SMOOTH GROUP ACTIONS ON 4-MANIFOLDS AND 
THE SEIBERG-WITTEN INVARIANTS : II

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Abstract

In this paper we study the Seiberg-Witten invariants of 4-manifold with a finite group (or a compact Lie group) acting on. Among other things, we will prove the following result:

Let $X$ be a smooth closed 4-manifold. Suppose $H_1(X, \mathbb{R}) = 0$ and $b_2^+ \geq 2$, where $b_2^+$ is the rank of $H_2^+(X)$. Let $\mathcal{C}$ be a Spin$^c$-structure on $X$. Assume that $\mathcal{C}$ is equivariant with respect to an $\mathbb{Z}_p$ action on $X$, where $p$ is a prime. If $\mathbb{Z}_p$ acts on the space $H_2^+(X, \mathbb{R})$ of harmonic self dual 2-forms trivially. Then the Seiberg-Witten invariant

$$SW(\mathcal{C}) = 0 (\text{mod } p)$$

if $k_i \leq \frac{1}{2}(b_2^+ - 1)$ for $i = 0, 1, \ldots, p - 1$. Here $k_i = m_i - n_i$, $m_i$ and $n_i$ are the dimensions of the $\omega^i$-eigenspaces of the linear $\mathbb{Z}_p$ actions on $\ker D_A$ and $\text{coker } D_A$ respectively, $\omega = e^{2\pi i/p}$ is the $p$-th unit root. $D_A : \Gamma(W^+) \to \Gamma(W^-)$ is the Dirac operator corresponding to $\mathcal{C}$ and an equivariant connection $A$ on $\text{det } W^+$.

§1 Introduction

In this paper we study the Seiberg-Witten invariant of smooth 4-manifolds with the symmetry of some finite group or compact Lie group. This is a continuation of [3] where we have used the Seiberg-Witten moduli space of a Spin manifold in the presence of a finite group action. We will prove some mod $p$ vanishing theorems for the Seiberg-Witten invariants. The strategy of this work is to use the "finite dimensional approximation" technique due to Furuta to interpret the Seiberg-Witten invariant as certain equivariant degree. In this sequel equivariant K-theory especially the Adams $\psi$-operation plays a key role. In his celebrated paper [12], E.Witten proved that the Seiberg-Witten invariant for a Kähler manifold is $\pm 1$ for the standard Spin$^c$-structure. C.Taubes [11] has generalized this result to symplectic 4-manifolds. It is may be interesting to compare these with our vanishing theorem, because this would exclude the existence of certain group actions on symplectic 4-manifold.

To state our main results, we need to give some necessary preliminaries. Let $X$ be a closed Riemannian Spin$^c$ four manifold. Let $\mathcal{C}$ be a Spin$^c$ structure. A Spin$^c$ structure
consists of a principal $\text{Spin}^c$-bundle $P_{\text{Spin}^c}(X)$ over $X$ with a $\text{Spin}^c$-equivariant bundle map

$$P_{\text{Spin}^c}(X) \to P_{\text{SO}(X) \times P_{U(1)}(X)}$$

The first Chern class of the bundle $P_{U(1)}(X)$ is called the canonical class of the $\text{Spin}^c$-structure. Let us write $W^+$ and $W^-$ to denote the associated complex spinor bundles. Let $L = \det W^+$, the determinant line bundle on $X$. Note that $L$ is the associated line bundle of $P_{U(1)}(X)$.

Let $G$ be a finite group or a compact Lie group acting on $X$ preserving the orientation. We can assume that $G$ preserves the isometries without loss of generality. Hence $G$ acts on the frame bundle $P_{\text{SO}(X)}$. We say that the $\text{Spin}^c$-structure $C$ is preserved by the $G$-action if $P_{U(1)}(X)$ is a principal $G$-$U(1)$ bundle, and the product action of $G$ on $P_{\text{SO}(X)} = P_{U(1)}(X)$ lifts to the bundle $P_{\text{Spin}^c}(X)$. We say that $C$ is $G$-equivariant if the induced $G$-action on $P_{\text{SO}(X)} = P_{U(1)}(X)$ lifts to a $G$-action on $P_{\text{Spin}^c}(X)$. For such a $\text{Spin}^c$-structure, if $A$ is a $G$-connection on $L$, the Dirac operator $D_A$ is $G$-equivariant. Therefore the index

$$\text{ind}_G D_A = \ker D_A - \text{coker} D_A \in R(G)$$

In particular, if $G = \mathbb{Z}_p$ is a cyclic group and $C$ is $\mathbb{Z}_p$-equivariant. Then the eigenvalues of the $\mathbb{Z}_p$-actions on $\ker D_A$ and $\text{coker} D_A$ are the $p$-th unit roots of $1$. Let $\omega_p = e^{2\pi \sqrt{-1}/p}$ and let $k_i = m_i - n_i$, where $m_i$ and $n_i$ are the dimensions of the $\omega_p^i$-eigenspace of $\ker D_A$ and $\text{coker} D_A$ respectively. Obviously $k_0 + k_1 + \cdots + k_{p-1} = \text{ind} D_A$. For convenience we use $b_i$ to denote the $i$-th Betti number of $X$ and $b_i^+$ for the rank of $H^i(X, \mathbb{R})$.

Our main results are as follows

**Theorem 1** Let $X$ be a smooth closed $\text{Spin}^c$ four manifold with $b_1 = 0$ and $b_2^+ \geq 2$. Let $C$ be a $\text{Spin}^c$ structure equivariant with respect to a smooth $\mathbb{Z}_p$ action on $X$, where $p$ is a prime. If $b_2^+(X/\mathbb{Z}_p) = b_2^+$, then the Seiberg-Witten invariant

$$SW(C) = 0 \text{(mod} p)$$

when $k_i \leq \frac{1}{2}(b_2^+ - 1)$ for $i = 0, 1, \cdots, p - 1$.

Recall that for an oriented 4-manifold $X$ with $H_1(X, \mathbb{Z}_2) = 0$, a $\text{Spin}^c$ structure on $X$ is completely determined by its canonical class. In this case, $C$ is equivariant with respect to an $\mathbb{Z}_p$ action on $X$ for an odd $p$ if and only if $L$ is a $\mathbb{Z}_p$-$U(1)$-bundle. On the other hand, by [6] $L$ is an $\mathbb{Z}_p$-$U(1)$-bundle if and only if $c_1(L) \in H^2(X, \mathbb{Z})$ is $\mathbb{Z}_p$-invariant.

However, for $p$ even, sometimes the lifted action to $P_{\text{Spin}^c}(X)$ must have higher order. For example, if $p = 2$ and $X$ is spin, let $C$ be the $\text{Spin}^c$ structure with trivial canonical class, then $C$ is $\mathbb{Z}_2$-equivariant if and only if this $\mathbb{Z}_2$ action on $X$ has only isolated fixed points [1].

Let $\tau : X \to X$ be an involution preserving the $\text{Spin}^c$-structure $C$. We say that $\tau$ is of even type if $C$ is $\mathbb{Z}_2$-equivariant, where $\mathbb{Z}_2$ is generated by $\tau$. Otherwise, we say that $\tau$ is of odd type. In the latter case, it is easy to see that

$$\tau : P_{\text{SO}(X) \times P_{U(1)}(X)} \to P_{\text{SO}(X) \times P_{U(1)}(X)}$$
can be lifted to an fiberwise diffeomorphism $\hat{\tau} : P_{\text{Spin}^c}(X) \to P_{\text{Spin}^c}(X)$ such that $\hat{\tau}^2 = -1$. Therefore, the eigenvalues of $\hat{\tau}$ on the spinor bundles $W^+$ and $W^-$ are $\pm \sqrt{-1}$. If $A$ is a $\mathbb{Z}_2$-connection on $L$, the eigenvalues of $\hat{\tau}$ on $\ker D_A$ and $\coker D_A$ are $\pm \sqrt{-1}$. Define $k_+ := m_+ - n_+$ and $k_- := m_- - n_-$, where $m_\pm$ and $n_\pm$ denote the dimensions of $\sqrt{-1}$ and $-\sqrt{-1}$ eigenspaces of $\hat{\tau}$ on $\ker D_A$ and $\coker D_A$ respectively. We have the following

**Theorem 2** Let $X$ be a smooth closed $\text{Spin}^c$ four manifold. Suppose $b_1 = 0$ and $b_2^+ \geq 2$. Let $C$ be the $\text{Spin}^c$-structure on $X$. Suppose that $\tau : X \to X$ is a smooth odd type involution with respect to $C$ such that $b_2^+(X/\tau) = b_2^+$. Then the Seiberg-Witten invariant

$$SW(C) = 0(\text{mod}2)$$

if $k_+ \leq \frac{1}{2}(b_2^+ - 1)$.

