Abstract

A basic arbitrariness in the determination of the topology of a manifold at the Planck length is discussed. An explicit example is given of a ‘smooth’ change in topology from the 2-sphere to the 2-torus through a sequence of noncommuting geometries. Applications are considered to the theory of $D$-branes within the context of the proposed $M$(atrix) theory.

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1 Motivation and notation

Since the early efforts of Wheeler [36] in this direction it has often been speculated that the topology of space might not be a well defined dynamical invariant. There can of course be no smooth time evolution of a space of one topology into that of another; a classical space cannot change topology without the formation of a singularity. However the ‘true’ description of space and of space-time should reasonably include quantum fluctuations and it is possible that a quantum space-time exists which seems in a quasi-classical approximation to evolve from a space of one topology to that of another, the ‘exact’ quantum space-time being a sum over many topologies.

Noncommutative geometry furnishes a possible alternative mathematical language in which one can also discuss this question. A change in topology is possible simply because even classically the topology of a manifold is not a well defined quantity at all length scales. A change in space topology can occur if the dynamical evolution of space-time is such that the space enters a regime in which its description requires the use of noncommutative geometry. This could be expected to occur near a classical singularity. We are able to treat this problem only in 2+1 dimensions where we can identify space with a smooth compact surface of genus $g$. Even here we can give no reasonable field equations which would dynamically implement the change of space topology which we consider.

In Section 2 we propose a definition of a fuzzy surface of ill-defined topology. In Section 3 we discuss the differential structures of fuzzy surfaces and we describe in detail an explicit example of topology change from the 2-sphere to the 2-torus through a sequence of such fuzzy surfaces. We speculate on the analogous transition between two compact surfaces of arbitrary genus. Finally in Section 5 we discuss our results in light of the recent fuzzy description of $D$-branes.

When using in the noncommutative context a word which is usually defined only for an ordinary differential manifold we enclose it in quotation marks if there is a chance of ambiguity.

2 Topological fuzzy surfaces

The general definition of a fuzzy surface has been given elsewhere [27, 29]. Points are replaced by elementary cells (Planck cells) of a quantum of area. If the ‘surface’ is to be in some sense compact then there can only be a finite number of cells and the structure algebra must be of finite dimension. It is usually taken to be the algebra $M_n$ of $n \times n$ complex matrices. The ‘topology’ is encoded in a filtration $P_h$ of the algebra which we shall introduce for each genus $h$. The differential structure is encoded in the differential calculus over the algebra. We shall suppose that there exist fuzzy versions of compact surfaces $\Sigma_h$ of arbitrary genus $h$ although we know of explicit constructions only in the particular cases $h = 0$ and $h = 1$.

Let $f$ be an element of $M_n$. For each $h$ we shall introduce a norm $\|f\|_{h,n}$ If the sequence $\|f\|_{h,n}$ has a limit for some value of $h$ then we consider $f$ to be a matrix approximation to a function $\tilde{f}$ on $\Sigma_h$. The limiting procedure is rather obscure and we shall consider it only on the algebra of polynomials in a set of basic matrices, the ‘coordinates’. The choice of this set will define the filtration and hence the value of $h$. Since $M_n$ is a simple $*$-algebra a morphism is necessarily of the form $f \mapsto u^{-1}fu$ for some hermitian $u$. In the commutative limit these maps tend to symplectomorphisms of $\Sigma_h$ into itself [9, 23, 15, 17, 7, 26]. A general ‘coordinate transformation’ would
be a map of the form \( f \mapsto \phi(f) \) which respects the algebraic structure only in the commutative limit. If such a map is singular in the commutative limit then the resulting transformation is a change of topology.

### 2.1 Genus zero

Let \( \tilde{x}^a \) be the cartesian coordinates of \( \mathbb{R}^3 \) and \( g_{ab} \) the euclidean metric. The ordinary round sphere \( S^2 \) of radius \( r \) is defined by the constraint \( g_{ab}\tilde{x}^a\tilde{x}^b - r^2 = 0 \). The algebra \( \mathcal{C}(S^2) \) of smooth functions on \( S^2 \) is a completion of the quotient of the algebra of polynomials in \( \tilde{x}^a \) by the ideal generated by those which contain \( g_{ab}\tilde{x}^a\tilde{x}^b - r^2 \) as a factor.

A fuzzy version of the sphere \([5, 9, 23, 15, 17, 7, 27]\) is constructed using an \( n \)-dimensional irreducible representation of the Lie algebra of the group \( SU_2 \). We let \( \tilde{x}^a \), for \( 1 \leq a \leq 3 \), be the generators and we raise and lower indices using the Killing metric \( g_{ab} \). We introduce a macroscopic length scale \( r \), the radius of the sphere, and a microscopic area scale \( \bar{k} \) which are related, for large \( n \), by the equation

\[
\frac{4\pi r^2}{2\pi \bar{k}} = n.
\]  

(2.1.1)

The integer \( n \) counts the number of elementary cells of area \( 2\pi \bar{k} \). The Casimir relation is written as \( g_{ab}\tilde{x}^a\tilde{x}^b = r^2 \) and the commutation relations of the ‘coordinates’ \( x^a \) are given by

\[
[x_a, x_b] = ikC_{ab}^c x_c, \quad C_{abc} = r^{-1}\epsilon_{abc}.
\]  

(2.1.2)

We shall consider the length scale \( r \) as fixed and so \( \bar{k} \to 0 \) in the limit as \( n \to \infty \). We can identify therefore

\[
\lim_{n \to \infty} x^a = \tilde{x}^a.
\]  

(2.1.3)

Any matrix \( f \) can be written as a polynomial \( f(x^a) \) in the \( x^a \),

\[
f = \sum_{l=0}^{l} \frac{1}{k!} f_{a_1 \ldots a_k} x^{a_1} \ldots x^{a_k},
\]  

