Green’s Function for Nonlocal Potentials

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(August 11, 1998)

Abstract

The single-particle nuclear potential is intrinsically nonlocal. In this paper, we consider nonlocalities which arise from the many-body and fermionic nature of the nucleus. We investigate the effects of nonlocality in the nuclear potential by developing the Green’s function for nonlocal potentials. The formal Green’s function integral is solved analytically in two different limits of the wavelength as compared to the scale of nonlocality. Both results are studied in a quasi-free limit. The results illuminate some of the basic effects of nonlocality in the nuclear medium.

24.10.-i, 03.65.Nk

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I. GENERAL ASPECTS OF NONLOCAL POTENTIALS

A nonlocal term can be introduced to the non-relativistic Schrödinger equation as follows:

\[-\frac{\hbar^2}{2m} \nabla^2 \Psi(x) + [V_L(x) - E] \Psi(x) = -\int d^3x' V_{NL}(x, x') \Psi(x'). \]  

(1.1)

Generally, such a nonlocal potential arises when one attempts to write the equation of motion for a single particle interacting with a many-body system. One source of nonlocality is the coupling of the elastic channel to other degrees of freedom. The potential in the elastic channel is modified by the contribution from excitation out of the elastic channel at a point \(x\), propagation in an intermediate state, and then de-excitation back to the elastic channel at a point \(x'\). This type of nonlocality is called Feshbach nonlocality. In addition, for a fermionic system, one must also account for exchange effects, which require antisymmetrization of the wave function between the projectile and the appropriate components of the target. This type of nonlocality is called Pauli nonlocality.

The form of \(V(x, x')\) will not be considered in full mathematical generality. Nevertheless, this does not seem to be a significant restriction for the physical theory. We consider the nonlocality to be characterized by a finite range \(b\). In particular, we consider nonlocal potentials which limit to local potentials as the range \(b \to 0\). This form allows sufficient richness in the theory, but allows us to stay in contact with the familiar case of local quantum mechanics. This requirement excludes, for example, nonlocal potentials of the form \(V_1(x)V_2(x')\), which have no classical analog as a limiting case in the sense above [1].

In the development below, we consider a nonlocal potential operator defined as

\[ \hat{V}_{NL} = \int \int d^3x d^3x' V_{NL}(x, x') |x\rangle \langle x'|. \]  

(1.2)

For calculating the Green’s function, we need the Fourier transform of \(V_{NL}(x, x')\), i.e.,

\[ \tilde{V}_{NL}(p, p') = \langle p|\hat{V}_{NL}|p'\rangle. \]  

(1.3)

Here and below a tilde is used to indicate the Fourier transform. Calculation of these matrix elements reveals some other general properties of the nonlocal potential. To begin with, consider the following form:

\[ V_{NL}(x, x') = V_1(x, x')V_2(x - x'). \]  

(1.4)

We require that the second term limit to a delta function as the range of nonlocality \(b \to 0\). That insures that the potential has a definite local limit. The first term is allowed for now to have an arbitrary dependence on \(x\) and \(x'\). Then

\[ \tilde{V}_{NL}(p, p') = \frac{1}{(2\pi\hbar)^3} \int \int d^3x d^3x' \]

\[ \times \exp \left( \frac{i p' \cdot x'}{\hbar} \right) \exp \left( - \frac{i p \cdot x}{\hbar} \right) V_1(x, x')V_2(x - x'). \]  

(1.5)

We change variables to
\[ Y = \frac{x + x'}{2} \]  

(1.6) and

\[ y = x - x' \]  

(1.7)

so that the integrals can be partially separated:

\[
\tilde{V}_{NL}(p, p') = \frac{1}{(2\pi\hbar)^3} \int d^3y \, V_2(y) \exp \left( -\frac{i(p + p') \cdot y}{2\hbar} \right) \times \int d^3Y \, V_1(Y, y) \exp \left( -\frac{i(p - p') \cdot Y}{\hbar} \right).
\]  

(1.8)

In principle, the second integrand depends on \( y \). Since we assume that the potential is almost local, \( V_2(y) \) is very sharply peaked in \( y \). Then the \( y \)-dependence of \( V_1(Y, y) \) will not be seen at leading order in \( b \). In that case, \( V_1(Y, y) \approx V_1(Y) \) alone. This is true to the extent that the range \( b \) of the nonlocality is small on the scale in which \( V_1 \) is changing. For applications in nuclear physics, where the width \( b \) is of order 1 fm or less, this should be a very reasonable approximation. We have shown that the assumption of a small nonlocality range \( b \) motivates the following factorized form

\[
V_{NL}(x, x') = V_a \left( \frac{x + x'}{2} \right) V_w \left( x - x' \right),
\]  

(1.9)

where \( V_a \) (called \( V_1 \) above) is the “average” local value of the nonlocal potential and \( V_w \) (called \( V_2 \) above) governs the “width” of the nonlocality. For a sufficiently small range of nonlocality, the particular functional form of the width term should not be too important. What is important is that some nonlocality is allowed, with a characteristic scale \( b \).

