Renormalization Group Scaling of the $1/m^2$ HQET Lagrangian

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Abstract

The operator mixing matrix for the dimension six operators in the heavy quark effective theory Lagrangian is computed at one loop. The results are shown to be consistent with constraints from the equations of motion, and from reparametrization invariance.

Heavy quark effective theory (HQET) [1,2] is a useful tool for studying the physics of hadrons containing a single heavy quark. The HQET Lagrangian has an expansion in powers of derivatives divided by the heavy quark mass $m$, which translates into an expansion of hadronic quantities in powers of $\Lambda_{\text{QCD}}/m$, where $\Lambda_{\text{QCD}}$ is the non-perturbative scale parameter of the strong interactions.

The HQET Lagrangian can be computed by matching with the full QCD Lagrangian at a scale $\mu \approx m$. This has been done at one loop to order $1/m^3$ for the two-Fermion terms in the Lagrangian [3–5], and at one loop to order $1/m^2$ for the two-Fermion terms [6]. The renormalization group running of the dimension five ($1/m$) operators in the HQET Lagrangian has been computed [3,4]. There are several computations of the running of the dimension six ($1/m^2$) operators [6–10] in the literature, but the various papers disagree with each other. In Refs. [7–10] the authors do not take into account the effect of a four fermion operator which is present in the dimension six operator basis and is related to the Darwin term by the equations of motion. A complete calculation including all dimension six operators was given in Ref. [6], but we disagree with this calculation in a few entries of the anomalous dimension matrix.

In this paper we compute the running of the dimension six operators of the HQET Lagrangian. The coefficients are computed using one-loop running and tree-level matching, which makes the calculation particularly simple. We will use the notation of Ref. [6], to
make it easier to compare with previous results. The computations are done in a theory with one heavy quark with velocity \( v \), \( h_v \), and \( n_f \) flavors of massless quarks, \( q_i \). The covariant derivative is chosen to be \( \partial_\mu + igA^A_\mu T^A \). With this convention, the HQET Lagrangian to order \( 1/m^2 \) is

\[
\mathcal{L} = -\frac{1}{4} G^{\mu\nu} G^A_{\mu\nu} + \sum_i \bar{q}_i i\gamma_5 q_i + \bar{h} (iv \cdot D) h - \frac{c_k}{2m} \bar{h} D^2 h - \frac{c_F}{4m^2} \bar{h} \sigma_{\mu\nu} G^{\mu\nu} h
+ c_D O_D + c_S O_S + \sum c_i O_i,
\]

where the Darwin and spin-orbit operators are defined as

\[
O_D = \frac{g}{8m^2} \bar{h} (D^\mu G^{\mu\nu}) v_\rho h, \quad O_S = i \frac{g}{8m^2} \bar{h} \sigma_{\mu\nu} \{ D^\mu, G^{\rho\nu} \} v_\rho,
\]

respectively. The remaining operators \( O_i \) are four-Fermion operators involving two heavy and two light-quark fields,

\[
O_1^{hl} = \frac{g^2}{8m^2} \sum_i \bar{h} T^A h \bar{q}_i \gamma_5 T^A q_i, \quad O_2^{hl} = \frac{g^2}{8m^2} \sum_i \bar{h} \gamma^\mu \gamma_5 T^A h q_i \gamma_\mu \gamma_5 T^A q_i,
\]

\[
O_3^{hl} = \frac{g^2}{8m^2} \sum_i \bar{h} h \bar{q}_i \gamma_5 q_i, \quad O_4^{hl} = \frac{g^2}{8m^2} \sum_i \bar{h} \gamma^\mu \gamma_5 h q_i \gamma_\mu \gamma_5 h q_i,
\]

four-Fermion operators involving four light quark fields,

\[
O_1^{ll} = \frac{g^2}{8m^2} \sum_{i,j} \bar{q}_i T^A \gamma^\mu q_j \bar{q}_j T^A \gamma_\mu q_j, \quad O_2^{ll} = \frac{g^2}{8m^2} \sum_{i,j} \bar{q}_i T^A \gamma^\mu \gamma_5 q_i \bar{q}_j T^A \gamma_\mu \gamma_5 q_j,
\]

