A Quantum Mechanical Model of the Reissner-Nordström Black Hole

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Abstract

We consider a Hamiltonian quantum theory of spherically symmetric, asymptotically flat electrovacuum spacetimes. The physical phase space of such spacetimes is spanned by the mass and the charge parameters \(M\) and \(Q\) of the Reissner-Nordström black hole, together with the corresponding canonical momenta. In this four-dimensional phase space, we perform a canonical transformation such that the resulting configuration variables describe the dynamical properties of Reissner-Nordström black holes in a natural manner. The classical Hamiltonian written in terms of these variables and their conjugate momenta is replaced by the corresponding self-adjoint Hamiltonian operator, and an eigenvalue equation for the ADM mass of the hole, from the point of view of a distant observer at rest, is obtained. Our eigenvalue equation implies that the ADM mass and the electric charge spectra of the hole are discrete, and the mass spectrum is bounded below. Moreover, the spectrum of the quantity \(M^2 - Q^2\) is strictly positive when an appropriate self-adjoint extension is chosen. The WKB analysis yields the result that the large eigenvalues of the quantity \(\sqrt{M^2 - Q^2}\) are of the form \(\sqrt{2n}\), where \(n\) is an integer. It turns out that this result is closely related to Bekenstein’s proposal on the discrete horizon area spectrum of black holes.

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1. Introduction

Electrically charged, externally static black holes are very interesting objects. One of the main interests of these so called Reissner-Nordström black holes lies in the peculiar properties of their radiation. For example, the Hawking temperature of a black hole with mass $M$ and electric charge $Q$ is, in natural units where $\hbar = c = G = k_B = 1$[1],

$$T_H = \frac{\sqrt{M^2 - Q^2}}{2\pi(M + \sqrt{M^2 - Q^2})^2}.$$  \hspace{1cm} (1.1)

Hence, we see that when $M^2 = Q^2$, i.e. when the hole is extreme, the temperature of the hole goes to zero, and no radiation comes out from the hole. Moreover, although an evaluation of the Bekenstein-Hawking entropy of a non-extreme black hole by means of Euclidean methods implies that the entropy is exactly one quarter of the area of the apparent horizon of the hole, the same analysis implies zero entropy for an extreme black hole[2-4].

What, then, is the difference between extreme and non-extreme black holes? Of course, one of the fundamental differences lies in their different topological properties in Euclidean spacetime: the topology of a non-extreme black hole in Euclidean spacetime is $\mathbb{R}^2 \times S^2$, whereas that of an extreme black hole is $S^2 \times \mathbb{R} \times S^1$. However, there is yet another difference which is the primary object of interest in this paper. Non-extreme black holes have dynamics, whereas extreme black holes, in a certain sense, have not. More precisely, inside the apparent horizon of a non-extreme black hole there is a region which does not admit a timelike Killing vector field, from which it follows that it is impossible to choose a timelike coordinate in such a way that spacetime metric with respect to this coordinate would be static everywhere in that region. However, an extreme black hole is static everywhere in its interior as well as in its exterior regions, which means that everywhere in these domains there is a timelike Killing vector field, which is orthogonal to a spacelike hypersurface of spacetime. Naively, this can be seen by considering the spacetime metric of an extreme Reissner-Nordström hole, written in the curvature coordinates $T$ and $R$ as:

$$ds^2 = -\left(1 - \frac{M}{R}\right)^2 dT^2 + \frac{dR^2}{\left(1 - \frac{M}{R}\right)^2} + R^2(d\theta^2 + \sin^2 \theta d\phi^2).$$ \hspace{1cm} (1.2)

We find that the spacetime metric is static inside as well outside of the horizon $R = M$ with respect to the time $T$. Could this difference in the dynamical properties be related to the result that non-extreme holes radiate, whereas extreme do not?

In this paper we shall address this question by means of a simple quantum mechanical model of the Reissner-Nordström black hole. Our model gives a Hamiltonian quantum theory of spherically symmetric, asymptotically flat electrovacuum
spacetimes. The model is based on the study of the Hamiltonian dynamics of such spacetimes by Louko and Winters-Hilt in Ref.[5]. In Ref.[5] it was found that after the classical constraints of the ADM formulation of the dynamics of such spacetimes have been solved, only two independent degrees of freedom, together with the corresponding canonical momenta, are left. In other words, the real physical phase space of spherically symmetric, asymptotically flat electrovacuum spacetimes is four-dimensional. Because of that, the Hamiltonian quantum theory of such spacetimes can be addressed by means of a finite-dimensional quantum mechanics.

After performing in Section 2 a minor modification of the analysis made in Ref.[5] at the classical level, we shall in Section 3 construct a Hamiltonian quantum theory of electrovacuum spacetimes under study. To make things simple, the electric charge is kept fixed. As a consequence, the phase space contains geometro-dynamical variables only. The crucial point is the choice of phase space coordinates in such a way that they reflect the dynamical properties of non-extreme Reissner-Nordström black hole in a natural manner. To put it simply, we shall use the radius of the wormhole throat of the Reissner-Nordström hole, from the point of view of an observer in a radial free fall through the bifurcation two-sphere, as the configuration variable of our theory. (For the conformal diagram of maximally extended Reissner-Nordström spacetimes see, for example, Refs.[6] and [7].) That radius takes its minimum value $M - \sqrt{M^2 - Q^2}$ at the past $R = r_-$-hypersurface, then attains its maximum value $M + \sqrt{M^2 - Q^2}$ at the bifurcation two-sphere, and finally shrinks back to its minimum value $M - \sqrt{M^2 - Q^2}$ at the future $R = r_-$-hypersurface. As one can see, the domain of the classically allowed values of the throat radius includes just one value, $M$, in the limit of extremality. Hence, extreme black holes have no dynamics with respect to our dynamical variable.

It turns out that the choice of the momentum variable conjugate to the throat radius is related to the choice of the foliation of spacetime into space and time. In this paper we choose a foliation in which the time coordinate at the right infinity is chosen to be an asymptotic Minkowski time, and the time evolution at the left infinity is frozen. The time coordinate at the throat is chosen in such a way that the proper time of an observer in a free fall through the bifurcation two-sphere agrees with the asymptotic Minkowski time. With this choice of foliation, the classical Hamiltonian is quadratic in momenta and its numerical value agrees, when the electric potential is assumed to vanish at infinity, with the ADM mass of the hole, from the point of view of a distant observer at rest with respect to the hole. The classical Hamiltonian is then replaced by the corresponding self-adjoint Hamiltonian operator, and the spectrum of that operator, which gives the ADM mass spectrum of the hole, is analyzed.

The Hamiltonian quantization outlined above was performed for the Schwarzschild black hole in Ref.[8], where also the Hamiltonian quantum theory of Reissner-Nordström black holes of Section 3 of the present paper was outlined on a qualitative level. In this paper, however, that Hamiltonian quantum theory is developed in full details. We shall see that the mass spectrum is discrete and bounded below.
Moreover, we shall see that with an appropriate choice of a self-adjoint extension, the spectrum of the quantity

$$M^2 - Q^2$$

is strictly positive. Regarding Hawking radiation, this is a very interesting result. If we think of Hawking radiation as an outcome of a chain of transitions from higher to lower energy eigenstates of the hole, the positivity of the spectrum of the quantity $M^2 - Q^2$ implies that a non-extreme black hole with non-zero temperature can never become, through Hawking radiation, an extreme black hole with zero temperature. This result is in harmony with the third law of black hole thermodynamics as well as with the qualitative difference between extreme and non-extreme black holes.

The WKB analysis of the eigenvalue equation for the ADM mass of the hole yields the result that for macroscopic black holes the eigenvalues of the quantity $\sqrt{M^2 - Q^2}$ are of the form $\sqrt{2n}$, where $n$ is an integer. The physical interest of this result lies in its relationship with the proposal made first by Bekenstein in 1974[9]. He argued that the possible eigenvalues of the black hole horizon area are of the form:

$$A_n = \gamma n l_{Pl}^2,$$

where $\gamma$ is a pure number of order one, $n$ is an integer, and $l_{Pl} := (\hbar G/c^3)^{1/2}$ is the Planck length. In Ref.[8] the spectrum (1.3) was, in effect, obtained for the area of the Schwarzschild black hole with $\gamma = 32\pi$, by means of a model similar to the one used here. In this paper we shall see that although the spectrum of the area of the outer horizon of the Reissner-Nordström black hole fails to coincide with Eq.(1.3), the eigenvalues of the sum of the areas of the inner and the outer horizons of the hole—which we shall refer, for the sake of convenience, as the total area of the horizons—are, in the semi-classical limit, of the form:

$$A_{n}^{tot} \approx 32\pi n l_{Pl}^2 + 2A^{ext},$$

where $A^{ext} := 4\pi Q^2$ is the area of an extreme black hole. Hence, our result on the spectrum of the total area of the Reissner-Nordström black hole is closely related to Bekenstein’s proposal: according to Bekenstein’s proposal, the spectrum of the area of the outer horizon of a black hole is of the form (1.3), whereas our model implies that it is the total area of the hole which is quantized in that manner. Moreover, our model implies that, up to the factor $2A^{ext}$, the spectrum of the total area of the horizons of the hole is exactly the same for the Reissner-Nordström and Schwarzschild black holes.

Finally, at the end of this paper, we shall relax the requirement that the electric charge is a mere external parameter of our theory. Instead, we shall consider the electric charge as a dynamical variable accounting for the electromagnetic degrees of freedom of the hole. The resulting quantum theory implies that the spectrum of the electric charge of the hole is discrete.
2. Hamiltonian Reduction

In the curvature coordinates $R$ and $T$ the spacetime metric corresponding to the Reissner-Nordström black hole can be written as:

$$ds^2 = -(1 - \frac{2M}{R} + \frac{Q^2}{R^2})dT^2 + \frac{dR^2}{1 - \frac{2M}{R} + \frac{Q^2}{R^2}} + R^2 d\Omega^2. \quad (2.1)$$

In this expression, $M$ is the mass and $Q$ is the electric charge of the hole. $d\Omega^2$ is the line element on the unit two-sphere. As it is well known, the Reissner-Nordström metric is the only spherically symmetric, asymptotically flat electrovacuum solution to the combined Einstein-Maxwell equations. In the coordinates $R$ and $T$ the only non-zero component of the electromagnetic potential $A_{\mu}$ is

$$A_T = \frac{Q}{R}. \quad (2.2)$$

Before going into the canonical quantization of Reissner-Nordström spacetimes we must investigate the Hamiltonian dynamics of such spacetimes. An extensive study of this problem, along the lines first shown by Kuchař[10] in the case of Schwarzschild spacetimes, was performed by Louko and Winters-Hilt in Ref.[5]. However, the considerations of those authors were thermodynamically motivated, and so they restricted their investigations to the exterior regions of the hole, whereas our interest, in contrast, lies basically in the interior regions of the hole. In what follows, we shall briefly review the analysis of Ref.[5], and consider an extension of that analysis to include also the interior regions of the hole.

The starting point in Ref.[5] was to write the general spherically symmetric Arnowitt-Deser-Misner (ADM) metric in the form:

$$ds^2 = -N^2 dt^2 + \Lambda^2 (dr + N^r dt)^2 + R^2 d\Omega^2. \quad (2.3)$$

In this equation, the lapse $N$ and the shift $N^r$, as well as the variables $\Lambda$ and $R$, which are considered as the dynamical variables of the spacetime geometry, are assumed to be functions of the time coordinate $t$ and the radial coordinate $r$ only. The electromagnetic potential $A_{\mu}$, also, is assumed to be spherically symmetric such that its only non-zero components are

$$A_t := \phi, \quad (2.4.a)$$
$$A_r := \Gamma, \quad (2.4.b)$$

where $\phi$ and $\Gamma$ are assumed to be functions of $t$ and $r$ only.

