Form factors of the XXZ model and the affine quantum group symmetry

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ABSTRACT

We present new expressions of form factors of the XXZ model which satisfy Smirnov’s three axioms. These new form factors are obtained by acting the affine quantum group $U_q(\widehat{\mathfrak{sl}_2})$ to the known ones obtained in our previous works. We also find the relations among all the new and known form factors, i.e., all other form factors can be expressed as kind of descendents of a special one.

* Supported by the Grant-in-Aid for Scientific Research from the Ministry of Education, Science and Culture.
1 Introduction

In [1] we presented integral formulae to the quantum Knizhnik–Zamolodchikov (q-KZ) equation [2] of level 0 associated with the vector representation of the affine quantum group $U_q(\widehat{\mathfrak{sl}_2})$. Those solutions satisfy Smirnov’s three axioms of form factors [3].

Throughout the study of form factors of the sine-Gordon model, Smirnov [3] found that his three axioms are sufficient conditions of local commutativity of local fields of the model. Smirnov also constructed the space of local fields of the sine-Gordon model [4], in terms of the form factor bootstrap formalism. Smirnov’s formulae for form factors of the sine-Gordon model are expressed in terms of the deformed Abelian integrals, or deformed hyper-elliptic integrals [5].

Babelon, Bernard and Smirnov [6] computed form factors of the restricted sine-Gordon model at the reflectionless point, by quantizing solitons of the model. They also found null vectors of the model[7], which leads to a set of differential equations in terms of form factors.

A form factor is originally defined as a matrix element of a local operator. In this paper, however, we call any vector valued function a ‘form factor’ that satisfies Smirnov’s three axioms. In this sense, the integral formulae given in [1] are form factors of the XXZ model. Furthermore, we wish to consider the space of form factors of the XXZ model, or solutions of Smirnov’s three axioms. Our earlier motivation is the question if the space of form factors is invariant under the action of the affine quantum group $U_q(\widehat{\mathfrak{sl}_2})$, which is a symmetry of the XXZ model.

Let us consider the spin 1/2 XXZ model with the nearest neighbor interaction:

$$H_{XXZ} = -\frac{1}{2} \sum_{n=-\infty}^{\infty} (\sigma_n^x \sigma_{n+1}^x + \sigma_n^y \sigma_{n+1}^y + \Delta \sigma_n^z \sigma_{n+1}^z),$$

where $\Delta = (q + q^{-1})/2$ and $-1 < q < 0$.

Let $V = \mathbb{C}^2$ be a vector representation of the affine quantum group $U_q(\widehat{\mathfrak{sl}_2})$. The XXZ Hamiltonian $H_{XXZ}$ formally acts on $\cdots \otimes V \otimes V \otimes \cdots$. This Hamiltonian commutes with $U_q(\widehat{\mathfrak{sl}_2})$. In [8, 9] the space of states $V^\otimes \infty$ was identified with the tensor product of level 1 highest and level $-1$ lowest representations of $U_q(\widehat{\mathfrak{sl}_2})$.

Since the XXZ model possesses $U_q(\widehat{\mathfrak{sl}_2})$-symmetry, any physical quantities of the model are expected to also possess the same symmetry. The following question is thus natural: Let $G(\zeta_1, \ldots, \zeta_N)$ be a $V^\otimes N$-valued form factor of the XXZ model. Then does $\pi_\zeta(y) G(\zeta)$, where $y \in U_q(\widehat{\mathfrak{sl}_2})$, again satisfy Smirnov’s axioms?

The answer is as follows: It is not always true that $\pi_\zeta(y) G(\zeta)$ solves Smirnov’s axioms even if $G(\zeta)$ does. However, for a form factor that satisfies Smirnov’s axioms, there exists $y \in U_q(\widehat{\mathfrak{sl}_2})$ such that $\pi_\zeta(y) G(\zeta)$ again satisfies the axioms.

The present paper is organized as follows. In section 2 we summarize the results obtained in the previous paper [1]. In section 3 we present new form factors which satisfy Smirnov’s three axioms by acting the affine quantum group $U_q(\widehat{\mathfrak{sl}_2})$ to form factors given in section 2. In section 4 we show the relations among form factors obtained in sections 2 and 3. In section 5 we give some remarks.
2 Integral formula of form factors of the XXZ model

In this section we review Smirnov’s three axioms of form factors [3] and the integral formula of form factors of the XXZ model given in [1]. See [10, 11] as for explicit expressions of some scalar functions and homogeneous functions below.

