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Abstract

The Matrix Darboux–Toda Mapping is represented as a product of a number of commutative mappings. The Matrix Davey–Stewartson Hierarchy is invariant with respect to each of these mappings. We thus introduce an entirely new type of discrete transformation for this hierarchy. The discrete transformation for the Vector Nonlinear Shrödinger System coincides with one of the mappings under necessary reduction conditions.

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1 Introduction

Integrable mappings are an important tool for integrable system investigation. It has been suggested that the theory of integrable systems is in a tight connection with a representation theory of the group of integrable mappings [1]. This viewpoint continues to get many independent confirmations. In approach like this a classification of integrable mappings plays the key role. A mapping (V-mapping in future references) for the Vector Nonlinear Shrödinger System (VNLSS) [2] has recently been introduced by Aratyn’s group [3]. To find it, they considered the transformations that preserve the form of the corresponding Lax operator and equation (this technique can be applied to the (1 + 1)-dimensional case only).

In the present paper, we reveal new discrete symmetry of the (1 + 2)-dimensional Matrix Nonlinear Shrödinger System (MNLSS) [4], [5]. We also show that the V-mapping (generalized to two space dimensions) is a particular case of this symmetry. Additional reduction from two to one dimension gives transformations considered in [3].

2 Multi–Soliton Solutions of the MNLSS

In the next two sections we discuss one of the possible ways of deriving discrete transformations for MNLSS in the (1 + 1)-dimensional case. The V-mapping [3] is gotten as a particular case of these mappings. In section 4, the results are generalized to the two–dimensional case. In this section, we represent explicit expressions for multi–soliton type solutions of MNLSS (We did not meet this sort of expressions in the available literature on the subject.) Proofs and details could be found in [5]. The MNLSS reads

\[\begin{align*}
-v_t + v_{xx} + 2uvv &= 0 \\
u_t + u_{xx} + 2uu &= 0
\end{align*}\] (1)

where \( u \) and \( v \) are \( k \times k \) matrices of an arbitrary rank. Particularly, when the ranks of the matrices equal 1, non–zero part of \( v \) is a single column and of \( u \) a single line, the system (1) coincides with the VNLSS. The system (1) can be obtained with the help of the Maurer-Cartan identity as applied to the following equations:

\[g_x g^{-1} = \begin{pmatrix} \lambda E & u \\ v & -\lambda E \end{pmatrix} g g^{-1} = \begin{pmatrix} 2\lambda^2 E - uv & u_x + 2\lambda u \\ -v_x + 2\lambda v & -2\lambda^2 E + vv \end{pmatrix}\] (2)

where \( E \) denotes the \( k \times k \) unity matrix and \( \lambda \) is a spectral parameter. A soliton–like solution of (1) is described by a pair of vectors \((n_i, m_j)\), where \( n_i \) and \( m_j \) are natural numbers, \( n_i \geq -1 \) and...
\[ m_j \geq -1. \]

\[ u_{i,j} = -\frac{[n_1, \ldots, n_i + 1, \ldots, n_k; m_1, \ldots, m_j - 1, \ldots, m_k]}{[n_1, \ldots, n_k; m_1, \ldots, m_k]} \]

\[ v_{i,j} = \frac{[n_1, \ldots, n_j - 1, \ldots, n_k; m_1, \ldots, m_i + 1, \ldots, m_k]}{[n_1, \ldots, n_k; m_1, \ldots, m_k]} \]

Here \([n_1, \ldots, n_k; m_1, \ldots, m_k]\) stands for the determinant of the matrix whose lines consist of sub-lines of lengths \(n_1 + 1, \ldots, n_k + 1; m_1 + 1, \ldots, m_k + 1\), respectively. The sublines from the \(s\)th line, corresponding to \(n_i + 1\) and \(m_j + 1\), are

\[ n_i : \quad e^{2r_s}, e^{2r_s} \lambda_s, \ldots, e^{2r_s} \lambda_{n_i} \]

\[ m_j : \quad 1, \lambda_s, \ldots, \lambda_{n_j} \]

where \(\tau_s = \lambda_s x/2 - \lambda_s^2 t/4 + c_s\), \(\lambda_s\) and \(c_s\) are sets of arbitrary parameters. For example

\[ [0; 1] = \begin{vmatrix} e^{2r_1} & 1 & \lambda_1 \\ e^{2r_2} & 1 & \lambda_2 \\ e^{2r_3} & 1 & \lambda_3 \end{vmatrix} \]

\[ [1; -1] = \begin{vmatrix} e^{2r_1} & e^{2r_1} \lambda_1 \\ e^{2r_2} & e^{2r_2} \lambda_2 \end{vmatrix} \]

One can directly check that (3) are indeed solutions of (1) using the identity (9) from the appendix. The solutions of VNLSS can be derived from (3) by inserting \(m_j = -1\) for \(j \geq 2\).

