda$ give a map

$$\mathcal{F}_\lambda : U_\lambda \oplus V_\lambda \rightarrow U'_\lambda \oplus V'_\lambda$$

Observe that $\mathcal{F}_\lambda$ is a S^1 -equivariant map, where S^1 acts on Ω^i trivially and on $\Gamma(W^\pm)$ by the complex multiplication.

Using compactness of the moduli space (more precisely the boundness), Furuta proved that

Lemma 2.1: (Furuta [4]) *Let μ_0 and ν be as above. For sufficiently large $R \in (0, \infty)$, there exists a real number $\Lambda \in (0, \infty)$ such that for $\lambda \geq \Lambda$, $\mathcal{F}_\lambda^{-1}(\mu_0 + \nu)$ does not intersect with the sphere of radius R in $U_\lambda \oplus V_\lambda$.*

Recall that the moduli space $\mathcal{M}_c(\nu)$ has no reducible point for a generic ν . In other words, $\mathcal{F}_0^{-1}(\mu_0 + \nu)$ does not intersect with $0 \times V$. The similar idea leads to

Lemma 2.2: (Furuta) *For a generic parameter ν and sufficiently large $R \in (0, \infty)$, there exists a real number $\Lambda \in (0, \infty)$ such that $\mathcal{F}_\lambda^{-1}(\mu_0 + \nu)$ does not intersect with $0 \times V_\lambda \cap B_\lambda$ for $\lambda \geq \Lambda$, where B_λ is the ball of radius R at the origin in $U_\lambda \oplus V_\lambda$.*

For the sake of simplicity, we write $W_\lambda = U'_\lambda \oplus V'_\lambda$ and $\nu_0 = \mu_0 + \nu$. From the above lemmas, one can see that $\mathcal{F}_\lambda$ gives a S^1 -equivariant map

$$\mathcal{F}_\lambda : (B_\lambda, \partial B_\lambda \cup 0 \times V_\lambda \cap B_\lambda) \rightarrow (W_\lambda, W_\lambda - \nu_0)$$

Note that the quotient $B_\lambda / (\partial B_\lambda \cup 0 \times V_\lambda)$ is equivariantly homotopic to the Thom complex of the product bundle $S(U_\lambda) \times (V_\lambda \oplus \mathbb{R}) \rightarrow S(U_\lambda)$. The pair $(W_\lambda, W_\lambda - \nu_0)$ is equivariantly homotopic to $(W_\lambda, W_\lambda - 0)$. Thus passing to the quotient we get a S^1 -map

$$f : S^{V_\lambda \oplus \mathbb{R}} \wedge S(U_\lambda) \rightarrow S^{W_\lambda}$$

Let $\Phi \in H_{S^1}^*(S^{W_\lambda}, \mathbb{Z})$ denote the equivariant Thom class. By the Thom isomorphism theorem there exists an $\theta \in H_{S^1}^{2(m-1-d)}(S(U_\lambda)) \approx H^{2(m-1-d)}(\mathbb{C}P^{m-1})$, $m = \dim U_\lambda$ and $2d = \dim \mathcal{M}_c$, such that $f^*(\Phi) = \sigma(\theta)$, where σ is the suspension isomorphism. As $H^{2(m-1-d)}(\mathbb{C}P^{m-1}) \cong \mathbb{Z}$ with a generator x^{m-1-d} where $x \in H^2(\mathbb{C}P^{m-1-d})$ is a generator. Hence we can regard θ as an integer given by its coefficient of x^{m-1-d} . This integer can also be viewed as the S^1 -equivariant "degree" of f .

Theorem 2.3(Furuta): *Let X be a $Spin^c$ 4-manifold. Suppose that $b_1(X) = 0$ and $b_2^+ \geq 2$. Let $\mathcal{C}$ denote the $Spin^c$ structure. Then the Seiberg-Witten invariant $SW(\mathcal{C}) = \theta$ for sufficiently large λ .*

Let G be a finite group (or a compact Lie group). Suppose G acts on a Riemannian manifold X by isometries preserving the $Spin^c$ -structure $\mathcal{C}$. Thus G acts on $P_{SO}(X) \times$

$P_{U(1)}(X)$ and this action lifts to an action of a group $\hat{G}$ on the bundle $P_{Spin^c}(X)$ and hence on $W^\pm$. Here $\hat{G}$ is an extension of G .

Without loss of generality we assume that the connection A_0 on L are G -invariant. Hence, $\mu_0 = -F_{A_0}^+ \in \Omega_+^2$ is also G -invariant.

Note that $\hat{G}$ acts on Ω^1 , Ω_+^2 factoring through a G action. $\hat{G}$ also acts on $\Gamma(W^\pm)$. It is easy to see that both $\mathcal{D}$ and $\mathcal{Q}$ are equivariant with respect to the $S^1 \times \hat{G}$ actions.

Theorem 2.4: *Let X be a $Spin^c$ 4-manifold. Assume that $b_2^+ \geq 2$ and $b_1 = 0$. Suppose G acts on X preserving the $Spin^c$ -structure $\mathcal{C}$. If $H_+^2(X/G, \mathbb{R}) \neq 0$. Then there is a $S^1 \times \hat{G}$ -map $f : S^{V_\lambda \oplus \mathbb{R}} \wedge S(U_\lambda) \rightarrow S^{W_\lambda}$ such that $SW(\mathcal{C}) = \theta$ for sufficiently large λ .*

Proof: First we warn that, in general it is impossible to choose a self dual G -invariant 2-form ν so that the moduli space $\mathcal{M}_c(\nu)$ is regular(i.e, every point is smooth). However, if $b_2^+(X/G) \geq 1$, we can choose a G -invariant $\nu \in i\Omega_+^2$ such that $\mathcal{M}_c(\nu)$ has no reducible point.

To see this, note that $(A, 0)$ is a point of $\mathcal{M}_c(\nu)$ if and only if $d^+(ia) = \nu + \mu_0$, where $ia = A - A_0$, and that $d^+(i\Omega^1) \subset i\Omega_+^2$ is a codimension $b_2^+(X)$ subspace, say $\mathcal{H}$. Hence for any ν such that $\nu + \mu_0$ is not in $\mathcal{H}$, the moduli space $\mathcal{M}_c(\nu)$ has no reducible solution. Note that μ_0 is G -invariant as A_0 is a G -invariant connection.