Let $\mathcal{M}_C$ denote the Seiberg-Witten moduli space. Define $2d$ to be the dimension of $\mathcal{M}_C$. It is well known [12] that $2d = \frac{1}{4}(c_1(L)^2 - 2\chi(X) - 3\sigma(X))$. By the definition of the Seiberg-Witten invariant, if $d \not\in \mathbb{Z}$ or negative, $SW(C) = 0$. It is conjectured that $SW(C) = 0$ if $d \geq 1$, $X$ is simply connected or $b_1(X) = 0$. Furuta announced an interesting partial result towards this conjecture

**Theorem 3** (Furuta) Let $X$ be a $\text{Spin}^c$ 4-manifold. Assume that $b_1 = 0$, $b_2^+ \geq 2$. Let $C$ denote the $\text{Spin}^c$ structure. Let $2d$ denote the dimension of the moduli space $\mathcal{M}_C$. Then the Seiberg-Witten invariant $a_iSW(C) \in \mathbb{Z}$ for $1 \leq i \leq d$, where $a_i$ is given by the following power series $\left\{ \frac{\log(1+x)}{x} \right\} = 1 + a_1x + a_2x^2 + \cdots$. Here $l = \frac{b_2^+ - 1}{2}$.

So far we have not seen the details for the proof. For convenience we will include some details of the proof.

If $d = 0$, by the dimension formula it is easy to check that $\text{ind} D_A = l + 1$, where $l = \frac{1}{2}(b_2^+ - 1)$ as above. On the other hand, $\text{ind} D_A \in R(G)$ can be calculated in terms of the fixed point data using the Atiyah-Singer index theorem. Thus $k_0, k_1, \cdots, k_{p-1}$ and $k_\pm$ in theorems 1 and 2 are determined by the local data about the fixed point set.

**Corollary 4** Let $(X, \omega)$ be a closed simply connected symplectic 4-manifold with $\Lambda^{0,2}T^*(X)$ trivial. Let $\tau$ be an involution on $X$ preserving the symplectic structure. If $b_2^+(X) \geq 2$ and $\tau$ has no isolated fixed point, then $b_2^+(X/\tau) < b_2^+(X)$.

**Proof:** By [11] there is a canonical $\text{Spin}^c$-structure on $X$ say $C$, with determinant line bundle $L = \Lambda^{0,2}T^*(X)$. By assumption, we conclude that $C$ is $\mathbb{Z}_2$-equivariant and $X$ is a spin manifold.

Suppose $b_2^+(X/\tau) = b_2^+(X)$, in other words $\tau$ acts trivially on $H_2^2(X, \mathbb{R})$. By the Atiyah-Singer $G$-index theorem it is easy to verify that $\text{ind} D_A = 0$, because that $\tau$ has no isolated fixed point. Thus $k_+ = k_- = \frac{1}{2} \text{ind} D_A$. Since the moduli space $\mathcal{M}_C$ is zero dimensional, $\text{ind} D_A = l + 1$. By Theorem 2 the Seiberg-Witten invariant $SW(C) = 0(\text{mod}2)$. This contradicts with Taubes’ result [11] that $SW(C) = \pm 1$. Thus we conclude that $b_2^+(X/\tau) < b_2^+(X)$, q.e.d.
To conclude this section, let us give a remark. It is quite obvious that our results can be applied to study $\mathbb{Z}_p$ actions on 4-manifolds with nontrivial Seiberg-Witten invariant. For example, it may be applied to study group actions on $K_3$-surface. This is related to a question, namely whether there is homologically trivial $\mathbb{Z}_p$ action on $K_3$ surface. For $p = 2$, it is already shown [10] that no such nontrivial involution even topologically. However, it is not hard to construct such a homologically trivial topological action for odd $p$.

The rest of this paper is organized as follows. In §2 we briefly recall some necessary details about the Seiberg-Witten theory and the finite dimensional approximation method. A key result for the rest, Theorem 2.4, will be proved. In §3 we give a proof of Theorem 3 following Furuta. In §4 we study the Seiberg-Witten invariant of 4-manifolds with an involution action on. The proof theorem 1 for $p = 2$ and theorem 2 will be given there. In §5 we carry out the analogue of §4 for odd order group action and prove the rest of theorem 1.

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§2 Seiberg-Witten Theory

In this section we first give a brief review on Seiberg-Witten theory and Furuta’s “finite dimensional approximation” method. Using this method, we will interpret the Seiberg-Witten invariant of a 4-manifold with a group action as certain equivariant degree.

Throughout the rest of this paper let $X$ be a smooth, closed, connected, oriented Riemannian 4-manifold satisfying $b_1(X) = 0$ and $b_2^+(X) \geq 2$. Recall that there always exist $Spin^c$ structures on $X$, which are one to one correspondent to classes in $H^2(X, \mathbb{Z})$. Let $\mathcal{C}$ be a $Spin^c$ structure on $X$. As above we use $W^+$ and $W^-$ to denote the associated complex spinor bundles. Let $L = \det W^+$, the determinant line bundle on $X$. Recall that when $H^1(X, \mathbb{Z}_2) = 0$, $C$ is entirely determined by the topology of $L$.

Let $i = \sqrt{-1}$. Using the Clifford multiplication, we will identify $i\Omega^+$ with the Lie algebra bundle $i su(W^+)$. The Seiberg-Witten’s monopole equations are a pair of equations for a unitary connection $A$ on $L$ and a section $\phi \in \Gamma(W^+)$:

\begin{align*}
D_A \phi &= 0 \\
F_A^+ &= q(\phi)
\end{align*}

Here $q(\phi) := \phi \otimes \phi^* - \frac{|\phi|^2}{2} Id$, $D_A : \Gamma(W^+) \to \Gamma(W^-)$ is the twisted Dirac operator. $F_A^+$ is the self dual part of the curvature $F_A$.

The gauge group $\mathcal{G}_L = Map(X, U(1))$ acts on the set of solutions. Let $\mathcal{M}_c$ denote the moduli space, the quotient of the set of solutions by the gauge group.

Let $\mathcal{A} := \{\text{unitary connections on } L\}$. $\Omega^+ := \{\text{self dual part of harmonic 2-forms}\}$. 

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The above equations give a $G_L$-equivariant map $\mathcal{F}$

$$
\mathcal{A} \times \Gamma(W^+) \to i\Omega^+ \times \Gamma(W^-) \\
(A, \phi) \mapsto (F_A^+, -q(\phi), D_A \phi)
$$

The moduli space $\mathcal{M}_c = \mathcal{F}^{-1}(0)/G_L$.

To study the moduli space and its property, one has to work in the completed Banach space with suitable Sobolev norm. By [4], one needs to complete $\Gamma(W^+)$ and $\Omega^1$ with the $L^2_4$-norm and complete $\Gamma(W^-)$ and $\Omega^2$ with the $L^2_3$-norm. For the sake of simplicity we use the same notations for these completed spaces.

Instead of viewing the moduli space as the orbits space of the gauge group $G_L$, we consider the restriction of the map $\mathcal{F}$ to a slice at a base point $A_0 \in \mathcal{A}$. Note that $\mathcal{A} = A_0 + i\Omega^1$. The slice is given by $T_{A_0} := \{ A_0 + ia|d_{A_0}^* a = 0 \} \subset \mathcal{A}$. Let $\Omega^1_{c}$ denote the space of co-closed 1-forms, i.e., $\Omega^1_{c} = \text{ker}\{d^* : \Omega^1 \to \Omega^0\}$. Observe that $T_{A_0}$ is an affine space diffeomorphic to $i\Omega^1_{c}$. The stabilizer of the gauge group $G_L$ action at $A_0$ is $S^1$ since $b_1(X) = 0$.

Let $A - A_0 = ia$. Note that $D_A \phi = D_{A_0} \phi + ia \cdot \phi$ where $i\Omega^1$ acts on $\Gamma(W^+)$ by the Clifford multiplication. Define a map

$$
\mathcal{F}_0 : \quad i\Omega^1_{c} \times \Gamma(W^+) \to i\Omega^+ \times \Gamma(W^-) \\
(ia, \phi) \mapsto (d^+(ia) - q(\phi), D_{A_0} \phi + ia \cdot \phi)
$$

In view of this point, the moduli space $\mathcal{M}_c$ can be naturally identified with $\mathcal{F}_0^{-1}(\mu_0)/S^1$, where $\mu_0 = -F_{A_0}^+ \in i\Omega^+$, $S^1$ acts on $\Gamma(W^+)$ by the complex multiplication and acts trivially on $\Omega^1$.

