Scattering of Four-Dimensional Black Holes

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(August 16, 1997)

Abstract

The moduli space metric for an arbitrary number of extremal black holes in four dimensions with arbitrary relatively supersymmetric charges is found.
Black holes have proven to be an excellent testing ground for theories of gravity. In particular, one of the recent exciting developments in string theory has been the reproduction of many of the macroscopic black hole properties from the microscopic D-brane picture—for reviews and references, see e.g. [1,2]. At the same time, one of the current puzzles is the failed attempt in [3] to obtain, from a microscopic calculation, the macroscopic scattering of a D-string probe off a five-dimensional supersymmetric black hole carrying the maximum three charges. Specifically, the interaction that is quadratic in the charges was reproduced exactly, but the cubic term, which was seen in [4] to be a degeneration of a three point interaction, was not at all reproduced by the microscopic calculation.

In [4], the macroscopic scattering of an arbitrary number of the triply-charged supersymmetric five-dimensional black holes was given. A proposal for a microscopic calculation, based on the just-mentioned observation of the origin of the three-point interaction, was also given. In this paper, scattering of quadrupally-charged supersymmetric four-dimensional black holes will be discussed. The motivation here rests on the fact that these non-singular black holes can be made purely out of D-brane [5,6]. (If a supersymmetric four-dimensional black hole has fewer than four charges, then it will be singular at the horizon.) In principle, this makes the microscopic structure more transparent. This is in distinction to the five-dimensional case, where despite requiring only three charges for non-singularity at the horizon, there is no U-dual basis in which the charges are pure D-brane; the usual description is as collections of parallel 5-branes and strings, with momentum along the strings. The difference in four dimensions is due to the additional internal direction allowing the conditions for preservation of a supersymmetry to be satisfied by a more general brane configuration.

In section II we construct and discuss the black hole solution. The black hole solution that we use is actually familiar from the heterotic string—see e.g. [7,8]. We rederive the solution in a way that makes explicit its Type II origin; this complements the discussion in [9]. In section III we give the effective action that describes the scattering of several of
these black holes, and give a lengthy discussion of its U-dual generalization. In particular, the U-dual formulation of the three-point function is rather technical. While we only explicitly calculate the effective action for quadrupally charged black holes, we explain at the end of section III why the U-duality invariant formula should hold for arbitrary supersymmetric black holes including the black holes of [8] that carry five charges. In section IV we conclude with a discussion of the scattering of two black holes.

II. THE BLACK HOLE SOLUTION

In [5], black holes were constructed purely out of e.g. several D-4-branes, intersecting at arbitrary $U(3)$ angles in the compact torus, and a D-0-brane. In [6], the black holes were constructed out of orthogonally intersecting D-3-branes. However, because we will be doing the macroscopic calculation, it will be convenient to use neither of these descriptions in this paper. Instead, we would like to find an NS-NS description of the black holes, so we can use the formulas of [10,11] for the dimensionally reduced supergravity lagrangian. This can be obtained, for example, via the following series of dualities from the D-3-brane configuration of [6]:

\[ \begin{array}{c|cccccccc} & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ \hline \text{D-3} & X & X & X \\ \text{D-3} & X & X & X & \rightarrow & \text{T4,T5} & \text{D-1} & X & X \\ \text{D-3} & X & X & X & \rightarrow & \text{D-5} & X & X & X & X & X \\ \text{D-3} & X & X & X & X & \rightarrow & \text{D-3} & X & X & X & X \\ \text{D-3} & X & X & X & X & \rightarrow & \text{D-3} & X & X & X & X \\ \end{array} \right. \\

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\[ \begin{array}{c|cccccccc} & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ \hline \text{NS-1} & X & X & \rightarrow & \text{T5,T9} & \text{NS-1} & X & X \\ \text{ETN} & X & X & X & \rightarrow & \text{NS-5} & X & X & X & X & X \\ \text{D-1} & X & X & X & \rightarrow & \text{D-3} & X & X & X & X \\ \text{D-3} & X & X & X & X & \rightarrow & \text{D-3} & X & X & X & X \\ \end{array} \right. \\