As $H_+^2(X/G, \mathbb{R}) \neq 0$, we can choose a G -invariant self dual harmonic 2-form ν such that $\nu + \mu_0 \notin \mathcal{H}$. Thus $\mathcal{M}_c(\nu)$ consists of irreducible solutions. Moreover, G acts on the moduli space $\mathcal{M}_c(\nu)$. Notice that $\mathcal{M}_c(\nu)$ is not necessary regular.

Next we choose a perturbation ν' such that the moduli space $\mathcal{M}_c(\nu')$ consists of irreducible smooth points. Hence $\nu' + \mu_0 \notin \mathcal{H}$. By assumption $b_2^+(X) \geq 2$, we conclude that $\Omega_+^2 - \mathcal{H}$ is connected. Choose an arc in $\Omega_+^2 - \mathcal{H}$, say $\nu(t)$ with $0 \leq t \leq 1$, to join $\nu' + \mu_0$ and $\nu + \mu_0$. For the same reason as above, $\mathcal{M}_c(\nu(t))$ has no reducible solution for all t . In other words, $\mathcal{F}_0^{-1}(\nu(t)) \subset (U - 0) \times V$, where U and V are as above.

For each t , $\mathcal{M}_c(\nu(t))$ is compact. That is, $\mathcal{F}_0^{-1}(\nu(t))$ is compact. It is easy to adapt the proofs of Lemmas 2.1 and 2.2 to show that there is a map

$$\mathcal{F}_\lambda(t) : (B_\lambda, \partial B_\lambda \cup 0 \times V_\lambda \cap B_\lambda) \rightarrow (W_\lambda, W_\lambda - \nu(t))$$

Note that S^1 acts on $\nu(t)$ trivially. The pair $(W_\lambda, W_\lambda - \nu(t))$ has the same S^1 -equivariant homotopy type as $(W_\lambda, W_\lambda - 0)$. Thus as above, by passing to the Thom complex, $\mathcal{F}_\lambda(0)$ and $\mathcal{F}_\lambda(1)$ give two maps $f, f' : S^{V_\lambda \oplus \mathbb{R}} \wedge S(U_\lambda) \rightarrow S^{W_\lambda}$. Moreover, f and f' are S^1 -equivariant homotopic. f is a $S^1 \times \hat{G}$ -equivariant map since ν is G -invariant.

By regarding f as an S^1 -equivariant map, we get an integer θ as before, which is essentially the equivariant degree of f . On the other hand, by Theorem 2.3 the Seiberg-Witten invariant $SW(\mathcal{C})$ is equal to θ' . Clearly $\theta = \theta'$ because $f \simeq_{S^1} f'$. This proves the theorem. q.e.d.

§3 Seiberg-Witten Invariant of 4-manifolds of Non-simple Type

A smooth closed $Spin^c$ four manifold is called of *simple type* if its Seiberg-Witten invariant is zero, provided the moduli space is of nonzero dimensional. It is still a wide

open problem that whether every $Spin^c$ four manifold is of simple type. In [5] Furuta announced an interesting theorem, Theorem 3, which asserts that the Seiberg-Witten invariant is divisible by certain integer related to the dimension of the moduli space. In this section we give a detailed proof of his theorem, Theorem 3. We include a proof here since so far we have not yet seen the details.

Proof of Theorem 3: Let $f : S^{V_\lambda \oplus \mathbb{R}} \wedge S(U_\lambda) \rightarrow S^{U'_\lambda \oplus V'_\lambda}$ be the S^1 -equivariant map in §2. Note that S^1 acts on V_λ and V'_λ trivially. Let $ES^1 \rightarrow BS^1$ denote the universal circle bundle. There is a map

$$\hat{f} : ES^1 \times_{S^1} (S^{V_\lambda \oplus \mathbb{R}} \wedge S(U_\lambda)) \rightarrow ES^1 \times_{S^1} S^{U'_\lambda \oplus V'_\lambda}$$

Notice that the S^1 action on $S^{U'_\lambda \oplus V'_\lambda}$ has a fixed point ∞ . Hence the sphere bundle $ES^1 \times_{S^1} S^{U'_\lambda \oplus V'_\lambda} \rightarrow BS^1$ has a section sending every point $p \in BS^1$ to ∞_p , say $s(BS^1)$. Clearly the quotient

$$ES^1 \times_{S^1} S^{U'_\lambda \oplus V'_\lambda} / s(BS^1) = T(\xi)$$

is the Thom complex of ξ . Here $\xi = ES^1 \times_{S^1} (U'_\lambda \oplus V'_\lambda) \rightarrow BS^1$ is the associated vector bundle.

It is easy to see that $\xi = m'H \oplus \varepsilon^{n'}$, where H is the Hopf complex line bundle on BS^1 , and ε is the trivial real line bundle. m' is the complex dimension of U_λ and n' the real dimension of V_λ .

On the other hand, because that S^1 acts freely on $S(U_\lambda)$,

$$ES^1 \times_{S^1} (S^{V_\lambda \oplus \mathbb{R}} \wedge S(U_\lambda)) \simeq S^{V_\lambda \oplus \mathbb{R}} \wedge \mathbb{C}P^{m-1}$$

where $m = \dim U_\lambda$.

By Theorem 2.3 we have

$$\sigma(SW(\mathcal{C})x^{m-d-1}) = \hat{f}^*(\Phi_\xi)$$

where σ is the suspension isomorphism, $\Phi_\xi \in H^{2m'+n'}(T(\xi), \mathbb{Z})$ is the Thom class and $x \in H^2(\mathbb{C}P^{m-1}, \mathbb{Z})$ is a generator.

