Integrability of the Pairing Hamiltonian

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Abstract

We show that a many-body Hamiltonian that corresponds to a system of fermions interacting through a pairing force is an integrable problem, i.e. it has as many constants of the motion as degrees of freedom. At the classical level this implies that the Time-dependent Hartree-Fock- Bogoliubov dynamics is integrable and at the quantum level that there are conserved operators of two-body character which reduce to the number operators when the pairing strength vanishes. We display these operators explicitly and study in detail the three-level example.

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1 Introduction

Two kinds of simple models are commonly used in nuclear physics for displaying the essential properties of the nuclear interaction, the particle-hole and the particle-particle models. The simplest one, the Lipkin model[1], consists of two levels of equal degeneracy and fermions interacting through a particle-hole force. It has been extensively used to test various approximation schemes and its classical version, provided by the Time-Dependent Hartree-Fock (TDHF) approach, is integrable. When this model is extended to three or more levels [2, 3, 4] one finds that the TDHF approximation yields a classical problem which is non integrable, displays various degrees of chaotic behavior and has an intricate and interesting phase-space structure [4].

In principle the analogous situation for the particle-particle interaction can be thought to behave in a similar way. A two-level model with a pairing force [5] is integrable [6, 7] and one would expect that the extension to three or more levels would yield non-integrable TDHF-Bogoliubov (TDHFB) dynamics. In this paper we report the fact that this is not so and that the problem of a pairing force acting in a restricted shell model space with \( L \) single-particle levels turns out to be integrable both classically and quantum mechanically. We display explicitly the constants of the motion involved and we study their properties, their group structure and their classical limits.

The outline of the paper is as follows. In Section 2 we review the pairing model, in both its quantum and classical aspects. The classical limit is obtained by a large scale degeneracy argument that leads to the TDHFB equations of motion. Section 3 is devoted to the search for new constants of the motion, i.e. new operators commuting with the Hamiltonian. These constants of the motion, which are not unique, turn out to be non trivial two body operators involving the coupling constant and the single particle energies. A set of commuting operators is thus constructed that renders the problem integrable. In Section 4 we treat the three-
level case and show explicitly the consequences of this integrability both at the quantum and classical levels. The last Section is devoted to conclusions and final remarks.

2 The model

The pairing force is a very general interacting mechanism that has an ubiquitous role in the quantum many body problem. In electron systems it leads to the superconducting mechanism and in nuclear physics to the collectivity associated to the pairing degree of freedom. In this latter case it provides a simplified description of the short range part of the nuclear interaction [8]. A schematic model that incorporates this basic mechanism can be defined by interacting fermions that can occupy $L$ different single-particle shells of degeneracy $2\Omega_i$ and single-particle energies $\epsilon_i$. The fermions interact via a monopole pairing force. The Hamiltonian of such a system is

$$H = \sum_{i=1}^{L} 2\epsilon_i K_i^0 - \frac{G}{2}(K^+K^- + K^-K^+),$$

(1)

where

$$K_i^0 = \frac{1}{2} \sum_{m_i} (b_{j,m_i}^\dagger b_{j,m_i} - \frac{1}{2}),$$

(2)

$$K^+ = \sum_{i=1}^{L} K_i^+ = \sum_{i=1}^{L} \frac{1}{2} \sum_{m_i} (-1)^{j_i-m_i} b_{j,m_i}^\dagger b_{j,-m_i}^\dagger,$$

(3)

and

$$K^- = (K^+)^\dagger = \sum_{i=1}^{L} K_i^-.$$

(4)

The $(b_{j,m_i}^\dagger, b_{j,m_i})$ are the usual fermion operators obeying anticommutation relations which create or annihilate a fermion on the i-th shell which has degeneracy $2j_i + 1 = 2\Omega_i$. The $K_i^0$ operators count pairs of particles $N_i$ in each shell by $K_i^0 = (N_i - \frac{1}{2}\Omega_i)$. The operators $(K_i^+, K_i^-, K_i^0)$ conform an $SU(2)$ algebra whose Casimir operator is

$$k_i^2 = K_i^{02} + \frac{1}{2}(K_i^+K_i^- + K_i^-K_i^+).$$

(5)
The full dynamics of the system occurs in the group space of $[SU(2)]_1 \times [SU(2)]_2 \times \ldots \times [SU(2)]_L$.