(2.1.4)

where the \( f_{a_1 \ldots a_k} \) are completely symmetric and trace-free. We can associate to \( f \) the function \( \tilde{f} = f(\tilde{x}^a) \). Set \( \tilde{f} = \phi_n(f) \). We have defined then a vector-space map

\[
M_n \xrightarrow{\phi_n} \hat{P}(S^2).
\]

This map cannot of course respect the product structures of the respective algebras but if \( f \) and \( g \) are two polynomials of order less than some integer \( l \) then one can show that

\[
\phi_n(fg) - \phi_n(f)\phi_n(g) = o(l/n).
\]  

(2.1.5)

For each integer \( 0 \leq l \leq n - 1 \) introduce the vector space \( \mathcal{P}_{0,l} \) of symmetric polynomials of order \( l \) in the \( x^a \). Obviously

\[
\mathcal{P}_{0,l} \subset \mathcal{P}_{0,l+1}, \quad \bigcup_{l=0}^{n-1} \mathcal{P}_{0,l} = M_n.
\]  

(2.1.6)

The filtration of the algebra \( M_n \) which defines the sphere is given by the \( \mathcal{P}_{0,l} \) and \( \phi(\mathcal{P}_{0,l}) \) is a filtration of the polynomials of order \( n \) on the sphere.
We define the norm $\|f\|_n$ of an element $f \in M_n$ as

$$\|f\|_n^2 = \frac{1}{n} \text{Tr} (f^* f).$$  \hfill (2.1.7)

In particular we find that

$$\|x^a\|_n^2 = \frac{1}{3} r^2. \hfill (2.1.8)$$

We introduce the norm of an element $\tilde{f} \in \mathcal{C}(S^2)$ as

$$\|\tilde{f}\|_n^2 = \frac{1}{4\pi r^2} \int_{S^2} |\tilde{f}|^2.$$  \hfill (2.1.9)

Then if $f \in \mathcal{P}_{0,l}$ we have

$$\|f\|_n^2 - \|\tilde{f}\|_n^2 = o(l/n).$$

The norm of a generic element of $\mathcal{P}_{0,l}$ grows as $l$.

### 2.2 Genus one

Let $r$ be again a length scale and consider the torus $\mathbb{T}^2$ defined to be the subset of $\mathbb{R}^2$ with coordinates $(\tilde{x}, \tilde{y})$ subject to the conditions $0 \leq \tilde{x}, \tilde{y} \leq 2\pi r$. Consider the two functions

$$\tilde{u} = e^{i\tilde{x}/r}, \quad \tilde{v} = e^{i\tilde{y}/r}. \hfill (2.2.1)$$

The algebra $\mathcal{C}(\mathbb{T}^2)$ of smooth functions on $\mathbb{T}^2$ is a completion of the algebra of polynomials in $\tilde{u}$ and $\tilde{v}$.

A fuzzy version of the torus was constructed by Weyl [35], Schwinger [33] and others [16, 1, 2] to describe a finite version of quantum mechanics. One introduces elements $u$ and $v$ which satisfy the Weyl relation

$$uv = qvu$$  \hfill (2.2.2)

as well as the constraints

$$u^n = 1, \quad v^n = 1, \quad q = e^{2\pi i/n}. \hfill (2.2.3)$$

The algebra generated by $u$ and $v$ is isomorphic then to the matrix algebra $M_n$. Define the area parameter $\kappa$ by the relation

$$\frac{(2\pi r)^2}{2\pi \kappa} = n. \hfill (2.2.4)$$

This is the same as (2.1.1) if one replaces the area $4\pi r^2$ of the sphere of radius $r$ by the area $(2\pi r)^2$ of the torus. It is worth noticing that when described by the algebra $M_n$ the torus has $n$ ‘cells’; each observable can take $n$ possible values. It is therefore to be compared with an approximation on a lattice with

$$\frac{2\pi r}{\sqrt{n}} = \sqrt{2\pi \kappa}$$

as unit of length.

An explicit form for $u$ and $v$ can be easily found [33]. There is an orthonormal basis $|j\rangle_1$, $0 \leq j \leq n - 1$, of $\mathbb{C}^n$ such that $u$ and $v$ are given by

$$v|j\rangle_1 = |j + 1\rangle_1, \quad u|j\rangle_1 = q^j|j\rangle_1.$$  \hfill (2.2.5)
and such that the cyclicity condition
\[ v|n - 1\rangle_1 = |0\rangle_1 \] (2.2.6)
holds. One can introduce matrices \( x \) and \( y \) defined by the relations
\[ u = e^{ix/r}, \quad v = e^{iy/r}. \] (2.2.7)
In the basis \( |j\rangle_1 \) it is obvious that one can choose \( x \) such that
\[ x|j\rangle_1 = \frac{k}{r} j|j\rangle_1. \] (2.2.8)

There is also an orthonormal basis \( |j\rangle_2 \) in which the \( v \) is diagonal. The two bases are related by the ‘Fourier transformation’ [33]
\[ |l\rangle_2 = \frac{1}{\sqrt{n}} \sum_{j=0}^{n-1} q^{-jl}|j\rangle_1, \quad |j\rangle_1 = \frac{1}{\sqrt{n}} \sum_{l=0}^{n-1} q^{jl}|l\rangle_2. \] (2.2.9)
If we define, for arbitrary \( z \in \mathbb{C} \),
\[ f(z) = \frac{1}{n} \sum_{l=0}^{n-1} q^{-zl} = \frac{1}{n} \frac{q^{-zn}-1}{q^{-z}-1} \] (2.2.10)
then we can write
\[ 1\langle j'|j\rangle_1 = f(j' - j). \] (2.2.11)
The Fourier transformation is unitary because of the relations \( f(0) = 1 \) and \( f(j) = 0 \), \( j \neq 0 \). A short calculation yields the relation
\[ 1\langle j'|[x,y]|j\rangle_1 = ik(j' - j)f'(j' - j). \] (2.2.12)
In the \( n \to \infty \) limit we must have \( f(z) \to \delta(z) \) and therefore \( zf' \to -\delta \). We recover then the commutation relation
\[ [x,y] = -ik \] (2.2.13)
which is equivalent to the Weyl relation (2.2.2).