Without loss of generality we assume that n' is even. Otherwise, we can suspend the map $\hat{f}$ once. Note that the S^1 -equivariant degree is the same under the suspension. For an even n' , we can regard the real trivial bundle $\varepsilon^{n'}$ as a trivial complex line bundle of dimension $\frac{n'}{2}$. By the definition, for $d \notin \mathbb{Z}$, $SW(\mathcal{C})$ is defined to be zero. Thus we assume that $d \in \mathbb{Z}$. It follows that $l = \frac{1}{2}(b_2^+ - 1) \in \mathbb{Z}$.

Let $\tau_\xi \in K(T\xi)$ denote the Thom class of K-theory. By [8] the Chern character

$$ch(\tau_\xi) = \left\{ \frac{1 - e^x}{x} \right\}^{m'} \Phi_\xi$$

Recall that $K(\mathbb{C}P^{m-1}) \approx \mathbb{Z}[T]/T^m = 0$, where $T = H - 1 \in K(\mathbb{C}P^{m-1})$ and H is the Hopf line bundle. Let $y = \log(1 + T) \in K(\mathbb{C}P^{m-1})$. It is easy to show that $ch(H) = e^x$ and $ch(y) = x$.

Consider the homomorphism

$$\hat{f}^* : K(T\xi) \rightarrow K(S^{V_\lambda \oplus \mathbb{R}} \wedge \mathbb{C}P^{m-1})$$

As the Chern character commutes with f^* ,

$$ch(\hat{f}^* \tau_\xi) = \left\{ \frac{1 - e^x}{x} \right\}^{m'} \hat{f}^*(\Phi_\xi) = SW(\mathcal{C})(1 - e^x)^{m'} x^l = (-1)^{m'} SW(\mathcal{C}) ch(T^{m'} y^l)$$

Here $l = \frac{1}{2}(b_2^+ - 1) = m - m' - d - 1$ by the dimension formula for the moduli space.

Since $ch : K(\mathbb{C}P^{m-1}) \rightarrow H^*(\mathbb{C}P^{m-1})$ is a rational isomorphism. By the above identity we conclude that

$$\hat{f}^*(\tau_\xi) = (-1)^{m'} SW(\mathcal{C}) \left\{ \frac{\log(1+T)}{T} \right\}^l T^{m-d-1} \in \frac{\mathbb{Z}[T]}{T^m = 0}$$

Thus the coefficients of the right side of the above equality must be integers. It follows that $a_i SW(\mathcal{C}) \in \mathbb{Z}$ for $i \leq d$, where a_i is the coefficient of T^i of the power series $\left\{ \frac{\log(1+T)}{T} \right\}^l$. q.e.d.

§4 Seiberg-Witten Invariant and Involutions on 4-manifolds

Let X be a $Spin^c$ 4-manifold and let $\mathcal{C}$ be the $Spin^c$ -structure. Suppose $b_1(X) = 0$ and $b_2^+(X) \geq 2$. For convenience we assume that the signature of X , $\sigma(X)$, is nonpositive. In this section we will apply Theorem 2.4 to study the Seiberg-Witten invariant of X in the case when there exists an involution on X preserving $\mathcal{C}$. We assume that the involution acts on the space of self dual harmonic 2-forms, $H_+^2(X, \mathbb{R})$, trivially throughout the rest. However, the argument of this section can be easily adapted to consider other cases.

As before we let $\mathcal{M}_c$ to denote the Seiberg-Witten moduli space. $\dim \mathcal{M}_c = 2d$. By the definition, if $d \notin \mathbb{Z}$, $SW(\mathcal{C}) = 0$. Thus we only need to consider the case of $d \in \mathbb{Z}$.

Our goal is to prove Theorem 2 and Theorem 1 in the case of $p = 2$ advertised in the introduction.

As in §2, for any $\Lambda \in \mathbb{R}$, let U_Λ , U'_Λ denote the direct sums of the λ -eigenspaces of $D_{A_0}^* D_{A_0}$ and $D_{A_0} D_{A_0}^*$ for $\lambda \leq \Lambda$ respectively, A_0 is an equivariant connection with respect to the $\mathbb{Z}_2$ -action on the bundle L .

4.0 Even Type Involution Let us consider the case of an even type involution acting on X as above. In this case, $\mathbb{Z}_2$ acts on the bundle $P_{Spin^c}(X)$. Hence U_Λ , U'_Λ are complex $\mathbb{Z}_2$ -modules. We define $m := \dim_{\mathbb{C}} U_\Lambda$ and m_+ , m_- to be the dimensions of the $+1$ and -1 eigenspaces. Similarly $n := \dim_{\mathbb{C}} U'_\Lambda$ and n_+ , n_- for the $+1$ and -1 eigenspaces of the $\mathbb{Z}_2$ -action on U'_Λ . Clearly $m_+ + m_- = m$ and $n_+ + n_- = n$.

By Theorem 2.4 there exists a $S^1 \times \mathbb{Z}_2$ -equivariant map $f : S^{V_\Lambda \oplus \mathbb{R}} \wedge S(U_\Lambda) \rightarrow S^{W_\Lambda}$. Here $W_\Lambda = U'_\Lambda \oplus V'_\Lambda$. By the construction in §2 we have that $V'_\Lambda \cong H_+^2(X, \mathbb{R}) \oplus V_\Lambda$ as a real $\mathbb{Z}_2$ -module. By assumption $\mathbb{Z}_2$ acts on $H_+^2(X, \mathbb{R})$ trivially. Thus, as a real $\mathbb{Z}_2$ -module, $V'_\Lambda \oplus V_\Lambda \oplus \mathbb{R} \cong V_\Lambda \otimes \mathbb{C} \oplus \mathbb{C}^{l+1}$ where $l = \frac{1}{2}(b_2^+ - 1)$ is an integer. This is because that d is an integer.

Obviously $V_\Lambda \oplus V_\Lambda \oplus \mathbb{R} \oplus \mathbb{R}$ is the realization of the complex $S^1 \times \mathbb{Z}_2$ -module $(V_\Lambda \oplus \mathbb{R}) \otimes \mathbb{C}$. $V_\Lambda \oplus \mathbb{R} \oplus W_\Lambda$ is the realization of the complex $S^1 \times \mathbb{Z}_2$ -module $U_\Lambda \oplus V_\Lambda \otimes \mathbb{C} \oplus \mathbb{C}^{l+1}$. Here S^1 acts on $(V_\Lambda \oplus \mathbb{R}) \otimes \mathbb{C}$ and $V_\Lambda \otimes \mathbb{C} \oplus \mathbb{C}^{l+1}$ trivially.

