Conformally Covariant Differential Operators: 
Symmetric Tensor Fields

Johanna Erdmenger
Institut für Theoretische Physik, Universität Leipzig, Augustusplatz 10/11, 
D - 04109 Leipzig, Germany. e-mail: erd@tph204.physik.uni-leipzig.de

and

Hugh Osborn
Department of Applied Mathematics and Theoretical Physics, University of Cambridge, 
Silver Street, Cambridge CB3 9EW, England. e-mail: ho@damtp.cam.ac.uk

Abstract

We extend previous work on conformally covariant differential operators to consider the case of second order operators acting on symmetric traceless tensor fields. The corresponding flat space Green function is explicitly constructed and shown to be in accord with the requirements of conformal invariance.

PACS: 03.70.+k; 11.10.Kk; 11.25.Hf; 11.30.Ly
Keywords: Conformal invariance, Weyl invariance, Quantum field theory.
Conformal differential operators $\Delta$ are covariant differential operators acting on tensor fields, or more generally sections of some vector bundle, over a curved manifold with metric $g_{\mu\nu}$ which also transform covariantly under local Weyl rescalings of the metric

$$\delta_\sigma g^{\mu\nu} = 2\sigma \, g^{\mu\nu},$$

so that $\delta_\sigma \Delta = r\Delta + (s - r)\sigma \Delta$ for some $r$ if $\Delta$ is $s$-th order. Such operators are generalisations of the well known operator $-\nabla^2 + \frac{1}{6}R$ acting on scalars in four dimensions and have been classified by Branson [1]. Except for special values of the dimension $d$ such operators exist for general tensor fields belonging to representations of the tangent space group $O(d)$, or $O(d - 1, 1)$, or their spinor counterparts. The Greens functions associated with such conformal differential operators also transform covariantly under local rescalings of the metric and they may have a role in constructing forms for the quantum field theory effective action on curved manifolds.

Recently one of us discussed conformal differential operators and their associated Green functions from the point of view of the reduction to flat space [2] (this paper is subsequently referred to as I). In this case the form of the flat space Green function is unique up to an overall constant due to the restrictions imposed by the flat space conformal group $O(d + 1, 1)$ or $O(d, 2)$ [3, 4]. In I the general analysis was applied to conformal differential operators acting on totally antisymmetric $k$-index tensor fields, or $k$-forms, and also for 4-index tensor fields with the symmetries of the Weyl tensor, in both cases for arbitrary dimension $d$ when the general results were explicitly verified.

In this follow up we extend the discussion to totally symmetric, traceless $p$-index tensor fields and again find a result for the flat space Green function which is in accord with general theory, although the combinatorics are more involved in this case. The corresponding conformal differential operator was apparently first constructed by Wünsch [5] and also found as part of his general theory by Branson [1]. For $p = 2$ a particular conformally covariant differential operator was found by Gusynin and Roman’kov [6] (the general case involves a term proportional to the Weyl tensor with an arbitrary coefficient), for $d = 4$ see also [7]. The case of general $p$ has also been discussed more recently from a rather different point of view by O’Raifeartaigh et al [8], the results agree with ours when $d = 4$. For completeness we here follow, for arbitrary $d$, the general method of [8] which determines the conformal differential operator $\Delta^S$ by first constructing a Weyl invariant quadratic action $S^S(g, \omega)$ for the symmetric traceless tensor field $\omega_{\mu_1...\mu_p}$. With a convenient overall normalisation a general expression with manifest coordinate invariance which is second order in covariant derivatives has the form

$$S_0[g, \omega] = \frac{1}{2p!} \int d^d x \sqrt{g} \left[ \nabla^\lambda \omega_{\mu_1...\mu_p} \nabla_\lambda \omega_{\mu_1...\mu_p} + a \nabla_\rho \omega_{\mu_1...\mu_{p-1}\rho} \nabla^\lambda \omega_{\mu_1...\mu_{p-1}\lambda} \right],$$

for $a$ an arbitrary parameter. Since from (1) $\delta_\sigma \sqrt{g} = -d\sigma \sqrt{g}$ it is easy to see that this is invariant under constant Weyl rescalings if

$$\delta_\sigma \omega_{\mu_1...\mu_p} = \frac{1}{2}(d - 2p - 2)\sigma \omega_{\mu_1...\mu_p}.$$
In general an action which is invariant under rigid scale transformations has a variation under local $\sigma(x)$ linear in derivatives of the form

$$\delta_{\sigma}S_0[g,\omega] = \int d^d x \sqrt{g} \partial_{\lambda}\sigma J^\lambda. \quad (4)$$