In Ref.[5], the radial coordinate $r$ was taken to be within the interval $[0, \infty)$. To understand the meaning of this semi-boundedness of the allowed values of $r$, recall the conformal diagram of the maximally extended Reissner-Nordström spacetime (such a diagram can be seen, for example, in Refs.[6] and [7]). The
maximally extended Reissner-Nordström spacetime has, in the interior regions of the hole, a periodic geometrical structure. We may choose one such period and pick up from the conformal diagram a bifurcation point corresponding to that period. In the bifurcation point, two lines with the curvature coordinate $R = r_+$, where

$$r_+ := M + \sqrt{M^2 - Q^2}$$

(2.5)

is the radius of the outer horizon of the hole, intersect each other. In Ref.[5], the point $r = 0$ corresponds to the bifurcation point, and $r \to \infty$ corresponds to the right hand side infinity in the conformal diagram. Thus, the spacelike hypersurfaces where the time coordinate $t$ is a constant begin from the bifurcation point and extend to the asymptotic infinity. Hence, these hypersurfaces can never cross the apparent horizon $R = r_+$, and the investigation performed in Ref.[5] is restricted to the exterior regions of the hole.

In this paper, we take the range of $r$ to be from negative to positive infinity. The region $r \to -\infty$ corresponds to the left, and the region $r \to +\infty$ to the right hand side asymptotic infinity in the conformal diagram. The spacelike hypersurfaces where the time $t$ is a constant, are assumed to go through the interior regions of the hole in arbitrary ways. However, the requirement that the hypersurfaces $t = \text{constant}$ are spacelike and extendible from left to right asymptotic infinities imposes an important restriction: if we look at the conformal diagram of the Reissner-Nordström spacetime, we find that it is not possible to push these hypersurfaces beyond the inner horizons, where

$$R = r_- := M - \sqrt{M^2 - Q^2},$$

(2.6)

since otherwise the hypersurfaces would necessarily fail to be spacelike. Hence, our study of the Hamiltonian dynamics of Reissner-Nordström spacetimes must be restricted to include, in addition to the left and right exterior regions of the hole, only such an interior region of the hole which lies between two successive $R = r_-$-hypersurfaces in the conformal diagram. Our spacelike hypersurface $t = \text{constant}$ therefore begins its life at the past $R = r_-$-hypersurface, then goes through the bifurcation point $R = r_+$, and finally ends its life at the future $R = r_-$-hypersurface. Bearing this restriction in mind, we shall now go into the Hamiltonian dynamics of Reissner-Nordström spacetimes. In most technical details, the discussion goes exactly as in Ref.[5]. The major difference comes from the boundary terms.

To begin with, one writes the action. For the Einstein-Maxwell theory the action is, in general,

$$S = \frac{1}{16\pi} \int d^4 x \sqrt{-g} \left( (4) R - F_{\mu\nu} \right) + \text{(boundary terms)}. \quad (2.7)$$

In this equation, the integration is performed over the whole spacetime under consideration, $(4) R$ is the four-dimensional Riemann scalar of spacetime, and

$$F_{\mu\nu} := \partial_\mu A_\nu - \partial_\nu A_\mu$$

(2.8)
is the electromagnetic field tensor. As it was shown in Ref.[5], the action takes, when the ansätze (2.3) and (2.4) are used, the form:

\[ S = S_\Sigma + S_{\partial \Sigma}, \]  

(2.9)

where

\[ S_\Sigma := \int dt \int_{-\infty}^{+\infty} dr \left( P_R \dot{R} + P_\Lambda \dot{\Lambda} + P_\Gamma \dot{\Gamma} - N\mathcal{H} - N^r \mathcal{H}_r - \hat{\phi} G \right), \]  

(2.10)

and \( S_{\partial \Sigma} \) is a boundary term corresponding to the left and right hand side asymptotic infinities. In Eq.(2.10), the overdot means time derivative, and \( P_R, P_\Lambda, P_\Gamma \), respectively, are the canonical momenta conjugate to the variables \( R, \Lambda, \Gamma \). \( \mathcal{H}, \mathcal{H}_r \) and \( G \), respectively, are the Hamiltonian, diffeomorphism and Gaussian constraints. For the explicit expressions for the momenta \( P_R \) and \( P_\Lambda \) as well as the constraints \( \mathcal{H} \) and \( \mathcal{H}_r \) we refer the reader to the Ref.[5]. Of particular interest is the momentum \( P_\Gamma \). It can be written as:

\[ P_\Gamma = N^{-1} \Lambda^{-1} R^2 (\dot{\Gamma} - \phi'), \]  

(2.11)

where \( \phi' := \frac{\partial}{\partial \Gamma} \phi \). The Gaussian constraint is

\[ G = -P_\Gamma', \]  

(2.12)

and the quantity \( \tilde{\phi} \), which appears as a Lagrangian multiplier of the theory, is defined as

\[ \tilde{\phi} := \phi - N^r \Gamma. \]  

(2.13)

As to the momenta \( P_R \) and \( P_\Lambda \), and the constraints \( \mathcal{H} \) and \( \mathcal{H}_r \), it should be noted that since the analysis made in Ref.[5] was thermodynamically motivated, the authors of that paper included a negative cosmological constant to improve the convergence of certain integrals. In the absence of the cosmological constant, however, the expressions for the momenta and the constraints can be obtained from those in Ref.[5] simply by putting the cosmological constant to zero.

What about the boundary term \( S_{\partial \Sigma} \) corresponding to the asymptotic spacelike infinities? To find an expression to the boundary term one must first specify the fall-off conditions for the canonical variables and the Lagrangian multipliers of the theory. We shall adopt the same asymptotic boundary conditions as in Ref.[5] in the absence of a cosmological constant:

\[ \Lambda(t, r) = 1 + M_\pm(t) |r|^{-1} + O^\infty(|r|^{-1-\epsilon}), \]  

(2.14.a)

\[ R(t, r) = |r| + O^\infty(|r|^{-\epsilon}), \]  

(2.14.b)

\[ P_\Lambda(t, r) = O(|r|^{-\epsilon}), \]  

(2.14.c)

\[ P_R(t, r) = O^\infty(|r|^{-1-\epsilon}), \]  

(2.14.d)

\[ N(t, r) = N_\pm(t) + O^\infty(|r|^{-\epsilon}), \]  

(2.14.e)

\[ N^r(t, r) = O^\infty(|r|^{-\epsilon}), \]  

(2.14.f)

\[ \Gamma(t, r) = O^\infty(|r|^{-1-\epsilon}), \]  

(2.14.g)

\[ P_\Gamma(t, r) = Q_\pm(t) + O^\infty(|r|^{-\epsilon}), \]  

(2.14.h)

\[ \tilde{\phi}(t, r) = \tilde{\phi}_\pm(t) + O^\infty(|r|^{-\epsilon}), \]  

(2.14.i)
when \( r \to \pm \infty \). In these equations, \( \epsilon > 0 \), and \( O(|r|^{-\epsilon}) \) denotes a term that falls off at infinity as \( |r|^{-\epsilon} \) and whose derivatives with respect to \( |r| \) fall accordingly as \( |r|^{-\epsilon-k} \), where \( k = 1, 2, 3, \ldots \). Our fall-off conditions ensure that spacetime is asymptotically flat at asymptotic infinities. Of particular interest is the fall-off condition \((2.14.f)\), which states that the shift vanishes at infinities. This means that our asymptotic coordinate systems are at rest with respect to the hole.

The boundary term \( S_{\partial \Sigma} \) has now two parts. The first of them is:

\[
S_{\partial \Sigma}^{ADM} := - \int dt \left( N_+(t)E_+(t) + N_-(t)E_-(t) \right),
\]

(2.15)

where

\[
E_{\pm}(t) := \frac{1}{16\pi} \int_{S^2_\pm} dS^a \delta^{bc}(g_{ab,c} - g_{bc,a}),
\]

(2.16)

are the ADM energies at the right and left infinities. In Eq.(2.16), the integral is taken over two-spheres \( |r| = \text{constant} \) at infinities. The term (2.15) is needed in order to cancel the terms appearing at spatial infinities when the action \( S_{\Sigma} \) is varied with respect to the variables \( R \) and \( \Lambda \) and their canonical momenta.

One can see from Eqs.(2.14.a) and (2.14.b) that the boundary term \( S_{\partial \Sigma}^{ADM} \) in Eq.(2.15) takes the form:

\[
S_{\partial \Sigma}^{ADM} = - \int dt \left( N_+(t)M_+(t) + N_-(t)M_-(t) \right).
\]

(2.17)

The second part of the boundary term is related to electromagnetism. If one writes down all of the electromagnetic terms in the action \( S_{\Sigma} \) of Eq.(2.10), one finds that the only term involving spatial derivatives of the quantities \( \Gamma, \phi \) and \( P_T \) is:

\[
\int dt \int_{-\infty}^{+\infty} dr P_T' \phi.
\]

Varying the action \( S_{\Sigma} \) with respect to \( P_T \) therefore brings along a term

\[
\int dt \int_{-\infty}^{+\infty} dr (\delta P_T \phi)'.
\]

Unless the variation of \( P_T \) is assumed to vanish at spatial infinities, we are thus compelled to bring along a boundary term

\[
S_{\partial \Sigma}^{em} = - \int dt \left( Q_+(t)\phi_+(t) - Q_-(t)\phi_-(t) \right).
\]

(2.18)

The whole boundary term \( S_{\partial \Sigma} \) is the sum the terms \( S_{\partial \Sigma}^{ADM} \) and \( S_{\partial \Sigma}^{em} \), and we find that the whole action finally becomes to:

\[
S = \int dt \int_{-\infty}^{+\infty} dr \left( P_R\dot{R} + P_A\dot{A} + P_T\dot{T} - N\mathcal{H} - N^r\mathcal{H}_r - \dot{\phi}G \right)
\]

\[
- \int dt \left( N_+M_+ + N_-M_- + \phi_+Q_+ + \phi_-Q_- \right).
\]

(2.19)
As it was mentioned before, the Reissner-Nordström solution is the only spherically symmetric, asymptotically flat solution to Einstein-Maxwell equations. This solution, moreover, is completely characterized by the mass and the charge parameters $M$ and $Q$. It was found in Ref. [5] that one can read off these parameters from any small piece of spacetime if one knows the values of the phase space coordinates $\Lambda, R, \Gamma, P_\Lambda, P_R$ and $P_\Gamma$ in that region. More precisely,

$$M = \frac{1}{2} \frac{P_\Lambda^2}{R} + \frac{1}{2} \frac{P_\Gamma^2}{R} + \frac{1}{2} R - \frac{1}{2} \frac{R (R')^2}{\Lambda^2},$$  \hspace{1cm} (2.20.a)  

$$Q = P_\Gamma.$$  \hspace{1cm} (2.20.b)

Although the derivation of these equations in Ref. [5] was performed in the exterior region of the hole, the same arguments as in Ref. [5] go through in any region of spacetime. In curvature coordinates, the reader may check the validity of Eq. (2.20.b) by using Eqs. (2.2), (2.4) and (2.11).

In addition to the parameters $M$ and $Q$, one can also read off how the spacelike hypersurface has been embedded into the Reissner-Nordström spacetime from the values of the phase space coordinates in any point of the hypersurface. In Ref. [5] it was found that if the curvature coordinate $T$ is considered as an arbitrary function of the coordinates $t$ and $r$, then

$$-T' = R^{-1} F^{-1} \Lambda P_\Lambda P_\Gamma,$$  \hspace{1cm} (2.21)

where

$$F := \left( \frac{R'}{\Lambda} \right)^2 - \left( \frac{P_\Lambda}{R} \right)^2.$$  \hspace{1cm} (2.22)

From Eq. (2.21) one can solve $T$ as a function of $r$ provided that one knows $T$ for one value of $r$. Keeping $t$ as a constant one can read off the position of the $t = constant$ hypersurface of spacetime in the Reissner-Nordström manifold from that solution.