(2.1)

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\[ \text{IIt is also possible to obtain the NS-NS black hole from the IIA NS-5-brane, D-6-brane and D-2-brane with momentum configuration of [12].} \]
Note that under the T-duality in the 6-direction (T6), the fundamental string parallel to the 6-direction transformed into a unit of Kaluza-Klein momentum. This is just the well-known momentum–winding exchange. Recalling that the Kaluza-Klein monopole is essentially the product of time and Euclidean Taub-NUT (ETN) [13,14], and that the NS-5-brane is the magnetic dual of the NS-string (c.f. equation (2.2c) below) the magnetic-dual of this phenomenon is the NS-5-brane–ETN transformation under the T9 perpendicular to the NS-5-brane [15] (compare also with [9]).

Now applying the harmonic function rule for orthogonally intersecting branes in ten-dimensions [16–18] gives (relabeling 9 → 4 and 6 → 9)

\[
\begin{align*}
    ds_{str}^2 &= \psi_1^{-1}[-dt^2 + dx_5^2 + \frac{Q_R}{r}(dt - dx_9)^2] + \psi_5 \psi^{-1}_E(dx_4 + Q_E(1 - \cos \theta) d\phi)^2 \\
    &\quad + \psi_5 \psi_E(dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2) + dx_5^2 + dx_6^2 + dx_7^2 + dx_8^2, \\
    \varphi &= \frac{1}{2} \ln(\psi_5 \psi_1^{-1}), \\
    H &= -Q_5 \sin \theta d\theta \wedge d\phi \wedge dx_4 + \psi_1^{-2} \frac{d\psi_1}{dr} dt \wedge dr \wedge dx_9,
\end{align*}
\]

(2.2a)

\[
\begin{align*}
    \psi_1 &= 1 + \frac{Q_1}{r}, \\
    \psi_5 &= 1 + \frac{Q_5}{r}, \\
    \psi_E &= 1 + \frac{Q_E}{r}.
\end{align*}
\]

(2.2b, 2.2c, 2.2d, 2.2e)

Here we have postulated an obvious generalization of the harmonic function rule to configurations involving the ETN; in particular, the ETN does not contribute an overall conformal factor, in analogy to the Kaluza-Klein momentum. For notational simplicity, only the one-centred black hole has been written; the generalization to the multi-black hole is almost obvious—see, e.g. [19] for details on multi-centred ETN. We have also set the string coupling constant \( g = e^{\varphi_{\infty}} = 1 \), where the subscript denotes evaluation at spatial infinity. The
$Q_\alpha$s are constants; see also equations (2.6b)–(2.6e). It is readily verified that equation (2.2) satisfies the equations of motion of the (string-frame) NS-NS IIB action

$$S = \frac{1}{16\pi G_{10}} \int d^{10}x \sqrt{-g} e^{-2\varphi} [R + 4(\nabla \varphi)^2 - \frac{1}{12} H^2],$$

(2.3)

where $G_{10} = 8\pi^6 g^2 \alpha'{}^4$ is the ten-dimensional Newton constant. This action, of course, describes the universal sector of all the string theories, and, in fact, the solution of equation (2.2) is not new, having been discussed in the context of the heterotic string in e.g. [7].

Dimensional reduction on a $T^6$ now proceeds in the usual way [10]. Of course, the NS-5-brane and the ETN give rise to magnetic charges in 4-dimensions; it is therefore convenient to dualize the corresponding vectors, and write the theory in terms of the magnetic vector potentials and field strengths for which the Bianchi identity and equation of motion are interchanged, e.g. $d\tilde{A}^{(2)}_4 \equiv \tilde{F}^{(2)}_4 = e^{-2\varphi} G^{44}\star F^{(2)}_4$, using the notation of [10], and tildes to denote magnetic quantities. The four-dimensional action and quadrupally charged static multi-black hole solution in the Einstein frame are then,

