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A Study of Kantowski-Sachs Model in Ashtekar Variables

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Abstract

In this paper We shall study classical and quantum cosmology in Kantowski-Sachs model using Ashtekar variables. Classical solutions are obtained for the above model with a cosmological term and inflationary scenario is exhibited using Hamilton-Jacobi(H-J)equations. In quantum cosmology, the wave function of the Universe is obtained using path integral formalism as well as by solving Wheeler-Dewitt equation(W-D).

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1 Introduction

In canonical formulation of general relativity by Ashtekar^{1,2}, gravity is described as the singular system containing low-order polynomial constraints. But here the canonical variables are complex functions on the real phase space. Due to this simplified version of gravity using Ashtekar variables, a lot of works have been going on (since its discovery) in various forms^{2,3}, particularly in quantum gravity and in quantum cosmology. Using loop formalism, the resulting quantum constraint equations are then solved completely⁴ and the solutions are classified using topological knot and link invariants⁵⁻⁷. Though these developments are attractive and important but they fail to give any direct interpretation in quantum cosmology as the loop space representation are too abstract⁸.

Under this circumstances, we shall in this paper deal with Kantowski-Sachs (K-S) cosmological model and study this model both classically and quantum mechanically. The classical solutions are exhibited with a cosmological term and these solutions are in accord with those in metric formulation⁹. Also inflationary scenario has been obtained using H-J equations. Finally, we have evaluated wave function of the Universe both in path integral technique and also in canonical formulation.

2 Classical Solutions and Inflationary Scenario

We consider compact four geometries bounded by a three surface with topology $S^1 \times S^2$. The metric ansatz for the K-S Universe is given by⁹

$$ds^2 = \rho^2 \left[-N^2(t) dt^2 + \alpha^2(t) dr^2 + \tilde{g}^2(t) d\Omega_2^2 \right], \quad (1)$$

where the co-ordinate r is periodic with a period of 2π , the overall prefactor $\rho^2 = G/2\pi$, and G is the Newtonian constant. Now, the ADM Lagrangian can be written as

$$L = p_a \dot{a} + p_b \dot{b} - NH ,$$

where p_a, p_b are the ADM momentum conjugate to the metric variables a and b and the scalar constraint has the expression

$$H = a \left(1 + \frac{p_a^2}{4b^2} \right) - \frac{p_a p_b}{2b} + \lambda a b^2, \quad (2)$$

with λ as the cosmological constant. Using the canonical variables¹⁰ due to Ashtekar, the above Hamiltonian becomes

$$H = \sigma_2 \left(1 + \frac{A_2^2}{4} \right) - \sigma_1 A_1 A_2 + \lambda \sigma_1 \sigma_2, \quad (3)$$

where (σ_1, A_1) and (σ_2, A_2) are the two pairs of Ashtekar's canonical variables. Thus the corresponding Lagrangian is

$$L = - \frac{\dot{\sigma}_1 \dot{\sigma}_2}{n \sigma_1} + \frac{\sigma_2 \dot{\sigma}_1^2}{4n \sigma_1^2} - n \sigma_2 (1 + \lambda \sigma_1), \quad (4)$$

where n is rescaled lapse function. Thus the classical field equations are

$$\frac{\sigma_2 \ddot{\sigma}_1}{2n \sigma_1^2} - \frac{\ddot{\sigma}_2}{n \sigma_1} + \frac{\dot{\sigma}_2 \dot{\sigma}_1}{2n \sigma_1^2} - \frac{\sigma_2 \dot{\sigma}_1^2}{2n \sigma_1^3} + n \lambda \sigma_2 = 0, \quad (5)$$

$$\frac{\ddot{\sigma}_1}{n \sigma_1} - \frac{3}{4} \frac{\dot{\sigma}_1^2}{n \sigma_1^2} = n (1 + \lambda \sigma_1), \quad (6)$$

and the Hamiltonian constant equation is

$$\frac{\dot{\sigma}_1 \dot{\sigma}_2}{\sigma_1} - \frac{\sigma_2 \dot{\sigma}_1^2}{4 \sigma_1^2} = n^2 \sigma_2 (1 + \lambda \sigma_1). \quad (7)$$