If this factorized form is assumed, then the integrals in Eq. (1.8) separate, and then one finds for the Fourier transform

\[
\tilde{V}_{NL}(p, p') = U_a \left( \frac{p + p'}{2} \right) U_w (p - p'),
\]  

(1.10)

where, in the \( p \)-representation, \( U_a \) is the average potential term, and \( U_w \) is the width term. These are given by:

\[
U_a \left( \frac{p + p'}{2} \right) = \tilde{V}_w (x - x')
\]  

(1.11)

and

\[
U_w (p - p') = \tilde{V}_a \left( \frac{x + x'}{2} \right).
\]  

(1.12)

This illuminates an interesting property of the factorized form. Under the Fourier transform, the “average” term of the potential in the \( x \)-representation becomes the “width” term of
the potential in the \( p \)-representation, and vice versa. In particular, there are the following correspondences (neglecting factors):

\[
\begin{align*}
\mathbf{x} + \mathbf{x}' & \longleftrightarrow \mathbf{p} - \mathbf{p}' \\
\mathbf{x} - \mathbf{x}' & \longleftrightarrow \mathbf{p} + \mathbf{p}'.
\end{align*}
\] (1.13) (1.14)

These relations depend only on the particular factorization of \( V_{NL} \), and not upon the details of \( V_a \) and \( V_w \).

Let us consider a few special cases. Consider first the width term \( V_w \). For a small range of nonlocality \( b \), \( V_w \) is a narrow peak (say Gaussian) in the \( x \)-representation. Then in the \( p \)-representation, the average term \( U_a \) is a broad Gaussian. Thus nonlocality in the \( x \)-representation leads to a modified dispersion relation in the \( p \)-representation. The narrower the nonlocality, the flatter the dispersion relation. As \( b \to 0 \), the free-particle dispersion relation is recovered. Now consider the average term \( V_a \). If it is constant, then its Fourier transform is proportional to a delta function, leading to a local potential in the \( p \)-representation. To the extent that it is non-constant, it leads to nonlocality in the \( p \)-representation (via \( U_w = \tilde{V}_a \)).

Typically, a Gaussian form is taken for \( V_w \), with \( V_w = V_w(|\mathbf{x} - \mathbf{x}'|) \). Other forms could be chosen, but this seems adequate to describe the nuclear scattering data well [2]. Besides having an evident local limit, this particular factorized form meets some other requirements: \( V_{NL} \) should be symmetric under the exchange of \( \mathbf{x} \) and \( \mathbf{x}' \); and when \( V_a = \text{const.} \), as in infinite nuclear matter, \( V_{NL} \) should be translation invariant. In this paper, we use the Gaussian form for \( V_w \). In Fig. 1, a schematic matrix representation of the above discussion is given.

\section*{II. NONLOCALITY IN NUCLEAR REACTIONS}

As we have asserted in the previous section, nonlocality is a general characteristic of a single-particle description of a many-body system. It can be present, for example, in descriptions of systems in atomic physics [3], condensed-matter physics [4], and quantum optics [5]. Here, we are particularly interested in the specific nonlocal potentials which can appear in nuclear reactions. Below, we consider the nonlocality due to channel coupling (Feshbach nonlocality) and that due the fermionic character of the system (Pauli nonlocality).

It is well known that a given nonlocal potential \( V_{NL}(\mathbf{x}, \mathbf{x}') \) can have a corresponding energy-dependent local equivalent potential \( V_{LE}(\mathbf{x}, E) \) [6]. Such a local equivalent potential gives the same cross section as the nonlocal potential. If the local equivalent potential also gives the same wave function, then it is said to be trivially equivalent.

It was shown in a recent article [7] that most of the observed energy dependence in the inner region of the phenomenological local equivalent potential arises from the Pauli nonlocality. The Feshbach nonlocality would then be related to the energy dependence at the surface of that potential. Exchange effects are important at relatively short distances (inner region of the nuclear potential), where the density is high. Channel coupling effects are important at relatively large distances (nuclear surface), where the coupling form factors are peaked. As an example of this kind of long range effect we have, for example, the nuclear structure effects in Coulomb excitation. The relative contributions of the Feshbach and Pauli nonlocalities can also be studied using the technique of Ref. [8].
We cannot give a rigorous proof that the factorized form for $V_{NL}(x, x')$ in Eq. (1.9) is completely general. However, we will motivate this form for both the Feshbach and Pauli nonlocalities below.

A. Feshbach Nonlocality

Recent theoretical and experimental developments in the study of fusion reactions below the Coulomb barrier led to a renewed interest in the coupled-channels formalism [9]. For a long time it has been known that coupling to nonelastic channels leads to a nonlocality in the elastic channel [10]. The factorized nonlocal potential of Eq. (1.9) is sometimes adopted in channel coupling calculations, and seems to provide a good description of the energy dependence of the nuclear potential. The generation of a nonlocality in the elastic channel due to channel couplings has been described in detail in Ref. [11]. Here we consider simplified example, using only two channels. The coupled Schrödinger equations in this case are given by:

$$
\left[\begin{array}{cc}
\left(\frac{\hat{p}^2}{2m} + V_L(x)\right) & 0 \\
0 & \left(\frac{\hat{p}^2}{2m} + V_L(x)\right)
\end{array}\right] + \left[\begin{array}{cc}
0 & F(x) \\
F(x) & \varepsilon
\end{array}\right] \left[\begin{array}{c}
\Psi_0(x) \\
\Psi_1(x)
\end{array}\right] = E \left[\begin{array}{c}
\Psi_0(x) \\
\Psi_1(x)
\end{array}\right],
$$