\[
O_3^{ll} = \frac{g^2}{8m^2} \sum_{i,j} \bar{q}_i \gamma^\mu q_i \bar{q}_j \gamma_\mu q_j, \quad O_4^{ll} = \frac{g^2}{8m^2} \sum_{i,j} \bar{q}_i \gamma^\mu \gamma_5 q_i \bar{q}_j \gamma_\mu \gamma_5 q_j,
\]

the light quark penguin operator

\[
O_p = \frac{1}{8m^2} \sum_i \bar{q}_i \gamma_\mu D_\mu G^{\mu\nu} q_i,
\]

and three-gluon operators

\[
O_1^g = \frac{1}{4m^2} g f_{ABC} G^A_{\mu\nu} G^B_{\alpha\delta} G^{C\nu\alpha}, \quad O_2^g = \frac{1}{4m^2} D^\mu G^A_{\mu\alpha} D_\nu G^{\nu\alpha A}.
\]

The identity

\[
0 = \int 2 \, D^\mu G^A_{\mu\alpha} D_\nu G^{\nu\alpha A} + 2 \, g f_{ABC} G^A_{\mu\alpha} G^B_{\mu\delta} G^{C\nu\alpha} + G^A_{\mu\nu} D^2 G^{A\mu\nu}
\]

has been used to eliminate \( G^A_{\mu\nu} D^2 G^{A\mu\nu} \) from the effective Lagrangian. There are also operators such as \( \bar{h} (iv \cdot D)^3 h \) which vanish by the heavy quark equations of motion, and can be eliminated by a field redefinition of the Lagrangian. Operators \( O^{hh} \) involving four heavy quark fields do not contribute in the single heavy quark sector, and will be omitted from the Lagrangian in this analysis.
The equation of motion for the gluon field,
\[ D_\mu G^{\mu\nu A} = g v^\nu \bar{h}_v T^A h_v + g \sum_i \bar{q}_i T^A \gamma_\nu q_i, \]  
(8)
can be used to further simplify the effective Lagrangian. It allows one to rewrite \( O_2^g \) and \( O^p \) in terms of four-Fermion operators. It also gives the relation
\[ O_D = O^{hl}_1 + \frac{g^2}{8m^2} \bar{h} T^a h \bar{h} T^a h. \]  
(9)

The second term on the right hand side can be omitted in the single heavy quark sector, so the equation of motion reduces to
\[ O_D = O_1^{hl}. \]  
(10)

This relation can be used to further simplify the effective Lagrangian by eliminating \( O_1^{hl} \). For the moment, we retain \( O_1^{hl} \), to make it easier to compare the results with Ref. [6].

The renormalization group scaling of the \( 1/m^2 \) terms in the HQET Lagrangian involves the operators described above, as well as the time ordered product of two \( 1/m \) operators. The time ordered products will be denoted by
\[ O_{kk} = \frac{i}{2} \int d^4 x \ T [O_k (x) \ O_k (0)], \]
\[ O_{km} = i \int d^4 x \ T [O_k (x) \ O_m (0)], \]
\[ O_{mm} = \frac{i}{2} \int d^4 x \ T [O_m (x) \ O_m (0)], \]  
(11)
where the kinetic and magnetic moment operators are
\[ O_k = -\frac{1}{2m} \bar{h} D^2 h, \quad O_m = \frac{1}{4m} g \bar{h} \sigma_{\mu\nu} G^{\mu\nu} h. \]  
(12)
The coefficient of \( O_k \), \( c_k \), is fixed to unity by reparametrization invariance [11]. The coefficient of the magnetic moment operator satisfies the renormalization group equation [3,4]
\[ \mu \frac{d c_F}{d \mu} = 2C_A c_F \frac{g^2}{16\pi^2}, \]  
(13)
where \( C_A = 3 \) is the Casimir of the adjoint representation. We will later also use \( C_F = 4/3 \), the Casimir of the fundamental representation (not to be confused with \( c_F \)), \( T_F = 1/2 \), the index of the fundamental representation, \( N = 3 \), the number of colors, and \( n_f \), the number of light quark flavors. Note that the coefficient of \( O_m \) in the Lagrangian is \(-c_F\).