$$S = \frac{1}{16\pi G_4} \int d^4x \sqrt{-g} \left\{ R - 2(\partial_\mu \phi)^2 - \frac{1}{4} (\partial_\mu \ln G_{44})^2 - \frac{1}{4} (\partial_\mu \ln G_{99})^2 - \frac{1}{4} e^{2\varphi} G^{-1}_{44}(\tilde{F}^{(1)4})^2 - \frac{1}{4} e^{-2\varphi} G_{99}(F^{(2)}_{9\mu\nu})^2 - \frac{1}{4} e^{2\varphi} G^{44}(\tilde{F}^{(2)}_{4\mu\nu})^2 - \frac{1}{4} e^{-2\varphi} G^{-1}_{44}(F^{(2)}_{4\mu\nu})^2 \right\},$$

(2.4)

$$ds_4^2 = -(\psi_1 \psi_5 \psi_R \psi_E)^{-\frac{1}{2}} dt^2 + (\psi_1 \psi_5 \psi_R \psi_E)^{\frac{1}{2}} d\vec{x}^2,$$

(2.5a)

$$\varphi = \ln(\psi_1^{-\frac{1}{4}} \psi_5^{-\frac{1}{4}} \psi_R^{-\frac{1}{4}} \psi_E^{\frac{1}{4}}),$$

(2.5b)

$$G_{44} = \psi_5 \psi_E^{-1},$$

(2.5c)

$$G_{99} = \psi_1^{-1} \psi_R,$$

(2.5d)

$$A^{(2)}_9 = \psi^{-1}_1 dt,$$

(2.5e)

$$\tilde{A}^{(2)}_4 = \psi^{-1}_5 dt,$$

(2.5f)

$$A^{(1)9} = \psi^{-1}_R dt,$$

(2.5g)

$$\tilde{A}^{(1)4} = \psi^{-1}_E dt,$$

(2.5h)
where the $\psi_\alpha$, $\alpha \in \{1, 5, R, E\}$ are harmonic functions,

$$\psi_\alpha = 1 + \sum_{a=1}^{N} \frac{Q_{\alpha a}}{r_a}, \quad (2.6a)$$

$$Q_{1a} = \frac{4G_4 R_9}{\alpha'} n_{1a}, \quad (2.6b)$$

$$Q_{5a} = \frac{\alpha'}{2R_4} n_{5a}, \quad (2.6c)$$

$$Q_{Ra} = \frac{4G_4}{R_9} n_{Ra}, \quad (2.6d)$$

$$Q_{Ea} = \frac{R_4}{2} n_{Ea}, \quad (2.6e)$$

where the $n_{\alpha a}$ are non-negative integers. $N$ is the number of black holes, and $\vec{r}_a$ is their positions. The radii of the internal circles are $R_4, \ldots, R_9$ and the four-dimensional Newton constant is $G_4 = \frac{g_2^{\alpha'^4}}{R_4 R_5 R_6 R_7 R_8 R_9}$.  

### III. THE EFFECTIVE ACTION AND U-DUALITY

The Manton-type scattering calculation [20] proceeds exactly as in [21,22,4], so we leave out all the details here. The result to $O(\vec{v}^2)$ is

$$S_{\text{eff}} = \int dt \left\{ -\sum_a m_a + \frac{1}{2} \sum_a m_a \vec{v}_a^2 + \frac{1}{2l_p^2} \sum_{\alpha < \beta} \sum_a Q_{\alpha a} Q_{\beta b} |\vec{v}_a - \vec{v}_b|^2 \right\}$$

$$+ \frac{1}{4l_p^2} \sum_{\alpha < \beta} \sum_{a,b,c} Q_{\alpha a} Q_{\beta b} Q_{\gamma c} |\vec{v}_a - \vec{v}_b|^2 \left( \frac{1}{r_{ab} r_{ac}} + \frac{1}{r_{ab} r_{bc}} - \frac{1}{r_{ac} r_{bc}} \right)$$