Because \( q \to 1 \) as \( n \to \infty \) we can identify
\[ \lim_{n \to \infty} u = \tilde{u}, \quad \lim_{n \to \infty} v = \tilde{v}. \]
Introduce \( u^\alpha = (u,v) \). Any matrix \( f \) can be written as a polynomial \( f(u^\alpha) \) in the \( u^\alpha \):
\[ f = \sum_{\alpha_1,\ldots,\alpha_j} \frac{1}{j!} f_{\alpha_1\ldots\alpha_j} u^{\alpha_1} \ldots u^{\alpha_j} \] (2.2.14)
where the \( f_{\alpha_1\ldots\alpha_j} \) are completely symmetric. We can associate to \( f \) the function \( \tilde{f} = f(\tilde{u}^\alpha) \). Set \( \tilde{f} = \phi_n(f) \). We have defined then a vector-space map
\[ M_n \phi_n \mapsto \tilde{P}(\mathbb{T}^2). \]
As above, if \( f \) and \( g \) are two polynomials of order less than some integer \( l \) then one can show that
\[ \phi_n(fg) - \phi_n(f)\phi_n(g) = o(l/n). \] (2.2.15)
For each integer $0 \leq l \leq n - 1$ introduce the vector space $\mathcal{P}_{1,l}$ of symmetric polynomials of order $l$ in the $u^a$. Obviously

$$\mathcal{P}_{1,l} \subset \mathcal{P}_{1,l+1}, \quad \bigcup_{l=0}^{n-1} \mathcal{P}_{1,l} = M_n. \quad (2.2.16)$$

The filtration of the algebra $M_n$ which defines the torus is given by the $\mathcal{P}_{1,l}$ and $\phi_n(\mathcal{P}_{1,l})$ is a filtration of the polynomials of order $n$ on the torus.

We define again the norm $\|f\|_n$ of an element $f \in M_n$ by (2.1.7). In particular we find that

$$\|u^a\|_n^2 = 1. \quad (2.2.17)$$

We define the norm of an element $\tilde{f} \in \mathcal{C}(\mathbb{T}^2)$ as

$$\|\tilde{f}\|^2 = \frac{1}{(2\pi r)^2} \int_{\mathbb{T}^2} |\tilde{f}|^2.$$

Then if $f \in \mathcal{P}_{1,l}$ we have

$$\|f\|_n^2 - \|\tilde{f}\|^2 = o(l/n). \quad (2.2.18)$$

The norm of a generic element of $\mathcal{P}_{1,l}$ grows as $l$.

### 2.3 Higher genera

We conjecture that the construction of Section 2.1 and Section 2.2 can be extended to arbitrary genus. Let $\Sigma_h$ be a surface of genus $h$ and choose generators $x^i$ of $M_n$ which define in the limit $n \to \infty$ coordinates $\tilde{x}^i$ on $\Sigma_h$. There might be a large number $d$ of the $x^i$ which satisfy $d - 2$ relations. For each integer $0 \leq l \leq n - 1$ introduce the vector space $\mathcal{P}_{h,l}$ of symmetric polynomials of order $l$ in the $x^i$ such that

$$\mathcal{P}_{h,l} \subset \mathcal{P}_{h,l+1}, \quad \bigcup_{l=0}^{n-1} \mathcal{P}_{h,l} = M_n. \quad (2.3.1)$$

The filtration of the algebra $M_n$ which defines $\Sigma_h$ is given then by the $\mathcal{P}_{h,l}$.

We define again the norm $\|f\|_n$ of an element $f \in M_n$ as (2.1.7). We introduce the norm of an element $\tilde{f} \in \mathcal{C}(\Sigma_h)$ as

$$\|\tilde{f}\|^2 = \frac{1}{\text{Vol}(\Sigma_h)} \int_{\Sigma_h} |\tilde{f}|^2.$$

Then, if $f \in \mathcal{P}_{h,l}$ we should have

$$\|f\|_n^2 - \|\tilde{f}\|^2 = o(l/n). \quad (2.3.2)$$

The norm of a generic element of $\mathcal{P}_{h,l}$ grows as $l$. We shall return to this in Section 3.3.

### 2.4 Continuous transitions

From the point of view of noncommutative geometry a transition is possible between space-times of different topology simply because an individual space-time is never completely in a ‘pure’ topological state. As long as $\bar{k}$ is not equal to zero the correct description of every surface is given in terms of a filtration of the matrix algebra $M_n$ for some (very large) integer $n$. A transition occurs when one filtration becomes more
appropriate than another. Below we shall introduce differential calculi on $M_\hbar$ and we shall be in a position to speak of the noncommutative analog of a smooth scalar field. A generic such field $\tilde{f}$ on a surface $\Sigma_\hbar$ of genus $h$ must have finite action $\tilde{S}_h(\tilde{f})$ and every other action $\tilde{S}_{\hbar'}(\tilde{f})$ must be ‘almost always’ infinite. If during the time evolution the action changes so that $\tilde{f}$ has finite action for the genus $h' \neq h$ then this means that the surface has evolved towards a different topology.

The difference in topology between the sphere and the torus is expressed in a discontinuity in the functions $\tilde{x}^a = \tilde{x}^a(\tilde{u}^a)$. These discontinuities will not show up in the norm (2.1.7) we have put on $M_\hbar$. They do show up however if we use the action as norm since it contains derivatives. We shall discuss in the following section how a topological transition can be induced using the partition function after we have introduced differential calculi and the associated scalar-field actions.