(2.1)

where $\Psi_0(x)$ and $\Psi_1(x)$ are the wave functions of the ground and first excited states, respectively. $V_L(x)$ is a local potential, $F(x)$ is the coupling potential and $\varepsilon$ is the excitation energy to the first state. These differential equations can be solved by the Green’s function method. We can decouple these equations by eliminating either channel. We first solve for $\Psi_1(x)$:

$$
\Psi_1(x) = \frac{2m}{\hbar^2} \int d^3x' \ G(x, x') \ F(x') \ \Psi_0(x'),
$$

(2.2)

with the Green’s function $G(x, x')$ is given by

$$
G(x, x') = \frac{\hbar^2}{2m} \left\langle x \left| \frac{1}{E - \frac{\hat{p}^2}{2m} - V_L(x) - \varepsilon \pm i\eta} \right| x' \right\rangle.
$$

(2.3)

Then the equation for $\Psi_0(x)$ can be written as

$$
\left[\frac{\hat{p}^2}{2m} + V_L(x) - E\right] \Psi_0(x) = - \int d^3x' \ F(x) \ G(x, x') \ F(x') \ \Psi_0(x').
$$

(2.4)

Now that the $\Psi_1$ channel has been eliminated, this is an equation for $\Psi_0$ alone. By comparing Eq. (2.4) to the general form in Eq. (1.1), we find that the channel coupling generates a nonlocal potential

$$
V_{NL}(x, x') = F(x) \ G(x, x') \ F(x').
$$

(2.5)

The nonlocality in $G(x, x')$ arises because the kinetic energy is not diagonal in the $x, x'$ representation. This and the elimination of all but the elastic channel leads to the nonlocality in the elastic channel.
As a simple example, we can assume that $F(x)$ has an approximately exponential shape and that $V_L(x)$ has a slow variation. Then we can write Eq. (2.5) as (keeping in mind that in this situation we have $G(x,x') \sim G(x-x')$ [12]):

$$V_{NL}(x,x') \approx F(x+x') G(x-x'),$$

which has the same form as Eq. (1.9).

**B. Pauli Nonlocality**

The simple factorized form as given by Eq. (1.9) has a clear local limit when the nonlocality range $b$ goes to zero. There are good reasons to consider such a simple form for the nonlocal potential $V_{NL}(x,x')$. For example, Bauhoff et al. [13] examined Pauli nonlocality in nucleon-nucleus interactions by using a folding approach. Adopting a nucleon-nucleon interaction written as the sum of a direct term and an exchange term, these authors obtained a nonlocal nucleon-nucleus potential from the microscopic description. Aoki and Horiuchi [14], using the Resonating Group Method, verified that the microscopic interaction can be decomposed as a sum of direct and exchange parts. A comparison of the results for the trivially and local equivalent potentials and those potentials from the Perey-Buck analyses [2] was made in Ref. [15]. It was shown that a large portion of the nonlocality present in the Perey-Buck potential is due to exchange effects. The microscopic approach of Ref. [13] was found to produce very similar results to those using the Perey-Buck factorized form given above in Eq. (1.9).

We conclude that the effects of the Pauli nonlocality in nucleon-nucleus interactions can be successfully described by a nonlocal potential given by Eq. (1.9). The same conclusion can be reached in the case of nucleus-nucleus interactions [7,16].

**C. Nonlocality and the Nuclear Dispersion Relation**

Here we note a consequence of allowing a nonlocal potential, as in Eq. (1.1) to describe the nuclear interaction [17,18]. For example, it is well known that the interaction described by Eq. (1.1) is dependent on the energy of the interacting system. In order to highlight the effects of nonlocality, we consider as an example an infinite nuclear medium (average term $V_a$ constant) and a Gaussian form for $V_w(x-x')$. Further, we neglect any isolated local potential $V_L(x)$. We can then rewrite Eq. (1.9) as

$$V_{NL}(x,x') = \frac{V_0}{\pi^{3/2}b^3} \exp \left[ -\frac{(x-x')^2}{b^2} \right].$$

Here $b$ is the parameter describing the range of the nonlocal effects and $V_0$ is the depth of the nuclear potential. In this way, we may rewrite Eq. (1.1) as

$$\left[ \frac{\hbar^2}{2m} \nabla^2 + E \right] \Psi(x) = \frac{V_0}{\pi^{3/2}b^3} \int d^3x' \exp \left[ -\frac{(x-x')^2}{b^2} \right] \Psi(x').$$

We first change the integration variable to $s$, defined by $bs = x' - x$. Then using the relation $\exp(bs \cdot \nabla) \Psi(x) = \Psi(x + bs)$, we can write
\[
\left[ \frac{\hbar^2}{2m} \nabla^2 + E \right] \Psi(x) = V_0 \exp \left( \frac{b^2}{4} \nabla^2 \right) \Psi(x). \tag{2.9}
\]