$$+ \frac{1}{2l_p^2} \sum_{\alpha < \beta; \gamma < \delta} \sum_{a,b,c,d} Q_{\alpha a} Q_{\beta b} Q_{\gamma c} Q_{\delta d} |\vec{v}_a - \vec{v}_b|^2 \int d^3x \frac{\vec{r}_a \cdot \vec{r}_b}{4\pi r_a r_b r_c r_d} \right\}, \quad (3.1)$$

where saturation of the Bogomol’nyi bound gives [12]

$$m_a = \frac{1}{l_p^2} (Q_{1a} + Q_{5a} + Q_{Ra} + Q_{Ea}), \quad (3.2)$$

\[2\]For details on deriving the quantization of the charges and the value of the $D$-dimensional Newton constant, see e.g. [12]. In particular, we obtained the quantum of $Q_E$ by T-dualizing the quantum of $Q_5$. 

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and the four-dimensional Planck constant is \( l_p = \sqrt{\frac{4G_4}{\sqrt{2 R_4...R_9}}} \).

Note that the \( a = b \) terms in the multiple sums clearly don’t contribute. Furthermore, in the triple sum, the singular terms for \( a = c \) or \( b = c \) cancel, and in the quadruple sum, the integral converges, even when two or more of the coordinates coincide. When one of the charges, say \( Q_{E_a} \) vanishes for every black hole, then equation (3.1) reduces to the result of [4] when the latter is reduced from five to four dimensions, as required by the arguments of [23].

We would now like to make equation (3.1) U-duality invariant. The U-dual expression for the terms linear and quadratic in the charges follow exactly as in [3,4]. In particular the mass is (in a sense elaborated below) already invariant, and the quadratic term involves the masses and contraction of two factors of the \( E_{7(7)} \) charge vector \( q_{\Lambda a} \) with the inverse of the matrix of moduli, \((M^{-1}_{\infty})^{\Lambda \Sigma}\) (see equation (3.9)); \( M_{\Lambda \Sigma} \) is the matrix which multiplies the kinetic term for the vector fields in the \( E_{7(7)} \) invariant action. The quartic term is clearly proportional to the quartic invariant of the U-duality group \( E_{7(7)} \). However, while in [4] the cubic term was proportional to the cubic invariant of the five dimensional U-duality group \( E_{6(6)} \), we can not directly associate such an interpretation to it in this case, since \( E_{7(7)} \) has no cubic invariant, nor does \( E_{6(6)} \) imbed itself into \( E_{7(7)} \) in an intrinsically natural way. Furthermore, it can be checked that we cannot use the invariants made out of the matrix of moduli, the charge vectors and the masses to obtain the cubic term; we can understand this because an expression involving the matrix of moduli could not reduce to the moduli independent \( E_{6(6)} \) formula.\(^3\)

Instead, we recall, following [24,11,25], that there is an intrinsically natural way of dissecting the \( D = 4, N = 8 \) central charge matrix. Specifically, we note that for each black hole the moduli-dependent central charge matrix \( Z_a \) can be \( SU(8) \subset E_{7(7)} \) rotated into the

\(^3\) There will actually be overall matrix of moduli factors that arise during the compactification from five to four dimensions.
form

$$Z_a = \text{diag}\{z_{1a}, z_{2a}, z_{3a}, z_{4a}\} \otimes \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad (3.3)$$

with the $z_a$'s the (possibly complex) "eigenvalues". The largest eigenvalue, which we choose to be $z_{1a}$, is the mass of the $a$th black hole, by the BPS condition. In fact, since $z_{1a} = l_p^2 m_a$, it, and more technically an $SU(2) \subset SU(8)$, is singled out. This was explained in [25] as the $SU(2)$ corresponding to the supercharges (which transform linearly under the $SU(8)$ automorphism) for which a complex linear combination annihilates the state. This is just the statement that it corresponds, by the BPS condition, to the unbroken supersymmetry.