## 3 Smooth fuzzy surfaces

Every surface can of course be endowed with a differential structure and the associated de Rham calculus of differential forms. To speak of a smooth fuzzy surface we must be able to define a differential calculus on each fuzzy $\Sigma_\hbar$ which in some sense has the de Rham calculus as a limit. Since the de Rham calculus is based on the derivations of the algebra of functions it is natural to require that for each $h$ the differential calculus over $M_\hbar$ be based on derivations. This idea was first suggested by Dubois-Violette [12] and developed by Dubois-Violette et al. [13]. We shall use a modified version proposed later by Dimakis [11]. Since we shall restrict our considerations here to scalar fields and shall not therefore need explicitly the differential calculus we shall not enter into the details of its construction. Some more details will be given where necessary in Section 4. We recall that a classical scalar field defined on a fuzzy surface of any genus is an element of $M_\hbar$. The form of the matrix determines the genus of the surface on which it is to be considered an approximation to a regular function.

For each $h$ we shall define a differential structure over $M_\hbar$ in order to be able to speak of the noncommutative analogue of a smooth scalar field. We can then define an action $S_{h,n}$ which tends to the action $\tilde{S}_h$ of a complex classical field on $\Sigma_\hbar$. Let $f$ be an element $f$ of $M_\hbar$ which tends to a function $\tilde{f}$ on $\Sigma_\hbar$. Then we have

$$\lim_{n \to \infty} S_{h,n}(f) = \tilde{S}_h(\tilde{f}).$$

We shall use the action to define a Sobolov-like norm on the matrices and a Sobolov norm on the limit functions. We shall return to this in Section 3.4.

### 3.1 Genus zero

The derivations

$$e_a = \frac{1}{ik} \text{ad} x_a$$

satisfy the commutation relations

$$[e_a, e_b] = ik C^{ab}_{\ c} e_c.$$  

(3.1.2)

In the commutative limit these derivations tend towards vector fields $\tilde{e}_a$ on the sphere defined by the action of the Lie algebra of $SO_3$. The relation $\tilde{x}^a \tilde{e}_a = 0$ defines the space
\( \mathcal{X}_0 \) of vector fields on the sphere as a (projective) submodule of the free \( \mathcal{C}(S^2) \)-module generated by the \( \tilde{e}_a \).

We choose the differential calculus \( \Omega^*(M_n) \) defined in terms of the \( e_a \), that is, with the 1-forms defined by the relation
\[
\text{df}(e_a) = e_a f. \tag{3.1.3}
\]
Since the sphere \( S^2 \) is not parallelizable the differential calculus must be defined on a parallelizable bundle over it. The details of this have been described elsewhere [26, 21, 8]. It is important only to recall the existence of a special basis or frame \( \theta^a \) which is dual to the derivations and which commutes with the elements of the algebra.

We define the action \( S_{0,n}(f) \) of the matrix \( f \) on the surface \( \Sigma_0 \) as the trace
\[
S_{0,n}(f) = \frac{1}{n} \text{Tr} \left( f^*(\Delta_0 f + \mu^2)f + V(f^*f) \right) \tag{3.1.4}
\]
where the laplacian is the covariant laplacian with respect to the geometry we have put on the sphere and \( V(f^*f) \) is an arbitrary (positive) potential function. The normalization has been chosen so that
\[
\lim_{n \to \infty} S_{0,n}(f) = \tilde{S}_0(\tilde{f}) \tag{3.1.5}
\]
where \( \tilde{S}_0(\tilde{f}) \) is the usual action of the classical complex scalar field \( \tilde{f} \). Obviously we shall have
\[
S_{0,n}(f) = 0(n)
\]
for almost all elements \( f \in M_n \).

It is of interest to note that because of the identity
\[
df^* * df = \frac{1}{2} e_a f^* e^a f \theta^1 \theta^2 \theta^3 \tag{3.1.6}
\]
it is possible to write the action (3.1.4) without explicitly using the derivations. The 2-form \( *df \) is defined using a straightforward generalization of the standard duality in forms which relies on the existence of the preferred frame.

### 3.2 Genus one

The vector fields
\[
\tilde{e}_1 = \partial_{\tilde{x}}, \quad \tilde{e}_2 = \partial_{\tilde{y}}
\]
form a basis of the free \( \mathcal{C}(T^2) \)-module \( \mathcal{X}_1 \) of vector fields on the torus. Their action on the generators is given by
\[
\begin{align*}
\tilde{e}_1 \tilde{u} &= ir^{-1} \tilde{u}, & \tilde{e}_1 \tilde{v} &= 0, \\
\tilde{e}_2 \tilde{v} &= ir^{-1} \tilde{v}, & \tilde{e}_2 \tilde{u} &= 0.
\end{align*} \tag{3.2.1}
\]
and of course they commute:
\[
[\tilde{e}_1, \tilde{e}_2] = 0.
\]

The dual de Rham 1-forms \( \tilde{\theta}^a \) are given by
\[
\tilde{\theta}^1 = d\tilde{x} = -ir \tilde{u}^{-1} d\tilde{u}, \quad \tilde{\theta}^2 = d\tilde{y} = -ir \tilde{v}^{-1} d\tilde{v}. \tag{3.2.2}
\]
Because the torus does not have as large an invariance group as the sphere it is more difficult to find a differential calculus over \( M_n \) which tends to the de Rham calculus.
This fact leads us to believe that the introduction of appropriate noncommutative
differential calculi over fuzzy surfaces of higher genera will be a delicate matter.

Were it not for the extra constraints (2.2.3) which distinguish the ‘quantum’ torus
from the ‘quantum’ plane we could have used the ‘quantum’ analog of (3.2.1) and
introduced a differential calculus based on the outer derivations \( \delta_\alpha \) defined by

\[
\begin{align*}
\delta_1 u &= ir^{-1}u, \\
\delta_2 v &= ir^{-1}v,
\end{align*}
\]

If we extend formally the algebra and admit hermitian elements \( x = -ir \log u \) and \( y = -ir \log v \) then these derivations become inner and can be written, using the relation

\[
(3.2.3)
\]

The associated frame is formally identical to (3.2.2):

\[
\begin{align*}
\theta^1 &= -iru^{-1}du, \\
\theta^2 &= -ivr^{-1}dv.
\end{align*}
\]

It is easy to see [11] that the associated differential calculus admits a flat metric-
compatible torsion-free linear connection.