### 3 Discrete Transformations for the MNLSS

Now consider solutions with \(n_\alpha - 1\) and \(m_\beta + 1\). Let us denote them by \(\tilde{u}_{ij}\) and \(\tilde{v}_{ij}\) and call transformed solutions. First of all from (3) we notice that

\[ \tilde{u}_{\alpha\beta} = \frac{1}{v_{\beta\alpha}} \]

Using only the identity (9) from the appendix, one can prove the following relations between initial and transformed functions:

\[ (\tilde{u}_{i,j} v_{j\beta})_x = -(uv)_{\alpha} \quad (\tilde{u}_{\alpha,j} v_{j\beta})_x = -(vu)_{\beta,i} \quad \tilde{u}_{\alpha\beta} = \frac{1}{v_{\beta\alpha}} \]

\[ \left( \frac{v_{\beta i}}{v_{\beta\alpha}} \right)_x = (\tilde{u} v)_{\alpha i} \quad \left( \frac{v_{j\alpha}}{v_{\beta\alpha}} \right)_x = (\tilde{v} u)_{j\beta} \]

\[ \tilde{v}_{ji} = v_{ji} - \frac{v_{\beta i} v_{j\alpha}}{v_{\beta\alpha}} \quad \tilde{u}_{ij} = u_{ij} + \tilde{u}_{i\beta} \tilde{u}_{\alpha j} v_{j\beta} \]

\[ v_{\beta \alpha}^2 (\tilde{u} v u)_{\alpha\beta} - (v u v)_{\beta\alpha} = v_{\beta \alpha} (ln v_{\beta\alpha})_{xx} \]

where \(i \neq \alpha\) and \(j \neq \beta\). Note that there are \(k^2\) basic commutative mappings since \(\alpha\) and \(\beta\) are arbitrary. Relations (4) establish connection between different definite type (soliton–like) solutions of the system (1). It turns out that the transformation (4) works not only for this definite type of solutions, but for arbitrary solutions. That is, if \(u\) and \(v\) obey the system (1), \(\tilde{u}\) and \(\tilde{v}\) obey it as well, no matter whether \(u\) and \(v\) are soliton–like or not. At the moment, we can check this only by direct substitution of (4) into (1). About the connection between (4) and Darboux–Toda substitution see section 5. The product of an arbitrary number of mappings (4) is, obviously, a discrete symmetry of (1) again.
4 Two–Dimensional Case

In this case, the MNLSS (two–dimensional matrix Davey–Stewartson system) reads

\[
\begin{cases}
  u_t + au_{xx} + bu_{yy} + 2au \int dy (vu)_x + 2b \int dx (uv)_y u = 0 \\
  -v_t + av_{xx} + bv_{yy} + 2a \int dy (vu)_x v + 2bv \int dx (uv)_y = 0
\end{cases}
\] (5)

where \( a \) and \( b \) are arbitrary numerical parameters. The system (5) is the third term of the Matrix Nonlinear Shrödinger Hierarchy (MNLSH) [6]. Now we generalize (4) to the two space dimensions.

\[
(\tilde{u}_{i\beta}v_{\beta\alpha})_x = -(uv)_{i\alpha} \quad (\tilde{u}_{\alpha j}v_{\beta\alpha})_y = -(vu)_{\alpha j} \quad \tilde{u}_{\alpha\beta} = \frac{1}{v_{\beta\alpha}}
\]

\[
\left( \frac{v_{\beta i}}{v_{\beta\alpha}} \right)_x = (\tilde{u}v)_{i\alpha} \quad \left( \frac{v_{j\alpha}}{v_{\beta\alpha}} \right)_y = (\tilde{v}u)_{j\beta}
\]

\[
\tilde{v}_{ji} = v_{ji} - \frac{v_{\beta i}v_{j\alpha}}{v_{\beta\alpha}} \quad \tilde{u}_{ij} = u_{ij} + \tilde{u}_{i\beta}u_{\alpha j}v_{\beta\alpha} \quad v_{\beta\alpha}^2(\tilde{u}\tilde{v}u)_{\alpha\beta} - (vu)_{\beta\alpha} = v_{\beta\alpha}(\ln v_{\beta\alpha})_{xy}
\] (6)

Within the scope of the present paper the above form of the two–dimensional mapping is a suggestion that should be checked independently. Substituting transformed functions \( \tilde{u} \) and \( \tilde{v} \) in (5), we directly prove that the system (5) is invariant with respect to the mapping (6). In this paper, we do not consider the problem of constructing the hierarchy corresponding to the isolated mapping from (6). But finding the hierarchy invariant with respect to all mappings (6) is not a problem. Indeed, the Matrix Darboux–Toda Substitution can be represented as a product of mappings (6) (see the next section). Hence, it commutes with any transformation from (6). Therefore, all systems of MNLSH are invariant with respect to any transformation from (6).