For a fixed complex parameter $q$ such that $0 < q < 1$, let $U$ be the affine quantum group $U_q'$($\hat{sl}_2$) generated by $e_i, f_i, t_i (i = 0, 1)$ [11]. Set $V = v_+ \oplus v_-$ and let $(\pi_, V) \in \{0\}$ denote the vector representation of $U$ defined by

$$\pi_0(e_i)(v_+, v_-) = \zeta(0, v_+), \quad \pi_0(f_i)(v_+, v_-) = \zeta^{-1}(v_-, 0), \quad \pi_0(t_i)(v_+, v_-) = (qv_+, q^{-1}v_-),$$

$$\pi_0(e_0)(v_+, v_-) = \zeta(v_-, 0), \quad \pi_0(f_0)(v_+, v_-) = \zeta^{-1}(0, v_+), \quad \pi_0(t_0)(v_+, v_-) = (q^{-1}v_+, qv_-). \tag{2.1}$$

Let $R(\zeta), S(\zeta) = S_0(\zeta)R(\zeta) \in \text{End}(V \otimes V)$ be the $R$ and $S$ matrix of the XXZ model, where the ratio $S_0(\zeta)$ is a scalar function, which satisfy the intertwining property [11]:

$$X(\zeta_1/\zeta_2)(\pi_{\zeta_1} \otimes \pi_{\zeta_2}) \circ \Delta(y) = (\pi_{\zeta_1} \otimes \pi_{\zeta_2}) \circ \Delta'(y)X(\zeta_1/\zeta_2), \tag{2.2}$$

for $X = R$ or $S$; $\Delta$ and $\Delta' = \sigma \circ \Delta$ are composites of $U$.

For $n \geq 0, l \geq 0, n + l = N$, let $V^{(nl)}$ be a subspace of $V^{\otimes N}$ such that

$$V^{(nl)} = \bigoplus_{\varepsilon_i = \bar{l} - n} v_{\varepsilon_1} \otimes \cdots \otimes v_{\varepsilon_N}.$$

Let $G^{(nl)}_\varepsilon(\zeta_1, \cdots, \zeta_N) \in V^{(nl)}$ with $\varepsilon = \pm$ be a form factor that satisfies the following three axioms:

1. **S-matrix symmetry**

$$P_{j+1}G^{(nl)}_\varepsilon(\cdots, \zeta_j, \zeta_{j+1}, \cdots) = S_{j+1}(\zeta_j/\zeta_{j+1})G^{(nl)}_\varepsilon(\cdots, \zeta_j, \zeta_{j+1}, \cdots) \quad (1 \leq j \leq N - 1), \tag{2.3}$$

where $P(x \otimes y) = y \otimes x$ for $x, y \in V$.

2. **Deformed cyclicity**

$$P_{12} \cdots P_{N-1}NG^{(nl)}_\varepsilon(\zeta_2, \cdots, \zeta_N, \zeta_1q^{-2}) = e\varepsilon\varepsilon(\zeta_1)D\varepsilon G^{(nl)}_\varepsilon(\zeta_1, \cdots, \zeta_N), \tag{2.4}$$

where $r(\zeta)$ is an appropriate scalar function and $D = q^{-N/2}\text{diag}(q^n, q^l)$.

3. **Annihilation pole condition** The $G^{(nl)}_\varepsilon(\zeta)$ has simple poles at $\zeta_N = \sigma\zeta_{N-1}x^{-1}$ with $\sigma = \pm$, and the residue is given by

$$\text{Res}_{\zeta_N/\sigma\zeta_{N-1}x^{-1}} G^{(nl)}_\varepsilon(\zeta) = \frac{1}{2}(I - \sigma^{N+1}x(\sigma\zeta_{N-1}x)D_N S_{N-1|N-2}(\zeta_{N-1}|\zeta_{N-2}) \cdots S_{N-1|1}(\zeta_{N-1}|\zeta_1)) G^{(n-l-1)}_\varepsilon(\zeta') \otimes u_\sigma, \tag{2.5}$$

where $(\zeta) = (\zeta_1, \cdots, \zeta_N), (\zeta') = (\zeta_1, \cdots, \zeta_{N-2})$, and $u_\sigma = v_+ \otimes v_- + \sigma v_- \otimes v_+$.