The solutions to these equations are the following

$$\sigma_1 = \frac{3}{\lambda} \operatorname{cosech}^2(nt), \quad (8)$$

$$\sigma_2 = \sigma_{20} \tanh(nt) \sinh(nt),$$

where σ_{20} is a constant of integration. It is to be noted that these results are in agreement with our previous solutions⁹ in metric formulation by rotating to imaginary time. The action at the above classical solution has the expression

$$I_0 = 2 \sigma_{20} \sinh(n). \quad (9)$$

3 Inflationary Scenario:

If we calculate the action at the classical solution for the time scale ranging over 0 to 1, then I_0 can be written as

$$S_0 = -2 e_2 \left(1 + \frac{\lambda e_1}{3}\right)^{1/2}, \quad (10)$$

where e_1, e_2 are the values of σ_1 and σ_2 at the final time i.e. at $t = 1$. This action S can be identified as the H-J function. So the gradient of S will characterise the classical trajectories in superspace¹¹. Now, expressing momenta in terms of H-J function, the first integral of the system is

$$p_{e_1} = \frac{\partial S}{\partial e_1} = \frac{\partial L}{\partial \dot{e}_1}, \quad p_{e_2} = \frac{\partial S}{\partial e_2} = \frac{\partial L}{\partial \dot{e}_2}, \quad (11)$$

where L is given by (4) (with a change of the variables i.e. replacing (σ_1, σ_2) by (e_1, e_2)). So we obtain (with $n=1$) the differential equations

$$\frac{\dot{e}_1}{e_1} = 2 \left(1 + \frac{\lambda e_1}{3}\right)^{1/2}$$

and

$$\frac{\dot{e}_2}{e_2} = \frac{(1 + \frac{2\lambda e_1}{3})}{(1 + \frac{\lambda e_1}{3})^{1/2}} \quad (12)$$

The solutions can be presented as

$$\begin{aligned} e_1 &= \frac{12}{\lambda} \cdot \frac{e^{2t}}{(e^{2t} - 1)^2} , \\ e_2 &= \frac{e^t(1 + e^{2t})}{(e^{2t} - 1)^2} . \end{aligned} \quad (13)$$

Now, if we assume λ to be so small that $\frac{\lambda e_1}{3}$ is negligible compare to unity, then the H-J equations give identical equations for e_1 and e_2 and as a result both e_1 and e_2 have exponential behaviour of the form $\exp(2t)$. Consequently, both the scale factors grow exponentially with time. Further, if we assume λ to be small such that square and higher powers of λ are negligible then we can approximate the H-J function as

$$S \simeq -2e_2(1 + \frac{\lambda e_1}{6}) .$$

So in this case the first integrals read

$$\begin{aligned} \frac{\dot{e}_1}{e_1} &= 2(1 + \frac{\lambda}{6} e_1) , \\ \frac{\dot{e}_2}{e_2} &= (1 + \frac{\lambda}{2} e_1) . \end{aligned}$$

Hence the solutions can be expressed as

$$\begin{aligned} e_1 &= \frac{3}{\lambda} e^{-t} \operatorname{cosech}(t) \\ e_2 &= 4 e^{t/2} \operatorname{cosech}^{3/2}(t) \end{aligned} \quad (14)$$

Thus for infinitesimally small λ it is possible to obtain inflationary solutions using Ashtekar variables.

4 Quantum Cosmology

a) Path integral approach:

In this section we shall evaluate quantum propagator (which is a general concept than wave function) between fixed values of σ_1 and σ_2 as¹²

$$G = \int dn \int D\sigma_1 D\sigma_2 \cdot \exp\{-I(\sigma_1, \sigma_2)\} \quad (15)$$

We note that the lapse function n is in general unrestricted and then G is a solution to the W-D equation. But if N is restricted to half of an infinite range then G is a Green's function to the W-D operator¹². As the action I is not positive definite so complex contours are needed for the convergence of the path integral. Further, the classical solutions of given by equation (8) represent saddle points of the above path integral¹² and the action at the saddle point has the expression equation (9).

Now to evaluate the path integral, we shall perform a shift of integration variables¹². Let

$$\sigma_i(t) = \bar{\sigma}_i(t) + x_i(t), \quad i = 1, 2$$

where $\bar{\sigma}_i$ are the classical solutions and x_i 's vanish at the two end points. The action ($= \int L dt$), in terms of new variables takes the form

$$I = I_0 + I_2(x_i(t), n),$$

where I_2 is the part of the action containing square power of x_i and $\dot{x}_i$. Hence the above path integral becomes¹²

$$G = \int dn \cdot \exp(-I_0) \int Dx_1 Dx_2 \exp(-I_2). \quad (16)$$

As the functional integrals over x_i 's are Gaussian in form so using the standard results¹³ one obtain (except for irrelevant factors)¹⁴