For the action given in (2) the derivatives and Christoffel connections generate an explicit expression for $J^\lambda$,

$$p! J^\lambda = \frac{1}{2} (d - 2) \nabla^\lambda \omega^{\mu_1 \cdots \mu_p} \omega_{\mu_1 \cdots \mu_p} + p \nabla^\rho \omega^{\mu_1 \cdots \mu_{p-1}} \omega_{\mu_1 \cdots \mu_{p-1} \rho} \omega^{\mu_1 \cdots \mu_{p-1}}$$

$$- \left(p + \frac{1}{2} (d + 2p - 2) a\right) \omega^{\mu_1 \cdots \mu_{p-1}} \nabla^\rho \omega_{\mu_1 \cdots \mu_{p-1} \rho}. \quad (5)$$

In order to achieve a Weyl invariant action it is essential to be able to re-express the variation in terms of second derivatives of $\sigma$. To achieve this it is necessary that $J^\lambda = \nabla_\rho J^{\lambda \rho} \Rightarrow \delta_{\sigma}S_0[g,\omega] = -\int d^d x \sqrt{g} \nabla_\rho \partial_{\lambda}\sigma J^{\lambda \rho}, \quad (6)$

where clearly we may assume $J^{\lambda \rho} = J^{\rho \lambda}$. In this case the variation in (6) may be cancelled by an additional curvature dependent action. From (5) it is easy to see that the result in (6) is possible only if $a = -\frac{4p}{d + 2p - 2}$, (7)

and then $p! J^{\lambda \rho} = \frac{1}{2} (d - 2) \delta^{\lambda \rho} \omega^{\mu_1 \cdots \mu_p} \omega_{\mu_1 \cdots \mu_p} + p \omega^{\mu_1 \cdots \mu_{p-1}} \omega_{\mu_1 \cdots \mu_{p-1} \rho}. \quad (8)$

To exhibit the required curvature dependent terms it is convenient to define in terms of the Ricci tensor $R_{\mu \nu}$ and scalar curvature $R$

$$J = \frac{1}{2(d - 1)} R, \quad K_{\mu \nu} = \frac{1}{(d - 2)} (R_{\mu \nu} - J g_{\mu \nu}), \quad (9)$$

since these transform under local Weyl rescalings as in (1) according to

$$\delta_{\sigma}J = 2\sigma J + \nabla^2 \sigma, \quad \delta_{\sigma}K_{\mu \nu} = \nabla_{\mu} \nabla_{\nu} \sigma. \quad (10)$$

It is then evident that the action

$$S_1[g,\omega] = \frac{1}{2p!} \int d^d x \sqrt{g} \left[\frac{1}{2} (d - 2) J \omega^{\mu_1 \cdots \mu_p} \omega_{\mu_1 \cdots \mu_p} + 2p K_{\rho \lambda} \omega^{\mu_1 \cdots \mu_{p-1}} \omega_{\mu_1 \cdots \mu_{p-1} \rho} \right]. \quad (11)$$

has a variation which exactly compensates that given by (6) and (8). We may also add a contribution which is separately invariant under local Weyl rescalings

$$S_2[g,\omega] = A \frac{1}{2p!} \int d^d x \sqrt{g} C_{\lambda \rho \eta \lambda} \omega^{\mu_1 \cdots \mu_{p-2}} \omega_{\mu_1 \cdots \mu_{p-2} \rho \eta} \omega^{\mu_1 \cdots \mu_{p-2}} \eta \omega_{\mu_1 \cdots \mu_{p-2} \rho \eta}, \quad (12)$$

which depends on the Weyl tensor which may be defined by

$$C_{\alpha \beta \gamma \delta} = R_{\alpha \beta \gamma \delta} - 2 \left(g_{\alpha [\gamma} K_{\delta] \beta} - g_{\beta [\gamma} K_{\delta] \alpha} \right), \quad (13)$$
The invariance of (12) follows simply since \( \delta_\sigma C_{\lambda p q} = -2\sigma C_{\lambda p q} \).

