Stable Topologies of Event Horizon

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Abstract

In our previous work, it was shown that the topology of an event horizon (EH) is determined by the past endpoints of the EH. A torus EH (the collision of two EH) is caused by the two-dimensional (one-dimensional) set of the endpoints. In the present article, we examine the stability of the topology of the EH. We see that a simple case of a single spherical EH is unstable. Furthermore, in general, an EH with handles (a torus, a double torus, ...) is structurally stable in the sense of catastrophe theory.

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I. INTRODUCTION

The topology of an event horizon (EH) is very important when one investigate the various properties of the EH, and is sometimes considered to be trivial. For example, one may assume that the topology of the event horizon (TOEH) is a sphere for the uniqueness theorem of a black hole. On the other hand, it is natural that the TOEH is a sphere in an astrophysical sense. Furthermore, many authors [2] proved that the TOEH is a sphere under some conditions.

On the contrary, the present author have shown the TOEH is determined by the structure of the endpoints of the EH [3]. From this, the two-dimensional (one-dimensional) set of the endpoints is related to an EH with a torus topology (the collision of the EH). Therefore the
question what determines the structure of the endpoints, arises. To discuss this problem, we need to study the dynamics of this structure. For this reason, it is worth determining the stability of the structure of the endpoints.

Hence, the purpose of the present article is to investigate the stability of the structure of the endpoints. From this investigation, we will find the stability of the TOEH.

First we investigate the stability of a spherical EH under linear perturbation. Especially, we examine the causal structure of a perturbed Oppenheimer-Snyder spacetime to discuss the stability of its endpoint. Second, catastrophe theory is applied to the EH for more general discussion. We argue the structural stability of more general cases than the spherical EH.

In the next section, we briefly introduce the our previous work [3], while the proof of the result are not given in this article. The third section shows the discussion of linear perturbation in a spherically symmetric spacetime. In the section 4, the structural stability of the endpoints is investigated with the base of catastrophe theory. The final section provides summary and discussions.

II. THE TOPOLOGY OF EVENT HORIZONS

In this section, we briefly introduce only the result of our previous work [3]. Now we apply the theories of topology change [5] [6] to EHs. Let \((M, g)\) be a four-dimensional \(C^\infty\) spacetime whose topology is \(\mathbb{R}^4\). In the rest of this article, the spacetime \((M, g)\) is supposed to be strongly causal. Furthermore, for simplicity the topology of the EH (TOEH\(^*\)) is assumed to be a smooth \(S^2\) far in the future and the EH is not eternal one (in other words, the EH begins somewhere in the spacetime, and is open to the infinity in the future direction with a smooth \(S^2\) section). These assumptions will be valid when we consider only one regular \((\sim R \times S^2)\) asymptotic region, namely the future null infinity \(\mathcal{J}^+\), to define the

\(^*\)The TOEH means the topology of the spatial section of the EH. Of course, it depends on a timeslicing.
EH, and the formation of a black hole.

In our investigation, the most important concept is the existence of the endpoints of null geodesics $\lambda$ which completely lie in the EH and generate it. We call them the endpoints of the EH. To generate the EH the null geodesics $\lambda$ are maximally extended to the future and past as long as they belong to the EH. Then the endpoint is the point where such null geodesics are about to come into the EH (or go out from the EH), though the null geodesic can continue to the outside or the inside of the EH through the endpoint in the sense of the whole spacetime. We consider a null vector field $K$ on the EH which is tangent to the null geodesics $\lambda$. $K$ is not affinely parametrized but parametrized so as to be continuous even on the endpoint where the caustic of $\lambda$ appears. Then the endpoints of $\lambda$ are the zeros of $K$, which can become only past endpoints since $\lambda$ must reach to infinity in the future direction.

First we pay attention to the relation between the endpoint and the differentiability of the EH. We see that the EH is not differentiable at the past endpoint.