Suspending f by $S^{V_\Lambda \oplus \mathbb{R}}$ we have a map $\sigma(f) : S^{(V_\Lambda \oplus \mathbb{R}) \otimes \mathbb{C}} \wedge S(U_\Lambda) \rightarrow S^{V_\Lambda \oplus \mathbb{R}} \wedge S^{W_\Lambda}$. Notice that the S^1 -equivariant degree of f and $\sigma(f)$ are the same. But for the latter it is more convenient to use K-theory.

By the Thom isomorphism theorem $K_{S^1 \times \mathbb{Z}_2}(S^{W_\Lambda \oplus V_\Lambda \oplus \mathbb{R}}) \cong R(S^1 \times \mathbb{Z}_2)$. Let $\tau \in K_{S^1 \times \mathbb{Z}_2}(S^{W_\Lambda \oplus V_\Lambda \oplus \mathbb{R}})$ denote the K-theory Thom class. Similarly, we have

$$K_{S^1 \times \mathbb{Z}_2}(S^{(V_\Lambda \oplus \mathbb{R}) \otimes \mathbb{C}} \wedge S(U_\Lambda)) \cong K_{S^1 \times \mathbb{Z}_2}(S(U_\Lambda))$$

Applying the $K_{S^1 \times \mathbb{Z}_2}$ -functor to the equivariant map $\sigma(f)$, we get a class $\beta_f \in K_{S^1 \times \mathbb{Z}_2}(S(U_\Lambda))$ such that

$$\sigma(f)^*(\tau) = \beta_f \tau_{V_\Lambda \otimes \mathbb{C} \oplus \mathbb{C}} \quad (4.1)$$

Note that $R(S^1) \cong \mathbb{Z}[t, t^{-1}]$, where t corresponds to the standard 1-dimensional complex representation. Let $\xi \in R(\mathbb{Z}_2)$ denote for the irreducible 1-dimensional complex representation.

Lemma 4.1: $K_{S^1 \times \mathbb{Z}_2}(S(U_\Lambda)) \cong R(S^1 \times \mathbb{Z}_2)/(1-t)^{m+}(1-\xi t)^{m-}$.

Proof: Since the unit ball $B(U_\Lambda)$ is equivariantly contractible to the origin, $K_{S^1 \times \mathbb{Z}_2}(D(U_\Lambda)) \cong R(S^1 \times \mathbb{Z}_2)$. By the Thom isomorphism theorem, $K_{S^1 \times \mathbb{Z}_2}(S^{U_\Lambda}) \cong R(S^1 \times \mathbb{Z}_2)$. On the other hand, the K-theoretical Euler class for the bundle $U_\Lambda \rightarrow 0$ is $(1-t)^{m+}(1-\xi t)^{m-}$. By the exact sequence

$$0 \rightarrow K_{S^1 \times \mathbb{Z}_2}(S^{U_\Lambda}) \rightarrow K_{S^1 \times \mathbb{Z}_2}(D(U_\Lambda)) \rightarrow K_{S^1 \times \mathbb{Z}_2}(S(U_\Lambda)) \rightarrow 0$$

for the cofibration $S(U_\Lambda) \rightarrow D(U_\Lambda) \rightarrow S^{U_\Lambda}$ it follows that $K_{S^1 \times \mathbb{Z}_2}(S(U_\Lambda))$ is the cokernel of the homomorphism

$$(1-t)^{m+}(1-\xi t)^{m-} : R(S^1 \times \mathbb{Z}_2) \rightarrow R(S^1 \times \mathbb{Z}_2)$$

This completes the proof. q.e.d.

Now we want to use the Adams ψ -operation to the both sides of the equation (4.1). Recall that for a complex vector bundle γ on Y , the Thom class $\tau_\gamma \in K(T\gamma)$,

$$\psi^q(\tau_\gamma) = \rho^q(\gamma)\tau_\gamma$$

Here $\rho^q(\gamma) \in K(Y)$ is the Bott *canibalistic* class of γ . Recall that $\rho^q(1) = q$ and

$$\rho(\eta) = (1 + \eta + \cdots + \eta^{q-1})$$

if η is a line bundle.

Applying ψ^q to the equality (4.1) it follows that

$$\psi^q(\beta_f)\rho^q(V_\Lambda \otimes \mathbb{C} \oplus \mathbb{C})\tau_{V_\Lambda \otimes \mathbb{C} \oplus \mathbb{C}} = \beta_f \rho^q(U'_\Lambda \oplus V_\Lambda \otimes \mathbb{C} \oplus \mathbb{C}^{l+1})\tau_{V_\Lambda \otimes \mathbb{C} \oplus \mathbb{C}}$$

Hence

$$\psi^q(\beta_f) = q^l \beta_f \rho^q(U'_\Lambda)$$

On the other hand, because that $\mathbb{Z}_2$ acts on U'_Λ with n_+ -times $(+1)$ -eigenvalues and n_- times (-1) -eigenvalues. S^1 acts on U'_Λ by the complex multiplication. Therefore $U'_\Lambda \cong n_+ t + n_- t \xi$ as a $S^1 \times \mathbb{Z}_2$ -module. From the multiplicative property of Bott class it follows that

$$\rho^q(U'_\Lambda) = (1 + t + \cdots + t^{q-1})^{n_+} (1 + t\xi + \cdots + t^{q-1}\xi^{q-1})^{n_-}$$

Substituting this into the above equation we have

$$\psi^q(\beta_f) = q^l \beta_f (1 + t + \cdots + t^{q-1})^{n_+} (1 + t\xi + \cdots + t^{q-1}\xi^{q-1})^{n_-} \quad (4.2)$$

It is easy to verify that:

Lemma 4.2: *There is a short exact sequence*

$$0 \rightarrow R(S^1 \times \mathbb{Z}_2)/(1 - \xi t)^{m_-} \xrightarrow{i} R(S^1 \times \mathbb{Z}_2)/(1 - t)^{m_+} (1 - \xi t)^{m_-} \xrightarrow{j} R(S^1 \times \mathbb{Z}_2)/(1 - t)^{m_+} \rightarrow 0$$

Here $i(z) = (1 - t)^{m_+} z$ and $j(z) = z$ the forgetful homomorphism.