Note that the consistency of these three axioms implies the relation $r_\varepsilon(\zeta) r_\varepsilon(\sigma x) = \sigma^N$.

Set $m = n - 1$ for $n = l$ and $m = \min(n, l)$ for $n \neq l$. Let $\Delta^{(nl)}(x_1, \cdots, x_m|z_1, \cdots, z_n|z_{n+1}, \cdots, z_N)$ be a homogeneous polynomial of $x$'s and $z$'s, antisymmetric with respect to $x$'s and symmetric with $z_j$'s $(1 \leq j \leq n)$.
and $z_i$'s $(n + 1 \leq i \leq N)$, respectively. For such a polynomial, let us define $\langle \Delta^{(nl)} \rangle (x_1, \ldots, x_m|\zeta_1, \ldots, \zeta_N) \in V^{\otimes N}$ by

$$
\langle \Delta^{(nl)} \rangle (x_1, \ldots, x_m|\zeta_1, \ldots, \zeta_N) = \Delta^{(nl)} (x_1, \ldots, x_m|z_1, \ldots, z_n|z_{n+1}, \ldots, z_N) \prod_{j=1}^n \zeta_j \left( \prod_{j=n+1}^N \frac{1}{z_j - z_j^2} \right),
$$

$$
P_{j+1}(\Delta^{(nl)}) (x_1, \ldots, x_m|\zeta_1, \ldots, \zeta_{j+1}, \zeta_j, \cdots) = R_{j+1}(\zeta_j/\zeta_{j+1})(\Delta^{(nl)}) (x_1, \ldots, x_m|\zeta_1, \ldots, \zeta_{j+1}, \zeta_j, \cdots),
$$

where $z_j = \zeta_j^2 (1 \leq j \leq N)$.

Assume that $n \leq l$ for a while. Then an integral formula that solves all the three axioms (2.3–2.5) is given as follows:

$$
G^{(nl)}_z(\zeta) = \frac{G_0(\zeta)}{m!} \prod_{\mu=1}^m \int_C \frac{dx_n}{2\pi i} \Psi^{(N)} (x_1, \ldots, x_m|\zeta_1, \ldots, \zeta_N) \langle \Delta^{(nl)} \rangle (\zeta_1, \ldots, \zeta_N),
$$

where $G_0(\zeta)$ is an appropriate scalar function.

The path of integral $C$ and the explicit expression of the integral kernel $\Psi^{(N)}$ are not important in this paper. See [10, 1] as for details. Note that (2.6) ensures the S-matrix symmetry (2.3). The second axiom (2.4) and the third one (2.5) imply the transformation properties and the recursion relation of the kernel $\Psi^{(N)} (x|\zeta)$, respectively.

The explicit expression of $\Delta^{(nl)}$ is also unimportant in this paper. The essential point concerning $\Delta^{(nl)}$ is the following recursion relations

$$
\Delta^{(nl)} (x_1, \ldots, x_m|z_1, \ldots, z_n|z_{n+1}, \ldots, z_N) \mid_{z_N = z_n q^{-2}}
$$

$$
= \prod_{\mu=1}^m (x_\mu - z_n q^{-1}) \sum_{\nu=1}^n (-1)^{m+\nu} h^{(N-2)} (x_\nu, \cdot^n, z_{N-1}) \Delta^{(nl-1)} (x_1, \cdot^n, x_m|z_1, \ldots, z_{n-1}, z_{n+1}, \ldots, z_{N-1}),
$$

where $h^{(N)} (x|z_1, \ldots, z_N)$ is a homogeneous function of degree $N - 1$ [10], and the degree condition

$$
\deg \Delta^{(nl)} = \left( \frac{m}{2} \right) + nl - n.
$$

Note that one can determine $\Delta^{(nl)}$ recursively by using (2.8). From the antisymmetry with respect to $x$'s, $\Delta^{(nl)}$ has the factor $\prod_{\mu < \nu} (x_\mu - x_\nu)$. Hence the degree of $\Delta^{(nl)}/\prod_{\mu < \nu} (x_\mu - x_\nu)$ is equal to $nl - n$. From the symmetry property with respect to $z$'s, the recursion relation (2.8) gives values of $\Delta^{(nl)}$ at $nl$ points. Thus the polynomial $\Delta^{(nl)}$ can be determined from the initial conditions

$$
\Delta^{(0)} = \Delta^{(11)} = 1, \quad l > 0.
$$

## 3 Form factors and the action of the affine quantum group

In this section we discuss the transformation properties of form factors given in the last section under the action of the affine quantum group $U = U_q(\widehat{sl}_2)$.