In the case at hand, [11,24]

$$z_{1a} = Q_{1a} + Q_{5a} + Q_{Ra} + Q_{Ea} = l_p^2 m_a, \quad (3.4a)$$

$$z_{2a} = Q_{1a} - Q_{5a} + Q_{Ra} - Q_{Ea}, \quad (3.4b)$$

$$z_{3a} = Q_{1a} + Q_{5a} - Q_{Ra} - Q_{Ea}, \quad (3.4c)$$

$$z_{4a} = Q_{1a} - Q_{5a} - Q_{Ra} + Q_{Ea}. \quad (3.4d)$$

It is easily checked that

$$\sum_{\alpha \neq \beta \neq \gamma} Q_{\alpha a} Q_{\beta b} Q_{\gamma c} = \frac{1}{16} \left\{ z_{1a} \left( z_{1b} z_{1c} - \sum_{l=2}^{4} z_{lb} z_{lc}^* \right) + (5 \text{ perms}) \right\}$$

$$+ \frac{1}{8} \left\{ \text{Re}(z_{2a} z_{3b} z_{4c}) + (5 \text{ perms}) \right\}, \quad (3.5)$$

where again we have taken into account the fact that for the more general black holes, the $z_a$ are complex. Furthermore, if we set $Z_a$ real and traceless, then we can make contact with the real, traceless five dimensional central charge matrix. Specifically, in this case equation (3.5) is equivalent (up to a proportionality constant) to the 5-dimensional $E_{6(6)}$ symmetric invariant, written in the form [11] $\sum_{l=1}^{4} z_{la} z_{lb} z_{lc}$. In other words, when we

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But note that only the overall phase is invariant and not the individual phases.
restrict to black holes with three charges (or fewer), then we recover the five-dimensional U-duality invariant formula.

Of course, we still need to convert equation (3.5) into a “U-duality” invariant formula involving the charge vectors $q_{\lambda a}$. The formula won’t be truly U-duality invariant because we are decomposing $E_{7(7)} \supset SU(8) \supset SU(2) \times SU(6)$; it will only be invariant under the subgroup.\(^5\) The central charge matrix transforms linearly in the $28 \oplus \overline{28}$ of $SU(8)$; under the above decomposition \([26,25]\),

$$28 \oplus \overline{28} \rightarrow (1,15) \oplus (1,\overline{15}) \oplus (2,6) \oplus (2,\overline{6}) \oplus (1,1) \oplus (1,1).$$

(3.6)

Clearly, $z_{1a} \in (1,1)$ and the other $z_a \in (1,15)$. So, the first term of equation (3.5) is

$$(1,1) \otimes \{(1,1) \}^2 + (1,15) \otimes (1,\overline{15}) \}$$

(3.7a)

which indeed contains a singlet, as required. The second term of equation (3.5) is

$$[(1,15)]^3 + [(1,\overline{15})]^3,$$

(3.7b)

and again each term contains a singlet.\(^6\) Note that this explains our choices of complex conjugation on the right-hand side of equation (3.5); any other polynomial choice that reduces to the left-hand side, and treats the $z_a$s and $z^*_a$s symmetrically\(^7\), would not be a singlet. Thus, we have arrived at equation (3.5) uniquely.

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\(^5\)As the $SU(8)$ is the maximal compact subgroup of $E_{7(7)}$, this is (almost) the maximal decomposition of $E_{7(7)}$ involving our $SU(2)$ factor. There is also a possible $U(1)$ factor; however, we have fixed the $U(1)$ by demanding that $z_{1a} = l_p^2 m_a$, i.e. by fixing that $z_{1a}$ be real.

\(^6\)This follows since the $15 \in SU(6)$ is an antisymmetric product of two fundamentals. The antisymmetric product of six fundamentals is clearly a singlet; this is the symmetric product of three $15$s.