The classical limit of the model is obtained from the TDHFB approximation when the degeneracy $\Omega_i$ of each level goes to infinity [6]. $\Omega_i^{-1}$ is the semiclassical parameter analogous to $\bar{\hbar}$ in the usual semiclassical treatments.

One way of obtaining this limit is through the time dependent variational principle implemented through coherent states that are constructed from the vacuum (or minimal weight) state $|0\rangle$, characterized by

$$K_i^- |0\rangle = 0K_i^0 |0\rangle = -\frac{\Omega_i}{2} |0\rangle \quad \quad (6)$$

The coherent state in this representation is [10]

$$|\z\rangle = |z_1 \ldots z_L\rangle = e^{\sum_{i=1}^{L} \z_i K_i^+} |0\rangle \quad \quad (7)$$

The equations of motion obtained through the time-dependent variational principle with this state are equivalent to the TDHFB equations. To obtain them we use the variational principle appropriate for non-normalized states [9] with an action defined as (we set $\bar{\hbar} = 1$)

$$S = \int dt \left[ \frac{1}{2} \frac{\langle \psi | \dot{\psi} \rangle - \langle \dot{\psi} | \psi \rangle}{\langle \psi | \psi \rangle} - \frac{\langle \psi | H | \psi \rangle}{\langle \psi | \psi \rangle} \right] \quad \quad (8)$$

A detailed derivation is given in [6]. The variables $z_i$ are not canonical but the transformation

$$\omega_i = \sqrt{\Omega_i} z_i \quad 0 \leq \omega_i \bar{\omega}_i \leq \Omega_i \quad \quad (9)$$

yields canonical variables satisfying $\{\omega_i, \bar{\omega}_j\} = \delta_{ij}$. The finite range of these variables is a consequence of the Pauli exclusion principle between correlated fermion pairs. In terms of $\omega_i$ the variational equations become ordinary hamiltonian equations in complex form,

$$i \dot{\omega}_i = \frac{\partial H}{\partial \omega_i} \text{ and c.c.} \quad \quad (10)$$
where $H = \lim_{\Omega_i \to \infty} \frac{\langle \psi | H | \psi \rangle}{\langle \psi | \psi \rangle}$ is the classical hamiltonian associated to the problem. In the same way for any quantum operator $\hat{A}$ its classical limit in this representation is,

$$A = \lim_{\Omega_i \to \infty} \frac{\langle \psi | \hat{A} | \psi \rangle}{\langle \psi | \psi \rangle}.$$  \hspace{1cm} (11)

Therefore the operators $(K^+_i, K^-_i$ and $K^0_i)$ have their classical analogues, $(\mathcal{K}^+_i, \mathcal{K}^-_i, \mathcal{K}^0_i)$, which can be written in terms of the $\omega_i$ variables

$$\mathcal{K}^0_i = \omega_i \bar{\omega}_i - \frac{\Omega_i}{2}$$  \hspace{1cm} (12)

$$\mathcal{K}^+_i = \omega_i \sqrt{\Omega_i - \omega_i \bar{\omega}_i}$$  \hspace{1cm} (13)

$$\mathcal{K}^-_i = \bar{\omega}_i \sqrt{\Omega_i - \omega_i \bar{\omega}_i}.$$  \hspace{1cm} (14)

This last set of operators obey the classical $SU(2)$ Poisson bracket relations [9].