But the above derivations \( \delta_\alpha \) are not compatible with the constraints (2.2.3). With
these constraints the algebra is a matrix algebra and all derivations must be inner.
This leads to problems. It is of course in itself not surprising to encounter a situation
where ‘quantization’ is inconsistent with certain constraints; this feature of quantum
mechanics was known to Dirac. Using the representations of Section 2.2 the commu-
tation relations

\[
[x, v] = \frac{k}{r}v(1 - nP_2), \quad [y, u] = -\frac{k}{r}u(1 - nP_1)
\]

are easily derived. We have here introduced the projectors

\[
P_2 = |n - 1\rangle_1\langle n - 1|, \quad P_1 = |0\rangle_2\langle 0|.
\]

As every element of the algebra they can be expressed as polynomials in the generators:

\[
P_2 = \frac{1}{n} \sum_{l=0}^{n-1} q^l u^l, \quad P_1 = \frac{1}{n} \sum_{l=0}^{n-1} v^l.
\]

It follows that the action of the derivations (3.2.4) on the generators of the algebra is
given by

\[
\begin{align*}
e_1 u &= ir^{-1}u(1 - nP_2), \\
e_1 v &= 0, \\
e_2 v &= ir^{-1}v(1 - nP_1), \\
e_2 u &= 0,
\end{align*}
\]

The highly singular projector term on the right-hand side of each of these equations is
due to the constraints (2.2.3). It is because of these terms that we find \( e_1 u^n = 0 \) and
\( e_2 v^n = 0 \) as we must.

The 1-forms dual to the derivations (3.2.4) are given by

\[
\begin{align*}
\theta^1 &= -ir(1 - \frac{n}{n - 1}P_1)u^{-1}du, \\
\theta^2 &= -ir(1 - \frac{n}{n - 1}P_2)v^{-1}dv.
\end{align*}
\]

If we compare (3.2.10) with (3.2.2) we see that the \( \theta^\alpha \) could in a weak way be considered
to tend to the \( \tilde{\theta}^\alpha \). The problem of the singular limit of the differential calculus is hidden
however in the differentials $dP_\alpha$; the differential calculus based on the derivations (3.2.4) does not tend to the de Rham differential calculus on the torus.

It was of course not necessary to use a differential calculus based on derivations and one can introduce many another differential calculi over the ‘quantum’ torus. There are in fact many which can be constructed [11] based on derivations but which are not real. It is easy to see however that whatever the definition of $du$ and $dv$ the 1-forms (3.2.5) cannot commute with the elements of the algebra and that the resulting differential calculus will not have them as a preferred frame. Also to define the action we will have to be able to define a Laplace operator using the derivations.

To construct the torus we identified the points $\tilde{x} + 2\pi r$ with $\tilde{x}$ and $\tilde{y} + 2\pi r$ with $\tilde{y}$. In the ‘quantized’ version this becomes the cyclicity condition (2.2.6) which gives rise to the singular projector terms in the derivations (3.2.9). One can eliminate them by a procedure which is equivalent to folding, so to speak, the torus at $\tilde{x} = \pi r$ or $\tilde{y} = \pi r$. For this we suppose that $n = 2m$ is even in the formulae of Section 2.2 and we consider the possibility of a differential calculus based on the derivations of the form (3.2.4) with $x$ and $y$ replaced respectively by $x'$ and $y'$ defined by

$$x|j\rangle_1 = \frac{k}{r}(j + nF_j)|j\rangle_1, \quad y|j\rangle_2 = \frac{k}{r}(j + nG_j)|j\rangle_2.$$  \(3.2.11\)

We shall suppose that $F_j, G_j \in \mathbb{Z}$ so that we have

$$u = e^{ix'/r}, \quad v = e^{iy'/r}.$$  

The matrices $x$ and $y$ are not defined then uniquely by the Formulae (2.2.7). This fact is related to the fact that only by using additional topological conditions was von Neumann able to deduce the uniqueness of the representation of the Heisenberg commutation relations. For a discussion of this and an introduction to the problems connected with the quantization of the torus as a classical phase space as well as reference to the previous literature on the subject we refer to the lecture by Emch [14] or to the recent article by Narnhofer [30].

If we choose

$$F_j = -|m - j|, \quad G_j = |m - j - 1|, \quad n = 2m$$  \(3.2.12\)

and introduce the ‘step functions’

$$\epsilon_1|j\rangle_2 = \begin{cases} -|j\rangle_2, & j \leq m - 1, \\ +|j\rangle_2, & j \geq m, \end{cases} \quad \epsilon_2|j\rangle_1 = \begin{cases} +|j\rangle_1, & j \leq m - 1, \\ -|j\rangle_1, & j \geq m \end{cases}$$  \(3.2.13\)

we find

$$[x', v] = \frac{k}{r}v(1 + n\epsilon_2(1 + P_2)), \quad [y', u] = -\frac{k}{r}u(1 + n\epsilon_1(1 + P_1)).$$  \(3.2.14\)

The commutation relations (3.2.14) are almost as singular as (3.2.6). The presence however of the extra factors $\epsilon_\alpha$ permits us to ‘renormalize’ the $e_\alpha$ and define

$$e_1 = \frac{1}{n} \frac{1}{ik} \text{ad } y', \quad e_2 = -\frac{1}{n} \frac{1}{ik} \text{ad } x'.$$  \(3.2.15\)

We find then in the $n \to \infty$ limit

$$e_1u = ir^{-1}ue_1(1 + P_1), \quad e_1v = 0, \quad e_2v = ir^{-1}ve_2(1 + P_2), \quad e_2u = 0.$$  \(3.2.16\)
We introduce the step functions
\[ \tilde{\epsilon}_1 = \begin{cases} -1, & \tilde{x} < \pi r, \\ +1, & \tilde{x} > \pi r, \end{cases} \quad \tilde{\epsilon}_2 = \begin{cases} +1, & \tilde{y} < \pi r, \\ -1, & \tilde{y} > \pi r. \end{cases} \]

We can claim then that in a weak way
\[ \lim_{n \to \infty} e_\alpha (1 + P_\alpha) = \tilde{\epsilon}_\alpha \]
and comparing (3.2.1) with (3.2.16) we find that
\[ \lim_{n \to \infty} e_\alpha = \tilde{\epsilon}_\alpha \tilde{e}_\alpha. \quad (3.2.17) \]
The limit of the derivations $e_\alpha$ are vector fields on the torus which form a basis of $X_1$ but which are not continuous along the lines $\tilde{x} = \pi r, \tilde{y} = \pi r$.