$$G = \int dn \cdot \exp(-I_0). \quad (17)$$

Thus the path integral reduces to a single ordinary integration over the lapse which we shall evaluate by performing a steepest-descent analysis. The saddle points are given by $\frac{\partial I_0}{\partial n} = 0$ i.e. at

$$n = \left(\kappa + \frac{1}{2}\right) \pi i, \quad \kappa = 0, \pm 1, \pm 2, \dots$$

and the corresponding action is

$$S_0 = (-1)^\kappa 2i e_2 \sqrt{\frac{\lambda e_1}{3}} \left\{ 1 + \frac{\lambda e_1}{3} \right\}^{1/2}, \quad (18)$$

This action at the above saddle points will satisfy the time-independent H-J equation. The paths of steepest-descent through these saddle points are given by $I_m(S_0) = \text{constant}$ ¹². Since the actions at these saddle points are identical (except for sign), they are all dominant saddle points. Hence any infinite or half infinite contours will produce divergent result. Only contours with finite saddle points will give convergent value of the integral. For convenience, one may consider a loop around any singularity. Thus the steepest-descent result plus polynomial corrections at the saddle point yields a contribution to the value of the integral of the form¹²

$$\exp(\pm i 2 e_2 \sqrt{\frac{\lambda e_1}{3}} \left(1 + \frac{\lambda e_1}{3} \right)^{1/2}) \left\{ c_0 + c_1 \rho^{1/2} + c_2 \rho + \dots + c_k \rho^{k/2} + \dots \right\}, \quad (19)$$

where the coefficients $c_0, c_1, \dots$ are determined by higher derivatives of the action at the saddle point and $\rho = \sqrt{\frac{\lambda}{3}}$ is considered as small parameter in the asymptotic expansion. In this case also the results are very similar to those obtained in metric formulation¹².

b) Wheeler-Dewitt Equation: Canonical approach.

In canonical formulation as we have only the Hamiltonian constraint in the present case, so the quantum constraint in Ashtekar variables¹⁴ gives the corresponding W-D equation. The classical Hamiltonian constraint is

$$H = -\sigma_1 A_1 A_2 + \sigma_2 \left(1 - \frac{A_2^2}{4} \right) + \lambda \sigma_1 \sigma_2 = 0.$$

so the corresponding W-D equation is

$$\left[\sigma_1 \frac{\partial^2}{\partial \sigma_1 \partial \sigma_2} + \sigma_2 \left(1 + \frac{1}{4} \frac{\partial^2}{\partial \sigma_2^2} \right) + \lambda \sigma_1 \sigma_2 \right] \Psi(\sigma_1, \sigma_2) = 0, \quad (20)$$

in σ representation and

$$\left[\left(1 - \frac{A_2^2}{4} \right) \frac{\partial}{\partial A_2} - A_1 A_2 \frac{\partial}{\partial A_1} + i \lambda \frac{\partial^2}{\partial A_1 \partial A_2} \right] \phi(A_1, A_2) = 0, \quad (21)$$

in A-representation. Here we have followed the operator ordering due to Jacobson and Smolin⁴.

The W-D equation in A-representation (i.e. equation (21)) can be solved easily if we assume separable form for ϕ as $\phi(A_1, A_2) = \phi(A_1) + \phi(A_2)$.

Then the solution is

$$\phi(A_1, A_2) = \phi_0 \ln \left\{ c_0 A_1 / \left(1 - \frac{A_2^2}{4} \right)^2 \right\},$$

where ϕ_0 and c_0 are constants of integration.

However, to solve the W-D equation in σ -representation, we first make a substitution

$$\tau = \sigma_2^4 / \sigma_1,$$

to reduce the above equation in canonical form and the resulting solution is

$$\Psi(\sigma_1, \sigma_2) = \psi_0 \frac{(\sigma_2^2 - c_1 \sqrt{\sigma_1})}{\sigma_1 (1 + \lambda \sigma_1)}$$

with ψ_0 , the separation constant and c_1 , the constant of integration.

Thus with the use of Ashtekar variables we have studied K-S model both Classically and Quantum mechanically. The classical solutions to the field equations, inflationary scenario and wave function in path integral approach are very similar to metric formulation. However, we have obtained solutions to the W-D equation those have no analogue in metric formulation. Therefore, future

work, we shall try to relate wave functions in different approaches and in various formalism, to study whether Ashtekar variables will give any new information than metric formulation.

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