We replace \( \Psi(x) \) with its definition via the inverse Fourier transform:

\[
\Psi(x) = \left( \frac{1}{2\pi \hbar} \right)^{3/2} \int d^3p \exp \left( \frac{i \mathbf{p} \cdot \mathbf{x}}{\hbar} \right) \Phi(p). \tag{2.10}
\]

From this we obtain the momentum-space representation of Eq. (2.9):

\[
\left( -\frac{\mathbf{p}^2}{2m} + E \right) \Phi(p) = V_0 \exp \left( -\frac{p^2 b^2}{4 \hbar^2} \right) \Phi(p). \tag{2.11}
\]

Defining \( p = \hbar k \) and \( k = |k| \), the dispersion relation for a nucleon moving in infinite nuclear matter is given by

\[
\frac{\hbar^2 k^2}{2m} = E - V_0 \exp \left( -\frac{k^2 b^2}{4} \right). \tag{2.12}
\]

Equation (2.12) shows the dependence of the nuclear potential on momentum. Note that the strength of the potential decreases quickly with increasing momentum. For a local interaction \( (b = 0) \), we recover the standard dispersion relation:

\[
\frac{\hbar^2 k^2}{2m} = E - V_0. \tag{2.13}
\]

Since Eq. (2.12) is a transcendental equation, it cannot solved for general momenta. For small momenta, the exponential can be expanded in a Taylor series. Below, only the two lowest orders of this expansion are kept. Thus we require that \( kb \ll 1 \), which means the nucleon wavelength is much larger than the range of the nonlocality. That is reasonable for the small range of nonlocality that we assume. Then the solution to Eq. (2.12) can be rewritten

\[
\frac{\hbar^2 k^2}{2m^*} = (E - V_0), \tag{2.14}
\]

where the effective mass \( m^* \) is given by

\[
\frac{m^*}{m} = \frac{1}{1 - (mb^2V_0/2\hbar^2)}. \tag{2.15}
\]

Note that we have changed \( k \) to \( \kappa \) above to emphasize the fact that \( m^* \) is used in the definition of the latter. Since \( V_0 \) is negative, \( m^* < m \). The nonlocal interaction between the nucleon and the nuclear matter changes the dispersion relation, lowering the effective mass. In terms of the dispersion relation, a nucleon with a mass \( m \) and an nonlocal interaction is approximately equivalent to a nucleon with a mass \( m^* \) and a local interaction.
### III. DEVELOPMENT OF THE NONLOCAL GREEN’S FUNCTION

#### A. General Development

In this section, we study the Green’s function for a nonlocal potential and present our main results. This allows a unifying approach to various limiting cases, as will be shown below. Studies of the general properties of nonlocal potentials and some limits thereof have been made in other formalisms [19,20]. These analytic studies are important in part because of the difficult nature of solving these systems numerically (but note a new approach in Ref. [21]). The Green’s function may also be useful in calculating the transmission coefficient in heavy-ion collisions, where a nonlocal potential modifies the usual result [22].

We begin the development by defining the Green’s operator for a nonlocal potential as

\[
\hat{G}_{NL} = \frac{1}{E - \hat{H}_{NL} \pm i\eta},
\]

where

\[
\hat{H}_{NL} = \frac{\hat{p}^2}{2m} + \hat{V}_{NL},
\]

and \(\hat{V}_{NL}\) is the nonlocal potential operator given by Eq. (1.2). An additional local potential will be considered in a later section.

The Green’s function in the \(x\)-representation is given by

\[
G_{NL}(E; x_f, x_i) \equiv \frac{\pi^3}{\hbar^2} m \langle x_f \left| \hat{G}_{NL} \right| x_i \rangle = \frac{\pi^3}{\hbar^2} \frac{1}{E - \hat{H}_{NL} \pm i\eta} \langle x_f \left| x_i \rangle \right. \rangle.
\]

This can be rewritten as

\[
G_{NL}(E; x_f, x_i) = \frac{\pi^3}{\hbar^2} \int \int \frac{d^3 p \, d^3 p'}{(2\pi\hbar)^3/2} \times \langle x_f \left| p \right\rangle \left\langle p \left| \frac{1}{E - \hat{H}_{NL} \pm i\eta} \right| p' \right\rangle \left\langle p' \left| x_i \rangle \right. \rangle.
\]

with

\[
\langle x_f \left| p \right\rangle = \frac{\exp(ip \cdot x_f / \hbar)}{(2\pi\hbar)^3/2}.
\]

To proceed further with Eq. (3.4), we adopt a factorized form for the nonlocal potential \(V_{NL}\) as in Eq. (1.9).