Observe that $\mathcal{F}_0 = \mathcal{D} + \mathcal{Q}$, where

$$
\mathcal{D} = (d^+, D_{A_0}) : \quad i\Omega^1_{c} \times \Gamma(W^+) \to i\Omega^+ \times \Gamma(W^-) \\
(ia, \phi) \mapsto (-q(\phi), ia \cdot \phi)
$$

$$
\mathcal{Q} : \quad \Gamma(W^+) \to i\Omega^+ \times \Gamma(W^-)
$$

For $\nu \in i\Omega^+$, we define $\mathcal{M}_c(\nu) = \mathcal{F}_0^{-1}(\mu_0 + \nu)/S^1$ to be the perturbed moduli space. Under the assumption of $b_2^+(X) \geq 1$, by [7] there is a dense set of $\nu$ such that $\mathcal{M}_c(\nu)$ is a closed smooth manifold of dimension $2d = \frac{1}{4}(c_1(L))^2 - (2\chi(X) + 3\sigma(X))$. Moreover, this moduli space does not contain reducible point and nonsmooth point. Consequently $S^1$ acts freely on $\mathcal{F}_0^{-1}(\mu_0 + \nu)$. By [12] the moduli space $\mathcal{M}_c(\nu)$ has a natural orientation.

If $d = 0$, by Witten's definition, the Seiberg-Witten invariant $Sw(c)$ is the algebraic sum of points in $\mathcal{M}_c(\nu)$ with signs. If $d \geq 0$, $Sw(c) = \chi^d[\mathcal{M}_c(\nu)]$. Here $\chi$ is the Euler class of the free $S^1$ action.

When $b_2^+(X) \geq 2$, it is well known (c.f.[9]) that the Seiberg-Witten invariant is independent of the perturbation and the Riemannian metric on $X$.

Now let us review Furuta's finite dimensional approximation method. Let $U = \Gamma(W^+)$ and $U' = \Gamma(W^-)$. For each positive real number $\lambda \in \mathbb{R}$, we use $U_{\lambda}$ and $U_1'$ to denote the vector spaces spanned by the eigenvectors of the operator $D_{A_0}^* D_{A_0}$ and $D_{A_0} D_{A_0}^*$ with
eigenvalues less than or equal to \( \lambda \), respectively. Similarly, we define \( V_\lambda \) and \( V_\lambda' \) to be the vector spaces spanned by the eigenvectors of the operator \( d_+^* d_+ \) and \( d_+d_+^* \) with eigenvalues less than or equal to \( \lambda \), where \( d_+^* : i\Omega^1_c \to i\Omega^+ \). Recall that both \( d_+ \) and \( D_{A_0} \) are elliptic operators. \( U_\lambda, U_\lambda', V_\lambda \) and \( V_\lambda' \) are finite dimensional spaces.

We let \( p_\lambda : U' \times V' \to U'_\lambda \oplus V'_\lambda \) denote the orthogonal projection. The composition of \( p_\lambda \) and the restriction of \( \mathcal{F}_0 \) to \( U_\lambda \oplus V_\lambda \) give a map

\[
\mathcal{F}_\lambda : U_\lambda \oplus V_\lambda \to U'_\lambda \oplus V'_\lambda
\]

Observe that \( \mathcal{F}_\lambda \) is a \( S^1 \)-equivariant map, where \( S^1 \) acts on \( \Omega^1 \) trivially and on \( \Gamma(W^\pm) \) by the complex multiplication.

Using compactness of the moduli space (more precisely the boundness), Furuta proved that

**Lemma 2.1:** (Furuta [4]) Let \( \mu_0 \) and \( \nu \) be as above. For sufficiently large \( R \in (0, \infty) \), there exists a real number \( \Lambda \in (0, \infty) \) such that for \( \lambda \geq \Lambda \), \( \mathcal{F}_\lambda^{-1}(\mu_0 + \nu) \) does not intersect with the sphere of radius \( R \) in \( U_\lambda \oplus V_\lambda \).

Recall that the moduli space \( \mathcal{M}_c(\nu) \) has no reducible point for a generic \( \nu \). In other words, \( \mathcal{F}_0^{-1}(\mu_0 + \nu) \) does not intersect with \( 0 \times V \). The similar idea leads to

**Lemma 2.2:** (Furuta) For a generic parameter \( \nu \) and sufficiently large \( R \in (0, \infty) \), there exists a real number \( \Lambda \in (0, \infty) \) such that \( \mathcal{F}_\lambda^{-1}(\mu_0 + \nu) \) does not intersect with \( 0 \times V_\lambda \cap B_\lambda \) for \( \lambda \geq \Lambda \), where \( B_\lambda \) is the ball of radius \( R \) at the origin in \( U_\lambda \oplus V_\lambda \).

For the sake of simplicity, we write \( W_\lambda = U'_\lambda \oplus V'_\lambda \) and \( \nu_0 = \mu_0 + \nu \). From the above lemmas, one can see that \( \mathcal{F}_\lambda \) gives a \( S^1 \)-equivariant map

\[
\mathcal{F}_\lambda : (B_\lambda, \partial B_\lambda \cup 0 \times V_\lambda \cap B_\lambda) \to (W_\lambda, W_\lambda - \nu_0)
\]

Note that the quotient \( B_\lambda/(\partial B_\lambda \cup 0 \times V_\lambda) \) is equivariantly homotopic to the Thom complex of the product bundle \( S(U_\lambda) \times (V_\lambda \oplus \mathbb{R}) \to S(U_\lambda) \). The pair \( (W_\lambda, W_\lambda - \nu_0) \) is equivariantly homotopic to \( (W_\lambda, W_\lambda - 0) \). Thus passing to the quotient we get a \( S^1 \)-map

\[
f : S^{W_\lambda \oplus \mathbb{R}} \wedge S(U_\lambda) \to S^{W_\lambda}
\]

Let \( \Phi \in H^*_S(S^{W_\lambda}, \mathbb{Z}) \) denote the equivariant Thom class. By the Thom isomorphism theorem there exists an \( \theta \in H^{2(m-1-d)}_S(S(U_\lambda)) \approx H^{2(m-1-d)}(\mathbb{C}P^{m-1}) \), \( m = \text{dim} U_\lambda \) and \( 2d = \text{dim} \mathcal{M}_c \), such that \( f^*(\Phi) = \sigma(\theta) \), where \( \sigma \) is the suspension isomorphism. As \( H^{2(m-1-d)}(\mathbb{C}P^{m-1}) \cong \mathbb{Z} \) with a generator \( x^{m-1-d} \) where \( x \in H^2(\mathbb{C}P^{m-1}) \) is a generator. Hence we can regard \( \theta \) as an integer given by its coefficient of \( x^{m-1-d} \). This integer can also be viewed as the \( S^1 \)-equivariant "degree" of \( f \).

**Theorem 2.3** (Furuta): Let \( X \) be a \( \text{Spin}^c \) 4-manifold. Suppose that \( b_1(X) = 0 \) and \( b_2^+ \geq 2 \). Let \( C \) denote the \( \text{Spin}^c \) structure. Then the Seiberg-Witten invariant \( SW(C) = \theta \) for sufficiently large \( \lambda \).

Let \( G \) be a finite group (or a compact Lie group). Suppose \( G \) acts on a Riemannian manifold \( X \) by isometries preserving the \( \text{Spin}^c \)-structure \( C \). Thus \( G \) acts on \( P_{SO}(X) \times \)
$P_{U(1)}(X)$ and this action lifts to an action of a group $\hat{G}$ on the bundle $P_{\text{Spin}^c}(X)$ and hence on $\mathcal{W}^\pm$. Here $\hat{G}$ is an extension of $G$.

Without loss of generality we assume that the connection $A_0$ on $L$ are $G$-invariant. Hence, $\mu_0 = -F_{A_0}^+ \in \Omega_+^2$ is also $G$-invariant.

Note that $\hat{G}$ acts on $\Omega^1$, $\Omega_+^2$ factoring through a $G$ action. $\hat{G}$ also acts on $\Gamma(W^\pm)$. It is easy to see that both $\mathcal{D}$ and $\mathcal{Q}$ are equivariant with respect to the $S^1 \times \hat{G}$ actions.

**Theorem 2.4:** Let $X$ be a Spin$^c$ 4-manifold. Assume that $b_2^+ \geq 2$ and $b_1 = 0$. Suppose $G$ acts on $X$ preserving the Spin$^c$-structure $\mathcal{C}$. If $H_2^+(X/G, \mathbb{R}) \neq 0$. Then there is a $S^1 \times \hat{G}$-map $f : S^V_{\nu} \otimes \ast S(U_\nu) \to S^{W^\nu}$ such that $SW(C) = \theta$ for sufficiently large $\lambda$.

**Proof:** First we warn that, in general it is impossible to choose a self dual $G$-invariant 2-form $\nu$ so that the moduli space $\mathcal{M}_\nu(\nu)$ is regular (i.e., every point is smooth). However, if $b_2^+(X/G) \geq 1$, we can choose a $G$-invariant $\nu \in i\Omega_+^2$ such that $\mathcal{M}_\nu(\nu)$ has no reducible point.