Hence the combined action given by (2,11,12), with (7),

\[
S^S[g, \omega] = S_0[g, \omega] + S_1[g, \omega] + S_2[g, \omega] = \frac{1}{2p!} \int \! d^d x \sqrt{g} \omega^{\mu_1 \ldots \mu_p} (\Delta^S \omega)_{\mu_1 \ldots \mu_p} \tag{14}
\]
defines a conformally covariant differential operator \( \Delta^S \) on symmetric traceless tensors, depending on a single parameter \( A \), such that\(^1\)

\[
\delta_\sigma \Delta^S = \frac{1}{2} (d + 2p + 2) \sigma \Delta^S - \frac{1}{2} (d + 2p - 2) \Delta^S \sigma . \tag{15}
\]

The corresponding Green function is defined in general by

\[
\sqrt{g(x)} \left( \Delta^S \Sigma \right)_{\mu_1 \ldots \mu_p, \nu_1 \ldots \nu_p} (x, y) = \mathcal{E}^S_{\mu_1 \ldots \mu_p, \nu_1 \ldots \nu_p} \delta^d(x - y) , \tag{16}
\]
where \( \mathcal{E}^S \) is the projector onto totally symmetric traceless \( p \)-index tensors. This may be given explicitly by

\[
\mathcal{E}^S_{\mu_1 \ldots \mu_p, \nu_1 \ldots \nu_p} = \delta_\mu (\nu_1 \ldots \delta_\mu_p \nu_p) + \sum_{r=1}^{\lceil \frac{1}{2} p \rceil} \lambda_r g(\mu_1 \mu_2 \ldots \mu_{2r-1} \mu_{2r} g^{(\nu_1 \nu_2 \ldots \nu_{2r-1} \nu_{2r}) \delta_\mu \nu_{2r+1} \ldots \delta_\mu_p \nu_p) , \tag{17}
\]
for \( \lceil \frac{1}{2} p \rceil \) the integer part of \( \frac{1}{2} p \). Under Weyl rescalings this transforms as

\[
\delta_\sigma G^S_{\mu_1 \ldots \mu_p, \nu_1 \ldots \nu_p} (x, y) = \frac{1}{2} (d + 2p - 2) \sigma(x) G^S_{\mu_1 \ldots \mu_p, \nu_1 \ldots \nu_p} (x, y) + \frac{1}{2} (d - 2p - 2) \sigma(y) G^S_{\mu_1 \ldots \mu_p, \nu_1 \ldots \nu_p} (x, y) . \tag{18}
\]

We here determine the flat space form for \( G^S \) following similar procedures to I. In the flat space limit, \( g_{\mu \nu} \to \delta_{\mu \nu} \), and we may identify up and down indices. Explicitly \( \Delta^S \to \Delta^S \) which is given by

\[
(\Delta^S \omega)_{\mu_1 \ldots \mu_p} = -\partial^2 \omega_{\mu_1 \ldots \mu_p} + \frac{4p}{d + 2p - 2} \partial_\lambda \partial_\mu_1 \omega_{\mu_2 \ldots \mu_{p-1} \lambda} - \frac{4p(p - 1)}{(d + 2p - 2)(d + 2p - 4)} \partial_\mu \partial_\lambda \delta_{\mu_1 \mu_2 \omega_{\mu_3 \ldots \mu_p} \lambda_\rho} , \tag{19}
\]
where the last term serves to ensure that the r.h.s. is traceless. Finding the flat space Green function,