In order to understand the effects of nonlocality on the Green’s function, we consider a simple nonlocal potential. We take the local or average term in this potential to be constant. For now, the form of the nonlocal or width term will be left unspecified. Thus

\[
V_{NL}(x, x') = V_a \left( \frac{x + x'}{2} \right) V_w(x - x') = V_0 V_w(x - x'),
\]

(3.6)
where $V_0$ is a negative constant, and $V_w$ is peaked at $x = x'$. As discussed above, taking $V_a$ constant forces $U_w = V_a$ to be a delta function, so that the potential is local in the $p$-representation. That allows one of the integrals in each of Eq. (1.3) and Eq. (3.4) to be collapsed. Then Eq. (1.3) can be written as

$$
\langle p | \hat{V}_{NL} | p' \rangle = \delta^3(p - p') V_0 \int d^3y V_w(y) \exp [-i y \cdot p / 2\hbar],
$$
(3.7)

where $y = x - x'$. Then

$$
\left\langle p \right| \frac{1}{E - \hat{H}_{NL} + i\eta} \left| p' \right\rangle = \frac{\delta^3(p - p')}{E - p^2/2m - V_0 V_w(p) \pm i\eta},
$$
(3.8)

where a tilde indicates the Fourier transform. Using these results, Eq. (3.4) can be written

$$
G_{NL}(E; x_f, x_i) = \frac{\hbar^2}{2m} \int d^3p \frac{1}{(2\pi\hbar)^3} \frac{\exp[ip \cdot (x_f - x_i)/\hbar]}{E - p^2/2m - V_0 V_w(p) \pm i\eta}.
$$
(3.9)

For definiteness, we now choose for the nonlocal or width term

$$
V_w(y) = \frac{1}{b^3 \pi^{3/2}} \exp \left( -\frac{y^2}{b^2} \right).
$$
(3.10)

As noted above, this form has a simple local limit when $b \to 0$. This implies for the Fourier transform

$$
\tilde{V}_w(p) = \exp \left( -\frac{p^2 b^2}{4\hbar^2} \right),
$$
(3.11)

and thus for Eq. (3.9)

$$
G_{NL}(E; x_f, x_i) = \frac{\hbar^2}{2m} \int d^3p \frac{1}{(2\pi\hbar)^3} \frac{\exp[ip \cdot (x_f - x_i)/\hbar]}{E - p^2/2m - V_0 \exp(-p^2 b^2/4\hbar^2) \pm i\eta}.
$$
(3.12)

Defining new variables via $E = \hbar^2 k^2/2m$ and $p = \hbar q$ (and $q = |q|$), this becomes

$$
G_{NL}(E; x_f, x_i) = \frac{1}{(2\pi)^2} \int_0^\infty dq q^2 \int_{-1}^{+1} d(cos \theta) \frac{\exp(iq |x_f - x_i| \cos \theta)}{k^2 - q^2 - (2mV_0/\hbar^2) \exp(-q^2 b^2/4) \pm i\eta}.
$$
(3.13)

Making the integration over $\cos \theta$, we obtain

$$
G_{NL}(E; x_f, x_i) = \frac{1}{8\pi^2 |x_f - x_i|} \int_0^\infty dq q \frac{(\exp(iq |x_f - x_i|) - \exp(-iq |x_f - x_i|))}{q^2 + \alpha \exp(-\beta q^2) - (k^2 \pm i\eta)},
$$
(3.14)

where
\[
\alpha = \frac{2mV_0}{\hbar^2}
\]  
(3.15)

and

\[
\beta = \frac{b^2}{4}.
\]  
(3.16)

Equation (3.14) is the starting point for the limits of the subsequent sections. To make the integration over \( q \) in Eq. (3.14), we use the method of residues. This requires that we identify the poles of the integrand in the complex-\( q \) plane. Unfortunately, the equation

\[
q^2 + \alpha \exp(-(\beta q^2)) - (k^2 \pm i\eta) = 0
\]  
(3.17)

is a transcendental equation, and the roots cannot be found in the general case. Below, we consider two limiting approximations involving \( \beta \) in which the roots of this equation can be found and the integral in Eq. (3.14) done analytically. In Fig. 2, the problem of finding these roots is indicated schematically.

B. Digression on the Poles of the Integrand

The integral in Eq. (3.14) is completely specified by the residues at the roots in \( q \) of Eq. (3.17). We show here how to develop expansions for these roots. In two cases of interest, these expansions can be truncated with minimal (and calculable) error. The errors on the integral results can be calculated using the known errors on the roots. Since the final results are based on controlled approximations, it is easy to determine where to truncate the expansions.

Defining \( y = (q/k)^2 \), \( A = \alpha/k^2 \), and \( B = \beta k^2 \), Eq. (3.17) can be written in a completely dimensionless form as

\[
y + Ae^{-By} - 1 = 0.
\]  
(3.18)

For convenience, the \( i\eta \) term has been dropped. From Fig. 2, one can see that there are exactly two roots on the real axis. We ignore any possible complex roots as they would not lead to pure propagating wave solutions in the Green’s function. We exclude the case \(|A| \gg 1\); otherwise the value of \( A \) is unrestricted.

Limit of \( B = \beta k^2 \ll 1 \): Rewrite Eq. (3.17) as

\[
y = 1 - Ae^{-By}
\]

\[
y = (1 - A) + A \left( By - \frac{1}{2} B^2 y^2 + \ldots \right)
\]

\[
(1 - AB) y = (1 - A) + A \left( -\frac{1}{2} B^2 y^2 + \ldots \right).
\]  
(3.19)

These are exact if all terms are kept. For \( B \ll 1 \), the latter term is small. The corrections can be solved for by perturbation, assuming the form

\[
(1 - AB) y = y_0 + By_1 + B^2 y_2 \ldots,
\]  
(3.20)
and solving order by order. Then \( y_0 = 1 - A \), and the first correction (the \( y_1 \) term) vanishes. Then Eq. (3.17) can be written as

\[
(1 - \alpha \beta) q^2 - (k^2 - \alpha \pm i \eta) + \mathcal{O} \left( k^2 (\beta k^2)^2 \right) = 0. \tag{3.21}
\]

This is a direct replacement for the denominator of Eq. (3.14) near the poles; note that we did not divide out the factor \( (1 - \alpha \beta) \).