To see this, note that $(A, 0)$ is a point of $\mathcal{M}_\nu(\nu)$ if and only if $d^+ (i\omega) = \nu + \mu_0$, where $iA = A - A_0$, and that $d^+ (i\Omega^1) \subset i\Omega_+^2$ is a codimension $b_2^+(X)$ subspace, say $\mathcal{H}$. Hence for any $\nu$ such that $\nu + \mu_0$ is not in $\mathcal{H}$, the moduli space $\mathcal{M}_\nu(\nu)$ has no reducible solution. Note that $\mu_0$ is $G$-invariant as $A_0$ is a $G$-invariant connection.

As $H_2^+(X/G, \mathbb{R}) \neq 0$, we can choose a $G$-invariant self dual harmonic 2-form $\nu$ such that $\nu + \mu_0 \notin \mathcal{H}$. Thus $\mathcal{M}_\nu(\nu)$ consists of irreducible solutions. Moreover, $G$ acts on the moduli space $\mathcal{M}_\nu(\nu)$. Notice that $\mathcal{M}_\nu(\nu)$ is not necessary regular.

Next we choose a perturbation $\nu'$ such that the moduli space $\mathcal{M}_\nu(\nu')$ consists of irreducible smooth points. Hence $\nu' + \mu_0 \notin \mathcal{H}$. By assumption $b_2^+(X) \geq 2$, we conclude that $\Omega_+^2 - \mathcal{H}$ is connected. Choose an arc in $\Omega_+^2 - \mathcal{H}$, say $\nu(t)$ with $0 \leq t \leq 1$, to join $\nu' + \mu_0$ and $\nu + \mu_0$. For the same reason as above, $\mathcal{M}_\nu(\nu(t))$ has no reducible solution for all $t$. In other words, $\mathcal{F}_0^{-1}(\nu(t)) \subset (U - 0) \times \mathcal{V}$, where $U$ and $\mathcal{V}$ are as above.

For each $t$, $\mathcal{M}_\nu(\nu(t))$ is compact. That is, $\mathcal{F}_0^{-1}(\nu(t))$ is compact. It is easy to adapt the proofs of Lemmas 2.1 and 2.2 to show that there is a map

$$\mathcal{F}_\nu(3) : (B_\lambda, \partial B_\lambda \cup 0 \times V_\lambda \cap B_\lambda) \to (W_\lambda, W_\lambda - \nu(t))$$

Note that $S^1$ acts on $\nu(t)$ trivially. The pair $(W_\lambda, W_\lambda - \nu(t))$ has the same $S^1$-equivariant homotopy type as $(W_\lambda, W_\lambda - 0)$. Thus as above, by passing to the Thom complex, $\mathcal{F}_\nu(0)$ and $\mathcal{F}_\nu(1)$ give two maps $f, f' : S^V_{\nu} \otimes \ast S(U_\nu) \to S^{W_\lambda}$. Moreover, $f$ and $f'$ are $S^1$-equivariant homotopic. $f$ is a $S^1 \times \hat{G}$-equivariant map since $\nu$ is $G$-invariant.

By regarding $f$ as an $S^1$-equivariant map, we get an integer $\theta$ as before, which is essentially the equivariant degree of $f$. On the other hand, by Theorem 2.3 the Seiberg-Witten invariant $SW(C)$ is equal to $\theta'$. Clearly $\theta = \theta'$ because $f \simeq_{S^1} f'$. This proves the theorem. q.e.d.

§3 Seiberg-Witten Invariant of 4-manifolds of Non-simple Type

A smooth closed Spin$^c$ four manifold is called of simple type if its Seiberg-Witten invariant is zero, provided the moduli space is of nonzero dimensional. It is still a wide
open problem that whether every Spin\(^c\) four manifold is of simple type. In [5] Furuta announced an interesting theorem, Theorem 3, which asserts that the Seiberg-Witten invariant is divisible by certain integer related to the dimension of the moduli space. In this section we give a detailed proof of his theorem, Theorem 3. We include a proof here since so far we have not yet seen the details.

**Proof of Theorem 3:** Let \( f : S^{v_{\Lambda} \oplus \mathbb{R}} \wedge S(U_\Lambda) \to S^{U_{\Lambda} \oplus V_\Lambda} \) be the \( S^1 \)-equivariant map in \( \S 2 \). Note that \( S^1 \) acts on \( V_\Lambda \) and \( V_{\Lambda}' \) trivially. Let \( ES^1 \to BS^1 \) denote the universal circle bundle. There is a map

\[
\hat{f} : ES^1 \times_{S^1} (S^{V_\Lambda \oplus \mathbb{R}} \wedge S(U_\Lambda)) \to ES^1 \times_{S^1} S^{U_{\Lambda} \oplus V_\Lambda}
\]

Notice that the \( S^1 \) action on \( S^{U_{\Lambda} \oplus V_\Lambda} \) has a fixed point \( \infty \). Hence the sphere bundle 
\( ES^1 \times_{S^1} S^{U_{\Lambda} \oplus V_\Lambda} \to BS^1 \) has a section sending every point \( p \in BS^1 \) to \( \infty \), say \( s(\mathbb{S}^1) \).

Clearly the quotient 
\( ES^1 \times_{S^1} S^{U_{\Lambda} \oplus V_\Lambda}/s(\mathbb{S}^1) = T(\xi) \)

is the Thom complex of \( \xi \). Here \( \xi = ES^1 \times_{S^1} (U_{\Lambda}' \oplus V_{\Lambda}') \to BS^1 \) is the associated vector bundle.

It is easy to see that \( \xi = m' H \oplus \varepsilon n' \), where \( H \) is the Hopf complex line bundle on \( BS^1 \), and \( \varepsilon \) is the trivial real line bundle. \( m' \) is the complex dimension of \( U_{\Lambda} \) and \( n' \) the real dimension of \( V_{\Lambda} \).

On the other hand, because that \( S^1 \) acts freely on \( S(U_\Lambda) \),
\[
ES^1 \times_{S^1} (S^{V_{\Lambda} \oplus \mathbb{R}} \wedge S(U_\Lambda)) \simeq S^{V_{\Lambda} \oplus \mathbb{R}} \wedge \mathbb{C}P^{m-1}
\]
where \( m = \text{dim} U_{\Lambda} \).

By Theorem 2.3 we have
\[
\sigma(SW(C)x^{m-d-1}) = \hat{f}^*(\Phi_{\xi})
\]
where \( \sigma \) is the suspension isomorphism, \( \Phi_{\xi} \in H^{2m'+n'}(T(\xi), \mathbb{Z}) \) is the Thom class and \( x \in H^2(\mathbb{C}P^{m-1}, \mathbb{Z}) \) is a generator.

Without loss of generality we assume that \( n' \) is even. Otherwise, we can suspend the map \( \hat{f} \) once. Note that the \( S^1 \)-equivariant degree is the same under the suspension. For an even \( n' \), we can regard the real trivial bundle \( \varepsilon n' \) as a trivial complex line bundle of dimension \( \frac{n'}{2} \). By the definition, for \( d \not\equiv \mathbb{Z}, SW(C) \) is defined to be zero. Thus we assume that \( d \in \mathbb{Z} \). It follows that \( l = \frac{1}{2}(b^2_d - 1) \in \mathbb{Z} \).

Let \( \tau_{\xi} \in K(T\xi) \) denote the Thom class of K-theory. By [8] the Chern character
\[
ch(\tau_{\xi}) = \left\{ \frac{1 - \varepsilon x}{x} \right\}^{m'} \Phi_{\xi}
\]
Recall that \( K(\mathbb{C}P^{m-1}) \simeq \mathbb{Z}[T]/T^m = 0 \), where \( T = H - 1 \in K(\mathbb{C}P^{m-1}) \) and \( H \) is the Hopf line bundle. Let \( y = \log(1 + T) \in K(\mathbb{C}P^{m-1}) \). It is easy to show that \( ch(H) = e^x \) and \( ch(y) = x \).
Consider the homomorphism
\[ \hat{f}^* : K(T\xi) \to K(S^{V_{\Lambda}} \otimes \mathbb{C}P^{m-1}) \]
As the Chern character commutes with $f^*$,
\[ ch(\hat{f}^*\tau_{\xi}) = \left(1 - \frac{e^x}{x}\right)^{m'} \hat{f}^*(\Phi_{\xi}) = SW(C)(1 - e^x)^{m'}x^l = (-1)^{m'} SW(C)ch(T^{m'}y') \]
Here $l = \frac{1}{2}(b^+_2 - 1) = m - m' - d - 1$ by the dimension formula for the moduli space.