\[
G^S_{\mu_1 \ldots \mu_p, \nu_1 \ldots \nu_p} (x, y) \bigg|_{g=\delta} = G^S_{\mu_1 \ldots \mu_p, \nu_1 \ldots \nu_p} (x - y) , \tag{20}
\]
\(^1\)The result for the curvature dependent terms arising from (11) differs from that in the papers of Branson [1] but this seems to arise from a simple arithmetic error.
is equivalent to solving
\[ (\hat{\Delta}^S \omega)_{\mu_1 \cdots \mu_p} = \phi_{\mu_1 \cdots \mu_p}, \] (21)
for arbitrary \( \phi \) and to this end we may write the Fourier transform as
\[ \tilde{\omega}_{\mu_1 \cdots \mu_p} = a_0 \frac{1}{k^2} \phi_{\mu_1 \cdots \mu_p} + \sum_{r=1}^{p} a_r \frac{2^r}{k^{2(1+r)}} k(\mu_1 \cdots k_{\nu_r} \tilde{\phi}_{\mu_{r+1} \cdots \mu_p})_{\rho_1 \cdots \rho_r} k_{\rho_1} \cdots k_{\rho_r} - \text{traces}(\mu_1 \cdots \mu_p). \] (22)
The coefficients \( a_r \) are then determined by requiring
\[ k^2 \tilde{\omega}_{\mu_1 \cdots \mu_p} - \frac{4p}{d+2p-2} k(\mu_1 \tilde{\omega}_{\mu_2 \cdots \mu_{p-1}})k_{\lambda} - \text{traces}(\mu_1 \cdots \mu_p) = \tilde{\phi}_{\mu_1 \cdots \mu_p}. \] (23)

To analyse (22) and (23) we first consider a symmetric traceless \((p-r)\)-index tensor \( \psi_{\mu_1 \cdots \mu_{p-r}} \) and obtain
\[ \left[ k(\mu_1 \cdots k_{\nu_r} \psi_{\mu_{r+1} \cdots \mu_{p-1} \lambda}) - \text{traces}(\mu_1 \cdots \mu_{p-1} \lambda) \right] \]
\[ = r k(\mu_1 \cdots k_{\nu_{r-1}} \psi_{\mu_{r} \cdots \mu_{p-1} \lambda}) k_{\lambda} + (p-r) k(\mu_1 \cdots k_{\nu_r} \psi_{\mu_{r+1} \cdots \mu_{p-1} \lambda}) \]
\[ - r(r-1) \frac{k^2 \delta_{\lambda(\mu_1} k_{\mu_2} \cdots k_{\nu_{r-1}} \psi_{\nu_{r} \cdots \mu_{p-1} \lambda)} - \frac{2r(p-r)}{d+2p-4} \delta_{\lambda(\mu_1} k_{\mu_2} \cdots k_{\nu_r} \psi_{\nu_{r+1} \cdots \mu_{p-1} \lambda)} k_{\rho} \]
\[ - \text{traces}(\mu_1 \cdots \mu_{p-1}). \] (24)
Hence
\[ \left[ k(\mu_1 \cdots k_{\nu_r} \psi_{\mu_{r+1} \cdots \mu_{p-1} \lambda}) - \text{traces}(\mu_1 \cdots \mu_{p-1} \lambda) \right] k_{\lambda} \]
\[ = \frac{r}{p} \frac{d+2p-r-3}{d+2p-4} k^2 k(\mu_1 \cdots k_{\nu_{r-1}} \psi_{\nu_{r} \cdots \mu_{p-1} \lambda}) \]
\[ + \frac{p-r}{p} \frac{d+2p-2r-4}{d+2p-4} k(\mu_1 \cdots k_{\nu_r} \psi_{\nu_{r+1} \cdots \mu_{p-1} \lambda}) k_{\rho} \]
\[ - \text{traces}(\mu_1 \cdots \mu_{p-1}). \] (25)
Using this to calculate the result of inserting (22) into (23) we get
\[ a_0 = 1 \] (26)
and
\[ \left( 1 - \frac{2p}{d+2p-2} \frac{r}{p} \frac{d+2p-r-3}{d+2p-4} \right) a_r = \frac{2p}{d+2p-2} \frac{p-r}{p} \frac{d+2p-2r-4}{d+2p-4} a_{r-1}, \] (27)
which simplifies to
\[ a_r = \frac{p-r+1}{\frac{1}{2}d+p-r-2} a_{r-1}. \] (28)
It is then straightforward, with (26), to find
\[ a_r = \prod_{j=1}^{r} \frac{p-j+1}{\frac{1}{2}d+p-j-2}. \] (29)
We find from (34) the inversion of the Fourier transform in (30) may be found with the aid of
\begin{align*}
\mathcal{E}^{S}_{\mu_1...\mu_p,\nu_1...\nu_p}(k) &= \mathcal{E}^{S}_{\mu_1...\mu_p,\nu_1...\nu_p} \frac{1}{k^{2(1+r)}} \\
+ \sum_{r=1}^{p} a_r \mathcal{E}^{S}_{\mu_1...\mu_p,\epsilon_1...\epsilon_r,\lambda_{r+1}...\lambda_p} \mathcal{E}^{S}_{\eta_1...\eta_r,\nu_1...\nu_p} \frac{2^r}{k^{2(1+r)}} k_{\epsilon_1} ... k_{\epsilon_r} k_{\eta_1} ... k_{\eta_r}.
\end{align*}