**Limit of \( B = \beta k^2 \gg 1 \):** Rewrite Eq. (3.17) as

\[
y = 1 - A e^{-By} \quad \text{and} \quad y = \left( 1 - A e^{-B} \right) - A \left( e^{-By} - e^{-B} \right). \tag{3.22}
\]

These are exact. For \( B \gg 1, y \simeq 1 \), so the latter term is small. The corrections can be solved for by iteration to yield a power series in \( e^{-B} \). Note that \( B \gg 1 \) and \( e^{-B} \ll 1 \), so that \( Be^{-B} \ll 1 \). Then Eq. (3.17) can be written as

\[
q^2 - \left( k^2 - \alpha \exp(-\beta k^2) \pm i \eta \right) + \mathcal{O} \left( k^2 \exp(-2\beta k^2) \right) = 0. \tag{3.23}
\]

This is a direct replacement for the denominator of Eq. (3.14) near the poles.

**C. Low-Momentum (or Effective Mass) Limit**

If \( \beta k^2 \ll 1 \), then the denominator of of Eq. (3.14) can be rewritten as in Eq. (3.21) and so Eq. (3.14) becomes

\[
G_{NL}(E; x_f, x_i) = \frac{1}{8\pi^2 |x_f - x_i| m} \left( \frac{m^*}{m} \right) \int_{-\infty}^{\infty} dq q \times \frac{\left[ \exp (iq|x_f - x_i|) - \exp (-iq|x_f - x_i|) \right]}{q^2 - (\kappa^2 \pm i \eta)}, \tag{3.24}
\]

where

\[
\frac{m^*}{m} = \frac{1}{1 - \alpha \beta}, \tag{3.25}
\]

is the effective mass and

\[
\kappa^2 = \frac{k^2 - \alpha}{1 - \alpha \beta}, \tag{3.26}
\]

is a new wave number that is equal to the wave number of the nucleon in a constant local potential with the effective mass \( m^* \). By construction, these expressions agree with those found in Eqs. (2.14) and (2.15). Note that the corrections to Eq. (3.21) have been dropped.

The integral in Eq. (3.24) is done by the method of residues, with the contour closed in the upper-half or lower-half complex plane as necessary. We obtain the following expression for \( G_{NL}(E; x_f, x_i) \) in the limit \( \beta k^2 \ll 1 \):

\[
G_{NL}(E; x_f, x_i) = \frac{1}{4\pi} \left( \frac{m^*}{m} \right) \frac{\exp (\pm i \kappa |x_f - x_i|)}{|x_f - x_i|}. \tag{3.27}
\]
Nonlocality modifies the mass (making the term in parentheses different from unity) and modifies the wave number (which is also modified by the constant average term in the potential.) The local limit is immediately recovered as $m^* \rightarrow m$ and $\kappa \rightarrow k$. On the other hand, if additional orders in Eq. (3.21) are retained, then the $m^*/m$ prefactor is unchanged and only $\kappa$ is modified.

Note that this Green’s function is an eigenstate of $p^2/\hbar^2 = -\nabla^2$ with eigenvalue $\kappa^2$. Using the definition of $\kappa^2$ in Eq. (3.26), $k^2 = 2mE/\hbar^2$ and the definitions in Eqs. (3.15) and (3.16), this can be written as

$$\frac{p^2}{2m^*} = \frac{\hbar^2 \kappa^2}{2m^*} = E - V_0.$$  \hfill (3.28)

This check of the integration by direct differentiation returns the correct dispersion relation at the order of approximation used in this case.

**D. High-Momentum Limit**

If $\beta k^2 \gg 1$, then the denominator of of Eq. (3.14) can be rewritten as in Eq. (3.23) and so Eq. (3.14) becomes

$$G_{NL}(E; \mathbf{x}_f, \mathbf{x}_i) = \frac{1}{8\pi^2 |\mathbf{x}_f - \mathbf{x}_i|} \int_{-\infty}^{\infty} dq \, q \left[ \exp(iq|\mathbf{x}_f - \mathbf{x}_i|) - \exp(-iq|\mathbf{x}_f - \mathbf{x}_i|) \right] \frac{q^2 - (k^2 - \alpha \exp(-\beta k^2) \pm i\eta)}{q^2 - (k^2 - \alpha \exp(-\beta k^2) \pm i\eta)}.$$ \hfill (3.29)

This is again done by residues, and is found to be

$$G_{NL}(E; \mathbf{x}_f, \mathbf{x}_i) = -\frac{1}{4\pi} \exp \left( \pm i \sqrt{k^2 - \alpha \exp(-\beta k^2)} |\mathbf{x}_f - \mathbf{x}_i| \right).$$ \hfill (3.30)

The small term $\exp(-\beta k^2)$ damps the effect of the constant average potential. Other than the modified wavenumber, the result in Eq. (3.30) is exactly the usual free-particle Green’s function. Note that $k$ is the free-particle wave number, and is not modified by the constant average potential. If additional orders in Eq. (3.23) are retained, then the form is unchanged but the wavenumber is corrected by powers of $\exp(-\beta k^2)$.