For any $y \in U$, the tensor representation $(\pi_{(\zeta_1, \cdots, \zeta_N)} (y), V^{\otimes N})$ is defined as follows:

$$
\pi_{(\zeta_1, \cdots, \zeta_N)} (y) = (\pi_{\zeta_1} \otimes \cdots \otimes \pi_{\zeta_N}) \circ \Delta^{(N-1)} (y).
$$
Let us act $\pi_{(\zeta_1, \ldots, \zeta_N)}(y)$ to $G^{(n)}(\zeta_1, \ldots, \zeta_N)$. The action of $t_i$'s are trivial:

$$\pi_{\zeta}(t_0)G^{(n)}(\zeta) = q^{n-1}G^{(n)}(\zeta), \quad \pi_{\zeta}(t_1)G^{(n)}(\zeta) = q^{-n}G^{(n)}(\zeta).$$

The action of $f_0$ is non-trivial but the result is very simple:

$$\pi_{\zeta}(f_0)G^{(n)}(\zeta) = 0,$$

(3.2)

In order to prove (3.2) it is enough to show

$$\pi_{\zeta}(f_0)(\Delta^{(n)}) (x|\zeta) = 0.$$  

(3.3)

From the intertwining property (2.2) the $R$-matrix symmetry (2.6) of $(\Delta^{(n)})(x|\zeta)$ implies that of $\pi_{\zeta}(f_0)(\Delta^{(n)}) (x|\zeta)$. The arbitrary component of $\pi_{\zeta}(f_0)(\Delta^{(n)}) (x|\zeta)$ can be expressed in terms of linear combination of the extreme component $(\pi_{\zeta}(f_0)(\Delta^{(n)}) (x|\zeta))_{---+}$'s. The claim (3.3) thus follows from that $(\pi_{\zeta}(f_0)(\Delta^{(n)}) (x|\zeta))_{---+}$ vanishes.

Set $(\Delta^{(0)}(n-1,l+1))(x|\zeta) = \pi_{\zeta}(f_0)(\Delta^{(n)}) (x|\zeta)$. Then $(\Delta^{(0)}(n-1,l+1))(x_1, \ldots, x_m|z_1, \ldots, z_{n-1}|z_n, \ldots, z_N)$ is proportional to $(\pi_{\zeta}(f_0)(\Delta^{(n)}) (x|\zeta))_{---+}$. Thanks to the $R$-matrix symmetry we obtain

$$\Delta^{(0)}(n-1,l+1)(x_1, \ldots, x_m|z_1, \ldots, z_{n-1}|z_n, \ldots, z_N) = \sum_{k=n}^{N} \frac{\prod_{i=n}^{k-1}(z_i - z_{q^{-2}})\Delta^{(n)}(x_1, \ldots, x_m|z_1, \ldots, z_{n-1}, x_k, z_n, \ldots, z_N)}{\prod_{i=n}^{k-1}(z_i - z_k)q^{-1}}.$$  

(3.4)

Note that the singularity at $z_k = z_i$ in the RHS of (3.4) is spurious, and hence that $\Delta^{(0)}(n-1,l+1)$ is a homogeneous polynomial of degree $(\frac{m}{2}) + (n-1)(l+1) - n$, antisymmetric with respect to $x_\mu$'s and symmetric with respect to $\{z_1, \ldots, z_{n-1}\}$ and $\{z_n, \ldots, z_N\}$, respectively. The recursion relation

$$\Delta^{(0)}(n-1,l+1)(x_1, \ldots, x_m|z_1, \ldots, z_{n-1}|z_n, \ldots, z_N)|_{z_N = z_{n-1}q^{-2}} = \prod_{\nu=1}^{m}(x_\nu - z_n q^{-1})\sum_{\mu=1}^{m}(-1)^{m+n}\Delta^{(0)}(n-2,l)(x_1, \ldots, x_m|z_1, \ldots, z_{n-2}|z_n, \ldots, z_N),$$

is enough to determine $\Delta^{(0)}(n-1,l+1)$ recursively. From the power counting $\Delta^{(0)}(0,l+1) = 0$, so we obtain (3.3).