The solution (2.2) to Maxwell’s equations involves a specific fixing of the electromagnetic gauge. The general spherically symmetric solution can be written as

$$A_T = \frac{Q}{R} + \frac{\partial}{\partial T} \xi;$$  \hspace{1cm} (2.23.a)  

$$A_R = \frac{\partial}{\partial R} \xi;$$  \hspace{1cm} (2.23.b)

where $\xi$ is an arbitrary function of $T$ and $R$. Hence, the general solution in the coordinates $t$ and $r$ is

$$A_t = \frac{Q}{R} T' + \xi;$$  \hspace{1cm} (2.24.a)  

$$A_r = \frac{Q}{R} T' + \xi'.$$  \hspace{1cm} (2.24.b)
Comparing Eqs. (2.4.b), (2.20.b), (2.21) and (2.24.b) one finds that:
\[ \xi' = \Gamma + R^{-2}F^{-1}F_{\Lambda}P_{\Lambda}P_{\Gamma}. \] (2.25)

If one knows the gauge function \( \xi \) for one value of \( r \), one can solve \( \xi \) for any \( r \) from Eq. (2.25). Hence, information about the choice of the electromagnetic gauge is carried by the phase space coordinates of the theory.

At first sight, there might seem to be a difficulty with the horizons \( R = r_{\pm} \), because it follows from Eqs. (2.20) and (2.22) that
\[ F = 1 - \frac{2M}{R} + \frac{Q^2}{R^2}. \] (2.26)

Hence, the quantities \( T' \) and \( \xi' \) appear to have a singularity when \( R = r_{\pm} \). This problem was considered in the case \( Q = 0 \) by Kuchař[10]. His conclusion was that one can nevertheless propagate through horizon one’s knowledge of \( T \) by using Eq. (2.21). The very same arguments can be applied also when \( Q \neq 0 \). Hence, the event horizon does not pose a problem for our knowledge of \( T \) and \( \xi \) as functions of \( r \).

In Ref.[5], the idea was to ”forget” for a moment the fact that \( M \) and \( Q \) are the mass and the charge parameters of the Reissner-Nordström black hole, and instead consider \( M \) and \( Q \) as arbitrary functions of \( r \) and \( t \), whose relationship with the phase space coordinates \( R, \Lambda, \Gamma, P_R, P_\Lambda \) and \( P_\Gamma \) is given by Eq.(2.20). A canonical transformation was then performed from the original phase space coordinates to the new canonical variables such that \( M \) and \( Q \) were considered as the configuration variables, and the quantities \(-T'\) and \(-\xi'\) defined in terms of the original phase space coordinates as in Eqs.(2.21) and (2.25) were taken to be the corresponding canonical momenta \( P_M \) and \( P_Q \). A new canonical momentum \( P_R \) conjugate to \( R \) was also defined such that the whole transformation from the ”old” to the ”new” phase space variables was canonical.

What about the constraints? Varying the action (2.19) with respect to the functions \( N, N^r \) and \( \phi \) yields the following constraint equations:
\[ \mathcal{H} = 0, \] (2.27.a)
\[ \mathcal{H}_r = 0, \] (2.27.b)
\[ G = 0. \] (2.27.c)

It should be noted that one is not allowed to vary the action with respect to the functions \( N_{\pm} \) and \( \phi_{\pm} \) but these functions should be kept as prescribed functions of the time \( t \). That is because varying the action with respect to \( N_{\pm} \) would imply vanishing ADM mass—and hence flatness—of spacetime; varying action with respect to \( \phi_{\pm} \), in turn, would imply vanishing electric charge.

Using Eqs.(2.12) and (2.20), and the expressions of Ref.[5] for the Hamiltonian and the diffeomorphism constraints \( \mathcal{H} \) and \( \mathcal{H}_r \) in the absence of the cosmological
constant, one finds that the spatial derivatives of the mass and the charge functions $M$ and $Q$ can be written in terms of the constraints:

\begin{align}
M' &= -\Lambda^{-1}R'H - \Lambda^{-1}R^{-1}P_\lambda H_r + (\Lambda^{-1}R^{-1} \Gamma P_T - R^{-1} P_T)G, \quad (2.28.a) \\
Q' &= -G. \quad (2.28.b)
\end{align}

Hence, the constraint equations (2.27) imply that:

\begin{align}
M' &= 0, \quad (2.29.a) \\
Q' &= 0. \quad (2.29.b)
\end{align}

In other words, the mass and charge functions $M$ and $Q$ are constants with respect to $r$. Moreover, it was noted in Ref.[5] that

$$\mathcal{H}_r = P_M M' + P_Q Q' + P_R R'.$$

Eqs.(2.27) and (2.29) therefore imply:

$$P_R = 0. \quad (2.31)$$

Hence, the action written in terms of the variables $M$, $Q$, $R$ and their canonical momenta,

$$S = \int dt \int_{-\infty}^{+\infty} dr \left( P_M \dot{M} + P_Q \dot{Q} + P_R \dot{R} - N\mathcal{H} - N'\mathcal{H}_r - \bar{\phi} G \right)$$

\begin{align}
&- \int dt \left( N_+ M_+ + N_- M_- + \bar{\phi}_+ Q_+ - \bar{\phi}_- Q_- \right),
\end{align}

takes, when the constraint equations (2.27) are satisfied, the form

$$S = \int dt \left[ p_m \dot{m} + p_q \dot{q} - (N_+ + N_-) m - (\bar{\phi}_+ + \bar{\phi}_-) q \right], \quad (2.33)$$

where we have defined:

\begin{align}
m(t) &:= M(t, r), \quad (2.34.a) \\
q(t) &:= Q(t, r), \quad (2.34.b) \\
p_m(t) &:= \int_{-\infty}^{+\infty} dr P_M(t, r), \quad (2.34.c) \\
p_q(t) &:= \int_{-\infty}^{+\infty} dr P_Q(t, r). \quad (2.34.d)
\end{align}

When the constraint equations are satisfied, our infinite-dimensional phase space is thus reduced to a phase space which is spanned by just four canonical coordinates. These canonical coordinates are the variables $m$ and $q$ which can be identified as
the mass \( M \) and the charge \( Q \) of the hole when Einstein’s equations are satisfied—and the corresponding canonical momenta \( p_m \) and \( p_q \).

The momenta \( p_m \) and \( p_q \) have an interesting interpretation: because we have defined \( P_M := -T' \) and \( P_Q := -\xi' \), we find that \( p_m \) is simply the difference in the Minkowski time \( T \) at the left and right infinities on the spacelike hypersurface \( t = \text{constant} \). The momentum \( p_q \), in turn, tells the difference between the choices of the gauge function \( \xi \) at the asymptotic infinities of spacetime.

One can read off from Eq. (2.33) the true reduced Hamiltonian of the Reissner-Nordström hole in terms of the variables \( m \) and \( q \):

\[
H = (N_+ + N_-)m + (\dot{\phi}_+ - \dot{\phi}_-)q. \tag{2.35}
\]

The Hamiltonian equations of motion are therefore

\[
\dot{m} = \frac{\partial H}{\partial p_m} = 0, \tag{2.36.a}
\]

\[
\dot{q} = \frac{\partial H}{\partial p_q} = 0, \tag{2.36.b}
\]

\[
\dot{p}_m = -\frac{\partial H}{\partial m} = -(N_+ + N_-), \tag{2.36.c}
\]

\[
\dot{p}_q = -\frac{\partial H}{\partial q} = -(\dot{\phi}_+ - \dot{\phi}_-). \tag{2.36.d}
\]

As one can see from Eqs. (2.36.a) and (2.36.b), the mass \( m \) and the charge \( q \) are constants of motion of the system.

### 3. Quantum Theory with Charge as an External Parameter

After finding in Eq. (2.35) an expression to the classical reduced Hamiltonian of Reissner-Nordström spacetimes in terms of the variables \( m \) and \( q \) and the corresponding canonical momenta, we shall now proceed into a Hamiltonian quantization of such spacetimes. The aim of all physical theories, at least in principle, is to be able to predict the possible outcomes of measurements. When we talk about measurements, however, we need a reference to an observer performing these measurements: the possible outcomes of measurements are the possible outcomes of measurements as such as they can be measured, in principle, by a certain observer. The properties of the observer, in turn, motivate the structure of the theory.\(^1\)

---

\(^1\) Recently, an interesting point of view to the interpretation of quantum mechanics was suggested by Rovelli[11]. His idea was, in rough terms, that one is not justified to talk about any absolute quantum state of a physical system. Instead, one should talk about a quantum state relative to some observer. This idea has given some inspiration to the point of view adopted in this paper.
In this paper we choose the observer in a most simple manner: our observer is at the right hand side asymptotic infinity in the conformal diagram, at rest with respect to the Reissner-Nordström hole. Our aim is to construct a quantum theory of the Reissner-Nordström spacetime from the point of view of such an observer. To this end, we choose the lapse functions $N_\pm(t)$ at asymptotic infinities as:

\begin{align}
N_+(t) &\equiv 1, \\
N_-(t) &\equiv 0.
\end{align}

(3.1.a) (3.1.b)

In other words, we have chosen the time coordinate at the right infinity to be the proper time of our observer, and we have "frozen" the time evolution at the left infinity. This can be considered justified on grounds that our observer can make observations at just one infinity.

The next task is to fix the functions $\tilde{\phi}_\pm(t)$. As one can see from Eqs. (2.13) and (2.14), the functions $\tilde{\phi}_\pm(t)$ are just the electric potentials at asymptotic infinities. It is customary to choose the zero point of the electric potential in such a way that at asymptotic infinities the electric potential vanishes. As one can see from Eq. (2.2), this choice is compatible with the Reissner-Nordström solution to Einstein-Maxwell equations. Hence, we choose:

\begin{align}
\tilde{\phi}_+(t) &\equiv \tilde{\phi}_-(t) \equiv 0.
\end{align}

(3.2)

With these choices of the lapse functions and the electric potentials the reduced Hamiltonian (2.35) takes the form:

\begin{align}
H = m.
\end{align}

(3.3)

Because of that, the numerical value of our Hamiltonian is just the mass $M$ of the hole. This mass includes, from the point of view of our observer, all the energy of the system, gravitational as well as electromagnetic.

Now, one could, of course, use the variables $m$ and $q$ and their canonical momenta as the phase space coordinates of the system, and construct a Hamiltonian quantum theory of the Reissner-Nordström hole based on the use of these coordinates. There is, however, a very grave disadvantage with these coordinates: They describe the static aspects of the black hole spacetime only. Indeed, we saw in Eq. (2.36) that the variables $m$ and $q$ are constants of motion of the system. However, there is dynamics in the Reissner-Nordström spacetime in the sense that in the region where $r_+ < R < r_+$, there is no timelike Killing vector field orthogonal to a spacelike hypersurface. Our task is to find such phase space coordinates which describe the dynamics of spacetime in a natural manner.

When choosing the phase space coordinates we again refer to the properties of our observer. Our observer sees the exterior regions of the black hole as static, and he is an inertial observer. These properties prompt us to choose the phase space coordinates in such a manner that when the classical equations of motion are satisfied, all the dynamics is, in a certain sense, confined inside the apparent
horizon $R = r_+$ of the hole. Moreover, as we shall see in a moment, the choice of the phase space coordinates describing the dynamics of spacetime is related to the choice of slicing of spacetime into space and time. We choose a slicing where the proper time of an observer in a radial free fall through the bifurcation two-sphere coincides with the proper time of our faraway observer at rest. On grounds of the Principle of Equivalence one may view these kind of slicings to be in a preferred position in relating the physical properties of the black hole interior to the physics observed by our faraway observer.

The new phase space coordinates describing the dynamics of the black hole spacetime can be obtained from the phase space coordinates $m, q, p_m$ and $p_q$ by means of an appropriate canonical transformation. To make things simple, we shall in this Section consider the charge $q$ as a mere external parameter of the system, having a fixed value $Q$. In the next Section, the charge will be considered as a dynamical variable. Hence, the dimension of the phase space of our system is, in this Section, just two.