The running of the dimension six operators is
\[
\frac{d}{d\mu} \left( \begin{array}{c} O_D \\ O_S \\ O_{kk} \\ O_{km} \\ O_{mm} \\ O_{hl}^1 \\ O_{hl}^2 \\ O_{hl}^3 \\ O_{hl}^4 \\ O_{ll}^1 \\ O_{ll}^2 \\ O_{ll}^3 \\ O_{ll}^4 \\ O_{g1} \\ O_{hh} 
\end{array} \right) = -\frac{g^2}{16\pi^2} \left( \begin{array}{cccccccc}
\Gamma_{11} & \Gamma_{12} & 0 & 0 & \Gamma_{15} \\
\Gamma_{21} & \Gamma_{22} & 0 & 0 & \Gamma_{25} \\
0 & \Gamma_{32} & \Gamma_{33} & 0 & \Gamma_{35} \\
\Gamma_{41} & \Gamma_{42} & \Gamma_{43} & \Gamma_{44} & \Gamma_{45} \\
0 & 0 & 0 & 0 & \Gamma_{55} 
\end{array} \right) \left( \begin{array}{c} O_D \\ O_S \\ O_{kk} \\ O_{km} \\ O_{mm} \\ O_{hl}^1 \\ O_{hl}^2 \\ O_{hl}^3 \\ O_{hl}^4 \\ O_{ll}^1 \\ O_{ll}^2 \\ O_{ll}^3 \\ O_{ll}^4 \\ O_{g1} \\ O_{hh} 
\end{array} \right) .
\]

(14)

The zero entries follow because no one-loop Feynman graph exists for that term in the mixing matrix. \(O_{hh}^{12}\) represents all possible operators involving four heavy quark fields.

The tree-level matching conditions at the scale \(\mu = m\) are \(c_D = c_S = 1\), and that \(c_{hl}^i\), \(c_{ll}^i\) and \(c_g^1\) are all zero. The operator mixing equations Eq. (14) then imply that \(c_g^1\) and \(c_{ll}^i\) stay zero on scaling from \(m\) to \(\mu\). The values of \(c_D\) and \(c_S\) can be obtained by solving the renormalization group equations in the \(1-2\) sector. The equation of motion Eq. (10) allows one to eliminate \(O_{hl}^1\) and further simplify the calculation. The only \(O_{hl}^i\) operator that can mix with \(O_D\) or \(O_S\) is \(O_{hl}^1\); there are no penguin diagrams from the other \(O_{hl}^i\) operators.

The \(O_{hl}^i\) operators do not mix among themselves, so that \(\Gamma_{22}\) is diagonal [12]. Thus eliminating \(O_{hl}^1\) using the equations of motion gives a renormalization group equation for \(c_D\) and \(c_S\) only in the \(\{O_D, O_S, O_{kk}, O_{km}, O_{mm}\}\) sector. The computation of the coefficients is straightforward. The renormalization group equation for the spin-orbit coefficient \(c_S\) is

\[
\dot{c}_S = \frac{g^2}{16\pi^2} 4 C_A c_k c_F .
\]

(15)

The running of \(c_S\) is consistent with the reparametrization constraint \(c_S = 2c_F - 1\). The \(c_S\) equation can be solved when combined with \(c_k = 1\) and Eq. (13) for \(c_F\) [6–10],

\[
c_S (\mu) = 2 z^{-C_A} - 1, \quad c_F (\mu) = z^{-C_A},
\]

(16)

where

\[
z = \left[ \frac{\alpha_s (\mu)}{\alpha_s (m)} \right]^{1/\beta_0},
\]

(17)

\(b_0 = 11C_A/3 - 4T_F n_f/3\) is the first term in the \(\beta\)-function, and we have used the initial condition \(c_S (m) = c_F (m) = 1\).