The inversion of the Fourier transform in (30) may be found with the aid of
\begin{align*}
\frac{1}{(2\pi)^d} \int d^d k e^{-ik \cdot x} &\frac{1}{k^{2(1+r)}} k_{\alpha_1} ... k_{\alpha_{2r}} \\
= \frac{\Gamma(\frac{1}{2},d-1)}{4\pi \frac{1}{2} (x^2)^{\frac{1}{2}d-1}} \frac{(2r)!}{4^r r!} \sum_{s=0}^{r} \frac{(-4)^s}{(2s)!} \frac{1}{x^{2s}} x_{(\alpha_1} ... x_{(\alpha_{2s}} \delta_{\alpha_{2s+1} \alpha_{2s+2}} ... \delta_{\alpha_{2r-1} \alpha_{2r})}.
\end{align*}

for \((y)_s = \Gamma(y + s)/\Gamma(y),\) and
\begin{align*}
\mathcal{E}^{S}_{\mu_1...\mu_p,\alpha_1...\alpha_r,\lambda_{r+1}...\lambda_p} \mathcal{E}^{S}_{\alpha_{r+1}...\alpha_2,\lambda_{r+1}...\lambda_p} \mathcal{E}^{S}_{\nu_1...\nu_p} x_{(\alpha_1} ... x_{(\alpha_{2s}} \delta_{\alpha_{2s+1} \alpha_{2s+2}} ... \delta_{\alpha_{2r-1} \alpha_{2r})} \\
= 2^{r-s} \frac{(2s)!}{(2r)!} \frac{r!}{s!} \frac{1}{\mathcal{E}^{S}_{\mu_1...\mu_p,\epsilon_1...\epsilon_s,\lambda_{s+1}...\lambda_p} \mathcal{E}^{S}_{\eta_1...\eta_s,\lambda_{s+1}...\lambda_p} \mathcal{E}^{S}_{\nu_1...\nu_p} x_{\epsilon_1} ... x_{\epsilon_s} x_{\eta_1} ... x_{\eta_s}}.
\end{align*}

We therefore find
\begin{align*}
\mathcal{G}^{S}_{\mu_1...\mu_p,\nu_1...\nu_p}(x) &= \frac{\Gamma(\frac{1}{2},d-1)}{4\pi \frac{1}{2} (x^2)^{\frac{1}{2}d-1}} \left\{ b_0 \mathcal{E}^{S}_{\mu_1...\mu_p,\nu_1...\nu_p} \right. \\
+ \sum_{s=1}^{p} b_s \mathcal{E}^{S}_{\mu_1...\mu_p,\epsilon_1...\epsilon_s,\lambda_{s+1}...\lambda_p} \mathcal{E}^{S}_{\eta_1...\eta_s,\lambda_{s+1}...\lambda_p} \mathcal{E}^{S}_{\nu_1...\nu_p} \frac{(-2)^s}{x^{2s}} x_{\epsilon_1} ... x_{\epsilon_s} x_{\eta_1} ... x_{\eta_s} \left\} ,
\end{align*}

where
\begin{align*}
b_0 &= 1 + \sum_{r=1}^{p} a_r, \quad b_s = \frac{1}{2} \frac{d - 1}{s!} \sum_{r=s}^{p} \binom{r}{s} a_r, \quad s \geq 1.
\end{align*}

To calculate \(b_s\) we may use induction on \(p\). From (26,27) it is easy to see that
\begin{align*}
a_r^{(p)} &= \frac{p}{2d + p - 3} a_r^{(p-1)}, \quad r = 1, \ldots p.
\end{align*}