Since $ch : K(\mathbb{C}P^{m-1}) \to H^*(\mathbb{C}P^{m-1})$ is a rational isomorphism. By the above identity we conclude that
\[ \hat{f}^*(\tau_{\xi}) = (-1)^{m'} SW(C)\left(\frac{log(1 + T)}{T}\right)^{iT^{m'-d-1}} \in \frac{Z[T]}{T^m = 0} \]
Thus the coefficients of the right side of the above equality must be integers. It follows that $a_i SW(C) \in Z$ for $i \leq d$, where $a_i$ is the coefficient of $T^i$ of the power series $\left(\frac{log(1 + T)}{T}\right)^i$.
q.e.d.

§4 Seiberg-Witten Invariant and Involutions on 4-manifolds

Let $X$ be a $Spin^c$ 4-manifold and let $C$ be the $Spin^c$-structure. Suppose $b_1(X) = 0$ and $b^+_2(X) \geq 2$. For convenience we assume that the signature of $X$, $\sigma(X)$, is nonpositive. In this section we will apply Theorem 2.4 to study the Seiberg-Witten invariant of $X$ in the case when there exists an involution on $X$ preserving $C$. We assume that the involution acts on the space of self dual harmonic 2-forms, $H^2_\Sigma(X, \mathbb{R})$, trivially throughout the rest. However, the argument of this section can be easily adapted to consider other cases.

As before we let $\mathcal{M}_\Lambda$ to denote the Seiberg-Witten moduli space. $dim\mathcal{M}_\Lambda = 2d$. By the definition, if $d \not\in Z$, $SW(C) = 0$. Thus we only need to consider the case of $d \in Z$.

Our goal is to prove Theorem 2 and Theorem 1 in the case of $p = 2$ advertised in the introduction.

As in §2, for any $\Lambda \in \mathbb{R}$, let $U_\Lambda, U'_\Lambda$ denote the direct sums of the $\lambda$-eigenspaces of $D_{\Lambda_0}D_{\Lambda_0}$ and $D_{\Lambda_0}D_{\Lambda_0}$ for $\lambda \leq \Lambda$ respectively, $A_0$ is an equivariant connection with respect to the $Z_2$-action on the bundle $L$.

4.0 Even Type Involution Let us consider the case of an even type involution acting on $X$ as above. In this case, $Z_2$ acts on the bundle $P_{Spin^c}(X)$. Hence $U_\Lambda, U'_\Lambda$ are complex $Z_2$-modules. We define $m := dim_{\mathbb{C}} U_\Lambda$ and $m_+, m_-$ to be the dimensions of the $+1$ and $-1$ eigenspaces. Similarly $n := dim_{\mathbb{C}} U'_\Lambda$ and $n_+, n_-$ for the $+1$ and $-1$ eigenspaces of the $Z_2$-action on $U'_\Lambda$. Clearly $m_+ + m_- = m$ and $n_+ + n_- = n$.

By Theorem 2.4 there exists a $S^1 \times Z_2$-equivariant map $f : S^{V_{\Lambda}} \otimes S(U_\Lambda) \to S^{W_\Lambda}$. Here $W_\Lambda = U'_\Lambda \oplus V'_\Lambda$. By the construction in §2 we have that $V'_\Lambda \cong H^2_\Sigma(X, \mathbb{R}) \oplus V_\Lambda$ as a real $Z_2$-module. By assumption $Z_2$ acts on $H^2_\Sigma(X, \mathbb{R})$ trivially. Thus, as a real $Z_2$-module, $V'_\Lambda \oplus V_\Lambda \cong V_{\Lambda} \otimes \mathbb{C} \oplus \mathbb{C}^{l+1}$ where $l = \frac{1}{2}(b^+_2 - 1)$ is an integer. This is because that $d$ is an integer.
Obviously $V_A \oplus V_A \oplus \mathbb{R} \oplus \mathbb{R}$ is the realization of the complex $S^1 \times \mathbb{Z}_2$-module $(V_A \oplus \mathbb{R}) \otimes \mathbb{C}$. $V_A \oplus \mathbb{R} \oplus W_A$ is the realization of the complex $S^1 \times \mathbb{Z}_2$-module $U_A \oplus V_A \otimes \mathbb{C} \oplus \mathbb{C}^{+1}$. Here $S^1$ acts on $(V_A \oplus \mathbb{R}) \otimes \mathbb{C}$ and $V_A \otimes \mathbb{C} \oplus \mathbb{C}^{+1}$ trivially.

Suspending $f$ by $S^{V_A \oplus \mathbb{R}}$ we have a map $\sigma(f) : S^{(V_A \oplus \mathbb{R}) \otimes \mathbb{C}} \wedge S(U_A) \to S^{V_A \oplus \mathbb{R}} \wedge S^{W_A}$. Notice that the $S^1$-equivariant degree of $f$ and $\sigma(f)$ are the same. But for the latter it is more convenient to use $K$-theory.

By the Thom isomorphism theorem $K_{S^1 \times \mathbb{Z}_2}((S^{W_A \oplus V_A \oplus \mathbb{R}}) \otimes \mathbb{C}) \cong R(S^1 \times \mathbb{Z}_2)$. Let $\tau \in K_{S^1 \times \mathbb{Z}_2}((S^{W_A \oplus V_A \oplus \mathbb{R}}) \otimes \mathbb{C}) \cong K_{S^1 \times \mathbb{Z}_2}(S(U_A))$ denote the $K$-theory Thom class. Similarly, we have

$$K_{S^1 \times \mathbb{Z}_2}((S^{V_A \oplus \mathbb{R}}) \otimes \mathbb{C}) \wedge S(U_A)) \cong K_{S^1 \times \mathbb{Z}_2}(S(U_A))$$

Applying the $K_{S^1 \times \mathbb{Z}_2}$-functor to the equivariant map $\sigma(f)$, we get a class $\beta_f \in K_{S^1 \times \mathbb{Z}_2}(S(U_A))$ such that

$$\sigma(f)^*(\tau) = \beta_f \tau_{V_A \oplus \mathbb{R} \oplus \mathbb{C}} \quad (4.1)$$

Note that $R(S^1) \cong \mathbb{Z}[t, t^{-1}]$, where $t$ corresponds to the standard 1-dimensional complex representation. Let $\xi \in R(\mathbb{Z}_2)$ denote for the irreducible 1-dimensional complex representation.

**Lemma 4.1:** $K_{S^1 \times \mathbb{Z}_2}(S(U_A)) \cong R(S^1 \times \mathbb{Z}_2)/(1-t)^{m+}(1-\xi t)^{m-}$.

**Proof:** Since the unit ball $B(U_A)$ is equivariantly contractible to the origin, $K_{S^1 \times \mathbb{Z}_2}(D(U_A)) \cong R(S^1 \times \mathbb{Z}_2)$. By the Thom isomorphism theorem, $K_{S^1 \times \mathbb{Z}_2}(S^{U_A}) \cong R(S^1 \times \mathbb{Z}_2)$. On the other hand, the $K$-theoretical Euler class for the bundle $U_A \to 0$ is $(1-t)^{m+}(1-\xi t)^{m-}$. By the exact sequence

$$0 \to K_{S^1 \times \mathbb{Z}_2}(S^{U_A}) \to K_{S^1 \times \mathbb{Z}_2}(D(U_A)) \to K_{S^1 \times \mathbb{Z}_2}(S(U_A)) \to 0$$

for the cofibration $S(U_A) \to D(U_A) \to S^{U_A}$ it follows that $K_{S^1 \times \mathbb{Z}_2}(S(U_A))$ is the cokernel of the homomorphism

$$(1-t)^{m+}(1-\xi t)^{m-} : R(S^1 \times \mathbb{Z}_2) \to R(S^1 \times \mathbb{Z}_2)$$

This completes the proof. q.e.d.

Now we want to use the Adams $\psi$-operation to the both sides of the equation (4.1). Recall that for a complex vector bundle $\gamma$ on $Y$, the Thom class $\tau_\gamma \in K(T\gamma)$,

$$\psi^q(\tau_\gamma) = \rho^q(\gamma)\tau_\gamma$$

Here $\rho^q(\gamma) \in K(Y)$ is the Bott canibalistic class of $\gamma$. Recall that $\rho^q(1) = q$ and

$$\rho(\eta) = (1 + \eta + \cdots + \eta^{q-1})$$

if $\eta$ is a line bundle.