**Lemma II.1** Suppose that $H$ is a three-dimensional null surface imbedded into the spacetime $(M, g)$ by a function $F$ as

$$H : x^4 = F(x^i, i = 1, 2, 3),$$  \hspace{1cm} \text{(II.1)}

in a coordinate neighborhood $(U_\alpha, \phi_\alpha)$, $\phi_\alpha : U_\alpha \to \mathbb{R}^4$, where $\partial/\partial x^4$ is timelike. When $H$ is generated by the set of null geodesics whose tangent vector field is $K$, $H$ and the imbedding function $F$ are indifferentiable at the endpoint of the null geodesic (the zero of $K$).

Here, we assume that the EH is $C^r (r \geq 1)$-differentiable except on the endpoint of the null geodesics generating the EH and the set of the endpoints is compact. Thus we suppose that the EH is indifferentiable only on a compact subset.

Next, we prepare a basic proposition. Suppose there is no past endpoint of the null geodesic generator of an EH between $\Sigma_1$ and $\Sigma_2$. Then, Geroch’s theorem [3] [6] stresses the topology of the smooth EH does not change.
Proposition II.2 Let $H$ be the compact subset of the EH of $(M,g)$, whose boundaries are an initial spatial section $\Sigma_1$ and a final spatial section $\Sigma_2$, $\Sigma_1 \cap \Sigma_2 = \emptyset$. $\Sigma_2$ is assumed to be far in the future and a smooth sphere. Suppose that $H$ is $C^r (r \geq 1)$-differentiable. Then the topology of $\Sigma_1$ is $S^2$.

Now we discuss the possibilities of non-spherical topologies. From Sorkin’s theorem there should be any zero of null vector field $K$ in the interior of $H$ provided that the Euler number of $\Sigma_1$ is different from that of $\Sigma_2 \sim S^2$. Such a zero can only be the past endpoint of the EH since the null geodesic generator of the EH cannot have a future endpoint. About this past endpoint of the EH we state the following proposition.

**Proposition II.3** The set of the past endpoints (SOEP) of the EH is a connected spacelike set.

Then, we give theorems and corollaries about the topology of the spatial section of the EH on a timeslicing. First we consider the case where the EH has simple structure.

**Theorem II.4** Let $S_H$ be the section of an EH by a spacelike hypersurface. If the EH is $C^r (r \geq 1)$-differentiable at $S_H$, it is topologically $\emptyset$ or $S^2$.

On the other hand, we get the following theorem about the change of the TOEH with the aid of Sorkin’s theorem [5].

**Theorem II.5** Consider a smooth timeslicing $\mathcal{T} = \mathcal{T}(T)$ defined by a smooth function $T(p)$;

$$\mathcal{T}(T) = \{p \in M | T(p) = T = \text{const.}, \ T \in [T_1, T_2], \ g(\partial_T, \partial_T) < 0\}. \quad (\text{II}.2)$$

Let $H$ be the subset of the EH cut by $\mathcal{T}(T_1)$ and $\mathcal{T}(T_2)$, whose boundaries are the initial spatial section $\Sigma_1 \subset \mathcal{T}(T_1)$ and the final spatial section $\Sigma_2 \subset \mathcal{T}(T_2)$, and $K$ be the null vector field generating the EH. Suppose that $\Sigma_2$ is a sphere. If, in the timeslicing $\mathcal{T}$, the TOEH changes ($\Sigma_1$ is not homeomorphic to $\Sigma_2$) then there is the SOEP (the zeros of $K$) in $H$, and when the timeslice touches
the one-dimensional segment of the SOEP, it causes the coalescence of two spherical EHS.

the two-dimensional segment of the SOEP, it causes the change of the TOEH from a torus to a sphere.

This theorem needs the following remark.