From now on, we wish to consider $\pi_{\zeta}(y)G^{(n)}(\zeta)$ for $y \in U$. For that purpose, let us list the following formulae for $\langle \Delta^{(0)}(n+1,l)^{(n)}(x|\zeta) = \pi_{\zeta}(e_0)(\Delta^{(n)})(x|\zeta)$, $\langle \Delta^{(1)}(n+1,l)^{(n)}(x|\zeta) = \pi_{\zeta}(e_1)(\Delta^{(n)})(x|\zeta)$, and $\langle \Delta^{(1)}(n+1,l-1)^{(n)}(x|\zeta) = \pi_{\zeta}(f_1)(\Delta^{(n)})(x|\zeta)$:

$$\Delta^{(0)}(n+1,l+1)(x_1, \ldots, x_m|z_1, \ldots, z_{n+1}|z_{n+2}, \ldots, z_N) = \sum_{k=1}^{n+2} \prod_{i=k+1}^{n+2}(z_i - z_k q^{-2})\Delta^{(n)}(x_1, \ldots, x_m|z_1, \ldots, z_{n+1}, x_k, z_{n+2}, \ldots, z_N),$$

(3.5)

$$\Delta^{(1)}(n+1,l+1)(x_1, \ldots, x_m|z_1, \ldots, z_{n+1}|z_{n+1}, \ldots, z_N) = q^{l-n+1} \sum_{k=n}^{n+2} z_k \prod_{i=n}^{k-1}(z_i - z_k q^{-2})\Delta^{(n)}(x_1, \ldots, x_m|z_1, \ldots, z_{n+1}, z_k, z_{n+2}, \ldots, z_N),$$

(3.6)
Then the following relations hold:

$$ \Delta^{(1)}_{(n+1l-1)}(x_1, \cdots, x_m|z_1, \cdots, z_{n+1}|z_n, \cdots, z_N) = q^{n+1} \sum_{k=1}^{n+1} \sum_{j=1}^{n+1} \prod_{j \neq k}^{n+1} \frac{1}{(z_k - z_j)q^{-1}} \Delta^{(n)}(x_1, \cdots, x_m|z_1, \cdots, z_{n+1}|z_k, z_{n+2}, \cdots, z_N).$$ (3.7)

The expressions (3.5–3.7) can be proved in a similar manner for (3.4).

Let \( A_0(\xi) = G^{(n)}(\xi) \), and \( A_j(\xi) = \pi(\xi)A_{j-1}(\xi) \), where \( 1 \leq j \leq l \). Then \( A_j(\xi) = 0 \) for \( j > l - n \), and \( A_{l-1}(\xi) = \pi(\xi)e_0A_j(\xi) \) for \( 1 \leq j \leq l - n \). These \((l-n+1)\{A_j(\xi)\}_{0 \leq j \leq l-n}\) form a multiplet. Now the following natural question arises: Do all \( A_j(\xi) \)'s satisfy the three axioms (2.3–2.5) for a suitable choice of the diagonal operator \( D_l \)?

The S-matrix symmetry is apparently satisfied by any \( A_j(\xi) \). The second and third axioms are unfortunately invalid unless \( n = l \). (Since \( \pi(\xi)f_0G^{(nn)}(\xi) = 0 = \pi(\xi)e_0G^{(nn)}(\xi) \), the case \( n = l \) is trivial.)

However, we can consider if \( \pi(\xi(y)G^{(n)}(\xi)) \) do satisfy the three axioms, where \( y = e_1 \) or \( f_1 \), because \( \pi(\xi(y)G^{(n)}(\xi)) \) for any \( y \in U \) always satisfy the first axiom (2.3). We do not have to restrict ourselves to the case \( y = e_0 \). Actually, \( \tilde{G}^{(n+l-1)} := \pi(\xi)f_1G^{(n)}(\xi) \) solves all the three axioms with \( D \to D_{l_1}^{-1} \). The second axiom is proved as follows:

$$ P_{l_2} \cdots P_{N-1\xi} \pi(\xi_2, \cdots, \xi_{n-1}) \pi(\xi_1)G^{(n)}(\xi_2, \cdots, \xi_{n-1}, \xi_1q^{-2}) $$

$$ = (\pi(\xi_1q^{-1}) \circ \Delta'(f_1))P_{l_2} \cdots P_{N-1\xi} G^{(n)}(\xi_2, \cdots, \xi_{n-1}, \xi_1q^{-2}) $$