Applying $\psi^q$ to the equality (4.1) it follows that

$$\psi^q(\beta_f)\rho^q(V_A \otimes \mathbb{C} \oplus \mathbb{C})\tau_{V_A \otimes \mathbb{R} \oplus \mathbb{C}} = \beta_f\rho^q(U_A' \oplus V_A \otimes \mathbb{C} \oplus \mathbb{C}^{+1})\tau_{V_A \otimes \mathbb{R} \oplus \mathbb{C}}$$
Hence

$$\psi^q(\beta_f) = q^{l} \beta_f \rho^q(U'_A)$$

On the other hand, because that $\mathbb{Z}_2$ acts on $U'_A$ with $n_+$-times $(+1)$-eigenvalues and $n_-$-times $(-1)$-eigenvalues. $S^1$ acts on $U'_A$ by the complex multiplication. Therefore $U'_A \cong n_+ t + n_- t \xi$ as a $S^1 \times \mathbb{Z}_2$-module. From the multiplicative property of Bott class it follows that

$$\rho^q(U'_A) = (1 + t + \cdots + t^{q-1})^{n_+} (1 + t \xi + \cdots + t^{q-1} \xi^{q-1})^{n_-}$$

Substituting this into the above equation we have

$$\psi^q(\beta_f) = q^{l} \beta_f (1 + t + \cdots + t^{q-1})^{n_+} (1 + t \xi + \cdots + t^{q-1} \xi^{q-1})^{n_-} \quad (4.2)$$

It is easy to verify that:

**Lemma 4.2:** There is a short exact sequence

$$0 \to R(S^1 \times \mathbb{Z}_2)/(1-\xi t)^{m-} \to R(S^1 \times \mathbb{Z}_2)/(1-t)^{m+} (1-\xi t)^{m-} \to j \to R(S^1 \times \mathbb{Z}_2)/(1-t)^{m+} \to 0$$

Here $i(z) = (1-t)^{m-} z$ and $j(z) = z$ the forgetful homomorphism.

Now we want to study the class $\beta_f$ defined in the equation (4.1). By Lemma 4.1 we understand $\beta_f$ as an element of the ring $R(S^1 \times \mathbb{Z}_2)$ subject to the the relation $(1-t)^{m+}(1-\xi t)^{m-} = 0$. If we ignore the $\mathbb{Z}_2$ action, the image of $\beta_f$ in $R(S^1)/(1-t)^{m+} = 0$ gives an $\beta$. That is, $\beta = \beta_f(t,1)$, substitute 1 for $\xi$ in an expression of $\beta_f$. The following result is already proved in §3 during the proof of Theorem 3.

**Proposition 4.3:** $\beta = (-1)^n SW(C)(\frac{\log(1+T)}{T})^l \psi^{m-d-1}$, where $T = 1-t$.

Now let us first consider the case of $d = 0$, i.e., the moduli space is zero dimensional.

**Lemma 4.4:** If $d = 0$ and $0 < k^+ = m_+ - a_+ < l + 1$, then $j(\beta_f) = 0$.

**Proof:** Let $T = 1-t$ and $y = 1-\xi \in R(\mathbb{Z}_2)$. Recall that $\xi^2 = 1$.

Let $\beta_f = \sum_j (a_j + b_j \xi) T^j$. By Proposition 4.3, $\beta_f(t,1) = c T^{m-1}$ and up to sign, $c$ is the Seiberg-Witten invariant. Thus $a_j + b_j = 0$ if $j < m - 1$.

Now let us prove that $a_j = 0$ if $j \leq m_+ - 1$. Substituting $q = 3$, $t = 1-T$ and $\xi = 1-y$ to the equation (4.2), we get

$$\sum_j (a_j + b_j \xi) (3T - 3T^2 + T^3)^j = 3^l (3 - 3T + T^2)^{n_+} (3 - y - 3T + y T + T^2)^{n_-} \sum_j (a_j + b_j \xi) T^j$$

If $j < m - 1$ is the minimal number such that $a_j \neq 0$. The coefficients of $T^j$ in the both sides of the above equation give

$$3^l a_j y = 3^{l+n_+} (3 - y)^{n_-} a_j y$$

It is easy to show that for any nonnegative integer $r$, $(3 - y)^r y \equiv y$. Thus the above identity holds only if $a_j = 0$, provided $j < l + n_+$. 

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By $d = 0$ we get that $m - 1 = n + l$. On the other hand, by the assumption $m_{+} \leq l_{+} n_{+}$. Therefore $a_{j} = 0$ for $j \leq m_{+} - 1$ and hence the image of $\beta_{f}$ in $R(S^{1} \times \mathbb{Z}_{2})/(1 - t)^{m_{+}}$ is zero. q.e.d

From Lemmata 4.4 and 4.2 it follows that there is an $f(t, \xi) \in R(S^{1} \times \mathbb{Z}_{2})/(1 - t)^{m_{-}}$ such that $\beta_{f} = (1 - t)^{m_{+}} f(t, \xi)$.

**Lemma 4.5:** If $d = 0$ and $0 < k_{+} < l + 1$, then there is an integer $a$ such that $f(t, \xi) = a(1 + \xi)(1 - t)^{m_{-}}$.

**Proof:** Let $T = 1 - t \xi$ for this moment. Let us write $f(t, \xi) = \sum_j (a_{j} + b_{j} \xi)T_{j}$. By Proposition 4.3, $\beta_{f}(t, 1) = c(1 - t)^{m_{-}}$. Thus $a_{j} + b_{j} = 0$ for $j < m_{-} - 1$.

Let $j_{0} = m_{-} - 1$ and $\theta = (3 - 3T + T^{2})$. Applying the equation (4.2) with $q = 3$ we get

$$
(1 + t + t^{2})^{k_{+}} \left\{ \sum_{j \leq m_{-} - 2} a_{j} y(T\theta)^{j} + (a_{j_{0}} + b_{j_{0}} \xi)(T\theta)^{j_{0}} \right\} = 3^{L} \theta^{m_{-} - 1} \left\{ \sum_{j \leq m_{-} - 2} a_{j} y T^{j} + (a_{j_{0}} + b_{j_{0}} \xi) T^{j_{0}} \right\}
$$

Note that $1 + t + t^{2} = 3 - 3T + T^{2} + (T - 1)y$. Therefore, as in the proof of the above lemma, comparing the coefficients of $T^{j}$ in this equation we get $a_{j} = 0$ for $j \leq m_{-} - 2$. The rest of the equation reads

$$
3^{L} (3 - y)^{k_{+}} (a_{j_{0}} + b_{j_{0}} \xi) T^{j_{0}} = 3^{L} \theta^{m_{-} - 1} (a_{j_{0}} + b_{j_{0}} \xi) T^{j_{0}}
$$

(4.3)

Note that $c = a_{j_{0}} + b_{j_{0}}$ and $a_{j_{0}} + b_{j_{0}} \xi = c - b_{j_{0}} y$. It is easy to verify that

$$
(3 - y)^{k_{+}} = \frac{1}{2} (1 - 3^{k_{+}}) y + 3^{k_{+}}
$$

and $y(3 - y)^{k_{+}} = y$. The assumption $d = 0$ implies that $k_{+} + k_{-} = l + 1$ and so it is easy to verify that $l + n_{-} = j_{0} + k_{+}$. Substituting these to (4.3) we get

$$
(3^{L} - 3^{j_{0}}) b_{j_{0}} = 3^{j_{0}} c 3^{k_{+} - 1} - \frac{c}{2}
$$

Therefore $c = 2b_{j_{0}}$ if $k_{+} = 0$ and hence $a_{j_{0}} = b_{j_{0}}$. Let $a = a_{j_{0}}$. We get $f(t, \xi) = a(1 + \xi)(1 - t)^{m_{-}, -1}$. q.e.d

**Proof of Theorem 1 for $p = 2$:** By Lemmata 4.4 and 4.5, if $d = 0$ and $0 < k_{+} \leq l$, $\beta_{f} = a(1 + \xi)(1 - t)^{m_{+}}(1 - t \xi)^{m_{-}}$. Substituting $\xi = 1$ we get $\beta = 2a(1 - t)^{m_{-}}$. By Proposition 4.3, this implies that the Seiberg-Witten invariant $SW(C)$ is even. This proves Theorem 1 for $p = 2$ and $d = 0$. The proof for $d \geq 1$ is similar. One only needs to replace $m$ by $m - d$ in the proof of lemmanata 4.4 and 4.5 and consider the image of $\beta_{f}$ in the truncated ring $R(S^{1} \times \mathbb{Z}_{2})/(1 - t)^{m_{+} - d}(1 - t \xi)^{m_{-}}$. q.e.d

REMARK: It is possible to get a stronger conclusion for $d \geq 1$ by a more careful calculation. We will not do this here since $d = 0$ is the most interesting case in the Seiberg-Witten theory and also for the applications.
4.6 Odd Type Involutions. Let us consider a Spin$^c$ 4-manifold $X$ with an odd type involution. In this case, there is an $\mathbb{Z}_4$-action on the principal bundle $P_{\text{Spin}^c}(X)$ and the spinor bundles $W^\pm$, where the generator of $\mathbb{Z}_2 \subset \mathbb{Z}_4$ acts as $-id$ on $W^\pm$. Hence for a generator $g \in \mathbb{Z}_4$, the eigenvalues of the induced linear action of $g$ on the eigenspaces of $D_{A_0}D^*_{A_0}$ and $D^*_{A_0}D_{A_0}$ are $\pm \sqrt{-1}$. As in the subsection 4.0, for $\Lambda \geq 0$, let $U_\Lambda$ and $U'_\Lambda$ denote the direct sum of eigenspaces of $D_{A_0}D^*_{A_0}$ and $D^*_{A_0}D_{A_0}$ with eigenvalues no larger than $\Lambda$. Let $m_+$ and $m_-$ for the dimensions of $U_\Lambda$ and $U'_\Lambda$ respectively. Note that $m_+ - n_+$ and $m_- - n_-$ are independent of $\Lambda$. Let $k_+ = m_+ - n_+$, $k_- = m_- - n_-$. 