Since
\begin{align*}
\binom{r}{s} = \binom{r-1}{s-1} + \binom{r-1}{s}
\end{align*}

we find from (34)
\begin{align*}
b_0^{(p)} &= 1 + \frac{p}{2d + p - 3} b_0^{(p-1)}, \quad b_s^{(p)} = \frac{p}{s} \frac{1}{2d + p - 3} b_s^{(p-1)} + \frac{p}{2d + p - 3} b_s^{(p-1)}, \quad s = 1, 2, \ldots.
\end{align*}
It is then easy to verify the general result for any \( p \)

\[
b_s = \binom{p}{s} \frac{1}{2} d + p - 2 \frac{1}{2} d - 2 .
\]  

(38)

Applying (38) in (33) with the standard binomial theorem gives

\[
\hat{G}^S_{\mu_1 \ldots \mu_p, \nu_1 \ldots \nu_p}(x) = \frac{\Gamma(\frac{1}{2} d - 1)}{4 \pi^{\frac{1}{2} d} (x^2)^{d-1}} \frac{d + 2 p - 4}{d - 4} \mathcal{T}^S_{\mu_1 \ldots \mu_p, \nu_1 \ldots \nu_p}(x) ,
\]

(39)

where

\[
\mathcal{T}^S_{\mu_1 \ldots \mu_p, \nu_1 \ldots \nu_p}(x) = \mathcal{E}^S_{\mu_1 \ldots \mu_p, \epsilon_1 \ldots \epsilon_p} I_{\epsilon_1 \nu_1}(x) \ldots I_{\epsilon_p \nu_p}(x) , 
\]  

(40)

for

\[
I_{\epsilon \nu}(x) = \delta_{\epsilon \nu} - \frac{2}{x^2} x_\epsilon x_\nu .
\]  

(41)

\( I_{\epsilon \nu}(x) \) is the inversion tensor so that \( \mathcal{T}^S_{\mu_1 \ldots \mu_p, \nu_1 \ldots \nu_p}(x) \) as given by (40) is the inversion tensor for totally symmetric traceless \( p \)-index tensor fields and (39) is exactly of the form expected as a consequence of applying flat space conformal invariance in this case. Except when \( p = 0 \) the Green function does not exist for \( d = 4 \) reflecting the fact that \( d = 4 \) is the critical dimension for \( \Delta^S \).

To understand the role of the critical dimension \( d = 4 \) we may introduce a linear differential operator \( \mathcal{D} \) acting on symmetric traceless tensors, with index \( p \geq 1 \), which is defined by

\[
(\mathcal{D} \omega)_{\mu_1 \ldots \mu_p \lambda} = \nabla_{\lambda} \omega_{\mu_1 \ldots \mu_p} - \nabla_{(\mu_1} \omega_{\mu_2 \ldots \mu_p)\lambda}
\]

\[
- \frac{p - 1}{d + p - 3} (g_{\lambda \mu_1} \nabla^\rho \omega_{\mu_2 \ldots \mu_p)\rho} - g_{(\mu_1 \mu_2} \nabla^\rho \omega_{\mu_3 \ldots \mu_p)\lambda_\rho) .
\]

(42)

This satisfies the traceless conditions

\[
g^{\lambda \mu_1} (\mathcal{D} \omega)_{\mu_1 \ldots \mu_p \lambda} = g^{\lambda \mu_1} (\mathcal{D} \omega)_{\mu_1 \ldots \mu_p \lambda} = 0 ,
\]

(43)

and under local Weyl rescalings according to (1,3)

\[
\delta_\sigma (\mathcal{D} \omega)_{\mu_1 \ldots \mu_p \lambda} = \frac{1}{2} (d - 2 p - 2) \sigma (\mathcal{D} \omega)_{\mu_1 \ldots \mu_p \lambda}
\]

\[
+ \frac{1}{2} (d - 4) \left( \partial_\lambda \sigma \omega_{\mu_1 \ldots \mu_p} - \partial_{(\mu_1} \sigma \omega_{\mu_2 \ldots \mu_p)\lambda}
\]

\[
- \frac{p - 1}{d + p - 3} (g_{(\mu_1 \mu_2} \omega_{\mu_3 \ldots \mu_p)\rho} - g_{(\mu_1 \mu_2 \mu_3} \omega_{\mu_4 \ldots \mu_p)\lambda_\rho) \partial_\rho \sigma \right) .
\]

(44)