As in Ref.[8], we now perform a canonical transformation from the phase space variables $(m, p_m)$ to the new phase space variables $(a, p_a)$ such that the relationship between the "old" and the "new" phase space variables is ($q$ is now a constant, which we denote by $Q$):

$$|p_m| = \sqrt{2ma - a^2 - Q^2} + m \sin^{-1}\left(\frac{m-a}{\sqrt{m^2 - Q^2}}\right) + \frac{1}{2}\pi m,$$  \hspace{1cm} (3.4.a) 

$$p_a = sgn(p_m) \sqrt{2ma - a^2 - Q^2},$$  \hspace{1cm} (3.4.b) 

and we have imposed by hand a restriction:

$$-\pi m \leq p_m \leq \pi m. \hspace{1cm} (3.5)$$

With the restriction (3.5) the transformation (3.4) is well-defined and one-to-one. It follows from Eq.(3.4.b) that

$$m = \frac{p_a^2}{2a} + \frac{1}{2}a + \frac{Q^2}{2a}. \hspace{1cm} (3.6)$$

If one substitutes this expression for $m$ to Eq.(3.4.a), one gets $p_m$ in terms of $a$ and $p_a$. One finds that the fundamental Poisson brackets between $m$ and $p_m$ are preserved invariant, and hence the transformation (3.4) is canonical.

Eqs.(3.3) and (3.6) imply that the classical Hamiltonian takes, in terms of the variables $a$ and $p_a$, a form:

$$H = \frac{p_a^2}{2a} + \frac{1}{2}a + \frac{Q^2}{2a}. \hspace{1cm} (3.7)$$

The geometrical interpretation of the variable $a$ is extremely easy to find. We first write the Hamiltonian equation of motion for $a$:

$$\dot{a} = \frac{\partial H}{\partial p_a} = \frac{p_a}{a}, \hspace{1cm} (3.8)$$
and it follows from Eq.(3.6) and the fact that \( m = M \) when the classical equations of motion are satisfied that the equation of motion for \( a \) is

\[
\dot{a}^2 = \frac{2M}{a} - 1 - \frac{Q^2}{a^2}.
\]  

(3.9)

One can see from the Reissner-Nordström metric in Eq.(2.1) that the equation of motion for an observer in a radial free fall through the bifurcation two-sphere is

\[
\dot{R}^2 = \frac{2M}{R} - 1 - \frac{Q^2}{R^2},
\]  

(3.10)

where the overdot means proper time derivative. As one can see, Eqs.(3.9) and (3.10) are identical. Hence, we can interpret \( a \) as the radius of the wormhole throat of the Reissner-Nordström black hole, from the point of view of an observer in a radial free fall through the bifurcation two-sphere. Moreover, we see from Eq.(3.9) that \( a \) is confined to be, classically, within the region \([r_-, r_+]\). In other words, our variable \( a \) "lives" only within the inner and outer horizons of the Reissner-Nordström black hole, and this is precisely the region in which it is impossible to find a time coordinate in such a way that spacetime with respect to that time coordinate would be static. Hence, both of the requirements we posed for our phase space coordinates are satisfied: dynamics is confined inside the apparent horizon and the time coordinate on the wormhole throat is the proper time of an observer in a radial free fall.

With the interpretation explained above, the restriction (3.5) becomes understandable. One can see from Eq.(2.36,c) that when the lapse functions \( N_\pm \) at asymptotic infinities are chosen as in Eq.(3.1), the canonical momentum \( p_m \) conjugate to \( m \) is \(-t + constant\), where \( t \) is the time coordinate of our asymptotic observer. Now, the transformation (3.4) involves an identification of the time coordinate \( t \) with the proper time of a freely falling observer on the throat. However, as it was noted in the beginning of Section 2, it is impossible to to push the spacelike hypersurfaces \( t = constant \) beyond the \( R = r_- \) hypersurfaces in the conformal diagram. The proper time a freely falling observer needs to fall from the past \( R = r_- \) hypersurface to the future \( R = r_- \) hypersurface through the bifurcation two-sphere is, as it can be seen from Eq.(3.10),

\[
\Delta t = 2 \int_{r_-}^{r_+} \frac{R' \, dR'}{\sqrt{2MR' - R'^2 - Q^2}} = 2\pi M,
\]  

(3.11)

and hence the restriction (3.5) is needed. As one can see from Eq.(3.4,a), \(|p_m| = 0\), when \( a = r_+ \), and \(|p_m| = \pi M\), when \( a = r_- \). We have chosen \( p_m \) to be positive, when the hypersurface \( t = constant \) lies between the past \( R = r_- \) hypersurface and the bifurcation point, and negative when that hypersurface lies between the bifurcation point and the future \( R = r_- \) hypersurface.

As to the classical Hamiltonian theory, the only thing one still needs to check is, whether there exist such foliations of the Reissner-Nordström spacetime where
the Minkowski time $t$ at asymptotic infinity and the proper time of a freely falling observer at the throat through the bifurcation two-sphere really are the one and the same time coordinate. It is easy to see that time coordinates determining this sort of foliations really exist. As a concrete example, consider a generalization of the so called Novikov coordinates in the Schwarzschild geometry[7]. More precisely, one takes a collection of freely falling test particles whose initial three-velocity with respect to the curvature coordinates is zero when $T = 0$. If one relates the radial coordinate of spacetime to the positions of these test particles when $T = 0$, and takes the time coordinate to be at every spacetime point the proper time of the test particle falling through that point, one finds that the time coordinate of our distant observer at rest, the Minkowski time, and the proper time of a freely falling observer in the throat are the one and the same time coordinate. It should be noted, however, that all foliations in which the proper time in the throat and the asymptotic Minkowski time are identified, are incomplete, since such foliations, in addition of failing to cover regions outside the past and future $R = r_-$ hypersurfaces, also fail to cover the whole exterior regions of the hole. More precisely, these foliations are valid only when $-\pi M \leq t \leq \pi M$.

After finding a classical Hamiltonian reflecting the dynamical properties of Reissner-Nordström spacetimes, we are now prepared to go into the Hamiltonian quantization of such spacetimes. In what follows, we shall specify to a particular class of Hamiltonian quantum theories. More precisely, we choose our Hilbert space to be the space $L^2(\mathbb{R}^+, a^s da)$ with the inner product

$$
\langle \psi_1 | \psi_2 \rangle := \int_0^\infty \psi_1^* (a) \psi_2 (a) a^s da,
$$

where $s$ is some real number. Through the substitution $p_a \rightarrow -id/da$ we replace the classical Hamiltonian $H$ of Eq.(3.7) with the corresponding symmetric Hamiltonian operator

$$
\hat{H} := -\frac{1}{2} a^{-s} \frac{d}{da} \left(a^{s-1} \frac{d}{da} \right) + \frac{1}{2} a + \frac{Q^2}{2a}.
$$

Since the numerical value of the classical Hamiltonian $H$ is the total (ADM) energy of the Reissner-Nordström hole, we can view the eigenvalue equation

$$
\hat{H} \psi(a) = E \psi(a)
$$

as an eigenvalue equation for the total energy of the hole, from the point of view of a distant observer at rest.

Before going into the detailed analysis of Eq.(3.14), let us pause for a moment to investigate some qualitative aspects of that equation. One finds, by substituting $M$ for $E$ that Eq.(3.14) can be written in the form:

$$
a^{-s} \frac{d}{da} \left(a^{s-1} \frac{d}{da} \right) \psi(a) = \left( \frac{Q^2}{a} + a - 2M \right) \psi(a).
$$

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As one can see, the function \((\frac{Q^2}{a} + a - 2M)\) is negative, when \(r_- < a < r_+\), and positive (or zero) elsewhere. Semiclassically, one may therefore expect oscillating behaviour from the wave function \(\psi(a)\) when \(r_- < a < r_+\), and exponential behaviour elsewhere. Hence, our system is somewhat analogous to a particle in a potential well such that \(a\) is confined, classically, between the outer and the inner horizons of the black hole. What happens semiclassically is that the wave packet corresponding to the variable \(a\) is reflected from the future inner horizon. As a result we get, when the hole is in a stationary state, a standing wave between the outer and inner horizons. Thus, the classical incompleteness, associated with the fact that our foliation is valid only when \(-\pi M \leq t \leq \pi M\), is removed by quantum mechanics: in a stationary state there are no propagating wave packets between the horizons, and our quantum theory is therefore valid in any moment of time.

Let us now go into the detailed analysis of the eigenvalue equation (3.14). To begin with, we see, as in Ref.[8], that if we denote

\[
\begin{align*}
  x &:= a^{3/2}, \\
  \psi &:= x^{-r} \chi(x),
\end{align*}
\]

where we have defined

\[
\begin{align*}
  r &:= \frac{2s - 1}{6}; \quad s \geq 2, \\
  r &:= \frac{7 - 2s}{6}; \quad s < 2,
\end{align*}
\]

then Eq.(3.14) takes the form:

\[
\frac{9}{8} \left[ -\frac{d^2}{dx^2} + \frac{r(r - 1)}{x^2} + \frac{4}{9} \left( x^{2/3} + \frac{Q^2}{x^{2/3}} \right) \right] \chi(x) = E\chi(x).
\]

The Hilbert space is now \(L^2(\mathbb{R}^+, dx)\) with the inner product

\[
\langle \chi_1 | \chi_2 \rangle := \int_0^\infty \chi_1^*(x) \chi_2(x) \, dx.
\]

It was shown in Ref.[8] that the energy spectrum in Eq.(3.18) is discrete, bounded below, and can be made positive. From the physical point of view, the semi-boundedness and positivity (in some cases) of the spectrum are very satisfying results: the semi-boundedness of the spectrum implies that one cannot extract an infinite amount of energy from the system, whereas the positivity of the spectrum is in harmony with the well-known positive energy theorems of general relativity which state, roughly speaking, that the ADM energy of spacetime is always positive or zero when Einstein’s equations are satisfied.

However, one can prove even more than that. As it is shown in Appendix B, the eigenvalue equation (3.18) implies that when \(r \geq 3/2\), the eigenvalues of the quantity

\[
E^2 - Q^2
\]
are strictly positive, and when $1/2 \leq r < 3/2$, the eigenvalues of the quantity $E^2 - Q^2$ can, again, be made positive by means of an appropriate choice of the boundary conditions of the wave function $\chi(x)$ at the point $x = 0$ or, more precisely, by means of an appropriate choice of a self-adjoint extension. Moreover, the WKB analysis of Eq.(3.18) performed in Appendix A yields the result that when $|Q| \gg 1$ and $E^2 - Q^2 \gg 1$, such that $r_- \geq 1$, the WKB eigenenergies $E_n$ have a property

$$E_n^2 - Q^2 \sim 2n + 1 + o(1),$$

where $n$ is an integer and $o(1)$ denotes a term that vanishes asymptotically for large $n$. We have tested the accuracy of this WKB estimate numerically, and we have found that, up to the term 1 on the right hand side of Eq.(3.20), the WKB estimate (3.20) gives fairly accurate results even when $|Q|$ and $n$ are relatively small (i.e. of order ten).\(^1\) In other words, it seems that the eigenvalues of the quantity $\sqrt{E^2 - Q^2}$ are of the form $\sqrt{2n}$ in the semiclassical limit.

Now, how should we understand these results? As it was noted in the Introduction, the positivity of the spectrum of the quantity $E^2 - Q^2$ has an interesting consequence regarding Hawking radiation: if one thinks of Hawking radiation as an outcome of a chain of transitions from higher to lower energy eigenstates, the positivity of the spectrum of $E^2 - Q^2$ implies that a non-extreme Reissner-Nordstrom black hole with non-zero temperature can never become, through Hawking radiation, an extreme black hole with zero temperature—a result which is in harmony with both the third law of thermodynamics and the qualitative difference between extreme and non-extreme black holes. One may consider this result as a strong argument in favour of our choice of the phase space coordinates describing the dynamics of Reissner-Nordstrom spacetimes.

How about Eq.(3.20)? Since the area of the apparent horizon of the Reissner-Nordstrom hole is, in natural units,

$$A_+ := 4\pi (M + \sqrt{M^2 - Q^2})^2,$$

we find that, if $|Q|$ is negligible when compared to $M$, Eq.(3.20) yields the result that the possible eigenvalues of the area $A_+$ are, in the semiclassical limit,

$$A_{+n} \approx 32\pi nl_{Pl}^2 + \text{constant},$$

where $l_{Pl}^2 := (\hbar G/c^3)^{1/2}$ is the Planck length. This result is the same as the one obtained in Ref.[8] for a Schwarzschild black hole. Moreover, the result is in harmony with the proposal (1.3), first made by Bekenstein in 1974[9] and since then revived by many authors[12-32] on the discrete area spectrum of a black hole.