We have not succeeded in finding real derivations of $M_n$ which tend to real smooth vector fields on the torus. The limit $n \to \infty$ is a rather singular limit and it need not be true that an arbitrary vector field on the torus is the limit of a derivation. We constructed the algebra $M_n$ using generators and relations. This is the noncommutative version of the method of defining a curved manifold by an embedding in a higher-dimensional flat euclidean space. This procedure works well for the sphere but the flat torus possesses no such embedding. We refer to the book by Thorpe [34] for a discussion of this point.

The 1-forms dual to the derivations (3.2.16) are given by
\[ \theta^1 = -i r \epsilon_1 (1 - \frac{1}{2} P_1) u^{-1} du, \quad \theta^2 = -i r \epsilon_2 (1 - \frac{1}{2} P_2) v^{-1} dv. \quad (3.2.18) \]
These are almost as singular as the limit of the expressions given by (3.2.10). It is important however for us to have the derivations to define the Laplace operator.

We define the action $S_{1,n}(f)$ of the matrix $f$ on the surface $\Sigma_1$ as the trace
\[ S_{1,n}(f) = \frac{1}{n} \text{Tr} \left( f^* (\Delta_1 f + \mu^2) f + V(f^* f) \right) \quad (3.2.19) \]
where the laplacian is the covariant laplacian with respect to the geometry we have put on the torus and $V(f^* f)$ is an arbitrary (positive) potential function. The normalization has been chosen so that
\[ \lim_{n \to \infty} S_{1,n}(f) = \bar{S}_1(\bar{f}) \quad (3.2.20) \]
where $\bar{S}_1(\bar{f})$ is the usual action of the classical complex scalar field $\bar{f}$. From (3.2.16) we find
\[ \frac{1}{n} \text{Tr} (u^* \Delta_1 u) = \frac{1}{n} \text{Tr} (e_\alpha u^* e^* u) = \frac{1}{n^2} \text{Tr} (u^* u) = \left( 1 + \frac{1}{n} \right) \frac{1}{r^2} \]
and similarly for $v$.

Obviously we shall have
\[ S_{1,n}(f) = 0(n) \]
for almost all elements $f \in M_n$. As an example consider the ‘coordinate’ $x^3$ on the sphere. With the conventions we have been using one finds the expression
\[ x^3 = \frac{2r}{n} \sum_{l=1}^{n-1} \frac{u^l}{1 - q^{-l}}. \quad (3.2.21) \]
The numerical factor in this expression is valid only for large values of \( n \). Since
\[
\text{Tr} (e_1 u^* e_1 u^t) = o(l^2)
\]
there follows then the estimate
\[
S_{1,n}(x^3) = o(n).
\]  
(3.22)

At least one of the ‘coordinates’ of the fuzzy sphere becomes singular then when considered as an element of the fuzzy torus.

### 3.3 Higher genera

An introduction to general Riemann surfaces can be found for example in the lecture notes by Schlichenmaier [32]. The algebra of functions on each surface has been ‘quantized’ using general \( \mathbb{C}^* \)-algebras [24, 25]. We conjecture in fact that this can be done using matrix algebras and that differential calculi can be constructed over \( M_n \) which tend in some way to the de Rham differential calculus of \( \Sigma_h \) for each genus \( h \). The construction of Berezin [5] as well as the fact that each \( \Sigma_h \) can be endowed with a metric of constant Gaussian curvature is some encouragement. If the differential calculus is based on derivations then one can define a Laplace operator \( \Delta_h \) and an action
\[
S_{h,n}(f) = \frac{1}{n} \text{Tr} (f^* (\Delta_h f + \mu^2) f + V(f^* f))
\]  
(3.3.1)

with
\[
\lim_{n \to \infty} S_{h,n}(f) = \tilde{S}_h(\tilde{f})
\]

where \( \tilde{S}_h(\tilde{f}) \) is the usual action of the classical complex scalar field \( \tilde{f} \) on the Riemann surface \( \Sigma_h \).

### 3.4 Smooth transitions

A generic classical field \( \tilde{f} \) on a surface \( \Sigma_h \) of genus \( h \) must have finite action \( \tilde{S}_h(\tilde{f}) \) and every other action \( \tilde{S}_{h'}(\tilde{f}) \) must be infinite. If during the time evolution the action changes so that \( \tilde{f} \) has finite action for some other genus \( h' \) then this means that the surface has evolved towards a different topology. To describe a topological transition from the sphere to the torus one introduces a ‘temperature’ \( \beta \) and an action \( \tilde{S}_{h(\beta)} \) such that \( h(\beta) = 0 \) for \( \beta < \beta_c \) and \( h(\beta) = 1 \) for \( \beta > \beta_c \). The transition will be of first order. It can be made to be of infinite order by choosing \( h(\beta) = 0 \) for \( \beta < \beta_c - \epsilon \) and \( h(\beta) = 1 \) for \( \beta > \beta_c + \epsilon \) and choosing as action a smooth functional
\[
\tilde{S}_{h(\beta)} = (1 - p(\beta)) \tilde{S}_0 + p(\beta) \tilde{S}_1
\]
in the region \( \beta_c - \epsilon \leq \beta \leq \beta_c + \epsilon \). One task of a noncommutative version of gravity would be to motivate this \textit{ad hoc} change of action functional, to calculate, that is, the function \( p(\beta) \).