5 Different mappings and connection between them

First of all, we easily derive the V-mapping from the transformation (4). Let us take

\[
\alpha = \beta = r \quad u_{ir} \equiv u_i \quad v_{ri} \equiv v_i \\
\]

\[
u_{ij} = v_{ji} = 0 \quad j \neq r
\]

Now we consider connection between different discrete transformations corresponding to MNLSH. In [6], the (1+2)-dimensional MNLSH has been constructed as a consequence of its invariance with regard to the Matrix Darboux–Toda Transformation

\[
\tilde{u} = v^{-1} \quad \tilde{v} = [v(u - (v^{-1})_y)v \equiv v[vu - (v^{-1}v_y)_x]]
\] (7)

where \( u \) and \( v \) are an invertible \( k \times k \) matrices. Denote this transformation by \( M_k \) and the mapping (6) by \( T_{\alpha\beta} \). The following equality holds:

\[
T_{11}T_{22} \times \ldots \times T_{kk} = M_k
\]
The operators $T_{ij}$ are related by

$$T_{ij}T_{ji} = T_{ii}T_{jj}$$

and so on. The algebra of $T_{ij}$ generators may appear to be an important instrument to investigate MNLSS solutions.

In the one-dimensional case, there is another substitution corresponding to the MNLSS (i.e., the MNLSS is invariant with respect to that mapping)

$$\tilde{u}_x = u - \tilde{u}v\tilde{u} \quad v_x = v\tilde{u}v - \tilde{v}$$  \hspace{1cm} (8)

We have checked that in the scalar case (when $u$ and $v$ are scalar functions) the solutions produced by this mapping are the same as those produced by the Darboux–Toda transformation. The only difference is in choosing initial (starting) functions. However, it is not clear whether the substitution (8) has a two-dimensional analogue.

### 6 Outlook

The main result of the present paper consists in new discrete transformations for the (1+2)-dimensional MNLSS. Whereas the Darboux–Toda mapping (7) requires $Det v \neq 0$, mappings (4) are free from that restriction. This expands a possibility of MNLSS investigation. Especially, when ranks of the matrices equal 1, we get (1+2)-dimensional generalizations of the VNLSS and the corresponding mapping [3]. But this is far not all possible partial cases. At the moment, we do not know how to solve the symmetry equation for an isolated $T_{\alpha\beta}$ mapping from (6). Obviously, the conventional Lax technique does not work in the two-dimensional case. Solution of the symmetry equation is the most intrigue and perspective unsolved problem of the present paper. We hope to return to it in future publications.

### 7 Appendix

Consider a square matrix

$$F = \begin{pmatrix} A & a_1 & b_1 \\ a_2 & c_1 & d_1 \\ b_2 & c_2 & d_2 \end{pmatrix}$$

where $A$ is a square matrix; $a_1, b_1$ and $a_2, b_2$ are columns and lines of the corresponding dimension, respectively; $c_{1,2}$ and $d_{1,2}$ are scalars. We have

$$Det F = Det A \ Det \begin{pmatrix} E & A^{-1}a_1 & A^{-1}b_1 \\ a_2 & c_1 & d_1 \\ b_2 & c_2 & d_2 \end{pmatrix} =$$

$$Det A \ Det \begin{pmatrix} c_1 - a_2 A^{-1}a_1 & d_1 - a_2 A^{-1}b_1 \\ c_2 - b_2 A^{-1}a_1 & d_2 - b_2 A^{-1}b_1 \end{pmatrix}$$

where $E$ is the corresponding unity matrix. Using this, we readily prove the following identity:

$$|\Pi_{12}| |\Pi_{34}| + |\Pi_{23}| |\Pi_{14}| = |\Pi_{24}| |\Pi_{13}|$$  \hspace{1cm} (9)

Here $| \cdot |$ stands for determinant, $\Pi$ denotes $k \times (k - 2)$ matrix and $1, 2, 3$ and $4$ are columns of dimension $k$. 

4
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