The classical pairing hamiltonian can then be expressed, in analogy with the quantum one, directly in terms of the $SU(2)$ generators,

$$H = \sum_{i=1}^{L} 2\epsilon_i \mathcal{K}^0_i - G(\mathcal{K}^+ \mathcal{K}^- + \sum_{i=1}^{L} \frac{\Omega_i}{2}).$$  \hspace{1cm} (15)

Energy conservation is guaranteed by the time-dependent variational principle [11] so that the motion occurs in the $(2L-1)$-dimensional manifold defined by $H(\omega, \bar{\omega}) = E$. There is a further constant of the motion linked to the conservation of the total number of pairs [6]

$$\mathcal{N}(\omega, \bar{\omega}) = \sum_{i=1}^{L} \frac{<z|N_i|z>}{<z|z>} = \sum_{i=1}^{L} \omega_i \bar{\omega}_i.$$  \hspace{1cm} (16)

The TDHFB equations are then classical hamiltonian equations in a phase space of $2L$ variables. The conservation of $H$ and $\mathcal{N}$ implies that the case $L = 2$ is classically integrable, a fact that was exploited in [6] to compute energy levels and transition matrix elements semiclassically.

### 3 Search for additional constants of the motion

The proof of the integrability for $L > 2$ requires the existence of $L - 1$ independent, well-defined, global functions (constants of the motion), whose Poisson brackets with each other and
with the hamiltonian vanish. Following Hietarinta[12] a quantum mechanical hamiltonian of 
$L$ degrees of freedom is defined to be quantum integrable if there are $L − 1$ independent, well 
defined, global operators (quantum invariants) which commute with each other and with 
the hamiltonian. Then the energy spectrum of a quantum integrable hamiltonian system is 
naturally labeled by $L$ quantum numbers [13], which are the eigenvalues of the corresponding 
quantum invariants. Likewise the stationary states are simultaneous eigenfunctions of the $L$ 
corresponding operators.

For the cases $L = 1$ and $L = 2$ the integrability is trivial. $H$ and $K_0$ (or $N$) provide the 
commuting operators and $\mathcal{H}$ and $\mathcal{K}_0$ (or $\mathcal{N}$) the corresponding classical conserved quantities. 
But in the case of $L > 2$ no new obvious quantum invariants are present and we could expect 
the system to be non integrable and therefore display generically regions of chaotic behavior. 
However we now show how to construct non trivial operators, independent of the hamiltonian 
and the total number of pairs and commuting with them, which make the problem integrable.

For this purpose let us construct the more general two-body operator $O$ which is hermitian 
and conserves the total number of pairs

$$
R = \sum_{i,j} \left( \alpha_{ij} K_i^0 K_j^0 + \beta_{ij} K_i^+ K_j^- \right) + \sum_i \gamma_i K_i^0 
$$

(17)

with $\alpha_{ij} = \alpha_{ji}$ and $\beta_{ij} = \beta_{ji}$, and require it to commute with the hamiltonian

$$
[H, R] = 0.
$$

(18)

The following relations among its coefficients

$$
\alpha_{ij} = \beta_{ij} 
$$

(19)

$$
\alpha_{ij} = -\frac{G}{2} \frac{\alpha_{ii} + \gamma_i - \alpha_{jj} - \gamma_j}{(\epsilon_i - \epsilon_j)},
$$

(20)

are sufficient for the commutativity but they do not determine completely the coefficients. 
Thus several solutions can be found, and one should further check that the operator con- 
structed is independent of $H$ and $N$. 

6
For example if we choose

\[ \alpha_{ii} = 0 \text{and} \gamma_i = \epsilon_i \]  

we then would have

\[ R = \frac{H}{2} - \frac{G}{2} \left[ (K^0)^2 + \sum_{i=1}^{L} k_i^2 \right] \]  

where

\[ k_i^2 = (K_i^0)^2 + \frac{1}{2}(K_i^+ K_i^- + K_i^- K_i^+) \]  

is the Casimir of the SU(2) algebra in the ith-level. In this case \( R \) is not useful because it is not independent of the conserved magnitudes that we already know.