However, if we look at the expression (3.21) for the horizon area, we find that Eq.(3.20) does not produce the proposal (1.3) in the general case. The solution of this dilemma lies in the fact that the Reissner-Nordstrom black hole has, actually,

\(^1\) We thank Matias Aunola for performing this numerical analysis to us.
two horizons: the inner and the outer horizons. The sum of the areas of these two horizons—which we shall call, for the sake of convenience, the *total* area of the horizons—is

\[ A^{\text{tot}} := A_+ + A_-, \]  

(3.23)

where \( A_+ \) is the area of the outer horizon as given in Eq.\((3.21)\), and

\[ A_- := 4\pi (M - \sqrt{M^2 - Q^2})^2 \]  

(3.24)

is the area of the inner horizon. We get

\[ A^{\text{tot}} = 16\pi (M^2 - Q^2) + 2A^{\text{ext}}, \]  

(3.25)

where

\[ A^{\text{ext}} := 4\pi Q^2 \]  

(3.26)

is the area of an extreme black hole. Eq.\((3.20)\) now gives the spectrum of the total area of the horizons:

\[ A_n^{\text{tot}} \sim 32\pi (n + \frac{1}{2}) \ell_P^2 + 2A^{\text{ext}} + o(1). \]  

(3.27)

As one can see, we have obtained a result which is closely related to the proposal \((1.3)\). However, it must be emphasized that Eq.\((3.27)\) is not exactly the same as Bekenstein’s proposal \((1.3)\): according to Bekenstein, the area of the *outer* horizon of the hole has the spectrum \((1.3)\), whereas we obtained the same spectrum, with \( \gamma = 32\pi \), for the *total* area of the horizons. Nevertheless, it is most interesting that, according to our model, the spectrum of the total area of the horizons of the Reissner-Nordström black hole is, up to the term \( 2A^{\text{ext}} \), exactly the same as that of the Schwarzschild black hole.

The WKB spectrum \((3.20)\) has yet another property which is of some interest. It follows from the mass formula of black holes that the ADM mass of a non-rotating electrovacuum black hole is, from the point of view of a distant observer at rest \([33,34]\),

\[ M = \frac{\kappa}{4\pi} A + \Phi Q. \]  

(3.28)

In this equation, \( \kappa \) is the surface gravity, \( A \) is the area of the (outer) horizon, and \( \Phi \) is the electric potential of the hole. For a Reissner-Nordström black hole we have

\[ \kappa = \frac{4\pi \sqrt{M^2 - Q^2}}{A}, \]  

(3.29.a)

\[ \Phi = Q \frac{1}{r_+}. \]  

(3.29.b)

Eq.\((3.20)\) now implies that the WKB eigenvalues of the quantity

\[ \frac{\kappa}{4\pi} A \]  

is
are of the form $\sqrt{2n}$. The physical interest of this result lies in its charge independence. In other words, the spectrum of the quantity $\kappa A/4\pi$ is the same for the Reissner-Nordström and Schwarzschild black holes.

4. Quantum Theory with Charge as a Dynamical Variable

In the previous Section we considered the electric charge of the Reissner-Nordström black hole as a mere external parameter, with no dynamics whatsoever. In other words, we quantized only the gravitational degrees of freedom of Reissner-Nordström spacetimes. The object of this Section is to extend our quantum theory of such spacetimes to include the electromagnetic degrees of freedom as well. The guiding principle in our search for appropriate canonical variables describing the dynamics of the electromagnetic field is that since our distant observer at rest observes the electromagnetic field outside the event horizon as static, all the dynamics of the electromagnetic field must be confined, classically, inside the horizon.

To find appropriate canonical variables, recall that when the classical equations of motion are satisfied, the only non-vanishing component of the electromagnetic potential $A_\mu$ is, in curvature coordinates, the component $A_T = Q/R$. Now, this component is static with respect to the time $T$ everywhere outside the horizon. However, since $R$ becomes a timelike coordinate when $r_- < R < r_+$, we find that $A_T$ necessarily has dynamics between the inner and the outer horizons of the Reissner-Nordström black hole. In terms of $A_T$ we can write the Reissner-Nordström metric (2.1) as

$$ds^2 = -\left(1 - \frac{2M}{R} + A_T^2\right) dT^2 + \frac{dR^2}{1 - \frac{2M}{R} + A_T^2} + R^2 d\Omega^2. \quad (4.1)$$

In what follows, we shall "forget" the explicit dependence of $A_T$ on $R$ and $Q$, and instead treat $A_T$ as an independent dynamical variable of our theory. However, in all our investigations we shall assume that $A_T$ is independent of $T$ or, more precisely,

$$\frac{\partial A_T}{\partial T} \equiv 0. \quad (4.2)$$

From this restriction it follows that we can treat $A_T$ as a function of an appropriate time coordinate only. Hence, our phase space, which will be spanned by the throat variables $(a, p_a)$ and the electromagnetic variables $(A_T, p_{A_T})$, where $p_{A_T}$ is the canonical momentum conjugate to $A_T$, will be four-dimensional, which is in harmony with the results of Ref.[5] reviewed in Section 2. To complete the classical theory we must just find a canonical transformation from the "old" phase space variables $(m, p_m, q, p_q)$ to the "new" phase space variables $(a, p_a, A_T, p_{A_T})$, and write the classical Hamiltonian, from the point of view of our distant observer at rest, in terms of the variables $a, p_a, A_T$ and $p_{A_T}$. 

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To find a clue to the expression of the classical Hamiltonian in terms of the gravitational and electromagnetic variables, let us write the Hamiltonian constraint on the timelike geodesic going through the bifurcation two-sphere of the Reissner-Nordström spacetime, in the foliation used in Section 3. In that foliation the spacetime metric can be written as

\[ ds^2 = -dt^2 + \left( \frac{2M}{a} - 1 - A_T^2 \right) d\tau^2 + a^2 d\Omega^2. \]  

(4.3)

As one can see, \( T \) is now a spatial coordinate of spacetime. Hence, we can identify the expression with the variable \( \Lambda(r, t) \) of Section 2. Moreover, we can identify \( A_T \) with \( \Gamma \). The variable \( \phi(r, t) \) is assumed to vanish. With these identifications, and treating \( M \) as a constant, we find that the Hamiltonian constraint of Ref.\cite{5} written in terms of \( \Lambda, R \) and \( \Gamma \) and their time derivatives can be written in terms of \( a \) and \( A_T \) and their time derivatives as

\[ \mathcal{H} = \left( \frac{2M}{a} - 1 - A_T^2 \right)^{-1/2} \left[ \frac{1}{2} (1 + A_T^2) \dot{a}^2 + a A_T \dot{\Lambda}_T \dot{a} + \frac{1}{2} a^2 \ddot{A}_T^2 - \frac{M}{a} + \frac{1}{2} (1 + A_T^2) \right] = 0. \]  

(4.4)

From this equation one can solve \( M \):

\[ M = \frac{1}{2} a (1 + A_T^2) \dot{a}^2 + a^2 A_T \dot{A}_T^2 \dot{a} + \frac{1}{2} a (1 + A_T^2). \]  

(4.5)

It is easy to see that if one substitutes

\[ A_T = \frac{Q}{a}, \]  

(4.6)

and keeps \( Q \) as a constant, one gets

\[ M = \frac{1}{2} a \dot{a}^2 + \frac{1}{2} a + \frac{Q^2}{2a}. \]  

(4.7)

Hence, if one interprets the right hand side of Eq.\!(4.7)\! as the classical Hamiltonian of the system, one gets, with the substitution \!(4.6)\!, the same Hamiltonian as in Eq.\!(3.7)\!.

At this point we define a new variable

\[ b := a A_T. \]  

(4.8)

As a result, Eq.\!(4.5)\! becomes simplified to

\[ M = \frac{1}{2} a \dot{a}^2 + \frac{1}{2} a \dot{b}^2 + \frac{1}{2} a + \frac{b^2}{2a}. \]  

(4.9)
Because of that, we are prompted to write the classical Hamiltonian of the Reissner-Nordström black hole, from the point of view of our distant observer at rest, as

\[ H = \frac{p_a^2}{2a} + \frac{p_b^2}{2a} + \frac{1}{2}a + \frac{b^2}{2a}, \]  

(4.10)

where \( p_a \) is, as in Section 3, the canonical momentum conjugate to the throat radius \( a \), and

\[ p_b := ab \]  

(4.11)

is the canonical momentum conjugate to the variable \( b \).

We obtained the Hamiltonian (4.10) by means of a guesswork based on the study of the Hamiltonian constraint on the timelike geodesic going through the bifurcation two-sphere, in a specific foliation of the Reissner-Nordström spacetime. The real problem is to find out, whether there exists a well-defined one-to-one canonical transformation from the phase space coordinates \( m, p_m, q \) and \( p_q \), introduced in Section 2, to the phase space coordinates \( a, p_a, b \) and \( p_b \) such that the Hamiltonian takes the form (4.10) if we choose the lapse functions and the electric potentials at asymptotic infinities as in Section 3.

We shall perform such a transformation in two steps. We first define a canonical momentum \( p_w \) conjugate to a yet unknown variable \( w \) as

\[ p_w := q. \]  

(4.12)

With this choice the classical Hamiltonian takes the form

\[ H = \frac{p_a^2}{2a} + \frac{p_w^2}{2a} + \frac{1}{2}a. \]  

(4.13)

The variables \( m \) and \( p_m \) are expressed in terms of \( a \) and \( p_a \) as in Eqs.(3.4.a) and (3.6), but we have replaced \( Q \) with \( p_w \).

The next task is to find \( w \). One expects \( w \) to be related in one way or another to the momentum \( p_q \) conjugate to \( q \). Since \( p_q \) defines the electromagnetic gauge, we first write the Hamiltonian in a general gauge:

\[ H = \frac{p_a^2}{2a} + \frac{p_w^2}{2a} + \frac{1}{2}a + (\dot{\phi}_+ - \dot{\phi}_-)p_w, \]  

(4.14)

which follows from Eq.(2.35). Using Eq.(2.36.d) we find that the Hamiltonian equation of motion for \( w \) is

\[ \dot{w} = \frac{\partial H}{\partial p_w} = \frac{p_w}{a} - \dot{p}_q. \]  

(4.15)

An expression of \( p_q \) in terms of \( a, p_a, w \) and \( p_w \) can be gained by integrating the both sides of Eq.(4.15) along the classical trajectory in the phase space:

\[ p_q = \int_0^a \frac{p_w}{a_0} \, da - w, \]  

(4.16)
where we have substituted

$$\dot{a} = -\text{sgn}(p_w) \sqrt{\frac{2m}{a} - 1 - \frac{p_w^2}{a}}. \quad (4.17)$$

This substitution involves choosing \( \dot{p}_q = 0 \). When the electric potential is assumed to vanish at asymptotic infinities, this choice can be made. With an appropriate choice of the integration constant we get

$$p_q = \text{sgn}(p_m)p_w \left[ \sin^{-1} \left( \frac{p_a^2 + p_w^2 - a^2}{\sqrt{(p_a^2 + p_w^2 + a^2)^2 - 4a^2p_w^2}} \right) + \frac{\pi}{2} \right] - w, \quad (4.18)$$

where we have made the substitution

$$m = \frac{p_a^2}{2a} + \frac{p_w^2}{2a} + \frac{1}{2} a. \quad (4.19)$$

Eqs. (3.4), (4.12) and (4.18) now constitute a transformation from the phase space coordinates \( m, p_m, q \) and \( p_q \) to the phase space coordinates \( a, p_a, w \) and \( p_w \). It is easy to see that this transformation is well defined and canonical. Moreover, the transformation is one-to-one provided we impose a restriction

$$\left| \frac{p_q + w}{p_w} \right| \leq \pi. \quad (4.20)$$

This restriction is related to the fact that we are considering spacetime between two successive \( R = r_- \)-hypersurfaces. Since \( \dot{p}_q \) vanishes when the electric potentials are assumed to vanish at asymptotic infinities, we find that, classically, \( w = -q\pi + p_q \) at the past \( R = r_- \)-hypersurface, \( w = p_q \) at the bifurcation point, and \( w = q\pi + p_q \) at the future \( R = r_- \) hypersurface. In other words, the domain of \( w \) is bounded by the fact that the \( t = \text{constant} \)-hypersurfaces cannot be pushed beyond the \( R = r_- \) hypersurfaces.