The renormalization group equation for the Darwin term \(c_D\) is

\[
\dot{c}_D = \frac{g^2}{16\pi^2} \left[ \frac{13}{3} C_A c_D - \left( \frac{20}{3} C_A + \frac{32}{3} C_F \right) c_k^2 - \frac{1}{3} C_A c_F^2 \right] ,
\]

(18)
whose solution is
\[ c_D(\mu) = z^{-2C_A} + \left( \frac{20}{13} + \frac{32}{13} C_F \right) \left[ 1 - z^{-13C_A/6} \right], \] (19)
using the initial condition \( c_D(m) = 1 \), and Eq. (16). For QCD, Eqs. (16,19) reduce to
\[ c_S(\mu) = 2z^{-3} - 1, \quad c_F(\mu) = z^{-3}, \quad c_D(\mu) = z^{-6} + \frac{308}{117} \left[ 1 - z^{-13/2} \right]. \] (20)
As an example of numerical values, running the \( b \)-quark terms between \( m_b \) and \( m_c \) gives
\[ c_F = 0.83, \quad c_S = 0.65, \quad c_D = 1.57, \] (21)
where we have used \( n_f = 4 \) and \( \alpha_s(m_c)/\alpha_s(m_b) = 1.7 \), to be compared with the tree-level values \( c_F = c_S = c_D = 1 \).

To compute the full effective Lagrangian, including the four-Fermion operators, one needs the anomalous dimension matrix Eq. (14). The matrix is computed without using the equations of motion to eliminate \( O_{hi}^{hl} \), to make it easier to compare with earlier calculations. The entries of the anomalous dimension matrix are listed below. The second form of the matrix uses the explicit values of \( C_A \), etc. for QCD. In deriving these equations, we have used the identity
\[ \bar{\psi} \{ T^A, T^B \} \psi \bar{\chi} \{ T^A, T^B \} \chi = \left( 1 - 1/N^2 \right) \bar{\psi} \bar{\chi} + (N - 4/N) \bar{\psi} T^A \psi \bar{\chi} T^A \chi \] (22)
which is valid for Fermions in the fundamental representation of \( SU(N) \), and holds regardless of the \( \gamma \)-matrix structure of the fermion bilinears.

\[
\Gamma_{11} = \begin{pmatrix}
\frac{2}{3} C_A & 0 & 0 & 0 & 0 \\
0 & -\frac{2}{3} C_A - \frac{32}{3} C_F & 0 & 0 & 0 \\
0 & 0 & 4 C_A & 0 & 0 \\
-\frac{10}{3} C_A & 0 & 0 & 0 & 4 C_A
\end{pmatrix} = \begin{pmatrix}
4 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix}, \] (23)

\[
\Gamma_{12} = \begin{pmatrix}
3 C_A & 0 & 0 & 0 & 0 \\
6 C_A & 0 & 0 & 0 & 0 \\
0 & -3(1 - 1/N^2) & 0 & -3(1 - 1/N^2) & 0 \\
-8(1 - 1/N^2) & 0 & -8(1 - 1/N^2) & 0 & 0 \\
3 C_A & -(N - 4/N) & -64/9 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix} = \begin{pmatrix}
9 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix}, \] (24)

\[
\Gamma_{21} = \begin{pmatrix}
8 T_F n_f/3 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix} = \begin{pmatrix}
4 n_f/3 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix}, \] (25)

\[
\Gamma_{22} = \begin{pmatrix}
-3 C_A & 0 & 0 & 0 & 0 \\
0 & -3 C_A & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix} + 2b_0 = \begin{pmatrix}
13 - \frac{4}{3} n_f & 0 & 0 & 0 & 0 \\
0 & 13 - \frac{4}{3} n_f & 0 & 0 & 0 \\
0 & 0 & 22 - \frac{4}{3} n_f & 0 & 0 \\
0 & 0 & 0 & 22 - \frac{4}{3} n_f & 0 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix}, \] (26)
\[ \Gamma_{32} = \begin{pmatrix} 8C_F/3 - 4C_A/3 + 16T_Fn_f/3 & 0 & 0 & 0 \\ 8C_F/3 - 4C_A/3 & 0 & 0 & 0 \\ 8/3 & 0 & 0 & 0 \\ 8/3 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} -4/9 + 8n_f/3 & 0 & 0 & 0 \\ -4/9 & 0 & 0 & 0 \\ 8/3 & 0 & 0 & 0 \\ 8/3 & 0 & 0 & 0 \end{pmatrix}, \quad (27) \]