Remark: One may face special situations. The possibilities of the branching endpoints should be noticed. If the SOEP possesses a branching point, a special timeslicing can make the branching point into a point where the TOEH changes though such a timeslicing loses this aspect under the small deformation of the timeslicing. The index of this branching endpoint may deny a direct consideration. The situation, however, is regarded as the degeneration of the two distinguished SOEP. Imagine a little slanted timeslicing, and it will decompose the branching point into two distinguished (of course, there are the possibilities of the degeneration of three or more) SOEP. Some of examples are shown in the following. The first case is the branch of the one-dimensional SOEP\(^\dagger\). Then, three spheres coalesce there. The next case is a one-dimensional branch from the two-dimensional SOEP. This branching point is the degeneration of the one-dimensional SOEP and the two-dimensional SOEP. This decomposition tells that the TOEH changes at this point, for example, from a sphere and a torus to a sphere.

Incidentally, a certain timeslicing gives the further changes of the Euler number (see Fig.1).

**Corollary II.6** The topology changing processes of an EH from \(n \times S^2\) to \(S^2\) \((n = 1, 2, 3, ...\) can change each other, and from a surface with genus=\(n\) to \(S^2\) \((n = 1, 2, 3, ...\) can also change, under the appropriate deformation of their timeslicing.

As shown in the corollary II.6, the TOEH highly depends on the timeslicing. Nevertheless, the theorem II.5 tells that there is the distinct difference between the coalescence of \(n\)

\(^{\dagger}\)We can also treat the branching points of the two-dimensional SOEP in the same manner.
spheres where the Euler number decreases by the one-dimensional SOEP and the EH of a
surface with genus=n where the Euler number increases by the two-dimensional SOEP.

Finally we see that, in a sense, the TOEH is a transient term (see Fig.1).

**Corollary II.7** *All the changes of the TOEH are reduced to the trivial creation of an EH
which is topologically $S^2$.*

Thus we see that the change of the TOEH is determined by the topology of the SOEP
and the timeslicing way of it. To fix the TOEH we must only give the order to each vertex,
edge or face of the SOEP by a timeslicing.

III. THE LINEAR PERTURBATION OF AN EVENT HORIZON WITH A
SPHERICAL TOPOLOGY

The purpose of this section is to investigate the stability of the TOEH which always is
a spherical topology, under linear perturbation. From our work in the previous section [3],
such an EH has only one zero-dimensional SOEP (see Fig. 1). Then, we investigate whether
this zero-dimensional SOEP is stable under linear perturbation. Now, it should be noted
the ‘stable’ does not mean that the perturbation does not blow up but the TOEH is not
changed by the perturbation.

As a background spacetime, a spherically symmetric spacetime is appropriate. If the
spherical symmetric spacetime has a non-eternal EH, it has only one endpoint at the origin
and the TOEH is always a sphere. In this case, it is possible to study the linear perturbation
in an established framework [7].

Now we consider the Oppenheimer-Snyder spacetime as the familiar example of the EH
with a spherical topology. Its line element is given by

- interior:

\[
\begin{align*}
\text{interior:}
\quad ds^2 &= a(\eta)^2 \left( -d\eta^2 + d\chi^2 + \sin^2 \chi d\Omega^2 \right), \quad 0 \leq \chi \leq \chi_0 \\
\quad a(\eta) &= \frac{1}{2}a_m(1 + \cos \eta),
\end{align*}
\]

\text{(III.1)}

\text{(III.2)}
\[ ds^2 = -(1 - 2m/R) dt^2 + \frac{dR^2}{1 - 2m/R} + R^2 d\Omega^2, \quad R_B(t) \leq R \] (III.3)

\[ = \left(\frac{32m^3}{R}\right) e^{-R/m} (-dV^2 + dU^2) + R^2 d\Omega^2, \] (III.4)

where \( V \) and \( U \) are Kruskal-Szekeres coordinates. When these geometries are continued at \( \chi = \chi_0 \), the parameters of the exterior region are related to \( a_m \) and \( \chi_0 \) as

\[ m = \frac{1}{2}a_m \sin^3 \chi_0, \] (III.5)

\[ R_B = \frac{a_m \sin \chi_0}{2} (1 + \cos \eta). \] (III.6)