It only remains to find a canonical transformation from the variables \( w \) and \( p_w \) to the variables \( b \) and \( p_b \). We define

$$b := p_w \sin \left( \frac{w}{p_w} \right), \quad (4.21.a)$$

$$p_b := p_w \cos \left( \frac{w}{p_w} \right). \quad (4.21.b)$$

This transformation is well defined and canonical as well as, with the restriction (4.20), one-to-one. Because of that, we are justified to write the classical Hamiltonian as in Eq. (4.10). Moreover, since it follows from Eq. (4.21) that

$$p_w^2 = p_b^2 + b^2, \quad (4.22)$$
we can identify the quantity $p_b^2 + b^2$ as the square of the electric charge of the black hole.

We now proceed to quantization. We choose our Hilbert space to be the space $L^2(\mathbb{R}^+ \times \mathbb{R}, a^s \, da \, db)$ with the inner product

$$\langle \psi_1 | \psi_2 \rangle := \int_0^\infty da \, a^s \int_{-\infty}^{\infty} db \, \psi_1^*(a,b) \psi_2(a,b). \quad (4.23)$$

Through the substitutions $p_a \to -i \frac{\partial}{\partial a}$ and $p_b \to -i \frac{\partial}{\partial b}$ we replace the classical Hamiltonian $H$ of Eq.(4.10) with the corresponding symmetric operator

$$\hat{H} := -\frac{1}{2} a^{-s} \frac{\partial}{\partial a} \left( a^{s-1} \frac{\partial}{\partial a} \right) - \frac{1}{2a} \frac{\partial^2}{\partial b^2} + \frac{1}{2} a + \frac{b^2}{2a}. \quad (4.24)$$

As in Section 3, we can view the corresponding eigenvalue equation $\hat{H} \psi = E \psi$ as an eigenvalue equation for the total ADM energy of the Reissner-Nordström black hole, from the point of view of a distant observer at rest.

The eigenvalue equation $\hat{H} \psi = E \psi$ can be separated if we write

$$\psi(a,b) := \varphi(a) \beta(b), \quad (4.25),$$

and we get

$$\left[ -\frac{1}{2} a^{-s} \frac{d}{da} \left( a^{s-1} \frac{d}{da} \right) + \frac{1}{2} a + \frac{Q^2}{2a} \right] \varphi(a) = E \varphi(a), \quad (4.26.a)$$

$$\left( -\frac{d^2}{db^2} + \frac{b^2}{2a} \right) \beta(b) = Q^2 \beta(b). \quad (4.26.b)$$

Eq.(4.26.a) is now identical to Eq.(3.14), which is an eigenvalue equation for the total energy of the hole when $Q$ is treated as an external parameter, whereas Eq.(4.26.b) can be understood as an eigenvalue equation for the square of the electric charge $Q$ of the hole. When the eigenfunctions $\beta(b)$ are chosen to be the harmonic oscillator eigenfunctions, we find that the possible eigenvalues of $Q^2$ are of the form

$$Q_k^2 = 2k + 1, \quad (4.27)$$

or, in SI units:

$$Q_k^2 = (2k + 1) \frac{e^2}{\alpha}, \quad (4.28)$$

where $k = 0, 1, 2, 3, \ldots$. In this equation, $e$ is the elementary charge, and

$$\alpha := \frac{e^2}{4\pi\varepsilon_0 \hbar c} \approx \frac{1}{137} \quad (4.29)$$

is the fine structure constant. In other words, our theory implies that the electric charge of the Reissner-Nordström black hole has a discrete spectrum. It is interesting that the electric charge is quantized in terms of the "Planck charge" $e/\sqrt{\alpha}$.
in exactly the same way as the quantity $\sqrt{M^2 - Q^2}$ is quantized in terms of the Planck mass $m_{Pl} := (\hbar c/G)^{1/2}$. Moreover, we find that the possible eigenvalues of the area $A^\text{ext} := 4\pi Q^2$ of an extreme black hole are of the form

$$A^\text{ext}_k = 8\pi (k + \frac{1}{2}) l_{Pl}^2.$$  \hspace{1cm} (4.30)

In other words, we have recovered Bekenstein’s proposal (1.3), with $\gamma = 8\pi$, for a black hole near extremality. One can also see from Eq. (3.27) that even when the electric charge is made a dynamical variable, the spectrum of the total area of the horizons of the hole is the same as, according to Bekenstein, is the area spectrum of the outer horizon of the hole. However, it should be noted that when $k$ is very big, the difference between two successive charge eigenvalues is

$$Q_{k+1} - Q_k \approx \frac{e^2}{\alpha Q_k},$$  \hspace{1cm} (4.31)

and we find that we have, in practice, a continuous charge spectrum. One may therefore have very mixed feelings on the physical validity of the charge spectrum (4.27): for all known particles –in appropriate units– $Q$, rather than $Q^2$, is an integer. However, for all these particles we have, in natural units, $|Q| \gg M$, and nobody knows what happens when $M \geq |Q|$.

5. Concluding Remarks

In this paper we have presented a quantum mechanical model of the Reissner-Nordström black hole, paying particular attention to the dynamical properties of such black holes. More precisely, we used the throat radius of the hole from the point of view of an observer in a radial free fall through the bifurcation two-sphere of the hole, as the geometrodynamical variable of our model. Motivated by the Principle of Equivalence, we chose the foliation of spacetime in such a way that the proper time of a freely falling observer on the throat agrees with the asymptotic Minkowski time. We then performed a Hamiltonian quantization of our model. Our Hamiltonian quantum theory was based on the analysis of the Hamiltonian dynamics of asymptotically flat spherically symmetric electrovacuum spacetimes performed by Louko and Winters-Hilt. According to that analysis, the true phase space of spacetimes mentioned above is spanned by just four variables: the mass and the electric charge of the Reissner-Nordström hole, together with the corresponding canonical momenta. The dynamical variables of our model – which in addition of gravitational, also accounted for the electromagnetic degrees of freedom of the hole– were obtained by means of a canonical transformation from those used by Louko and Winters-Hilt.

Our model implied the results that the (ADM) mass and the electric charge spectra of the Reissner-Nordström black hole are discrete. Moreover, we saw that
the mass of the hole is bounded below and—with an appropriate choice of the inner product and boundary conditions of the wave function of the hole—the mass eigenvalues are always greater than the absolute values of the eigenvalues of the electric charge of the hole. Hence, if we think of black hole radiation as arising from a chain of transitions from higher to lower energy eigenstates of the hole, our model implies that a non-extreme black hole with non-zero Hawking temperature can never become, through black hole radiation, an extreme black hole with zero Hawking temperature. This result is compatible with the third law of thermodynamics as well as with the qualitative difference between extreme and non-extreme black holes.

At the high end of the mass spectrum, the WKB analysis yielded a result that the possible eigenvalues of the total area of the outer and the inner horizons of the hole are a constant plus an integer times $32\pi l_p^2$, where $l_p$ is the Planck length. Hence, our result is closely related, although not quite identical, to Bekenstein’s proposal (1.3). Moreover, we saw that no matter whether the hole is charged or not, the spectrum of the total area of its horizons is always essentially the same.

Even though our model gives a mathematically consistent quantum theory of spherically symmetric, asymptotically flat electrovacuum spacetimes and meets with some success—in particular in its prediction that a non-extreme black hole cannot become an extreme black hole by means of black hole radiation, and in its close relationship with Bekenstein’s proposal—it is possible to express serious objections against the physical relevance of our model.

The first objection is related to the mass spectrum given by our model. Recall that our model yields the result that the eigenvalues of the quantity $\sqrt{M^2 - Q^2}$ are of the form $\sqrt{2n}$, where $n$ is an integer. Now, this spectrum of the quantity $\sqrt{M^2 - Q^2}$ implies that when the hole performs a transition from a state with mass $M_{n+1}$ to the state with mass $M_n := M$, the angular frequency of a quantum emitted in this process is

$$\omega \approx \frac{1}{M}.$$  \hspace{1cm} (5.1)

Regarding Hawking’s results on black hole radiation, this kind of a spectrum appears very natural when $Q = 0$. In that case the expression (1.1) for Hawking temperature, together with Wien’s displacement law, implies that the maximum of the black-body spectrum of the black hole radiation, as predicted by Hawking and others, is proportional to $1/M$. In other words, the angular frequency associated with the discrete spectrum of black hole radiation, as predicted by our model, behaves, as a function of $M$, in the same way as does the angular frequency corresponding to the maximum of the black-body spectrum as predicted by Hawking and others. Unfortunately, this nice correspondence between Hawking’s results and our model breaks down when $Q \neq 0$. In that case one finds from Eq.(1.1) that the maximum of the black-body spectrum corresponds to the angular frequency

$$\omega_{\text{max}} \propto \frac{\sqrt{M^2 - Q^2}}{(M + \sqrt{M^2 - Q^2})^2}.$$  \hspace{1cm} (5.2)
In other words, the angular frequency (5.1) predicted by our model corresponds, when the hole is near extremality, to a temperature which is much higher than the Hawking temperature.

However, there is a possible way out from this problem. In all our investigations we have emphasized the importance of the dynamics of the intermediate region between the inner and the outer horizons of the Reissner-Nordström hole. The dynamics of the intermediate region is, in our model, responsible for the discrete energy states of the hole. Now, if we take this point of view to its extreme limits, we are prompted to speculate that it is the intermediate region of the hole which emits black hole radiation when the hole performs a transition from one energy eigenstate to another. Because of that, it is possible that not only does the intermediate region emit radiation outside the outer horizon but also inside the inner horizon. In other words, both the inner and the outer horizons may radiate. The radiation emitted by the inner horizon of the hole is not observed by an external observer, and presumably that radiation is either swallowed by the singularity or absorbed, again, by the intermediate region. Nevertheless, an emission of this radiation by the inner horizon is likely to reduce considerably the number of quanta, and hence the temperature, of the radiation coming out from the hole: the more the inner horizon radiates, the less quanta are left for the outer horizon. It remains to be seen whether this speculation of ours will be validated by a further study.

There is yet another, more fundamental, objection against our model. In our model we have used a very small number of dynamical degrees of freedom, together with the corresponding canonical momenta, to describe the quantum mechanics of black holes. However, the fact that the Bekenstein-Hawking entropy of a macroscopic black hole is very high suggests an enormous number of dynamical degrees of freedom. Hence, one may question the relevance of models with a small number degrees of freedom.

However, there are some very powerful theorems on our side. They are, of course, the black hole uniqueness theorems. According to these theorems, a black hole in a stationary spacetime is uniquely characterized by its mass, angular momentum and electric charge[35]. In other words, the number of physical degrees of freedom of a black hole is, in the classical level, very small. Hence, we seem to have a slight disharmony between the Bekenstein-Hawking entropy hypothesis and the black hole uniqueness theorems.

A somewhat analogous situation can be met with already in ordinary quantum mechanics. Consider a hydrogen atom. In elementary textbooks, the only degrees of freedom under consideration are the three degrees of freedom associated with the electron going around the proton. In more advanced textbooks, however, a student is revealed that not only should one quantize the degrees of freedom associated with the electron but also the degrees of freedom associated with the electromagnetic field. As a result, one gets an enormous number of degrees of freedom associated with the virtual photons and electron-positron pairs appearing as an outcome of the quantization of the electromagnetic field. In other words, although classically
we have, in effect, only the degrees of freedom associated with the electron, the full quantum theory with quantized electromagnetic field reveals an enormous number of particles and an enormous number of degrees of freedom. However, the whole contribution of all these additional degrees of freedom to the energy levels of the hydrogen atom is very small. Now, something similar may happen with black holes: classically, the number of relevant degrees of freedom is very small, but when the full quantum theory of gravity is employed, an enormous amount of degrees of freedom are likely to appear. Hence, one may feel tempted to regard the relationship between our model and the full quantum theory of black holes as somewhat analogous to the relationship between the treatments of a hydrogen atom in elementary and advanced textbooks: quantization of the two degrees of freedom of the classical Reissner-Nordström hole corresponds to the quantization of the three degrees of freedom associated with the electron going around the proton in a hydrogen atom. Whether the additional degrees of freedom appearing as a likely outcome of the full quantum theory of black holes have great or small effects to the energy levels of the hole is an open question. However, given the enormous pace of progress in the current research in black hole physics, one may hope for a definite answer in a not so distant future.