Note that this Green’s function is an eigenstate of $p^2/\hbar^2 = -\nabla^2$ with eigenvalue $k^2 - \alpha \exp(-\beta k^2)$. Using the definition $k^2 = 2mE/\hbar^2$ and the definitions in Eqs. (3.15) and (3.16), this can be written as

$$\frac{p^2}{2m} = E - V_0 \exp(-mb^2E/2\hbar^2).$$ \hfill (3.31)

This check of the integration by direct differentiation returns the correct dispersion relation at the order of approximation used in this case.
E. Quasi-Free Limit

Above, we simplified the integral by considering the limits of $B = \beta k^2 \ll 1$ and $B = \beta k^2 \gg 1$, respectively. The value of $A = \alpha/k^2$ was not specified, except for excluding the case $|A| \gg 1$. The case $|A| \ll 1$ was allowed, and to consider its consequences it is sufficient to apply this limit to the previous results. It is not necessary to develop a new expansion for the roots or to do another integral.

In the low-momentum case, we took $B \ll 1$ and the result was Eq. (3.27). If $|A| \ll 1$, then $|A|B = |\alpha|/\beta \ll 1$, so that $m*/m \rightarrow 1$ and $\kappa^2 \rightarrow k^2 - \alpha$.

\[ G_{NL}(E; x_f, x_i) \rightarrow -\frac{1}{4\pi} \exp \left( \pm i\sqrt{k^2 - \alpha} |x_f - x_i| \right). \quad (3.32) \]

This is of the same form as the free-particle Green’s function, with the wave number modified by the constant average potential. This is a local limit, since the scale $\beta$ has disappeared.

In the high-momentum case, we took $B \gg 1$ and the result was Eq. (3.30). In the case $|A| \ll 1$, the result is unchanged at the lowest nontrivial order:

\[ G_{NL}(E; x_f, x_i) \rightarrow -\frac{1}{4\pi} \exp \left( \pm i\sqrt{k^2 - \alpha} \exp (-\beta k^2) |x_f - x_i| \right). \quad (3.33) \]

This is also of the same form as the free-particle Green’s function, again with the wave number modified by a term depending on the interaction. In this case, the nonlocality still appears as part of the interaction. Note that it would appear that these two Green’s functions share a common limit if in the second we take $\beta k^2 \ll 1$. Strictly speaking, this should not be allowed since the derivation assumed $\beta k^2 \gg 1$.

F. Comparison to Data

From Eqs. (3.27), (3.30), (3.32), and (3.33) we can see some of the basic effects of nonlocality in the nuclear medium. Nonlocality, in the low-momentum limit, changes the constant mass of the system to an effective mass, which depends on the range of nonlocality and the strength of the potential. This effect, in the limit $|\alpha|/\beta \ll 1$, is negligible. In the high-momentum limit, nonlocality acts on the potential in a more complex way. However, this limit is satisfied in many cases of interest to nuclear physics [23]. In this situation, the static strength of the potential changes, under the effects of the nonlocality, to an exponentially energy-dependent potential, e.g., the equivalent local potential. Chamon et al. [23] verified such a dependence of the equivalent local potential on the energy for elastic scattering of many different systems at high energies.

We can define an energy-dependent local equivalent potential from Eq. (3.30) as

\[ V(E) = V_0 \exp(-\beta k^2) \]

or, substituting the expressions for $\beta$ and $k^2$, as

\[ V(E) = V_0 \exp \left( -\frac{mb^2}{2h^2} E \right). \quad (3.34) \]
As a test of our results, we compare Eq. (3.34) to the values of the inner region of the phenomenological potential extracted from data [7] for the systems $^{12}\text{C} + ^{12}\text{C}$ and $^{12}\text{C} + ^{16}\text{O}$. In these comparisons, we adopted for the range of the nonlocality $b = 0.14$ fm and $b = 0.11$ fm for the first and second systems, respectively. These values of $b$ are extracted from the data, and are in good agreement with those from a single-folding model. The values of $V_0$ also come from the data. For more details about the data used, see Ref. [7] and references therein.

In the upper panel of Fig. 3, $|V(E)|$ versus $E$ for the system $^{12}\text{C} + ^{12}\text{C}$ is shown. The circles represent the values of the phenomenological potential at $r = 4$ fm extracted from the data and the solid line represents the result of Eq. (3.34). In the lower panel of Fig. 3 the same quantities for the system $^{12}\text{C} + ^{16}\text{O}$ is shown. For the $^{12}\text{C} + ^{12}\text{C}$ system, $|\alpha|\beta = 0.085$; for the $^{12}\text{C} + ^{16}\text{O}$ system, $|\alpha|\beta = 0.075$. In both cases there is an impressive concordance between the extracted and predicted values for the potential.