The partition function for a complex scalar field over a surface of genus \( h = h(\beta) \) is given by
\[
\tilde{Z}_{h(\beta)} = \int e^{-\tilde{S}_{h(\beta)}(\tilde{f})} d\tilde{f}.
\]  
(3.4.1)
The matrix approximation [27] is given by

\[
Z_{h(\beta),n} = \int e^{-S_{h(\beta),n}[f]} df
\]

where the path integral is now a well-defined integration over matrices. We suppose that the ‘real’ value of \( n \) is ‘large’ but not infinite, given by (2.1.1) or (2.2.4). We can then claim that the expression (3.4.2) is the ‘correct’ one and (3.4.1) is the approximation. For \( \beta < \beta_c \), the contributions from almost all those matrices \( f \) which approximate functions on the torus (and other genera) are suppressed since \( S_{1,n}[f] = 0(n) \). On the other hand for \( \beta > \beta_c \), the contributions from almost all those matrices \( f \) which approximate functions on the sphere (and other genera) are suppressed since \( S_{0,n}[f] = 0(n) \).

4 D-branes

Matrices can also be used to give a finite ‘fuzzy’ description of the space complementary to a Dirichlet \( p \)-brane, a description which will allow one perhaps to include the reasonable property that points should be intrinsically ‘fuzzy’ at the Planck scale. Strings naturally play a special role here since they have a world surface of dimension two and an arbitrary matrix can always be written as a polynomial in two given matrices. We refer to the literature for a description of Dirichlet branes in general [31, 6, 10] and within the context of \( M \)(atrix)-theory [4, 18, 22, 3]. The action of the matrix description of the complementary space is conjectured [9] to be associated to the action in the infinite-momentum frame of a super-membrane of dimension \( p \). Since quite generally the compactified factors of the surfaces normal to the \( p \)-branes are of the Planck scale we conclude from the arguments of the previous sections that they have ill-defined topology and that a matrix description will include a sum over many topologies.

We consider a \( d \)-dimensional manifold \( V_d \) with a Kaluza-Klein reduction to a Dirichlet brane \( \Sigma_p \) of dimension \( p \). The \( \Sigma_p \) is known as a \((p-1)\)-brane. The manifold \( V_d \) is therefore a bundle over \( \Sigma_p \) with fibre an \((d-p)\)-dimensional manifold \( N_{d-p} \). We shall suppose for simplicity that the fibration is trivial, \( V_d = \Sigma_p \times N_{d-p} \), and that all manifolds are parallelizable. We shall suppose also, as is usual in Kaluza-Klein theory, that \( N_{d-p} \) is space-like. Let Greek indices \((\alpha, \beta, \ldots)\) take the values 1 to \( p \), Latin indices \((a, b, \ldots)\) the values \( p + 1 \) to \( d \) and Latin indices \((i, j, \ldots)\) the values 1 to \( d \). We introduce a moving frame \( \theta^i = (\theta^a, \theta^i) \) on \( V_d \) with \( \theta^a \) a moving frame on \( \Sigma_p \). Consider now an electromagnetic field on \( V_d \) and write the field strength \( F \) as

\[
F = \frac{1}{2} F_{ij} \theta^i \theta^j.
\]

Then the electromagnetic action in \( V_d \) takes the form

\[
S = \frac{1}{4g^2} \int_{\Sigma_p} \int_{N_{d-p}} F_{ij} F^{ij} d^{d-p}x d^p x.
\]

Although we have argued elsewhere [27] that the entire \( V_d \) should be described by a noncommutative algebra we shall suppose here that the \( D \)-brane can be described by an ordinary smooth manifold and that only the \( N_{d-p} \) need be ‘quantized’. This means that the algebra \( \mathcal{C}(N_{d-p}) \) of (smooth) functions on \( N_{d-p} \) is replaced by a finite-dimensional matrix algebra, the algebra \( M_n \) of \( n \times n \) matrices, a procedure which is analogous to the quantization of a compact phase space [5], for example spin. It means also that the algebra of de Rham differential forms \( \Omega^* (\mathcal{C}(N_{d-p})) \) on \( N_{d-p} \) must
be replaced by a differential calculus over $M_n$. This entire procedure has just been described for $d - p = 2$ in Section 4. For $d - p = 4$ we refer to Grosse et al. [20] and for general $d - p$ to Madore [28]. In the case $d - p = 2$ we have seen that only genus zero and one can be considered with any success.

The components $\theta^a$ of the moving frame introduced on $V_d$ are to be replaced by a noncommutative equivalent such that in some sense we have

$$\lim_{n \to \infty} \Omega^*(M_n) = \Omega^*(C(N_{d-p})).$$

We have seen in Section 4 how difficult this limit is to define even for $p = 2$. We shall use the same symbol to denote a de Rham form and the equivalent fuzzy form. Let $\omega$ be an element of $\Omega^1(C(V_d))$. Then in typical Kaluza-Klein fashion we can write $\omega$ as the sum of a ‘horizontal’ term $A = \omega_h$ in $M_n \otimes \Omega^1(C(\Sigma_p))$ and a ‘vertical’ term $\omega_v$ in $C(\Sigma_p) \otimes \Omega^1(M_n)$. More details of this can be found elsewhere [27].

### 4.1 Curved complements

The case in which the space $N_{d-p}$ complementary to a $p$-brane is curved and compact is the easiest to treat conceptually from a fuzzy point of view. It contains at least a simple generic example, $N_{d-p} = S^2$ which has been worked out in detail and although we formulate them more generally, most of the following calculations have been shown to be valid only in this one example. The case has however the drawback in that, being curved, the models have no immediate supersymmetric extension. A rudimentary version of ‘noncommutative supergravity’ would have to be developed for this purpose. This would involve introducing besides the noncommuting bosonic generators a set of non-anticommuting fermionic generators and defining a linear connection on the entire structure. This has yet to be done.