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Appendix A: Solution of the Energy Eigenvalue Equation for Large Energies and Charges

In this Appendix we evaluate the large eigenvalues of the Hamiltonian operator $\hat{H}$ which was written in Eq.(3.13). This leads us to the eigenvalue equation (3.14) as we have already seen. We shall find the large eigenvalue solutions of the eigenvalue equation by using the WKB approximation method when both $|Q|$ and $E^2 - Q^2$ are, in natural units, much greater than unity and, in addition, we demand that $r_+ \geq 1$ or, which is the same thing, $(2E - 1)/Q^2 \leq 1$. The basic idea is to match the WKB-approximation with an expression of the wave function in terms of modified Bessel functions close to the point where $a = 0$. The results used here on the matching of the WKB wave function and Bessel function approximations close to the turning points are widely known – see for example Ref.[34] – therefore we shall use them without any special review. Cases $r = 1/2$, $r \geq 3/2$, $r = 7/6$, $7/6 < r < 3/2$ and $1/2 < r < 7/6$ will be discussed separately, but we shall first look for the general solution $\psi(a)$, when the argument $a$ is very small i.e. $|Q|a \ll 1$. After that we shall search for the solution for ”slightly bigger” $a$ i.e $|Q|a \leq M$, where $M$ is an arbitrary positive number.
To begin with we deform the eigenvalue equation (3.18) in an appropriate manner. If we substitute in Eq. (3.18)

\[ a := x^{2/3}, \quad (A1.a) \]

\[ \chi := a^{-1/4}u(a), \quad (A1.b) \]

we get

\[ \frac{d^2}{da^2} - \left( \frac{3}{2}r - \frac{1}{4} \right) \left( \frac{3}{2}r - \frac{5}{4} \right) - a^2 - Q^2 + 2Ea \right] u(a) = 0, \quad (A2) \]

where \( r \) is defined in Eqs. (3.17). Eq. (A2) is now invariant under the transformation \( r \to 1 - r \); thus it is sufficient to consider solutions of the eigenvalue equation (A2) for \( r \geq 1/2 \). As a consequence the inner product of Eq. (3.19) becomes

\[ \langle u_1 | u_2 \rangle = \int_0^\infty u_1^*(a)u_2(a) a \, da. \quad (A3) \]

We shall now solve Eq. (A2) when \( E^2 - Q^2 \gg 1, |Q| \gg 1 \) and \( (2E - 1)/Q^2 \leq 1 \). For very small \( a \), the linearly independent solutions to Eq. (A2) are, when \( r > 1/2 \) and \( r \neq 7/6 \)

\[ u_1(a) = Aa^{(3/2)r}[a^{-1/4} + O(a^{7/4})], \quad (A4.a) \]

\[ u_2(a) = Ba^{-(3/2)r}[a^{5/4} + O(a^{13/4})], \quad (A4.b) \]

where \( A \) and \( B \) are constants. The case \( r = 1/2 \) will be considered later in Appendix A, and if \( r = 7/6 \) then the term proportional to \( a^{-(3/2)r+13/4} \) in Eq. (A4.b) must be multiplied by a term proportional to \( \ln(|1/2|)|Q|a) \). The leading term, however, is the same as in Eq. (A4.b) when \( r > 1/2 \).

By writing

\[ x = |Q|a, \quad (A5) \]

we get from Eq. (A2) an equation

\[ \left[ \frac{d^2}{dx^2} - \left( \frac{3}{2}r - \frac{1}{4} \right) \left( \frac{3}{2}r - \frac{5}{4} \right) - \frac{x^2}{Q^2} + \frac{2Ex}{|Q|^3} \right] u(x) = 0. \quad (A6) \]

Now the terms proportional to \( x^2 \) and \( x \) are asymptotically small at large \( |Q| \), whenever \( x \in (0,M] \), where \( M \) is an arbitrary positive constant. Omitting these last terms we get an equation

\[ \left[ \frac{d^2}{da^2} - \left( \frac{3}{2}r - \frac{1}{4} \right) \left( \frac{3}{2}r - \frac{5}{4} \right) - Q^2 \right] u(a) = 0, \quad (A7) \]

when the substitution (A5) is inversed. The general linearly independent solutions are, when \( \frac{3}{2}(r - 1/2) \) is not an integer, modified Bessel functions of the first kind, up to an overall normalization constant

\[ u_1(a) = a^{1/2}I_{\frac{3}{2}(r - 1/2)}(|Q|a), \quad (A8.a) \]

\[ u_2(a) = a^{1/2}I_{-\frac{3}{2}(2 - 1/2)}(|Q|a). \quad (A8.b) \]

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If \( \frac{3}{2}(r-1/2) \) is an integer then the general solutions are, similarly

\[
\begin{align*}
    u_1(a) &= a^{1/2}I_{\frac{3}{2}(r-1/2)}(|Q|a), \\
    u_2(a) &= a^{1/2}K_{\frac{3}{2}(r-1/2)}(|Q|a),
\end{align*}
\]

(A9.a) (A9.b)

where \( K_p \) is the modified Bessel function of the second kind of order \( p \).

1. Case \( r \geq 3/2 \)

We first consider the case \( r \geq 3/2 \). Throughout the discussion we shall assume that \( E^2 - Q^2 > 0 \). The solution (A4) to Eq.(A2) is normalizable with respect to the inner product (A3) only if the constant \( B \) vanishes. Now, a comparison with the asymptotic behaviour of the modified Bessel functions of Eq.(A8) for small \( a \) implies that the only normalizable solution for small \( a \) is

\[
u(a) = Ca^{1/2}I_{\frac{3}{2}(r-1/2)}(|Q|a)
\]

(A10)

when \( \frac{3}{2}(r-1/2) \) is not an integer. If \( \frac{3}{2}(r-1/2) \) is an integer, a comparison with Eq.(A9) gives similarly that the leading term is the same as in Eq.(A10). To verify this, the Bessel functions must be expanded as their small \( a \) series. If we fix \( \delta_1, \delta_2 > 0 \) such that \( \delta_1 \leq a \leq \delta_2 \), an asymptotic large \( |Q| \) behaviour of \( u(a) \) is, up to a normalization constant:

\[
u(a) \propto (2\pi |Q|)^{-1/2} \exp(|Q|a)
\]

(A11)

for all \( r \geq 3/2 \). From now on, the symbol \( \propto \) is used for the asymptotic form at large \( |Q| \), up to a possibly \( E, Q \)-dependent coefficient.

After very small, small, and slightly bigger argument \( a \) we enter into a region \( a \in (0, a_-) \), where \( a_- \) is the smaller turning point satisfying for large \( E \) and \( |Q| \) \( a_- \approx r_- \). Our object is now to use the WKB approximation method to the wave function in the region in question. The WKB approximation corresponding to such a wave function \( u(a) \) which decreases to the left of the turning point \( a_- \) is

\[
u_{\text{WKB}}(a) = [p_1(a)]^{-1/2} \exp \left[ - \int_{a_-}^a p_1(a')da' + \eta_1 \right],
\]

(A12)

where

\[
p_1(a) = \sqrt{a^2 + \frac{(3/2r - 1/4)(3/2r - 5/4)}{a^2} - 2Ea + Q^2}.
\]

(A13)

The major problem in the WKB approximation involves the evaluation of the integral (A12). It turns out to be, however, that it is not necessary to evaluate the WKB integral (A12) at all: we are interested in solutions for small \( a \) i.e. \( |Q|a \ll 1 \) and such solutions can be achieved easily by Taylor series. In the evaluation of the series it should be clear that \( E^2 - Q^2 \) and \( |Q| \) are assumed to be very large and
Furthermore, we assume that \( a \geq \delta_1 \) such that \( \delta_1^2 \) is negligible. The integral in the exponent in Eq. (A12) is

\[
S(a) := -\int_a^\infty da' \sqrt{a'^2 + Q^2 - 2Ea'} - \frac{(3/2r - 1/4)(3/2r - 5/4)}{a'^2}.
\]

(A14)

With the help of Eq. (A14) the exponent for small \( a \) in Eq. (A12) can be written as

\[
S'(\delta_1) (a - \delta_1) + \frac{1}{2} S''(\delta_1) (a - \delta_1)^2 + O(\delta_1^2),
\]

(A15)

where \( S' \) denotes \( \frac{dS}{da} \). Hence for small \( a \), we have the WKB wave function given by Eqs. (A12) and (A15)

\[
u_{WKB}(a) \approx |Q|^{-1/2} \exp(|Q|a),
\]

(A16)

where we have substituted the constant

\[
\eta_1 = Q\delta_1.
\]

(A17)

This connects the WKB solution with the asymptotic solution in Eq. (A11). In other words, the WKB solution decreases exponentially to the left of the turning point \( a_- \).

We next enter into an area with oscillations. We let the energy \( E \) to be so large that the eigenvalue equation has two turning points. We denote them as before as \( a_- \) and \( a_+ \). It should be clear that \( a_- \approx r_- \) and \( a_+ \approx r_+ \) when \( E^2 - Q^2 \) and \( |Q| \) are large enough. The region of oscillations is far right of \( r_- \) and far left of \( r_+ \). As the wave function decreases exponentially right of the larger turning point and left of the smaller turning point, the WKB approximation can be written far right of \( r_- \) as

\[
u_{WKB}^{r-}(a) = C_1 [p_2(a)]^{-1/2} \cos \left[ \int_{a_-}^{a} da' p_2(a') - \frac{\pi}{4} \right],
\]

(A18.a)

and far left of \( r_+ \) as

\[
u_{WKB}^{r+}(a) = C_1 [p_2(a)]^{-1/2} \cos \left[ \int_{a}^{a_+} da' p_2(a') - \frac{\pi}{4} \right],
\]

(A18.b)

where

\[
p_2(a) := \sqrt{-a^2 + 2Ea - Q^2} \sqrt{1 - \frac{(3/2r - 1/4)(3/2r - 5/4)}{a'^2(-a'^2 - Q^2 + 2Ea')}},
\]

(A19)

The wave functions above are equal if

\[
S := \int_{a_-}^{a_+} da p_2(a) = (n + \frac{1}{2})\pi,
\]

(A20)
where \( n \geq 0 \) is an integer. This integral fixes the levels of the spectrum of the Reissner–Nordström black hole.