Now we want to study the class β_f defined in the equation (4.1). By Lemma 4.1 we understand β_f as an element of the ring $R(S^1 \times \mathbb{Z}_2)$ subject to the relation $(1 - t)^{m_+} (1 - \xi t)^{m_-} = 0$. If we ignore the $\mathbb{Z}_2$ action, the image of β_f in $R(S^1)/(1 - t)^m = 0$ gives an β . That is, $\beta = \beta_f(t, 1)$, substitute 1 for ξ in an expression of β_f . The following result is already proved in §3 during the proof of Theorem 3.

Proposition 4.3: $\beta = (-1)^n SW(\mathcal{C}) \left(\frac{\log(1+T)}{T} \right)^l T^{m-d-1}$, where $T = 1 - t$.

Now let us first consider the case of $d = 0$, i.e, the moduli space is zero dimensional.

Lamma 4.4: *If $d = 0$ and $0 < k^+ = m_+ - n_+ < l + 1$, then $j(\beta_f) = 0$.*

Proof: Let $T = 1 - t$ and $y = 1 - \xi \in R(\mathbb{Z}_2)$. Recall that $\xi^2 = 1$.

Let $\beta_f = \sum_j (a_j + b_j \xi) T^j$. By Proposition 4.3, $\beta_f(t, 1) = c T^{m-1}$ and up to sign, c is the Seiberg-Witten invariant. Thus $a_j + b_j = 0$ if $j < m - 1$.

Now let us prove that $a_j = 0$ if $j \leq m_+ - 1$. Substituting $q = 3$, $t = 1 - T$ and $\xi = 1 - y$ to the equation (4.2), we get

$$\sum_j (a_j + b_j \xi) (3T - 3T^2 + T^3)^j = 3^l (3 - 3T + T^2)^{n_+} ((3 - y) - 3T + yT + T^2)^{n_-} \sum_j (a_j + b_j \xi) T^j$$

If $j < m - 1$ is the minimal number such that $a_j \neq 0$. The coefficients of T^j in the both sides of the above equation give

$$3^j a_j y = 3^{l+n_+} (3 - y)^{n_-} a_j y$$

It is easy to show that for any nonnegative integer r , $(3 - y)^r y \equiv y$. Thus the above identity holds only if $a_j = 0$, provided $j < l + n_+$.

By $d = 0$ we get that $m-1 = n+l$. On the other hand, by the assumption $m_+ \leq l+n_+$. Therefore $a_j = 0$ for $j \leq m_+ - 1$ and hence the image of β_f in $R(S^1 \times \mathbb{Z}_2)/(1-t)^{m_+}$ is zero. q.e.d

From Lemmata 4.4 and 4.2 it follows that there is an $f(t, \xi) \in R(S^1 \times \mathbb{Z}_2)/(1-t\xi)^{m_-}$ such that $\beta_f = (1-t)^{m_+} f(t, \xi)$.

Lemma 4.5: *If $d = 0$ and $0 < k_+ < l + 1$, then there is an integer a such that $f(t, \xi) = a(1 + \xi)(1 - t\xi)^{m_- - 1}$.*

Proof: Let $T = 1 - t\xi$ for this moment. Let us write $f(t, \xi) = \sum_j (a_j + b_j \xi) T^j$. By Proposition 4.3, $\beta_f(t, 1) = c(1 - t)^{m_- - 1}$. Thus $a_j + b_j = 0$ for $j < m_- - 1$.

Let $j_0 = m_- - 1$ and $\theta = (3 - 3T + T^2)$. Applying the equation (4.2) with $q = 3$ we get

$$(1 + t + t^2)^{k_+} \left\{ \sum_{j \leq m_- - 2} a_j y (T\theta)^j + (a_{j_0} + b_{j_0} \xi) (T\theta)^{j_0} \right\} = 3^l \theta^{n_-} \left\{ \sum_{j \leq m_- - 2} a_j y T^j + (a_{j_0} + b_{j_0} \xi) T^{j_0} \right\}$$

Note that $1 + t + t^2 = 3 - 3T + T^2 + (T - 1)y$. Therefore, as in the proof of the above lemma, comparing the coefficients of T^j in this equation we get $a_j = 0$ for $j \leq m_- - 2$. The rest of the equation reads

$$3^{j_0} (3 - y)^{k_+} (a_{j_0} + b_{j_0} \xi) T^{j_0} = 3^{l+n_-} (a_{j_0} + b_{j_0} \xi) T^{j_0} \quad (4.3)$$

Note that $c = a_{j_0} + b_{j_0}$ and $a_{j_0} + b_{j_0} \xi = c - b_{j_0} y$. It is easy to verify that

$$(3 - y)^{k_+} = \frac{1}{2} (1 - 3^{k_+}) y + 3^{k_+}$$

and $y(3 - y)^{k_+} = y$. The assumption $d = 0$ implies that $k_+ + k_- = l + 1$ and so it is easy to verify that $l + n_- = j_0 + k_+$. Substituting these to (4.3) we get

$$(3^{l+n_-} - 3^{j_0}) b_{j_0} = 3^{j_0} c \frac{3^{k_+} - 1}{2}$$

Therefore $c = 2b_{j_0}$ if $k_+ \neq 0$ and hence $a_{j_0} = b_{j_0}$. Let $a = a_{j_0}$. We get $f(t, \xi) = a(1 + \xi)(1 - t\xi)^{m_- - 1}$. q.e.d.