However another choice

\[ \alpha_{ii} = 0 \quad \text{and} \quad \gamma_i = \delta_{ii} \]  

leads to a set of operators

\[ R_i = K_i^0 - G \sum_{j \neq i}^{L} \frac{k_i \cdot k_j}{(\epsilon_i - \epsilon_j)} \]  

with \( k_i \cdot k_j = K_i^0 K_j^0 + \frac{1}{2}(K_i^+ K_j^- + K_i^- K_j^+) \).

In the non interacting limit (\( G = 0 \)) these operators become the natural set of commuting operators \{\( K_i^0; i = 1, \ldots, L \}\}. A straightforward but tedious calculation shows that

\[ [R_i, R_j] = 0 \quad i, j = 1 \ldots L \]  

for any value of \( G \) and \( \epsilon_i \). On the other hand, it is easy to see that \( H \) and \( N \) can be written in terms of the \( R_i \) as

\[ N = \sum_{i=1}^{L} R_i \]  

and

\[ H = \sum_{i=1}^{L} 2\epsilon_i R_i + G(\sum_{i=1}^{L} R_i)^2 - G \sum_{i=1}^{L} k_i^2. \]
Therefore we have constructed a set of $L$ commuting operators which also commute with the pairing hamiltonian and are number conserving. They extend to the fully interacting case the trivial properties of the number operators of the $G = 0$ system. Consideration of these operators then demonstrate the integrability of the quantum problem. The simultaneous eigenvalue equations

$$R_i |\psi(\lambda_1 \ldots \lambda_L)\rangle = \lambda_i |\psi(\lambda_1 \ldots \lambda_L)\rangle$$

(29)
gives the eigenvalues of the hamiltonian as

$$H = \sum_{i=1}^{L} 2\epsilon_i \lambda_i + G(\sum_{i=1}^{L} \lambda_i)^2 - \frac{G}{4} \sum_{i=1}^{L} (\Omega_i^2 - 1).$$

(30)

It should be noticed however that the actual solution is by no means simplified by this knowledge. The operators $R_i$ are two body operators as complicated in principle as the hamiltonian itself and they cannot be used (except by diagonalizing them) to separate the hamiltonian into invariant subspaces. however, as we will see, the simple fact of their existence has very drastic implications on the structure of both eigenvalues and eigenfunctions.

In the classical limit the associated operators for the set $\{R_i; i = 1 \ldots L\}$ written in terms of the classical operators ($K_i^+, K_i^-, K_i^0$) are

$$R_i = K_i^0 - G \sum_{j \neq i}^{L} \frac{K_i^j K_j^i}{(\epsilon_i - \epsilon_j)}$$

(31)

where $K_i^j, K_j^i = K_i^0 K_j^0 + \frac{1}{2}(K_i^+ K_j^- + K_i^- K_j^+)$.

Taking into account (1), (15), (25) and (31) we can see that the classical operators $R_i, N$ and $H$ have the same structure (in terms of the $SU(2)$ algebra generators ) as their quantum analogs, except for additive constants. It is then easy to see that

$$\{H, R_i\} = 0 \quad \{N, R_i\} = 0$$

(32)

and

$$\{R_i, R_j\} = 0.$$
Analogously to the quantum case, the mean number of pairs $N$ and the energy $H$ are

$$N(R_1 \ldots R_L) = \sum_{i=1}^{L} R_i$$

and

$$H(R_1 \ldots R_L) = \sum_{i=1}^{L} 2\epsilon_i R_i + G(\sum_{i=1}^{L} R_i)^2 - \frac{G}{4} \sum_{i=1}^{L} \Omega_i^2.$$ 

The functions $R_i(\omega, \bar{\omega})$ are surfaces on the $2L$ dimensional phase space. The trajectories lie in the intersection of these surfaces, which are $L$ dimensional tori labeled by the constants $R_i$. Chaotic motion therefore cannot occur in this system.