\(^7\)This is required since there is no invariant distinction between the complex representations of $SU(6)$ and their complex conjugates.
So, to write down a more invariant expression we decompose the integer-valued $E_7(7)$ charge vector $q_\Lambda a$. More precisely, since we were working with the central charge matrix, which is moduli dependent, it is convenient to raise the index using the matrix of moduli:

$$q^\Lambda a \equiv (M^{-1}_\infty)^{\Lambda \Sigma} q_{\Sigma a}.$$  \hspace{1cm} (3.8)

Then we can decompose $q^\Lambda a$ as \{m_a, q^A_a, q^{\bar{A}}_a, \ldots\} where $m_a = l_p^{-2} z_{1a}$ has been used for the $(1,1)$; the index $A, \bar{A} = 1, \ldots, 15$ labels respectively the $15, \overline{15} \in SU(6)$; and the ellipses denote the representations that have not been included. Then, we finally have the U-duality invariant version of equation (3.1).

$$S_{\text{eff}} = \int dt \left\{ - \sum a m_a + \frac{1}{2} \sum a m_a \vec{v}_a^2 + \frac{1}{2} \sum a \lt (l_p^2 m_a m_b - q_{\Lambda a}(M^{-1}_\infty)^{\Lambda \Sigma} q_{\Sigma b}) \rt |\vec{v}_a - \vec{v}_b|^2 \right. \frac{r_{ab}}{r_{ab}} \\
+ \frac{3}{32} \sum a \lt l_p^4 m_a m_b m_c - \frac{l_p^2}{6} (m_a q^b \delta^{AA} q^c + 5 \text{ perms}) + l_p d_{(6)ABC} q^A_a q^B_b q^C_c \rt |\vec{v}_a - \vec{v}_b|^2 \left[ \frac{1}{r_{ab} r_{ac}} + \frac{1}{r_{ab} r_{bc}} - \frac{1}{r_{ac} r_{bc}} \right] \\
+ \frac{l_p^2}{4} \sum a \lt d^{\Sigma \Pi} q_{\Lambda a} q_{\Sigma b} q_{\Pi c} q_{\Delta d} \rt |\vec{v}_a - \vec{v}_b|^2 \int d^3 x \frac{\vec{r}_a \cdot \vec{r}_b}{4 \pi r_a^2 r_b^2 r_c r_d} \right\} \hspace{1cm} (3.9)
$$

Here, $d_{(6)ABC}$ is proportional to the symmetric cubic invariant for the $15 \in SU(6)$, and $d^{\Sigma \Pi}$ is proportional to the $E_7(7)$ cubic invariant.

Two final comments are required regarding the decomposition $E_7(7) \supset SU(8) \supset SU(2) \times SU(6)$. First, it appears that we have assumed that the central charge matrices for the black holes can be simultaneously diagonalized (in the sense of equation (3.3)). However, all we really need to assume is that they can be simultaneously block-diagonalized into $SU(2)$ and $SU(6)$ subgroups; our final expression, equation (3.9) is $SU(6)$ invariant and so does not require that the matrices be diagonal. That the block-diagonalization is possible is simply the statement that the black holes preserve a common supersymmetry. Incidentally, the block-diagonalization implies that the charges that transform in the $(2,6)$ representation of $SU(2) \times SU(6)$ vanish; this is why there is no $(1,1) \otimes (2,6) \otimes (2,\overline{6})$ term in equation (3.5) or (3.9).
Second, it was implicitly assumed in the discussion that the solution preserves exactly $\frac{1}{8}$ of the supersymmetry. If the solution preserves more supersymmetry—i.e. if more than one $z_a = l^a_p m_a$—then there is no longer a natural $SU(2) \subset SU(8)$ but rather a larger subgroup that is selected. Nevertheless, it is easy from equation (3.5) to see that no matter how one chooses the $SU(2) \subset G$ (where, for a solution preserving $\frac{1}{4}$ of the supersymmetry, $G = SU(4)$, for example) one obtains the same answer for the cubic, namely zero, so there is no ambiguity when there is more supersymmetry.