\[ \Gamma_{33} = \begin{pmatrix} 8C_F/3 - 13/3C_A + 16T_Fn_f/3 & 3(N - 4/\pi \gamma) & 0 & 3(1 - 1/\pi \gamma) \\ 8C_F/3 - 4C_A/3 + 3(N - 4/\pi \gamma) & -3C_A & 0 & 0 \\ 8/3 & 12 & 0 & 0 \\ 44/3 & 0 & 0 & 0 \end{pmatrix} + 2b_0 \]

\[ = \begin{pmatrix} 113/9 + 4n_f/3 & 5 & 0 & 8/3 \\ 41/9 & 13 - 4n_f/3 & 8/3 & 0 \\ 8/3 & 12 & 22 - 4n_f/3 & 0 \\ 44/3 & 0 & 0 & 22 - 4n_f/3 \end{pmatrix}, \quad (28) \]

\[ \Gamma_{41} = \Gamma_{42} = \Gamma_{43} = 0, \quad \Gamma_{44} = 12C_A - 2b_0 = 14 + 4n_f/3. \quad (29) \]

In writing Eq. (14), penguin diagrams from \( O^{ii} \) have been rewritten in terms of \( O^{il} \) and \( O^{hl} \) using the gluon equation of motion. The Lagrangian in the single heavy quark sector does not depend on \( \Gamma_{5i} \), which have not been computed.

The anomalous dimension matrix Eq. (14) has been computed previously. \( \Gamma_{11} \) was computed in Refs. [6–10]. The submatrix in the 3–4 sector was computed in Ref. [6,13–15]. \( \Gamma_{44} \) was computed in Ref. [16,17]. The rest of the matrix was computed in Ref. [6]. We disagree with the previous computations in a few entries of the anomalous dimension matrix. One can check the consistency of Eq. (14) with the equation of motion Eq. (10). One finds, neglecting \( O^{hh} \) operators which were also neglected in Eq. (10),

\[ \mu \frac{d}{d\mu} (O_D - O_{11}^{hl}) = -\frac{g^2}{16\pi^2} \left( \frac{4}{3} C_A O_D + 3C_A O_{11}^{hl} \right) + \frac{g^2}{16\pi^2} \left( \frac{8}{3} T_F n_f O_D + [2b_0 - 3C_A] O_{11}^{hl} \right) \]

\[ = -\frac{g^2}{16\pi^2} \left( \frac{4}{3} C_A - \frac{8}{3} T_F n_f \right) (O_D - O_{11}^{hl}), \quad (30) \]

so that the equation of motion is multiplicatively renormalized. This is consistent with the result that equations of motion can only mix among themselves under renormalization [18]. One can then use the equations of motion to eliminate \( O_{11}^{hl} \). The resulting renormalization group equations have the same form as Eq. (14), with \( O_{11}^{hl} \) omitted in the second operator block. The new entries of the operator mixing matrix will be denoted by \( \Gamma_{ij}' \). The resulting matrix is (note that the zero entries are not the same as Eq. (14))

\[ \begin{pmatrix} \Gamma_{11}' & \Gamma_{12}' & 0 & 0 & \Gamma_{15}' \\ 0 & \Gamma_{22}' & 0 & 0 & \Gamma_{25}' \\ \Gamma_{31}' & 0 & \Gamma_{33}' & 0 & \Gamma_{35}' \\ 0 & 0 & 0 & \Gamma_{44}' & \Gamma_{45}' \\ 0 & 0 & 0 & 0 & \Gamma_{55}' \end{pmatrix}, \quad (31) \]

where the new entries are
\[ \Gamma'_{11} = \begin{pmatrix} \frac{13}{3} C_A & 0 & 0 & 0 & 0 \\ -20/3 C_A - \frac{32}{3} C_F & 0 & 0 & 0 & 0 \\ 0 & -4 C_A & 2 C_A & 0 & 0 \\ -\frac{1}{3} C_A & 0 & 0 & 4 C_A & 0 \end{pmatrix} = \begin{pmatrix} 13 & 0 & 0 & 0 & 0 \\ -308/9 & 0 & 0 & 0 & 0 \\ 0 & -12 & 6 & 0 & 0 \\ -1 & 0 & 0 & 0 & 12 \end{pmatrix}, \tag{32} \]