4 Manifestations of integrability in the three-level case

In this section we restrict our analysis to the three-level case ($L = 3$) and equal degeneracy $\Omega_i = \Omega$ and show the consequences of the integrable behavior, both in the quantum and in the classical solutions.

As in the previous section we will start with the quantum treatment. Although the problem has three degrees of freedom we have already seen that the total number of pairs $N$ in the system is a conserved magnitude. We can then use this fact to reduce explicitly the dimensionality to two degrees of freedom and therefore we only need to display another quantum invariant operator to have an integrable quantum problem. We then choose it as $O_2$ which is a linear combination of the $R_i$ operators defined in the previous section,

$$O_2 = \sum_{i=1}^{3} \epsilon_i^2 R_i.$$ 

The quantum integrability shows up clearly when we display a grid of the simultaneous eigenvalues of $H$ and $O_2$. This is done in Fig. 1 for the case where the total number of pairs is $N = \Omega = 14$, which corresponds to a set of 120 levels. We find that the eigenvalues lie on a regular grid that includes all eigenvalues. The fact that this grid is not parallel
and equally spaced reflects the fact that $O_2$ and $H$ are not action variables that quantize at integer spaced values. Of course a point transformation to a set $S_1$ and $S_2$ exists if $O_2$ and $H$ are independent. But we do not construct this variables explicitly. It is clear however that the grid we obtain is a smooth deformation of a regular one. This proves that the two operators ($O_2$ and $H$) are independent and that their simultaneous eigenvalues form a set of good quantum numbers[13].

For the classical analysis we introduce the non-interacting action-angle coordinates,

$$n_i = \omega_i \tilde{\omega}_i$$

(38)

$$\phi_i = \text{arg}(\omega_i)$$

(39)

where $n_i$ is the mean number of pairs in the level $i$ and is now a continuous classical variable.

In these variables the classical hamiltonian is,

$$\mathcal{H} = 2\epsilon_1 n_1 + 2\epsilon_3 n_3 - \Omega(\epsilon_1 + \epsilon_3) - G \left\{ n_1(\Omega - n_1) + n_2(\Omega - n_2) + n_3(\Omega - n_3) \right\}$$

$$+ 2\sqrt{n_1n_2}(\Omega - n_1)(\Omega - n_2) \cos(\phi_1 - \phi_2)$$

$$+ 2\sqrt{n_1n_3}(\Omega - n_1)(\Omega - n_3) \cos(\phi_1 - \phi_3)$$

$$+ 2\sqrt{n_2n_3}(\Omega - n_2)(\Omega - n_3) \cos(\phi_2 - \phi_3) \right\}$$

(40)

where we have taken $\epsilon_2 = 0$ as the energy reference.

The analysis is best performed by explicit elimination of the conserved quantity $N = \sum n_i$. We introduce the canonical transformation,

$$I_1 = \frac{n_1}{\Omega} \quad \theta_1 = \phi_1 - \phi_3$$

(41)

$$I_2 = \frac{n_2}{\Omega} \quad \theta_2 = \phi_2 - \phi_3$$

$$I_3 = \frac{1}{\Omega}(n_1 + n_2 + n_3) = 1 \quad \theta_3 = \phi_3.$$

We have scaled the variables so that they are all in the same range and adopted the mid-shell value, i.e. $N(\omega, \tilde{\omega}) = \Omega$. The problem is reduced to a space of two degrees of freedom whose
effective hamiltonian is,

\[
\mathcal{H}(I_1, I_2, \theta_1, \theta_2) = 2\epsilon_1 \Omega I_1 + 2\epsilon_3 \Omega (1 - I_1 - I_2) - \Omega (\epsilon_1 + \epsilon_3) \tag{42}
\]

\[
- G\Omega^2 \left\{ I_1 (1 - I_1) + I_2 (1 - I_2) + (I_1 + I_2) (1 - I_1 - I_2) \right. \\
+ 2\sqrt{I_1 I_2} \sqrt{(1 - I_1)(1 - I_2)} \cos(\theta_1 - \theta_2) \\
+ 2\sqrt{I_1 (1 - I_1 - I_2)} \sqrt{(1 - I_1)(I_1 + I_2)} \cos(\theta_1) \\
+ 2\sqrt{I_2 (1 - I_1 - I_2)} \sqrt{(1 - I_2)(I_1 + I_2)} \cos(\theta_2) \left. \right\} .
\]