In the background spacetime the equations of null geodesics are easily solved and integrated. The background values of an outgoing null geodesic \( \gamma \) in the direction \( \theta_0, \phi_0 \) and from the origin at \( \eta = \eta_0 \) are

\[ l_0^a = \left(\frac{\partial}{\partial \eta}\right)^a + \left(\frac{\partial}{\partial \chi}\right)^a \] (III.7)

\[ = \left(\frac{\partial}{\partial V}\right)^a + \left(\frac{\partial}{\partial U}\right)^a \] (III.8)

\[ \gamma(\eta_0, \theta_0, \phi_0) : \begin{align*}
\chi &= \eta - \eta_0, \\
U - U_0(\chi = \chi_0, \eta = \chi_0 + \eta_0) &= V - V_0(\chi = \chi_0, \eta = \chi_0 + \eta_0) \\
\theta &= \theta_0, \quad \phi = \phi_0, \\
\eta_{\text{crit}} &= \pi - 3\chi_0,
\end{align*} \] (III.9)

where \( l_0 \) is an outgoing null vector field and \( \eta_{\text{crit}} \) is the supremum of the time \( \eta \) when light ray emitted from the origin can reach to the future null infinity \( J^+ \). The SOEP of the EH is a point at the origin with \( \eta = \eta_{\text{crit}} \).

We expand the freedom of linear perturbation by spherical harmonics \( Y_{LM} \), and they are decomposed into odd parity \([(-1)^{L+1}]\) modes and even parity \([(-1)^L]\) modes. Since they are decoupled in the spherically symmetric background, we discuss the stability of the TOEH under each mode of the perturbation with a parity, \( L \), and \( M \). First we develop the property of null geodesics in a perturbed spacetime. The equations of null geodesics are given by,
\[0 = g_{ab} l^a l^b \]  
\[= (g_{0ab} + h_{ab}) (l^a_0 + \delta l^a) \left( l^b_0 + \delta l^b \right) \]  
\[= h_{ab} l^a_0 l^b_0 + 2g_{0ab}\delta l^a l^b_0,\]  

(III.12) 

(III.13) 

(III.14)

and

\[l^a \nabla_a l^b = \alpha l^b \]  
\[l^a \partial_a l^b + \Gamma_{ac}^b l^c = \alpha l^b \rightarrow \]  
\[l^a_0 \partial_a \delta l^b + \delta l^a \partial_a l^b_0 + 2\Gamma_{0ac}^b \delta l^c + \delta \Gamma_{ac}^b l^a_0 l^b_0 = \delta \alpha l^b_0 + \alpha_0 \delta l^b, \]  

(III.15) 

(III.16) 

(III.17)

where \(g_0, \Gamma_0\) is given by (III.1), (III.3) and \(l_0\) (III.7), (III.8). \(\delta \alpha\) corresponds to the parametrization of \(l\), and is set so that \(\delta l^b + \delta l^V (\delta l^V + \delta l^U)\) vanishes. The deformation \(\delta x^a\) of the light path \(\gamma\) fixing its end on the same position of the future null infinity \(\mathcal{J}^+\), is integrated backward along the background light path \(\gamma\) from the future null infinity to a point in the interior region as

\[\delta \alpha = \delta \Gamma_{ab}^{(V)} l^a_0 l^b_0 + \delta \Gamma^{(U)} l^a_0 l^b_0 \]  
\[\delta \eta = -\delta \chi = \int_{\chi = \chi_0}^{\chi} d\chi [\gamma] \frac{h_{\eta\eta} + 2h_{\eta\chi} + h_{\chi\chi}}{4a^2} + \delta \eta(\chi = \chi_0), \]  
\[\delta \eta(\chi = \chi_0) = \frac{\partial \eta}{\partial V} \delta V_0 + \frac{\partial \eta}{\partial U} \delta U_0, \]  
\[\delta V_0 = -\delta U_0 = \int_{U = \infty}^{U_0} dU[\gamma] \frac{(h_{VV} + 2h_{VU} + h_{UU}) Re^{R/m}}{32m^3}, \]  