Proof of Theorem 1 for $p = 2$: By Lemmata 4.4 and 4.5, if $d = 0$ and $0 < k_+ \leq l$, $\beta_f = a(1 + \xi)(1 - t)^{m_+}(1 - t\xi)^{m_- - 1}$. Substituting $\xi = 1$ we get $\beta = 2a(1 - t)^{m_- - 1}$. By Proposition 4.3, this implies that the Seiberg-Witten invariant $SW(\mathcal{C})$ is even. This proves Theorem 1 for $p = 2$ and $d = 0$. The proof for $d \geq 1$ is similar. One only needs to replace m by $m - d$ in the proof of lemmata 4.4 and 4.5 and consider the image of β_f in the truncated ring $R(S^1 \times \mathbb{Z}_2)/(1 - t)^{m_+ - d}(1 - t\xi)^{m_-}$. q.e.d

REMARK: It is possible to get a stronger conclusion for $d \geq 1$ by a more careful calculation. We will not do this here since $d = 0$ is the most interesting case in the Seiberg-Witten theory and also for the applications.

4.6 Odd Type Involutions. Let us consider a $Spin^c$ 4-manifold X with an odd type involution. In this case, there is an $\mathbb{Z}_4$ -action on the principal bundle $P_{Spin^c}(X)$ and the spinor bundles $W^\pm$, where the generator of $\mathbb{Z}_2 \subset \mathbb{Z}_4$ acts as $-id$ on $W^\pm$. Hence for a generator $g \in \mathbb{Z}_4$, the eigenvalues of the induced linear action of g on the eigenspaces of $D_{A_0} D_{A_0}^*$ and $D_{A_0}^* D_{A_0}$ are $\pm\sqrt{-1}$. As in the subsection 4.0, for $\Lambda \geq 0$, let U_Λ and U'_Λ denote the direct sum of eigenspaces of $D_{A_0} D_{A_0}^*$ and $D_{A_0}^* D_{A_0}$ with eigenvalues no larger than Λ . Let m_+ and m_- for the dimensions of U_Λ and U'_Λ respectively. Note that $m_+ - n_+$ and $m_- - n_-$ are independent of Λ . Let $k_+ = m_+ - n_+$, $k_- = m_- - n_-$.

The argument to show Theorem 2 is parallel to the case for an even type involution. Thus we only give a sketch of the proof. In this case, by the similar argument we get an

$$\beta_f \in K_{S^1 \times \mathbb{Z}_2}(S(U_\Lambda)) \cong R(S^1) \otimes R(\mathbb{Z}_4)/(1 - t\zeta)^{m_+}(1 - t\zeta^3)^{m_-}$$

Here $\zeta \in R(\mathbb{Z}_4)$ is an irreducible 1-dimensional representation. Moreover, β_f satisfies the following analogue equation of (4.2):

$$\psi^q(\beta_f) = q^l \beta_f (1 + t\zeta + \dots + t^{q-1} \zeta^{q-1})^{n_+} (1 + t\zeta^3 + \dots + t^{q-1} \zeta^{3(q-1)})^{n_-} \quad (4.4)$$

To understand this element β_f , it is convenient to consider the forgetful homomorphism

$$\iota : R(S^1) \otimes R(\mathbb{Z}_4)/(1 - t\zeta)^{m_+}(1 - t\zeta^3)^{m_-} \rightarrow R(S^1) \otimes R(\mathbb{Z}_2)/(1 - t\xi)^{m_+ + m_-}$$

Note that $\iota(\beta_f)$ can be regarded as the class corresponding to the trivial involution on X but acting as $-id$ on $W^\pm$. First let us present the following analogue of lemma 4.2:

Lemma 4.7: *There is an extension*

$$0 \rightarrow R(S^1 \times \mathbb{Z}_4)/(1 - \zeta^3 t)^{m_-} \xrightarrow{i} R(S^1 \times \mathbb{Z}_4)/(1 - \zeta t)^{m_+}(1 - \zeta^3 t)^{m_-} \xrightarrow{j} R(S^1 \times \mathbb{Z}_4)/(1 - \zeta t)^{m_+} \rightarrow 0$$

Here $i(z) = (1 - \zeta t)^{m_+} z$ and $j(z) = z$ the forgetful homomorphism.

Proof of Theorem 2: We claim that $j(\beta_f) = 0$ if $d = 0$. A useful point is to consider the image of β_f under ι as produced from an $\mathbb{Z}_2$ action on the spinor bundles $W^\pm$ by -1 but on X trivially. Thus most of the arguments of the last subsection can be carried over to show that $\iota(\beta_f) = (a + b\xi)(1 - t\xi)^{m_-}$ for some integers a, b . However, a and b are not necessary the same, since the final assumption on k_+ does not hold for the trivial involution.

Let $T = 1 - \zeta t$ and

$$\beta_f = \sum_i (a_i + b_i \zeta + c_i \zeta^2 + d_i \zeta^3) T^i$$

The above fact about $\iota(\beta_f)$ implies that $a_i + c_i = 0$ and $b_i + d_i = 0$ for $i < m_-$. To show that β_f has zero image under the homomorphism j , it suffices to show that $a_i = b_i = 0$ for $i < m_+ - 1$.

Similarly substituting $q = 3$ to the equation (4.4) we get

$$\psi^3(\beta_f) = 3^l \beta_f (1 + t\zeta + t^2 \zeta^2)^{n_+} (1 + t\zeta^{-1} + t\zeta^2)^{n_-}$$

The same argument of lemma 4.4 applies to prove that $a_i = b_i = 0$ for $i \leq l + n_+$. Hence $j(\beta_f) = 0$ for $d = 0$ because $m_+ - 1 \leq l + n_+$ in this case.

Consequently, we can write $\beta_f = (1 - t\zeta)^{m_+} f(t, \zeta)$ by Lemma 4.7. The argument of Lemma 4.5 can be modified to show that, if $d = 0$ and $0 < k_+ \leq l$, there are integers a, b such that $\beta_f = (a + b\zeta)(1 + \zeta^2)T^{m_+}(1 - \zeta^3 t)^{m_- - 1}$. Substituting $\zeta = 1$ to β_f we get $\beta_f(t, 1) = 2(a + b)T^{m_- - 1}$. Thus the Seiberg-Witten invariant $SW(\mathcal{C})$ is even by Proposition 4.3. The case of $d \geq 1$ is similar, just as in the case for even type involution. q.e.d.