We now claim that equation (3.9), which was really only derived for the special case of black holes with four charges, holds for general (e.g. five charge) supersymmetric black hole configurations. The forms of the two-point, three-point and four-point functions have already been fixed uniquely by equation (3.1) and the duality symmetries, so the only possible modification with the additional charges, is the appearance of higher-point functions. Since these higher-point functions vanish when only four charges are non-zero, they must be proportional to all five separate charges, and hence cannot be made out of invariants of order less than five. But the group theory that led us to the “U-duality” invariant form for the cubic term shows us that there are no such candidate invariants: all we can work with is the $(\mathbf{1}, \mathbf{15})$ and its complex conjugate, and since cubing one gives a singlet, and multiplying one by the other gives a singlet we can never get a higher-order invariant. This leaves the $E_{7(7)}$ invariants, and the only symmetric one is the quartic. Thus, equation (3.9) is the general result.

IV. DISCUSSION

We have given the effective action to $O(\vec{v}^2)$ for scattering of an arbitrary number of charged supersymmetric four dimensional black holes. The U-duality invariant form required a technical discussion of the natural $SU(2) \times SU(6)$ decomposition of the four-dimensional U-duality group $E_{7(7)}$. We would now like to give a slightly more detailed discussion of the asymptotically flat moduli space for two black holes. From equation (3.9), the moduli space
is
\[ ds^2 = \frac{1}{2} f(\tilde{r})(dr^2 + r^2 d\Omega^2), \] (4.1a)

where
\[ f(\tilde{r}) = \mu + \frac{\Gamma_{II}}{r} + \frac{\Gamma_{III}}{r^2} + \frac{\Gamma_{IV}}{r^3}, \] (4.1b)
\[ \Gamma_{II} = l_p^2 M \mu - q_{A1}(M^{-1}_\infty)^{\Lambda\Sigma} q_{\Sigma 2}, \] (4.1c)
\[ \Gamma_{III} = \frac{3}{16} \left\{ l_p^4 M^2 \mu + l_p d_{(6)ABCD} q_1^A q_2^B (q_1^C + q_2^C) + l_p d^*_{(6)ABCD} q_1^A q_2^B (q_1^C + q_2^C) \right. \\
- \left. \frac{l_p^2}{3} \left( M q_1^A \delta_{AA} q_2^\Lambda + M q_2^A \delta_{AA} q_1^\Lambda + m_1 q_2^A \delta_{AA} q_2^\Lambda + m_2 q_1^A \delta_{AA} q_1^\Lambda \right) \right\}, \] (4.1d)
\[ \Gamma_{IV} = \frac{l_p^2}{6} d^{\Lambda \Sigma \Pi} q_{A1} q_{\Sigma 2} (q_{\Gamma 1} q_{\Pi 1} + q_{\Gamma 2} q_{\Pi 2}), \] (4.1e)
and the centre of mass (relative mass) is \( M = m_1 + m_2 \) (\( \mu = \frac{m_1 m_2}{m_1 + m_2} \)); also the relative coordinate is \( \tilde{r} = r_2 - r_1 \). In equation (4.1a), we have subtracted away the centre of mass motion. We have also omitted the term
\[ \frac{\pi^3 l_p^4}{4} d^{\Lambda \Sigma \Pi} q_{A1} q_{\Sigma 1} q_{\Gamma 2} q_{\Pi 2} \delta^{(3)}(\tilde{r}), \] (4.1f)
from equation (4.1b) because this contact interaction only occurs at \( r = 0 \) by which point the moduli space approximation has presumably broken down. We recall that the moduli space approximation is only valid for small velocities and neglects radiation; in particular, in [21] it was argued that the moduli space approximation breaks down for \( r \lesssim v_\infty^2 M \).