In the above coordinates the classical version of \(O\) is

\[
\mathcal{O}_2(I_1, I_2, \theta_1, \theta_2) = \epsilon_1^2 \Omega I_1 - \epsilon_3^2 \Omega (I_1 + I_2) + \frac{\Omega}{2} (\epsilon_3^2 - \epsilon_1^2) \tag{43}
\]

\[
- G\Omega^2 \left\{ \epsilon_1 I_1 \left( \frac{1}{2} - I_1 \right) + \epsilon_3 (I_1 + I_2 - \frac{1}{2}) (1 - I_1 - I_2) \right. \\
+ \epsilon_1 \sqrt{I_1 I_2} \sqrt{(1 - I_1)(1 - I_2)} \cos(\theta_1 - \theta_2) \\
+ (\epsilon_1 + \epsilon_3) \sqrt{I_1 (1 - I_1 - I_2)} \sqrt{(1 - I_1)(I_1 + I_2)} \cos(\theta_1) \\
+ \epsilon_3 \sqrt{I_2 (1 - I_1 - I_2)} \sqrt{(1 - I_2)(I_1 + I_2)} \cos(\theta_2) \left. \right\} .
\]

It can be explicitly verified that \(\{\mathcal{O}_2, \mathcal{H}\} = 0\). We have also tested in the numerical integration of the equations of motion that both \(\mathcal{H}\) and \(\mathcal{O}_2\) are constants; and use this fact to control the numerical accuracy of the solutions. Integrability is also apparent in Poincaré sections, as Fig. 2 shows. Only separatrices (no chaotic layers) are observed for any value of the coupling constant. The salient feature of this figure is the appearance of forbidden regions, quite a common occurrence in spin systems\[4\].

5 Conclusions

Our analysis shows that the simple pairing force is integrable both at the quantum and at the classical levels. The reason why this simple feature has escaped attention can be found in the fact that the pairing force has been used almost exclusively in the context of quantum
many-body physics and for energies close to the ground state. Thus there are no significant consequences for ground state or RPA modes, which are in any case integrable. However the structure of the highly excited states and eigenfunctions should show this consequence. For example, the fluctuation properties of the eigenstates will follow a Poisson rather than a GOE statistics [14], while eigenfunctions will show the traces of conserved quantities and will not be ergodic.

There are very few models of interacting particles that are integrable[15] and the fact that this simple model belongs to this class comes as a surprise. We have reported some consequences in the TDHFB dynamics and on the spectrum. Other features should follow naturally, for example the statistics of levels should be very different than the particle-hole case[3]. Also the addition of a small perturbation to this hamiltonian should follow the KAM[16] perturbation scheme and not a fluctuating spectrum characteristic of the perturbation of chaotic systems. From the technical side it is quite useful to have at one’s disposal a model which takes into account important features of the nucleon-nucleon interaction and which remains integrable for arbitrary values of its strength. Thus perturbation expansions can have a steadying point which need not be the non interacting fermion system.

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References


FIG. 1. Simultaneous eigenvalues of the Hamiltonian \( H \) and \( O_2 \) for \( N = 14 \) and \( G = 0.1 \).
The variables have been scaled dividing them by \( N \).

FIG. 2. Poincaré surface of section for the three-level pairing hamiltonian. The mid-shell value \( \mathcal{N} = \Omega \) has been taken and the section is performed in the \( I_1 = 0 \) case (see text).