(III.18) 

(III.19) 

(III.20) 

(III.21)

since \(\delta V_0 = -\delta U_0\) implies \(\delta \eta(\chi_0) = -\delta \chi(\chi_0)\).

\[A. \text{ even parity mode}\]

The metric perturbation of the even parity mode is given by

\[h_{ab} = \begin{pmatrix} \eta, V, \chi, U & \theta & \phi \\ \bar{h}_{AB} Y_{LM} & \bar{h}_A Y_{LM,\alpha} & \text{Sym} \ \ r^2 (K_{\alpha\beta} Y_{LM} + G Y_{LM,\alpha\beta}) \end{pmatrix} \]  

(III.22)
where $r$ is a circumference radius and $\gamma_{\alpha\beta}$ is the metric of the unit sphere \cite{7}. For the even parity mode, the angular distribution of the $\delta \eta$ and $\delta \chi$ (III.19) is just the spherical harmonics $Y_{LM}$. So, it is helpful to discuss the symmetry of each $Y_{LM}$.

Since $Y_{00}$ is a spherically symmetric function it causes no change of the SOEP, unless the perturbation is unstable and destroy the whole of the EH. The even parity modes with $L = 1, M = \pm 1$, can change into the mode of $Y_{10}$ by a certain rotation, and we only consider $M = 0$ for $L = 1$ mode. By $Y_{10}$ perturbation, the wave front of light around the origin is shifted along the $z$-axis. Then, we only need to determine perturbed light paths starting from the origin for the north and the south. From eq. (III.22), we see

$$
\delta \chi(\gamma(\eta = \eta_0, \theta = 0)) = -\delta \eta(\gamma(\eta = \eta_0, \theta = 0)) = -\delta \chi(\gamma(\eta = \eta_0, \theta = \pi)) = \delta \eta(\gamma(\eta = \eta_0, \theta = \pi)).
$$

Furthermore, $\delta \theta$ and $\delta \phi$ for these light paths vanish because of axial-symmetry. These implies the intersection of $\gamma(\eta = \eta_{\text{crit}}, \theta = 0)$ and $\gamma(\eta = \eta_{\text{crit}}, \theta = \pi)$ does not change its time $\eta$ but position $\chi$ by $2\delta \chi$ along the $z$-axis. Since there is no peak of $Y_{10}$ between $\theta = 0, \pi$, also all the other $\gamma(\eta = \eta_{\text{crit}}, \theta, \phi)$’s should be shifted so as to pass the same position of $2\delta \chi$ on the $z$-axis at $\eta = \eta_{\text{crit}}$. Therefore the original zero-dimensional SOEP is only shifted in the $z$-direction by $2\delta \chi$. There is no change of the TOEH.

The even parity mode with $L = 2$ possesses reflection symmetries about three orthogonal planes. For small perturbation, these modes change the spherical wave front of light to the ellipsoidal one. By an appropriate rotation, the principal axes of the ellipsoidal wave front become $x$-, $y$-, and $z$-axis. Then, it is sufficient to determine light paths along these axes. By the symmetry, $\delta \theta$ and $\delta \phi$ vanish for these light paths. Since $\delta \eta = -\delta \chi$ means the change of $\eta_{\text{crit}}$ is given by $2\delta \eta(\gamma(\eta_{\text{crit}}))$, $\delta \eta_{\text{crit}}$’s of the light paths on the principal axes by each even parity $Y_{2M}$ mode are given by