The coordinate singularity at \( r = 0 \) of equation (4.1a) is removed, when \( \Gamma_{IV} \neq 0 \), by performing the coordinate transformation \( \xi = -2\frac{\alpha^{\Lambda}}{r^{\Lambda}} \). One then finds that as \( r \to 0 \), \( (\xi \to \infty) \), there is a second asymptotic region that is conical with deficit angle \( \pi \) [21]. For \( \Gamma_{IV} = 0 \) but \( \Gamma_{III} \neq 0 \), the coordinate singularity at \( r = 0 \) is removed by the coordinate transformation \( \xi = \ln \frac{r}{\sqrt{\alpha}} \) to find that the asymptotic region has topology \( \mathbb{R} \times S^2 \). If both \( \Gamma_{IV} = 0 \) and \( \Gamma_{III} = 0 \), then one removes the \( r = 0 \) coordinate singularity via \( \xi = \sqrt{\alpha^\Lambda} \) to again find a second asymptotic region that is conical with deficit angle \( \pi \) [22]. Thus, geodesics which extend to \( r = 0 \) enter this second asymptotic region; i.e. the black holes
coalesce [21,22,3]. (It might seem strange that, having just rejected the contact interaction for being at \( r = 0 \), we are exploring the \( r \to 0 \) behaviour. The point is that we can use equations (4.1), even as \( r \to 0 \) (but \( r \neq 0 \)), by taking \( v_\infty \to 0 \).)

By examining the geodesic equation as in [3], we find that the turning point \( r_c \) in the black hole motion is real and positive when the impact parameter \( b \) obeys

\[
b^2 > b_c^2 \equiv -\frac{\Gamma^2_{II}}{12\mu^2} + \frac{\Gamma_{II}}{\mu} + \frac{\Gamma^4_{II} + 18\Gamma_{II} \Gamma_{IV} \mu^2}{12 \left(-\left(\Gamma_{II}^5 \mu^5 + 540 \Gamma_{II}^3 \Gamma_{IV} \mu^3 + 243 \Gamma_{IV}^2 \mu^{10} + 540 \sqrt{\Gamma_{IV} \mu^{14} \left(-\Gamma_{II}^3 + 27 \Gamma_{IV} \mu^2 \right)^2} \right) \right)^{\frac{1}{3}}}
\]

(4.2)

In particular there is coalescence for \( b \leq b_c \). Naïvely, \( b_c \) is only well-defined if either \( \Gamma_{IV} = 0 \) or \( \Gamma_{II}^3 \leq 27 \Gamma_{IV} \mu^2 \); in fact, one must merely be careful about how one chooses the cube roots in equation (4.2).

As equation (4.2) is rather obscure, we point out some special cases. If we consider the black holes of section II, with \( Q_{1a} = Q_{5a} = Q_{Ra} = Q_{Ea} = l^2 m_a l^2 \mu \), then we have the Reissner-Nordström black holes of [21]\(^8\) for which the right-hand side of equation (4.2) is real and positive, and so there can be coalescence. Note that this is despite the fact that for Reissner-Nordström black holes, \( 27 \Gamma_{IV} \mu^2 - \Gamma_{II}^3 < 0 \). For two Reissner-Nordström black holes of equal mass (\( m_1 = m_2 \)), \( b_c \approx 2.3660 l^2 M \), in agreement with [21].

If the black holes only carry three charges, then \( \Gamma_{IV} = 0 \); we find \( b_c = \sqrt{\frac{\Gamma_{III}}{\mu}} \).\(^9\) In fact, if the black holes carry fewer than three charges, so that also \( \Gamma_{III} = 0 \), then there is never coalescence (except for the obvious case \( b = 0 \)); this is in agreement with the results of [22]. This is different from higher dimensions, where there is always a critical, non-zero impact parameter below which there is coalescence [22,4].

\(^8\)Equation (4.2) corrects the polynomial equation for \( b_c \) that was given in the reference.

\(^9\)To obtain this from equation (4.2) requires choosing \((-1)^{\frac{1}{3}} = \frac{1}{2} \pm i\frac{\sqrt{3}}{2}\).
I thank Vijay Balasubramanian, Eric Gimon, Gary Horowitz, John Pierre, Joe Polchinski, Andy Strominger and Haisong Yang for useful discussions. I also thank David Kaplan for collaboration on an earlier, related paper. I am grateful to Harald H. Soleng for making [27] available, which was very useful in the computation. The hospitality of the Physics Department at Harvard University is appreciated. Financial support from NSERC and NSF is gratefully acknowledged; this work was also supported in part by DOE Grant No. DOE-91ER40618.
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