$$ = (\pi(\xi_1q^{-1}) \circ \Delta'(f_1)) P_{l_2} \cdots P_{N-1\xi} G^{(n)}(\xi_2, \cdots, \xi_{n-1}, \xi_1q^{-2}) $$

$$ = (\pi(\xi_1q^{-1}) \circ \Delta'(f_1)) P_{l_2} \cdots P_{N-1\xi} G^{(n)}(\xi_2, \cdots, \xi_{n-1}, \xi_1q^{-2}) $$

$$ = (D_{l_1})_{(n+l-1)}(\xi_1f_1) \pi(\xi_2, \cdots, \xi_{n-1})G^{(n)}(\xi_2, \cdots, \xi_{n-1}, \xi_1q^{-2}) $$

$$ = (D_{l_1})_{(n+l-1)}(\xi_1f_1) \pi(\xi_2, \cdots, \xi_{n-1})G^{(n)}(\xi_2, \cdots, \xi_{n-1}, \xi_1q^{-2}) $$

One can similarly check that the third axiom for \( \tilde{G}^{(n+l-1)}(\xi) \) with \( D \to D_{l_1}^{-1} \).

4 Relations among form factors of the XXZ model

In this section we shall find further relations among \( G^{(n)}(\xi) \)’s and \( \tilde{G}^{(n+l-1)}(\xi) \)’s.

Put \( n = l \). Then it follows from (2.8) and (3.6) that \( \Delta^{(1)}_{(n-1l+1)} \) and \( (-1)^n q^{n+1} \Delta^{(n-1n+1)} \) have the same recursion relation and the same initial condition. Thus we have

$$ G^{(n-1n+1)}(\xi) = (-1)^n q^{-n-1} \pi(e_1)G^{(nn)}(\xi).$$ (4.1)

Two homogeneous polynomial \( \Delta^{(1)}_{(n+1l+1)} \) and \( \Delta^{(n-1l+1)} \) coincide up to a constant factor. The relation \( \Delta^{(1)}_{(n-1l+1)} \) and \( \Delta^{(n-1l+1)} \) for \( n < l \) is not so simple. In order to establish the relation, let us introduce the symbol \( \equiv \) as follows: We denote \( A(x_1, \cdots, x_m|z_1, \cdots, z_N) \equiv B(x_1, \cdots, x_m|z_1, \cdots, z_N) \) when

$$ \prod_{\mu=1}^{m} \int_{C} \frac{dx_{\mu}}{2\pi i} \Psi_{e}^{(N)}(x|\xi)A(x|z) = \prod_{\mu=1}^{m} \int_{C} \frac{dx_{\mu}}{2\pi i} \Psi_{e}^{(N)}(x|\xi)B(x|z).$$

Then the following relations hold:

$$ \Delta^{(1)}_{(n-1l+1)}((x_1, \cdots, x_m|z_1, \cdots, z_{n-1}|z_n, \cdots, z_N) \equiv n(l-n+2)(-1)^{l-n} q^{l+1}(1 - q^{-2(l-n)}) \prod_{j=1}^{n-1} (x_n - z_jq^{-1}) \Delta^{(n-l-1)}((x_1, \cdots, x_m|z_1, \cdots, z_{n-1}|z_n, \cdots, z_N).$$ (4.2)
The above relation (4.2) follows from the antisymmetry of $x$’s and the recursion relation of $\Delta^{(nl)}$.

Until now, we discuss the case $n \leq l$. Let us construct $G^{(nl)}(\zeta)$ with $n > l$ form $G^{(nn)}(\zeta)$, the spin 0 sector of form factors. Define $G^{(n+1n-1)}(\zeta) = \pi_\zeta(f_1)G^{(nn)}(\zeta)$. Then $G^{(n+1n-1)}(\zeta)$ also satisfies the three axioms with $D = q^{-N/2}\text{diag}(q^{n-1}, q^{n+1})$. By acting $f_1$ successively, we can obtain $G^{(n+kn-k)}(\zeta)$ for $n = 1, \cdots, n$; just like we construct $G^{(n-kn+k)}(\zeta)$ from $G^{(nn)}(\zeta)$ by acting $e_1$ successively. As for $G^{(nl)}(\zeta)$ with $n > l$, $\pi_\zeta(e_0)G^{(nl)}(\zeta) = 0$ holds. The proof is easy if you notice that $\pi_\zeta(e_0)G^{(nn)}(\zeta) = 0$ and $[e_0, f_1] = 0$.