§5 Odd Order Group Action and Seiberg-Witten Invariant

In this section we study the Seiberg-Witten invariants of $Spin^c$ 4-manifolds with some $\mathbb{Z}_p$ action for p an odd prime. We will give a proof of Theorem 1 for the case of p odd. Recall that U_Λ and U'_Λ are the direct sum of the eigenspaces of the Dirac operators $D_{A_0}^* D_{A_0}$ and $D_{A_0} D_{A_0}^*$ with eigenvalues less than or equal to Λ . Notice that $\mathbb{Z}_p$ acts linearly on U_Λ and U'_Λ . Similarly, $\mathbb{Z}_p$ acts linearly on V_Λ and V'_Λ . Let $\omega = e^{\frac{2\pi i}{p}}$ be the p -th unit root. Let $m_0, m_1, \dots, m_{p-1}$ denote the dimensions of $1, \omega, \dots, \omega^{p-1}$ eigenspaces of a generator of the $\mathbb{Z}_p$ action on U_Λ . Similarly define $n_0, n_1, \dots, n_{p-1}$ for U'_Λ .

As we have learned in §2, there exists an $S^1 \times \mathbb{Z}_p$ -equivariant map

$$\mathcal{F}_\Lambda : U_\Lambda \oplus V_\Lambda \rightarrow U'_\Lambda \oplus V'_\Lambda$$

such that the truncated moduli space $\mathcal{F}_\Lambda^{-1}(\nu_0)$ is compact in a sufficiently large ball. Moreover, $\mathcal{F}_\Lambda^{-1}(\nu_0)$ has no reducible points. Here $\nu_0 \in i\Omega_+^2$ is $\mathbb{Z}_p$ invariant. By Theorem 2.4, the Seiberg-Witten invariant $SW(\mathcal{C})$ is equal to the S^1 -equivariant degree of certain map constructed from $\mathcal{F}_\Lambda$. Let $\xi \in R(\mathbb{Z}_p)$ denote an irreducible complex $\mathbb{Z}_p$ -module. As an additive group, $R(\mathbb{Z}_p)$ is generated by $1, \xi, \dots, \xi^{p-1}$. For convenience we can add the $\mathbb{Z}_p$ -modules $1, \xi, \dots, \xi^{p-1}$ to U_Λ and U'_Λ and consider the map

$$id \oplus \mathcal{F}_\Lambda : \bigoplus_{k=0}^{p-1} \xi^k \oplus U_\Lambda \oplus V_\Lambda \rightarrow \bigoplus_{k=0}^{p-1} \xi^k \oplus U'_\Lambda \oplus V'_\Lambda$$

Notice that the preimages of $\mathcal{F}_\Lambda^{-1}(\nu_0)$ and $(id \oplus \mathcal{F}_\Lambda)^{-1}(\nu_0)$ are the same. Thus it is easy to show that the S^1 -degree of $id \oplus \mathcal{F}_\Lambda$ and $\mathcal{F}_\Lambda$ are the same. This is just the Seiberg-Witten invariant $SW(\mathcal{C})$.

In view of this, we may assume that $m_i \geq 1$ for $i = 0, \dots, p-1$ without loss of generality. Notice that the representation of $S^1 \times \mathbb{Z}_p$ on U_Λ splits as the orthogonal sum of $m_0 t + m_1 t\xi + \dots + m_{p-1} t\xi^{p-1}$. Let $k_0 = m_0 - n_0, k_1 = m_1 - n_1, \dots, k_{p-1} = m_{p-1} - n_{p-1}$. As in §4 we have

$$\textbf{Lemma 5.1: } K_{S^1 \times \mathbb{Z}_p}(S(U_\Lambda)) \cong \frac{R(S^1) \otimes R(\mathbb{Z}_p)}{(1-t)^{m_0} (1-t\xi)^{m_1} \dots (1-t\xi^{p-1})^{m_{p-1}}}$$

Proof of Theorem 1 for p odd: First let us assume that the moduli space is of zero dimension, i.e., $d = 0$. As in §4, there is a class $\beta_f \in K_{S^1 \times \mathbb{Z}_p}(S(U_\Lambda))$ satisfying the analogue equation of (4.2):

$$\psi^q(\beta_f) = q^l (1+t+\dots+t^{q-1})^{n_0} (1+t\xi+\dots+t^{q-1}\xi^{q-1})^{n_1} \dots (1+t\xi^{p-1}+\dots+t^{q-1}\xi^{q(p-1)})^{n_{p-1}} \quad (5.1)$$

Substituting $q = 2$ to this equality we get

$$\psi^2(\beta_f) = 2^l(1+t)^{n_0}(1+t\xi)^{n_1} \cdots (1+t\xi^{p-1})^{n_{p-1}} \quad (5.2)$$

We want to prove that there is an $a \in \mathbb{Z}$ such that

$$\beta_f = a(1 + \xi + \xi^2 + \cdots + \xi^{p-1})(1-t)^{m_0}(1-t\xi)^{m_1} \cdots (1-t\xi^{p-2})^{m_{p-2}}(1-t\xi^{p-1})^{m_{p-1}-1}$$

To show this, as in §4, we first prove that the image of β_f in the truncated representation ring $R(S^1) \otimes R(\mathbb{Z}_p)/(1-t)^{m_0}$ is zero if $k_0 \leq l$.

Let $T = 1 - t$ and let us write $\beta_f = \sum_i \sum_{j=0}^{p-1} (a_j^i \xi^j) T^i$ with integral coefficients.