We identify the gauge transformations on $M_n$ as the unitary elements $U_n$ of $M_n$. The way in which $U_n$ can be identified with the local $U_1$ transformations in the commutative limit has been explained by Grosse & Madore [19] in the case $p = 2$ and genus zero. A gauge transformation is therefore given by

$$\omega \mapsto g^{-1} \omega g + g^{-1} dg \tag{4.1.2}$$

with $g$ an element of $U_n$, the group of local gauge transformations on $\Sigma_p$.

Now $\Omega^1(M_n)$ has a preferred 1-form $\theta$ which is invariant [13] under a gauge transformation:

$$\theta \mapsto g^{-1} \theta g + g^{-1} d_\omega g = \theta. \tag{4.1.3}$$

We have here decomposed $d = d_\omega + d_h$. Choose an integer $m$ and an anti-hermitian basis $\lambda_a$ of the Lie algebra of $SU_m$. Restrict $n$ to those values such that $SU_m$ has an irreducible representation of dimension $n$ and restrict the $N_{d-p}$ to be an orbit of the adjoint representation of $SU_m$. For example if $m = 2$ then $n$ can take any values and $d - p = 2$ which is the dimension of $SU_m$ minus the number of Casimir operators. If $m = 3$ then $d - p = 6$ which is again the dimension of $SU_m$ minus the number of Casimir operators. The manifold $N_{d-p}$ is a manifold embedded in $S^7$ and defined by the cubic Casimir operator of $SU_3$. If we introduce the derivations $e_a = \text{ad} \lambda_a$ and the 1-forms $\theta_a$ dual to them then $\theta = -\lambda_a \theta^a$. (In the limit (4.1.1) $\theta$ is singular [26] in the case $p = 2$ and genus zero; this is because the sphere is not parallelizable and in the commutative limit the $\theta^a$ must be defined on a parallelizable bundle over it) We can decompose then

$$\omega = A + \theta + \phi \tag{4.1.4}$$

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where $\phi = \omega_v - \theta$ is the difference between two connections and so transforms under the adjoint representation of $U_n$. We write $\phi = \phi_a \theta^a$. Then a straightforward calculation leads to the identities

$$F_{ab} = [\phi_a, \phi_b] - C^c_{ab} \phi_c, \quad F_{aa} = D_a \phi_a$$  \hspace{1cm} (4.1.5)

for the ‘fuzzy’ and mixed components of the electromagnetic field strength. The structure constants $C^c_{ab}$ are defined with respect to the basis $\lambda_a$ of $SU_m$.

A rather dubious mathematical argument leads, at least in the case $d - p = 2$ and genus zero [19], to the limit

$$\lim_{n \to \infty} \frac{1}{n} S_n = S.$$  \hspace{1cm} (4.1.6)

The $S_n$ is given by

$$S_n = \frac{1}{4g^2} \int_{\Sigma_p} \text{Tr} (F_{ij} F^{ij}) = \frac{1}{4g^2} \int_{\Sigma_p} \text{Tr} (F_{\mu\nu} F^{\mu\nu}) - \frac{1}{2g^2} \int_{\Sigma_p} \text{Tr} (D_\mu \phi_a)(D^\mu \phi^a) + V(\phi)$$  \hspace{1cm} (4.1.7)

where

$$V(\phi) = -\frac{1}{4g^2} \int_{\Sigma_p} \text{Tr} (F_{ab} F^{ab})$$  \hspace{1cm} (4.1.8)

is the potential. We see that it can be thought of as the field strength in the fuzzy directions.

If we consider the $\phi_a$ as ‘coordinates’ on the fuzzy version of $N_{d-p}$ then $\Sigma_p$ is given by $\phi_a = 0$, which is a stable zero of the potential (4.1.8). Another obvious stable zero is given by $\phi = \lambda_a$. There are in general other stable zeros, the number of which increases with $n$. For example, in the case of the 2-sphere there are in all $p(n)$ (the partition function) zeros of $V(\phi)$. One can think of this as meaning that there are $p(n)$ possible ‘positions’ for the $D$-branes and they are all stable with the same potential energy. Energy is required however to transit from one state to another. In a complicated way the number of massless modes increases as one ‘approaches’ the vacuum $\phi_a = 0$. By this we mean that when $\phi_a = 0$ there is a $U_n$ multiplet of massless modes, when $\phi = \lambda_a$ there is only a $U_1$ multiplet and the number of massless modes in the $p(n) - 2$ vacua between these two extremes depends on the characteristics of the vacuum.

The $M_n$ are curved ‘manifolds’ in general and endowed with a linear connection. The covariant derivatives $D_a \phi_b$ of $\phi_a$ in the directions normal to $\Sigma_p$ are given by

$$D_a \phi_b = [\phi_a, \phi_b] - \frac{1}{2} C^c_{ab} \phi_c = F_{ab} + \frac{1}{2} C^c_{ab} \phi_c.$$  \hspace{1cm} (4.1.9)

It vanishes therefore on the stable vacuum given by $\phi_a = 0$ but not on the others. Except for the ‘curvature term’ in the expression of the ‘vertical’ components $F_{ab}$ of the Yang-Mills field strength the action (4.1.7) is identical to the bosonic part of the one which has been proposed in $M$-atrix theory. To see if it is possible to obtain exactly the $M$-atrix-theory action we turn our attention to flat complements $N_{d-p}$.

### 4.2 Flat complements

The simplest example of a compact manifold $N_{d-p}$ which could admit a flat metric is the $(d - p)$-torus. We have not succeeded in treating this case for general values of $d - p$ but it would seem from our considerations of Section 3.2 that it is very difficult if not impossible to define a differential calculus on a noncommutative version of the 2-torus which tends smoothly to the de Rham differential calculus and which admits a flat metric. The case of a flat $N_{d-p}$ is paradoxically more difficult to treat from a fuzzy point of view than the curved one.
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