$$
\delta \eta_{\text{crit}}(L = 2, M = 0) = \sqrt{\frac{5}{\pi}H}, \quad \frac{1}{2}\sqrt{\frac{5}{\pi}H}, \quad -\frac{1}{2}\sqrt{\frac{5}{\pi}H},
$$

$$
\delta \eta_{\text{crit}}(L = 2, M = 1) = \frac{3}{2} \sqrt{\frac{5}{6\pi}H}, 0, -\frac{3}{2} \sqrt{\frac{5}{6\pi}H},
$$

$$
\delta \eta_{\text{crit}}(L = 2, M = 1) = \frac{3}{2} \sqrt{\frac{5}{6\pi}H}, 0, -\frac{3}{2} \sqrt{\frac{5}{6\pi}H}.
$$
where $H$ (the factor not depending on $Y_{LM}$) is given by eq. III.19. By these results, we see the shape of the SOEP around the origin. Light paths from the latest direction (maximal $\delta\eta_{\text{crit}}$) form an endpoint at the origin (for example, see Fig.2). On the other hand, light paths on the other axes will cross a light path from another direction not passing the origin, at a position different from the origin, so that their intersections provide the dimensions of the SOEP to their directions. Thus, the case of $L = 2, M = \pm 1, 2$ provides two-dimensional SOEP. On the contrary, the SOEP with $L = 2, M = 0$ depends on the signature of $H$. If $H$ is negative (positive), the SOEP is one (two)-dimensional (see Fig.2). Since $H$ is generally not equal to zero, the TOEH is not stable under the perturbation with $L = 2$.

By the mode with $L > 2$, the wave front will experiences more complicated deformation. By such a deformation, the SOEP will get branching and become highly complicated as stated in the remark of the theorem II.5. For these modes, $\delta \theta$ and $\delta \phi$ will not be excluded from the discussion. A detailed investigation, however, would show the change of the structure of the SOEP occurs even with non-vanishing $\delta \theta$, $\delta \phi$.

**B. odd parity mode and higher order contribution**

The metric perturbation of the odd parity mode is given by

$$h_{ab} = \begin{pmatrix} \eta, V \chi, U & \theta & \phi \\ 0 & 0 & \tilde{h}_A S_{\alpha} \\ 0 & 0 & \text{Sym} \tilde{h} S_{(\alpha \beta)} \end{pmatrix},$$

(III.23)

where $S_{\alpha}$ is the transverse vector harmonics on the unit sphere [7]. From (III.19) and (III.23), it is clear that the odd parity mode does not affect $\delta \eta$, $\delta \chi$ in linear order. On the other hand, though $\delta \theta$ and $\delta \phi$ exist, they do not affect the structure of the SOEP. For, without $\delta \eta$ and $\delta \chi$, all the perturbed outgoing light paths whose original past endpoint in background is the origin at $\eta = \eta_0$, start the origin at the same time $\eta_0$. They still have only one endpoint at the origin with $\eta = \eta_{\text{crit}}$. 

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For the modes not changing the structure of the SOEP, it would be necessary to investigate contributions from higher order evaluation. The higher order contributions are contained in the back-reaction of the changes of the light path to the equation of null geodesics. Nevertheless, it is also necessary to include second order metric-perturbation. It will cause the difficulty of further investigations. In the second order, there should be mode coupling between different parities, $L$’s, and $M$’s. This fact implies generally the structure of the SOEP is unstable in the higher order. Even if so, however, there are the difference of the sensitivity of the SOEP among each mode. The TOEH is insensitive to odd parity mode and $L = 1$ even parity mode.

IV. THE STRUCTURAL STABILITY OF THE TOPOLOGY OF THE EVENT HORIZON

In the previous section, it is shown that the spherical TOEH is unstable under the linear perturbation. Since there is no appropriate example of a spacetime, however, with non-spherical topology, similar analysis is impossible for other TOEHs. Then, in this section we discuss the structural stability of the SOEP of the EH in catastrophe theory. As discussed in the section 2, it corresponds to the stability of the TOEH. First, we investigate it in a (2+1)-dimensional spacetime.