Assume that i is the minimal number so that the coefficient of T^i is nonzero in the above expression. Note that $\psi^2(\xi) = \xi^2$, $\psi^2(t) = t^2$ and ψ^2 is a ring homomorphism. By comparing the coefficients of T^i in the equation (5.2) we get an identity

$$2^i \sum_{j=0}^{p-1} (a_j^i \xi^{2j}) = 2^{l+n_0} (1+\xi)^{n_1} \cdots (1+\xi^{p-1})^{n_{p-1}} \left(\sum_{j=0}^{p-1} a_j^i \xi^j \right) \quad (5.3)$$

Substituting 1 for ξ , we obtain that $\sum_{j=0}^{p-1} a_j^i = 0$ if $i < l + n$, note that $n = \sum_{j=0}^{p-1} n_j$. When $d = 0$, the dimension formula for the moduli space implies $l + n = m - 1$. Thus $m_0 - 1 < l + n$ because $m_i \geq 1$ for all i . From these we conclude that $\sum_{j=0}^{p-1} a_j^i = 0$ for $i \leq m_0 - 1$.

Taking $\xi = \omega, \omega^2, \dots, \omega^{p-1}$ in (5.3) we get $p - 1$ equalities. As p is an odd prime, $(1+\omega)(1+\omega^2) \cdots (1+\omega^{p-1}) = 1$. Multiply these equations together and substituting the identity for ω we get

$$2^{(p-1)i} \prod_{k=1}^{p-1} \left(\sum_{j=0}^{p-1} a_j^i \omega^{2kj} \right) = 2^{(p-1)(l+n_0)} \prod_{k=1}^{p-1} \left(\sum_{j=0}^{p-1} a_j^i \omega^{kj} \right)$$

Notice that $\prod_{k=1}^{p-1} (\sum_{j=0}^{p-1} a_j^i \omega^{2kj})$ and $\prod_{k=1}^{p-1} (\sum_{j=0}^{p-1} a_j^i \omega^{kj})$ are the same. Thus, for $i < l + n_0$, there exists at least a zero term among this product. Hence there is an k with $1 \leq k \leq p-1$ such that

$$\sum_{j=0}^{p-1} a_j^i \omega^{kj} = 0$$

This implies that $a_0^i = a_1^i = \cdots = a_{p-1}^i$, since the polynomial $1 + x + x^2 + \cdots + x^{p-1}$ is irreducible over rational.

On the other hand, we have already proved that the sum of these a_j^i is zero. Thus $a_0^i = a_1^i = \cdots = a_{p-1}^i = 0$, provided $i < l + n_0$. If $k_0 \leq l$, then $m_0 \leq l + n_0$ and so

$m_0 - 1 < l + n_0$. Thus the image of β_f in the ring $R(S^1) \otimes R(\mathbb{Z}_p)/(1-t)^{m_0}$ is zero. Consequently $\beta_f = (1-t)^{m_0}\beta'_f$ for some element $\beta'_f \in R(S^1) \otimes R(\mathbb{Z}_p)$.

Next let us prove that the image of β'_f in $R(S^1) \otimes R(\mathbb{Z}_p)/(1-t)^{m_1}$ is zero if $k_1 \leq l$.

For the sake of simplicity, we let $T = (1 - t\xi)$ for this moment. Let us write $\beta'_f = \sum_i \sum_{j=0}^{p-1} (a_j^i \xi^j) T^i$. The equation (5.2) for $\psi^2(\beta_f)$ gives rise to the following identity

$$(1+t)^{k_0} \psi^2(\beta'_f) = 2^l \beta'_f (1+t\xi)^{n_1} \cdots (1+t\xi^{p-1})^{n_{p-1}} \quad (5.4)$$

If i is the minimal index so that the coefficient of T^i in the expression of β'_f nonzero. With the same argument as above, comparing coefficients in (5.4) we get

$$2^i (1 + \xi^{p-1})^{k_0} \sum_{j=0}^{p-1} a_j^i \xi^{2j} = 2^{l+n_1} \sum_{j=0}^{p-1} a_j^i \xi^j (1 + \xi)^{n_2} \cdots (1 + \xi^{p-2})^{n_{p-1}}$$

Taking $\xi = 1$, we get

$$2^{i+k_0} \left(\sum_{j=0}^{p-1} a_j^i \right) = 2^{l+n-n_0} \left(\sum_{j=0}^{p-1} a_j^i \right)$$

Therefore $\sum_{j=0}^{p-1} a_j^i = 0$ if $i + k_0 < l + n - n_0$. Hence $\sum_{j=0}^{p-1} a_j^i = 0$ for $i \leq m_1 - 1$ because $m_0 \geq 1, m_1 \geq 1, \dots, m_{p-1} \geq 1$.

Now taking $\xi = \omega, \omega^2, \dots, \omega^{p-1}$ and multiplying these identities, we get

$$2^{i(p-1)} \prod_{k=1}^{p-1} \left(\sum_{j=0}^{p-1} a_j^i \omega^{2ik} \right) = 2^{(l+n_1)(p-1)} \prod_{k=1}^{p-1} \left(\sum_{j=0}^{p-1} a_j^i \omega^{ik} \right)$$

For $i < l + n_1$, this implies that there is an $1 \leq k \leq p-1$ such that

$$\sum_{j=0}^{p-1} a_j^i \omega^{ik} = 0$$

As before this shows that $a_0^i = a_1^i = \dots = a_{p-1}^i$ and so they are all zero. Thus β'_f has zero image in the ring $R(S^1 \times \mathbb{Z}_p)/(1-t)^{m_1}$. Hence, $\beta_f = (1-t)^{m_0}\beta'_f$ is zero in the truncated ring $R(S^1) \otimes R(\mathbb{Z}_p)/(1-t)^{m_0}(1-t\xi)^{m_1}$.

Continuing this procedure, one can check that, under the assumption of Theorem 1 and $d = 0$,

$$\beta_f = a(1 + \xi + \xi^2 \cdots + \xi^{p-1})(1-t)^{m_0}(1-t\xi)^{m_1} \cdots (1-t\xi^{p-1})^{m_{p-1}-1}$$

for some $a \in \mathbb{Z}$. Therefore $\beta_f(t, 1) = pa(1-t)^{m-1}$. By Proposition 4.3, this proves that $SW(\mathcal{C}) = 0(\text{mod } p)$.

The proof for $d \geq 1$ is similar. One only needs to consider an over truncated ring by replacing m_0 by $m_0 - d$ and replacing m by $m - d$. •

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