\[ \Gamma'_{12} = \begin{pmatrix} 0 & 0 & 0 \\ -3(N - 4/N) & 0 & -3(1 - 1/N^2) \\ 0 & 0 & 0 \\ -8(N - 4/N) & 0 & -8(1 - 1/N^2) \\ -(N - 4/N) & 0 & -(1 - 1/N^2) \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ -5 & 0 & -8/3 \\ 0 & 0 & 0 \\ -40/3 & 0 & -64/9 \\ -5/3 & 0 & -8/9 \end{pmatrix}, \tag{33} \]

\[ \Gamma'_{22} = \begin{pmatrix} -3 C_A + 2 b_0 & 0 & 0 \\ 0 & 2 b_0 & 0 \\ 0 & 0 & 2 b_0 \end{pmatrix} = \begin{pmatrix} 13 - 4 n_f/3 & 0 & 0 \\ 0 & 22 - 4 n_f/3 & 0 \\ 0 & 0 & 22 - 4 n_f/3 \end{pmatrix}, \tag{34} \]

\[ \Gamma'_{31} = \begin{pmatrix} 8 C_F/3 - 4 C_A/3 + 16 T_F n_f/3 & 0 & 0 & 0 & 0 \\ 8 C_F/3 - 4 C_A/3 & 0 & 0 & 0 & 0 \\ 8/3 & 0 & 0 & 0 & 0 \\ 8/3 & 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} -4/9 + 8 n_f/3 & 0 & 0 & 0 & 0 \\ -4/9 & 0 & 0 & 0 & 0 \\ 8/3 & 0 & 0 & 0 & 0 \\ 8/3 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{35} \]

One can solve the renormalization group equations Eq. (14) with the initial conditions $c_F = c_D = c_S = 1$, and $c_i^{hl} = c_i^l = c_i^r = 0$ to obtain the coefficients in the effective Lagrangian with one-loop running, and tree-level matching. The results are Eq. (16) for $c_S$ and $c_F$, and

\[
c_2^{hl} = \left[ N - \frac{4}{N} \right] \left[ \frac{1}{2 b_0 - 7 C_A} \left( z^{-2 C_A} - z^{-b_0 + 3 C_A/2} \right) - \frac{2}{2 b_0 - 5 C_A} \left( z^{-C_A} - z^{-b_0 + 3 C_A/2} \right) \right] - \frac{3}{2 b_0 - 3 C_A} \left( 1 - z^{-b_0 + 3 C_A/2} \right)
\]

\[
c_3^{hl} = 0
\]

\[
c_4^{hl} = \left[ 1 - \frac{1}{N^2} \right] \left[ \frac{1}{2 b_0 - 4 C_A} \left( z^{-2 C_A} - z^{-b_0} \right) - \frac{1}{b_0 - C_A} \left( z^{-C_A} - z^{-b_0} \right) - \frac{3}{2 b_0} \left( 1 - z^{-b_0} \right) \right]
\]

\[
c_i^l = c_i^r = 0.
\tag{36} \]

One can solve Eq. (14) for $c_D$ and $c_1^{hl}$ separately. The result for the sum $c_D + c_1^{hl}$ is the same as the value of $c_D$ in Eq. (19), where the equation of motion was used to replace $O_1^{hl}$ by $O_D$. An independent linear combination of $c_D$ and $c_1^{hl}$ that has a relatively simple scaling law is

\[
c_D - \frac{8 T_F n_f}{9 C_A} c_1^{hl} = -\frac{5 C_A + 4 T_F n_f z^{-2 C_A}}{4 C_A + 4 T_F n_f} + \frac{C_A + 16 C_F - 8 T_F n_f}{2(C_A - 2 T_F n_f)}
\]

\[
+ \frac{-7 C_A^2 + 32 C_A C_F - 4 C_A T_F n_f + 32 C_F T_F n_f z^{4 T_F n_f/3 - 2 C_A/3}}{4(C_A + T_F n_f)(2 T_F n_f - C_A)}. \tag{37} \]

The numerical values for the coefficients with the same choice of parameters as before are $c_2^{hl} = -0.19$, $c_3^{hl} = 0$, $c_4^{hl} = -0.09$, $c_D = 1.61$ and $c_1^{hl} = -0.04$.

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