G. Born Series to Include a Local Potential

Referring back to Eq. (3.4), we note that in order to proceed we needed to collapse the double integral over $p$ and $p'$. To do that, we took the interaction to be local in the $p$-representation, i.e., assumed a constant average potential in the $x$-representation. (In the usual development of the Green’s function for a local interaction, any constant potential is defined away by shifting the energy.) One way to generalize the usual Green’s function integral is to allow a constant potential with a nonlocality, as done here. This introduces a $V(p)$, and the remaining single integral can be done in some cases.

To introduce an additional local potential, one can proceed as follows. We define a total Green’s function by

$$\hat{G} = \frac{1}{E - \hat{H}_{NL} - \hat{V}_L \pm i\eta}$$  (3.35)

where

$$\hat{H}_{NL} = \frac{\hat{p}^2}{2m} + \hat{V}_{NL}. $$  (3.36)

Here, $\hat{V}_{NL}$ is a nonlocal potential operator, as given by Eq. (1.2), and $\hat{V}_L$ is a local potential operator. Then

$$\hat{G} = \hat{G}_{NL} + \hat{G}_{NL} \hat{V}_L \hat{G}_L \hat{G}_L,$$  (3.37)

with

$$\hat{G}_{NL} = \frac{1}{E - \hat{H}_{NL} \pm i\eta}. $$  (3.38)

The Born series can be developed in the usual way:

$$\hat{G} = \hat{G}_{NL} + \hat{G}_{NL} \hat{V}_L \hat{G}_{NL} + \hat{G}_{NL} \hat{V}_L \hat{G}_{NL} \hat{V}_L \hat{G}_{NL} + \ldots. $$  (3.39)

This Born series for the operator $\hat{G}$ may or may not converge depending on the value of the energy of the system or on the strength of the local potential. The order by order contributions to the matrix element $\langle x_f \mid \hat{G} \mid x_i \rangle$ can then be developed once the potential is specified.
IV. CONCLUSIONS

In this article we presented a Green’s function approach to the study of nonlocal nuclear potentials. We showed that Green’s functions can be used to investigate various limits in a unified way. In the low-momentum limit the effects of the nonlocal interaction can be accounted for with an effective mass and a wavenumber modified by the strength of the potential and range of the nonlocality. In the high-momentum limit the Green’s function for the nonlocal potential also differs from the local one by a modified wavenumber. In this case, it is a function of the free wave number, i.e., the nonlocal potential is written in terms of an energy-dependent local equivalent potential. This behavior is consistent with the results recently obtained by Chamon, et al. [23], who showed that as energy goes up the equivalent local potential for the nucleus-nucleus interactions become shallower. It would also be interesting to generalize the Green’s function method to multidimensional systems where the effective mass becomes a tensor [24].

In the two main limits which we considered, we kept only the leading order results. However, in both cases we have shown how to easily generalize these results to any arbitrary order.

The results in this paper are readily applicable to nuclear matter. We expect our formalism to be useful to incorporate Pauli non-locality effects in the study of neutron stars.

ACKNOWLEDGMENTS

We thank L.C. Chamon for illuminating discussions, and K. Hagino for pointing out and discussing Ref. [11] with us. This work was supported in part by the U.S. National Science Foundation Grant No. PHY-9605140 at the University of Wisconsin, and in part by the University of Wisconsin Research Committee with funds granted by the Wisconsin Alumni Research Foundation. M.A.C.R. acknowledges the support of Fundação de Amparo à Pesquisa do Estado de São Paulo (Contract No. 96/3240-5).
REFERENCES


**Figure Captions**

FIG. 1. Schematic matrix representations of a nonlocal potential. The solid dots indicate nonzero matrix elements, and the vertical bars indicate the relative magnitude of those elements. In the matrix in the top left, only representative elements are indicated. In the top row, the potential has a constant local term multiplied by a Gaussian nonlocal term with a range $b$. On the left, the potential is represented in the $x, x'$ basis; on the right, in the $p, p'$ basis. The two bases are related by a Fourier transform (FT). In the bottom row, we consider the local limit ($b = 0$) of the top row.

FIG. 2. The problem of finding the root of the dispersion relation (as a function of momentum $q$) is indicated schematically in the figure. The figure is symmetric about $q = 0$. The parabolic curve indicates the kinetic energy term $q^2/2m$. The Gaussian curve indicates the nonlocal potential term $V_0 \exp(-q^2b^2/4)$, with $V_0 < 0$. The roots are at the points where the curves cross. When $bq \ll 1$, the Gaussian is extremely broad and the roots are near $q = \pm \sqrt{2m(E + |V_0|)/\hbar}$. When $bq \gg 1$, the Gaussian is very narrow and the roots are near $q = \pm \sqrt{2mE}/\hbar$.

FIG. 3. The energy dependence of the local equivalent potential $|V(E)|$ at $r = 4$ fm in the range $10 \leq E/A \leq 200$ MeV/nucleon. The circles are data, and the solid line is the result of Eq. (3.34). (See the text for details.) The upper panel is for $^{12}\text{C} + ^{12}\text{C}$, and the lower panel is for $^{12}\text{C} + ^{16}\text{O}$.