Note that $G^{(nl)}(\zeta)$ with $n > l$ is a form factor; i.e., $G^{(nl)}(\zeta)$ satisfies the three axioms of form factors with $D = q^{-N/2}\text{diag}(q^l, q^n)$. One can also show that $\tilde{G}^{(n-1l+1)}(\zeta) := \pi_\zeta(e_1)G^{(nl)}(\zeta)$ again satisfies the three axioms with $D = q^{-N/2}\text{diag}(q^{l-1}, q^{n+1})$.

Let us summarize the relations obtained until now.

\[
\begin{array}{ccc}
0 & f_0 & e_0 \\
\downarrow f_1 & \uparrow e_1 & \\
G^{(n-1n+1)}(\zeta) & G^{(nn)}(\zeta) & 0 \\
\downarrow f_1 & \uparrow e_1 & \\
G^{(n-2n+2)}(\zeta) & G^{(nn)}(\zeta) & G^{(n+2n-2)}(\zeta) \\
\downarrow f_1 & \uparrow e_1 & \\
G^{(n-3n+3)}(\zeta) & G^{(n-1n+1)}(\zeta) & G^{(n+1n-1)}(\zeta)
\end{array}
\]

It is evident from this relationship that $G^{(n-kn+k)}(\zeta)$ and $\tilde{G}^{(n-kn+k)}(\zeta)$ ($-n \leq k \leq n$) can be obtained from $G^{(nn)}(\zeta)$ by acting $e_i$ or $f_i$. We again notice that $\pi_\zeta(f_0)G^{(nn)}(\zeta) = \pi_\zeta(e_0)G^{(nn)}(\zeta) = 0$.

We naturally have a form factor $F^{(nn)}(\zeta)$ that belongs to $V^{(nn)}$ such that $\pi_\zeta(f_1)F^{(nn)}(\zeta) = \pi_\zeta(e_1)F^{(nn)}(\zeta) = 0$. We can obtain $F^{(nn)}(\zeta)$ from $G^{(nn)}(\zeta)$ by a simple transformation.

If $G(\zeta)$ solves the three axioms of form factors with the diagonal operator $D$, then $F(\zeta) = (\sigma^x)_{\otimes N}G(\zeta)$ solves them with the diagonal operator $\sigma^x D$. Hence $F^{(ln)}(\zeta) := (\sigma^x)_{\otimes N}G^{(nl)}(\zeta)$ and $\bar{F}(\zeta) := (\sigma^x)_{\otimes N}\tilde{G}^{(nl)}(\zeta)$ are also form factors of the XXZ model. We can further show that $\pi_\zeta(f_1)F^{(nn)}(\zeta) = \pi_\zeta(e_1)F^{(nn)}(\zeta) = 0$, and $\pi_\zeta(f_1)F^{(ln)}(\zeta) = 0$ for $n < l$, $\pi_\zeta(e_1)F^{(nn)}(\zeta) = 0$ for $n > l$.

Sum up the results obtained in this paper: For fixed $n < l$, we found four form factors which belongs to $V^{(nl)}$ sector; i.e., $G^{(nl)}(\zeta)$, $\tilde{G}^{(nl)}(\zeta)$, $F^{(nl)}(\zeta)$, and $\bar{F}(\zeta)$. Since we have had only one form factor when we fix $n < l$ at the stage of [1], we get four times solutions of the three axioms of form factor in this paper.
5 Concluding Remarks

In this paper, we have constructed new integral expressions of form factors of the XXZ model, by acting $U_q(\hat{sl}_2)$ to the form factors obtained in [1]. The spin 0 form factor $G^{(n\eta)}(\zeta)$ is a kind of singlet because $\pi_\zeta(f_0)G^{(n\eta)}(\zeta) = \pi_\zeta(e_0)G^{(n\eta)}(\zeta) = 0$. We tried in vain to decompose the space of form factors of the XXZ model into infinitely many multiplets of $U_q(\hat{sl}_2)$. In order to perform such a decomposition of the space of states, it may be useful to consider the XXX model limit $q \rightarrow -1$. At this limit the model has the Yangian $Y(sl_2)$-symmetry. The Yangian $Y(sl_2)$ is the minimal quantum group which includes $U(sl_2)$ as a sub-Hopf algebra, so that one can hope to perform the program of decomposition of the space of states of the massive integrable model.

Acknowledgments

The author would like to thank M. Jimbo, T. Miwa and A. Nakayashiki for useful discussion.

References