In the evaluation of the WKB integral it should, again, be clear that \( E^2 - Q^2 \) and \(|Q|\) are assumed very large. We can expand the second square root in its Taylor series as the first square root is of order \( \mathcal{O}(E^2 - Q^2) \) and the second term in the second square root is of order \( \mathcal{O}(1/(a_+ \sqrt{E^2 - Q^2})) \). Thus we can write the integral as

\[
S = \int_{a_-}^{a_+} da \left[ \sqrt{-a^2 - Q^2 + 2Ea} - \frac{(3/2r - 1/4)(3/2r - 5/4)}{2a^2\sqrt{-a^2 - Q^2 + 2Ea}} - \ldots \right]. \tag{A21}
\]

Now that \( a_- = r_- + \mathcal{O}(1/r_-^3) \) and \( a_+ = r_+ - \mathcal{O}(1/r_+^3) \), the evaluation of the integral \( S \) can be done in parts, by replacing the limits \( a_- \) and \( a_+ \) by \( r_- \) and \( r_+ \) the second integral gives us a term order of \( \mathcal{O}(E/Q^3) \), which, on the grounds of the requirement \( (2E - 1)/Q^2 \leq 1 \), is small compared to the first term. Thus the second term can be omitted from \( S \). In the evaluation of the third term from \( r_- \) to \( r_+ \) the integral does not converge. We therefore have to alter the integration region near the turning points by fixing a couple of constants, namely, \( \delta_3, \delta_4 > 0 \) such that we are able to restrict the argument \( a \) in the region \( a_- < a < a_+ \) and \( \delta_3/\sqrt{E_2 - Q_2} \) is of order \( \mathcal{O}(E_2 - Q_2)^{1/8} \). Then the limits can be replaced by \( r_- + \delta_3/\sqrt{E_2 - Q_2}^{1/8} \) and \( r_+ - \delta_4/\sqrt{E_2 - Q_2}^{1/8} \). Now the third term in the integral \( S \) gives us a term, which is at most of order \( \mathcal{O}(r_-^{-4}(E_2 - Q_2)^{-1/4}) \), which is small compared to the first term and can thus be omitted. The remaining integral is elementary and we obtain

\[
S \approx \int_{r_-}^{r_+} da \sqrt{-a^2 + 2Ea - Q^2} = \frac{\pi}{2}(E^2 - Q^2) = (n + \frac{1}{2})\pi. \tag{A22}
\]

This yields for large energies and charges, when \( r \geq 3/2 \) and \( r_- \geq 1 \), the WKB estimate

\[
E^2 - Q^2 \sim 2n + 1 + o(1). \tag{A23}
\]

2. Cases \( 7/6 < r < 3/2 \) and \( 1/2 < r < 7/6 \)

Now we are in a situation where we cannot just exclude either of the integration constants \( A \) or \( B \) in Eq.(A4) on the grounds of the normalizability of the wave function. However, the self–adjointness of the Hamiltonian operator implies the following boundary condition for the solutions \( u_{1,2}(a) \):

\[
\lim_{a \to 0} \left[ u_{1,2}^*(a) \frac{du_{1,2}(a)}{da} - \frac{du_{1,2}^*(a)}{da} u_{1,2}(a) \right] = 0. \tag{A24}
\]

Here \( u_1 \) and \( u_2 \) are two linearly independent, non–degenerate eigenfunctions. As shown in Eq.(A4) the differential equation \( \hat{H}u(a) = Eu(a) \) has two small \( a \) solutions which satisfy all those conditions stated above – at least when \( E^2 - Q^2 > 0 \).
Now it is easy to show that, for very small \( a \), the eigenfunctions of a self-adjoint Hamiltonian operator behave, up to normalization, as

\[ u(a) \approx \cos(\theta) a^{(3/2)r-1/4} + \sin(\theta) a^{-(3/2)r+5/4}, \]  

(\text{A25})

where \( \theta \in [0, \pi) \) is a parameter to be fixed later. Comparing the small \( a \) expansions of Eqs.(\text{A25}) and (\text{A8}) we can adjust the constants \( A \) and \( B \) such that \( u(a) \) behaves asymptotically

\[
\begin{align*}
  u(a) \propto a^{1/2} & \left[ 2^{3/2(r-1/2)} \Gamma(3/2r + 3/4) |Q|^{-\frac{5}{2}(r-1/2)} \cos(\theta) I_{\frac{3}{2}(r-1/2)}(|Q|a) \\
  & + 2^{-\frac{5}{2}(r-1/2)} \Gamma(-3/2r + 5/4) |Q|^{\frac{5}{2}(r-1/2)} \sin(\theta) I_{-\frac{3}{2}(r-1/2)}(|Q|a) \right].
\end{align*}
\]

(\text{A26})

When \( \theta = 0 \), the second term in Eq.(\text{A26}) vanishes and we can proceed just as in the case \( r \geq 3/2 \) from which it follows that the WKB estimate is given by Eq.(\text{A23}). When \( \theta \neq 0 \) the second term in Eq.(\text{A26}) dominates at large \( |Q| \) and the asymptotic behaviour is as Eq.(\text{A11}). Therefore the WKB estimate is again given by Eq.(\text{A23}).

3. Case \( r = 7/6 \)

When \( r = 7/6 \), the number \( \frac{3}{2} (r - 1/2) \) becomes an integer and the general solution of Eq.(\text{A7}) includes modified Bessel functions of the second kind as shown before. Furthermore, we cannot rule out either of the adjustable constants \( A \) or \( B \) and therefore we have to keep both the solutions in Eq.(\text{A9}). As before we get from the boundary condition (\text{A24}) that at least when \( E^2 - Q^2 > 0 \) the eigenfunction of a self-adjoint Hamiltonian operator is, for small \( a \),

\[ u(a) \approx \sin(\theta) a^{-1/2} + \cos(\theta) a^{3/2}, \]  

(\text{A27})

where again \( \theta \in [0, \pi) \) is a parameter.

After expanding the general solution of Eq.(\text{A7}) when \( a \) is small we have that \( u(a) \) is asymptotically

\[
\begin{align*}
  u(a) \propto a^{1/2} & \left[ (2 \cos(\theta) |Q|^{-1} - \sin(\theta) |Q|^\gamma I_1(|Q|a) + \sin(\theta) |Q| K_1(|Q|a) \right],
\end{align*}
\]

(\text{A28})

where \( \gamma \) is Euler’s constant. When \( \theta = 0 \) the term proportional \( a^{1/2} K_1(|Q|a) \) vanishes and we get the same WKB estimate as before in Eq.(\text{A23}). On the other hand, when \( \theta \neq 0 \), the term proportional \( a^{1/2} I_1(|Q|a) \) dominates for large \( |Q| \) and the situation is quite the same as before. The resulting WKB estimate is therefore given by Eq.(\text{A23}).

4. Case \( r = 1/2 \)

When \( r = 1/2 \) we can no more write the solutions of Eq.(\text{A7}) as powers of small \( a \), because the loss of the linear independency of the solutions (\text{A4.a}) and
(A4.b). By using the boundary condition (A24), however, and expanding the general solution \( u(a) = a^{1/2}[C I_0(|Q|a) + D K_0(|Q|a)] \) of Eq.(A7) for small \( a \) we notice that Eq.(A26) is replaced by

\[
u(a) \propto |Q|^{1/4} a^{1/2} \left[ (\cos(\theta) - \sin(\theta)(\gamma + \ln(\frac{1}{2}|Q|))) I_0(|Q|a) - \sin(\theta) K_0(|Q|a) \right]. \quad (A29)\]

For any \( \theta \) the term proportional to \( a^{1/2} I_0(|Q|a) \) dominates the term proportional to \( a^{1/2} K_0(|Q|a) \) for large charges, and the asymptotic behaviour is given by Eq.(A11). Thus the WKB result is, again, given by Eq.(A23).

It should be noted that we have not investigated what happens when the condition \((2E - 1)/Q^2 \leq 1 \) following from the requirement \( r_- \geq 1 \) does not hold; i.e. when \( Q \) is arbitrarily small when compared to \( E \). In that case, however, one expects that the WKB eigenenergies given by Eq.(A23) should be replaced by those given in Ref.[8] for the Schwarzschild black hole. On the grounds of the results of Ref.[8] it is likely that the eigenvalues of the quantity \( \sqrt{E^2 - Q^2} \) are of the form \( \sqrt{2n} \) even when \( |Q| \) is arbitrarily small compared to the black hole energy \( E \).

**Appendix B: Positiveness of the Spectrum of the Quantity \( E^2 - Q^2 \)**

In this appendix we shall investigate the possible positiveness of the spectrum of the quantity \( E^2 - Q^2 \). Cases \( r = 1/2 \) and \( r \geq 3/2 \) and \( 1/2 < r < 3/2 \) will be considered separately.

1. **Case \( r \geq 3/2 \)**

Let \( u(a) \) be an eigenfunction of Eq.(A2) with any eigenvalue \( E^2 - Q^2 \). Now, when \( r \geq 3/2 \), we have, up to a well chosen normalization constant, a small \( a \) expansion to the eigenfunction \( u(a) \) given by Eq.(A4.a) as \( u(a) = a^{(3/2)r} [a^{-\frac{3}{4}} + O(a^{7/4})] \). It is clear that \( u(a) \) and \( u'(a) \) are both real valued and positive. Therefore the eigenfunction is positive and real valued for sufficiently small \( a \). It is easy to see that the eigenvalue equation (A2) can be written as

\[
u''(a) = \left[ \left( \frac{3}{2} \right) r - \frac{1}{4} \right] \left( \frac{3}{7} \right) r - \frac{5}{4} \right) \frac{1}{a^2} + (a - E)^2 - (E^2 - Q^2) \right] u(a). \quad (B1)\]

Let us now assume \( E^2 - Q^2 \) is not strictly positive i.e. \( E^2 - Q^2 \leq 0 \). In that case Eq.(B1) implies that \( u''(a) > 0 \), for all \( a \) such that \( u(a) > 0 \). Since both \( u(a) \) and \( u'(a) \) are positive for sufficiently small \( a \), the positvity of \( u''(a) \) whenever \( u(a) \) is positive implies that \( u'(a) \), and hence \( u(a) \), are increasing functions of \( a \). Because of that, we have \( \lim a \to \infty u(a) > 0 \) and \( u(a) \) is not normalizable. Hence we must have \( E^2 - Q^2 > 0 \).
Here we shall show that it is possible to find such self-adjoint extensions of the Hamiltonian operator such that the spectrum is strictly positive. This corresponds to an appropriate choice of the parameter \( \theta \) introduced in Appendix A in Eq.(A25). We have already shown that when \( 7/6 < r < 3/2 \) and \( 1/2 < r < 7/6 \) the extensions take, up to an overall normalization constant, for small \( a \) the form given in Eq.(A25), where the parameter \( \theta \) specifies the self-adjoint extensions.

We consider first an extension with \( \theta \in [0, \pi/2] \). By taking \( E^2 - Q^2 \) in Eq.(B1) to be negative or zero, we get, when \( u(a) \) is positive,

\[
u''(a) \geq \frac{(3r - 1)}{2} \left( \frac{3r - 5}{4} \right) u(a) \quad \forall a > 0. \tag{B2}
\]

To find a possible lower bound for \( u(a) \) we therefore consider an equation

\[
f''(a) = \frac{(3r - 1)}{2} \left( \frac{3r - 5}{4} \right) f(a). \tag{B3}
\]

Now when \( r > 1/2 \) the general solution of Eq.(B3) is

\[
f(a) = A a^{(3/2)r-1/4} + B a^{-(3/2)r+5/4}, \tag{B4}
\]

and we see that if we choose \( A = \cos \theta \) and \( B = \sin \theta \) then \( f(a) \) coincides with the small \( a \) solution (A25). Since \( u(a) \) is positive for sufficiently small \( a \) when both \( \cos \theta \) and \( \sin \theta \) are positive, we find that the solution \( u(a) \) is equal or greater than the solution \( f(a) \) for all \( a \geq 0 \) when \( 0 \leq \theta \leq \pi/2 \). Hence the solution \( u(a) \) does not vanish exponentially as \( a \) goes to infinity. Yet, the potential increases without bound as \( a \) increases towards infinity; therefore any eigenfunction must vanish exponentially at large \( a \). Hence the spectrum must be strictly positive.

As to the remaining range \( \theta \in (\pi/2, \pi) \) we refer to the results of Ref.[8], which state that the energy spectrum is bounded below.

3. Case \( r = 1/2 \)

For \( r = 1/2 \) Eqs.(B2) and (B3) still hold and Eq.(B4) must be replaced by a solution

\[
f(a) = (\cos \theta - \sin \theta) a^{1/2} - \sin \theta a^{1/2} \ln a. \tag{B5}
\]

When \( \theta = 0 \), \( f(a) \) is positive for all \( a \geq 0 \) and one can argue as before.

In the remaining range \( 0 < \theta < \pi \) the energy spectrum is bounded below on the grounds of the results presented in Ref.[8].
References

Francisco, 1973)
[16] V. F. Mukhanov, in Complexity, Entropy and the Physics of Information,
City, California 1990)


[30] V. Berezin, gr-qc/9701017


[32] H. A. Kastrup, gr-qc/9707009

[33] See, for example, I. D. Novikov and V. P. Frolov, Physics of Balck Holes (Kluwer Academic Publishers, Dordrecht, Holland, 1989), and