Clearly when \( d = 4 \) \( \mathcal{D} \) is a first order conformally covariant differential operator. Moreover from the definition (42)

\[
\frac{p}{p + 1} (\mathcal{D} \omega)_{\mu_1 \ldots \mu_p \lambda} = \nabla^\lambda \omega_{\mu_1 \ldots \mu_p} \nabla_\lambda \omega_{\mu_1 \ldots \mu_p} - \nabla^\lambda \omega_{\mu_1 \ldots \mu_p - 1} \rho \nabla_\rho \omega_{\mu_1 \ldots \mu_p - 1 \lambda}
\]

\[
- \frac{p - 1}{d + p - 3} \nabla_{\rho} \omega_{\mu_1 \ldots \mu_p - 1 \rho} \nabla^\lambda \omega_{\mu_1 \ldots \mu_p - 1 \lambda} .
\]

(45)
By discarding total derivatives we may write
\[ \nabla^\lambda \omega^{\mu_1 \ldots \mu_{p-1} \rho} \nabla_{\rho} \omega_{\mu_1 \ldots \mu_{p-1} \lambda} \rightarrow \nabla_{\rho} \omega^{\mu_1 \ldots \mu_{p-1} \rho} \nabla^\lambda \omega_{\mu_1 \ldots \mu_{p-1} \lambda} - \omega^{\mu_1 \ldots \mu_{p-1} \rho} [\nabla_\lambda, \nabla_\rho] \omega_{\mu_1 \ldots \mu_{p-1} \lambda}, \tag{46} \]
where, using the definition of the Weyl tensor in terms of the curvature in (13) and also (10),
\[ \omega^{\mu_1 \ldots \mu_{p-1} \rho} [\nabla_\lambda, \nabla_\rho] \omega_{\mu_1 \ldots \mu_{p-1} \lambda} = -(p-1) C_{\lambda \epsilon \rho \eta} \omega^{\mu_1 \ldots \mu_{p-2} \lambda \epsilon \rho} \omega_{\mu_1 \ldots \mu_{p-2} \lambda} + (d + 2p - 4) K_{\lambda \rho} \omega^{\mu_1 \ldots \mu_{p-1} \lambda \epsilon \rho} \omega_{\mu_1 \ldots \mu_{p-1} \lambda} + J \omega^{\mu_1 \ldots \mu_{p} \omega_{\mu_1 \ldots \mu_{p}}} \tag{47} \]
With these results, if we choose for the parameter \( A \) in (12) \( A = -(p-1) \), we may obtain an alternative expression for the total action given by (14),
\[ S[g, \omega] = \frac{1}{2p!} \int d^d x \sqrt{g} \left[ \frac{p}{p+1} (\mathcal{D} \omega)^{\mu_1 \ldots \mu_p \lambda} (\mathcal{D} \omega)_{\mu_1 \ldots \mu_p \lambda} + \frac{(d-4)(d-2)}{(d+2p-2)(d+p-3)} \nabla^\rho \omega^{\mu_1 \ldots \mu_{p-1} \rho} \nabla^\lambda \omega_{\mu_1 \ldots \mu_{p-1} \lambda} - (d-4)(K_{\lambda \rho} \omega^{\mu_1 \ldots \mu_{p-1} \lambda \epsilon \rho} \omega_{\mu_1 \ldots \mu_{p-1} \lambda} + \frac{1}{2} J \omega^{\mu_1 \ldots \mu_p \omega_{\mu_1 \ldots \mu_p}} \right]. \tag{48} \]
This result demonstrates the importance of \( d = 4 \), in this case only the first term quadratic in operator \( \mathcal{D} \) is present, which is in accord with the results of Branson [1]. The above formula (48), along with (45), coincides with that given by O’Raifeartaigh et al [8] who required the absence of curvature dependent terms (although the motivation for such a condition is not clear). If \( p = 1 \) and \( d = 4 \) (48) is manifestly just the standard expression for conformally invariant Maxwell theory. On flat space if, for some scalar \( \rho \), \( \omega_{\mu_1 \ldots \mu_p} = \partial_{\mu_1} \ldots \partial_{\mu_p} \rho \) – traces then \((\mathcal{D} \omega)_{\mu_1 \ldots \mu_p \lambda} = 0\) which explains the absence of an inverse in the flat space limit when \( d = 4 \). Of course if \( p = 1 \) this is just a reflection of the usual gauge invariance of Maxwell’s equations.
References