The argument to show Theorem 2 is parallel to the case for an even type involution. Thus we only give a sketch of the proof. In this case, by the similar argument we get an equation

$$\beta_f \in K_{S^1 \times \mathbb{Z}_4}(S(U_\Lambda)) \cong R(S^1) \otimes R(\mathbb{Z}_4)/(1 - t \zeta)^{m_+}(1 - t \zeta^3)^{m_-}$$

Here $\zeta \in R(\mathbb{Z}_4)$ is an irreducible 1-dimensional representation. Moreover, $\beta_f$ satisfies the following analogue equation of (4.2):

$$\psi^3(\beta_f) = q t^s \beta_f(1 + t \zeta + \cdots + t^{s-1} \zeta^{s-1})^{n_+}(1 + t \zeta^3 + \cdots + t^{s-1} \zeta^{3(s-1)})^{n_-}$$

(4.4)

To understand this element $\beta_f$, it is convenient to consider the forgetful homomorphism

$$\iota : R(S^1) \otimes R(\mathbb{Z}_4)/(1 - t \zeta)^{m_+}(1 - t \zeta^3)^{m_-} \to R(S^1) \otimes R(\mathbb{Z}_2)/(1 - t \zeta)^{m_+ + m_-}$$

Note that $\iota(\beta_f)$ can be regarded as the class corresponding to the trivial involution on $X$ but acting as $-id$ on $W^\pm$. First let us present the following analogue of lemma 4.2:

**Lemma 4.7:** There is an extension

$$0 \to R(S^1 \times \mathbb{Z}_4)/(1 - \zeta)^{m_-} \to R(S^1 \times \mathbb{Z}_4)/(1 - \zeta t)^{m_+}(1 - \zeta^3 t)^{m_-} \to R(S^1 \times \mathbb{Z}_4)/(1 - \zeta t)^{m_+} \to 0$$

Here $i(z) = (1 - \zeta)^{m_+} z$ and $j(z) = z$ the forgetful homomorphism.

**Proof of Theorem 2:** We claim that $j(\beta_f) = 0$ if $d = 0$. A useful point is to consider the image of $\beta_f$ under $\iota$ as produced from an $\mathbb{Z}_2$ action on the spinor bundles $W^\pm$ by $-1$ but on $X$ trivially. Thus most of the arguments of the last subsection can be carried over to show that $\iota(\beta_f) = (a + b \zeta)(1 - t \zeta)^{m-1}$ for some integers $a, b$. However, $a$ and $b$ are not necessary the same, since the final assumption on $k_+$ does not hold for the trivial involution.

Let $T = 1 - \zeta t$ and

$$\beta_f = \sum_i (a_i + b_i \zeta + c_i \zeta^2 + d_i \zeta^3) T^i$$

The above fact about $\iota(\beta_f)$ implies that $a_i + c_i = 0$ and $b_i + d_i = 0$ for $i < m - 1$. To show that $\beta_f$ has zero image under the homomorphism $j$, it suffices to show that $a_i = b_i = 0$ for $i < m_+ - 1$.

Similarly substituting $q = 3$ to the equation (4.4) we get

$$\psi^3(\beta_f) = 3^i \beta_f(1 + t \zeta + t^2 \zeta^2)^{n_+}(1 + t \zeta^{-1} + t \zeta^2)^{n_-}$$
The same argument of lemma 4.4 applies to prove that $a_i = b_i = 0$ for $i \leq l + n_+$. Hence $j(\beta_f) = 0$ for $d = 0$ because $m_+ - 1 \leq l + n_+$ in this case.

Consequently, we can write $\beta_f = (1 - t^i t^m) f(t, \zeta) / (1 + \zeta^2)^m$ by Lemma 4.7. The argument of Lemma 4.5 can be modified to show that, if $d = 0$ and $0 < k_+ \leq l$, there are integers $a, b$ such that $\beta_f = (a + b \zeta)(1 + \zeta^2)^m f(t, \zeta)$. Substituting $\zeta = 1$ to $\beta_f$ we get $\beta_f(t, 1) = 2(a + b) T^{-m - 1}$. Thus the Seiberg-Witten invariant $SW(C)$ is even by Proposition 4.3. The case of $d \geq 1$ is similar, just as in the case for even type involution. q.e.d.

§5 Odd Order Group Action and Seiberg-Witten Invariant

In this section we study the Seiberg-Witten invariants of $\text{Spin}^c$ 4-manifolds with some $\mathbb{Z}_p$ action for $p$ an odd prime. We will give a proof of Theorem 1 for the case of $p$ odd. Recall that $U_\Lambda$ and $U'_\Lambda$ are the direct sum of the eigenspaces of the Dirac operators $D_{A_0} D_{A_0}$ and $D_{A_0} D'_{A_0}$ with eigenvalues less than or equal to $\Lambda$. Notice that $\mathbb{Z}_p$ acts linearly on $U_\Lambda$ and $U'_\Lambda$. Similarly, $\mathbb{Z}_p$ acts linearly on $V_\Lambda$ and $V'_\Lambda$. Let $\omega = e^{2\pi i p}$ be the $p$-th unit root. Let $m_0, m_1, \ldots, m_{p-1}$ denote the dimensions of $1, \omega, \ldots, \omega^{p-1}$ eigenspaces of a generator of the $\mathbb{Z}_p$ action on $U_\Lambda$. Similarly define $n_0, n_1, \ldots, n_{p-1}$ for $U'_\Lambda$.

As we have learned in §2, there exists an $S^1 \times \mathbb{Z}_p$-equivariant map

$$\mathcal{F}_{\Lambda} : U_\Lambda \oplus V_\Lambda \to U'_\Lambda \oplus V'_\Lambda$$

such that the truncated moduli space $\mathcal{F}_{\Lambda}^{-1}(\nu_0)$ is compact in a sufficiently large ball. Moreover, $\mathcal{F}_{\Lambda}^{-1}(\nu_0)$ has no reducible points. Here $\nu_0 \in \Omega^2_\Lambda$ is $\mathbb{Z}_p$ invariant. By Theorem 2.4, the Seiberg-Witten invariant $SW(C)$ is equal to the $S^1$-equivariant degree of certain map constructed from $\mathcal{F}_\Lambda$. Let $\xi \in R(\mathbb{Z}_p)$ denote an irreducible complex $\mathbb{Z}_p$-module. As an additive group, $R(\mathbb{Z}_p)$ is generated by $1, \xi, \ldots, \xi^{p-1}$. For convenience we can add the $\mathbb{Z}_p$-modules $1, \xi, \ldots, \xi^{p-1}$ to $U_\Lambda$ and $U'_\Lambda$ and consider the map

$$id \oplus \mathcal{F}_\Lambda : \oplus_{k=0}^{p-1} \xi^k \oplus U_\Lambda \oplus V_\Lambda \to \oplus_{k=0}^{p-1} \xi^k \oplus U'_\Lambda \oplus V'_\Lambda$$

Notice that the preimages of $\mathcal{F}_{\Lambda}^{-1}(\nu_0)$ and $(id \oplus \mathcal{F}_\Lambda)^{-1}(\nu_0)$ are the same. Thus it is easy to show that the $S^1$-degree of $id \oplus \mathcal{F}_\Lambda$ and $\mathcal{F}_\Lambda$ are the same. This is just the Seiberg-Witten invariant $SW(C)$.

In view of this, we may assume that $m_i \geq 1$ for $i = 0, \ldots, p - 1$ without loss of generality. Notice that the representation of $S^1 \times \mathbb{Z}_p$ on $U_\Lambda$ splits as the orthogonal sum of $m_0 l + m_1 l \xi + \cdots + m_{p-1} l \xi^{p-1}$. Let $k_0 = m_0 - n_0, k_1 = m_1 - n_1, \ldots, k_{p-1} = m_{p-1} - n_{p-1}$. As in §4 we have

**Lemma 5.1:** $K_{S^1 \times \mathbb{Z}_p}(S(U_\Lambda)) = \frac{R(S^1) \otimes \mathbb{R}(\mathbb{Z}_p)}{(1 - t)^{m_0} (1 - t \xi)^{m_1} \cdots (1 - t \xi^{p-1})^{m_{p-1}}}$

**Proof of Theorem 1 for $p$ odd:** First let us assume that the moduli space is of zero dimension, i.e., $d = 0$. As in §4, there is a class $\beta_f \in K_{S^1 \times \mathbb{Z}_p}(S(U_\Lambda))$ satisfying the analogue equation of (4.2):

$$\psi^t(\beta_f) = q^t (1 + t + \cdots + t^{p-1})^{n_0} (1 + t \xi + \cdots + t^{p-1} \xi^{p-1}) n_1 \cdots (1 + t \xi^{p-1} + \cdots + t^{p-1} \xi^{p(p-1)})^{n_{p-1}}$$

(5.1)
Substituting $q = 2$ to this equality we get

$$
\psi^2(\beta_f) = 2^l(1 + t)^{n_0}(1 + t\xi)^{n_1} \cdots (1 + t\xi^{p-1})^{n_{p-1}} \quad (5.2)
$$

We want to prove that there is an $a \in \mathbb{Z}$ such that

$$
\beta_f = a(1 + \xi + \xi^2 + \cdots + \xi^{p-1})(1 - t)^{m_0}(1 - t\xi)^{m_1} \cdots (1 - t\xi^{p-2})^{m_{p-2}} (1 - t\xi^{p-1})^{m_{p-1}}
$$

To show this, as in §4, we first prove that the image of $\beta_f$ in the truncated representation ring $R(S^1) \otimes R(\mathbb{Z}_p)/(1 - t)^{m_0}$ is zero if $k_0 \leq l$.

Let $T = 1 - t$ and let us write $\beta_f = \sum_{i=0}^{p-1} \sum_{j=0}^{\xi^j} (a_{i}^j \xi^j)T^i$ with integral coefficients.

Assume that $i$ is the minimal number so that the coefficient of $T^i$ is nonzero in the above expression. Note that $\psi^2(\xi) = \xi^2$, $\psi^2(t) = t^2$ and $\psi^2$ is a ring homomorphism. By comparing the coefficients of $T^i$ in the equation (5.2) we get an identity

$$
2^l \sum_{j=0}^{p-1} (a_{i}^j \xi^j) = 2^{l+n_0}(1 + \xi)^{n_1} \cdots (1 + \xi^{p-1})^{n_{p-1}} \left(\sum_{j=0}^{p-1} a_{i}^j \xi^j\right) \quad (5.3)
$$

Substituting 1 for $\xi$, we obtain that $\sum_{j=0}^{p-1} a_{i}^j = 0$ if $i < l + n$, note that $n = \sum_{j=0}^{p-1} n_j$.

When $d = 0$, the dimension formula for the moduli space implies $l + n = m - 1$. Thus $m_0 - 1 < l + n$ because $m_i \geq 1$ for all $i$. From these we conclude that $\sum_{j=0}^{p-1} a_{i}^j = 0$ for $i \leq m_0 - 1$.

Taking $\xi = \omega, \omega^2, \cdots, \omega^{p-1}$ in (5.3) we get $p - 1$ equalities. As $p$ is an odd prime, $(1 + \omega)(1 + \omega^2) \cdots (1 + \omega^{p-1}) = 1$. Multiply these equations together and substituting the identity for $\omega$ we get

$$
2^{(p-1)l} \prod_{k=1}^{p-1} \sum_{j=0}^{p-1} a_{i}^j \omega^{2kj} = 2^{(p-1)(l+n_0)} \prod_{k=1}^{p-1} \sum_{j=0}^{p-1} a_{i}^j \omega^{kj}
$$

Notice that $\prod_{k=1}^{p-1}(\sum_{j=0}^{p-1} a_{i}^j \omega^{2kj})$ and $\prod_{k=1}^{p-1}(\sum_{j=0}^{p-1} a_{i}^j \omega^{kj})$ are the same. Thus, for $i < l + n_0$, there exists at least a zero term among this product. Hence there is an $k$ with $1 \leq k \leq p-1$ such that

$$
\sum_{j=0}^{p-1} a_{i}^j \omega^{kj} = 0
$$

This implies that $a_{i}^0 = a_{i}^1 = \cdots = a_{i}^{p-1}$, since the polynomial $1 + x + x^2 + \cdots + x^{p-1}$ is irreducible over rational.

On the other hand, we have already proved that the sum of these $a_{i}^j$ is zero. Thus $a_{i}^0 = a_{i}^1 = \cdots = a_{i}^{p-1} = 0$, provided $i < l + n_0$. If $k_0 \leq l$, then $m_0 \leq l + n_0$ and so
\( m_0 - 1 < l + n_0 \). Thus the image of \( \beta_f \) in the ring \( R(S^1) \otimes R(\mathbb{Z}_p)/(1 - t)^{m_0} \) is zero. Consequently \( \beta_f = (1 - t)^{m_0} \beta'_f \) for some element \( \beta'_f \in R(S^1) \otimes R(\mathbb{Z}_p) \).

Next let us prove that the image of \( \beta'_f \) in \( R(S^1) \otimes R(\mathbb{Z}_p)/(1 - t)^{m_1} \) is zero if \( k_1 \leq l \).

For the sake of simplicity, we let \( T = (1 - t^\xi) \) for this moment. Let us write \( \beta'_f = \sum_{j=0}^{p-1} \sum (a^i_j \xi^j)T^i \). The equation (5.2) for \( \psi^2(\beta_f) \) gives rise to the following identity

\[
(1 + t)^{k_0} \psi^2(\beta'_f) = 2^l \beta'_f (1 + t^\xi)^{n_1} \cdots (1 + t^\xi^{p-1})^{n_{p-1}}
\]

(5.4)

If \( i \) is the minimal index so that the coefficient of \( T^i \) in the expression of \( \beta'_f \) nonzero. With the same argument as above, comparing coefficients in (5.4) we get

\[
2^i (1 + \xi^{p-1})^{k_0} \sum_{j=0}^{p-1} a^i_j \xi^{2j} = 2^l \sum_{j=0}^{p-1} a^i_j \xi^{j} (1 + \xi)^{n_j} \cdots (1 + \xi^{p-2})^{n_{j+1}}
\]

Taking \( \xi = 1 \), we get

\[
2^i + k_0 \left( \sum_{j=0}^{p-1} a^i_j \right) = 2^l \sum_{j=0}^{p-1} a^i_j
\]

Therefore \( \sum_{j=0}^{p-1} a^i_j = 0 \) if \( i + k_0 < l + n - n_0 \). Hence \( \sum_{j=0}^{p-1} a^i_j = 0 \) for \( i \leq m_1 - 1 \) because \( m_0 \geq 1, m_1 \geq 1, \ldots, m_{p-1} \geq 1 \).

Now taking \( \xi = \omega, \omega^2, \ldots, \omega^{p-1} \) and multiplying these identities, we get

\[
2^i (p-1) \prod_{k=1}^{p-1} \left( \sum_{j=0}^{p-1} a^i_j \omega^{2ik} \right) = 2^l (p_1)(p-1) \prod_{k=1}^{p-1} \left( \sum_{j=0}^{p-1} a^i_j \omega^{ik} \right)
\]

For \( i < l + n_1 \), this implies that there is an \( 1 \leq k \leq p - 1 \) such that

\[
\sum_{j=0}^{p-1} a^i_j \omega^{ik} = 0
\]

As before this shows that \( a^i_0 = a^i_1 = \cdots = a^i_{p-1} \) and so they are all zero. Thus \( \beta'_f \) has zero image in the ring \( R(S^1 \times \mathbb{Z}_p)/(1 - t)^{m_1} \). Hence, \( \beta_f = (1 - t)^{m_0} \beta'_f \) is zero in the truncated ring \( R(S^1) \otimes R(\mathbb{Z}_p)/(1 - t)^{m_0}(1 - t^\xi)^{m_1} \).

Continuing this procedure, one can check that, under the assumption of Theorem 1 and \( d = 0 \),

\[
\beta_f = a(1 + \xi + \xi^2 + \cdots + \xi^{p-1})(1 - t)^{m_0}(1 - t^\xi)^{m_1} \cdots (1 - t^\xi^{p-1})^{m_{p-1}} - 1
\]

for some \( a \in \mathbb{Z} \). Therefore \( \beta_f(t, 1) = pa(1 - t)^{m-1} \). By Proposition 4.3, this proves that \( SW(C) = 0(modp) \).

The proof for \( d \geq 1 \) is similar. One only needs to consider an over truncated ring by replacing \( m_0 \) by \( m_0 - d \) and replacing \( m \) by \( m - d \). \( \bullet \)
References


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