A. in (2+1)-dimensional spacetime

The plan of analysis is following. First of all, we consider the appropriate wave front of light in a flat spacetime. According to geometrical optics, the wave front produces backward caustics and the endpoints of a null surface related to the wave front. In the context of catastrophe theory, Thom’s theorem state that the structures of such caustics are classified [8], if they are structurally stable. So, we analyze the structure of the caustics and judge whether the SOEP is classified by Thom’s theorem. Here, we consider that the structural stability corresponds to the stability under the small change of the shape of the wave front.
and the local geometry around the endpoints. The shape of the wave front reflects the global structure of the spacetime between $\mathcal{J}^+$ and the wave front. Furthermore, the stability is that of only the local structure of the caustics. Therefore, to discuss it in the flat spacetime is valid as long as we deal with the structure of a small neighborhood.

For simplicity, we consider only the elliptical wave front,

$$E_2 : \left(\frac{x - x_0}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1 \quad \text{(IV.1)}$$

$$x_0 = -\frac{a^2 - b^2}{a}, \quad a \geq b. \quad \text{(IV.2)}$$

Then, the square of the distance between $(x, y)$ and $(X, Y)$ is given by

$$f_{XY}(x) = (X - x)^2 + (Y - y)^2 \quad \text{(IV.3)}$$

$$= \left(X - \left(a\sqrt{1 - (y/b)^2} + \frac{b^2 - a^2}{a}\right)\right)^2 + (Y - y)^2, \quad \text{(IV.4)}$$

where $(X, Y)$ is an arbitrary point and $x - x_0$ is positive. As known in geometrical optics, in a flat spacetime a light path through $(X, Y)$ is given by the stationary points of $f_{XY}(x, y)$;

$$\frac{\partial f_{XY}(x)}{\partial y} = 0 \quad \Rightarrow Y = A(y)X + B(y) \quad \text{(IV.5)}$$

$$A(y) = -\left(\frac{\partial x(y)}{\partial y}\right)_{E_2}, \quad B(y) = x \left(\frac{\partial x(y)}{\partial y}\right)_{E_2} + y, \quad \text{(IV.6)}$$

where $(\partial/\partial)_{E_2}$ means partial derivative with a constraint $E_2$. The light paths are drawn in Fig.3. From this figure, we see that they form a caustic at the origin, and the SOEP of the null surface concerning the wave front is a one-dimensional set, an interval on the $x$-axis $[2x_0, 0]$.

To see the structure of the caustic, we derive the Taylor series of $f_{XY}(x)$ around the origin,

$$\tilde{f}_{XY}(x) = \left(\frac{b^4}{a^2} - \frac{2b^2X}{a} + X^2 + Y^2\right) - 2Yy + \frac{aXY^2}{b^2} \quad \text{(IV.7)}$$

$$+ \left(\frac{a^2}{4b^4} - \frac{1}{4b^2} + \frac{aX}{4b^4}\right)y^4 + O(y^5). \quad \text{(IV.8)}$$

Hence, the light paths form a cusp (a type $A_3$ catastrophe) at $(X, Y) = (0, 0)$ because of $\tilde{f} \sim y^4$. From Thom’s theorem, it is structurally stable except for $a = b$. Of course, $a = b$ corresponds the circular wave front and the zero-dimensional SOEP.
For a (3+1)-spacetime, the investigation above can similarly be done, though its situation becomes a little complex. In this case, there are three possibilities of the SOEP of the EH, even after sufficient simplification. As shown in Fig.4, the endpoint forms a point, line, or surface. As the previous subsection, we consider the ellipsoidal wave front,

$$E_3 : \left( \frac{x}{a} \right)^2 + \left( \frac{y}{b} \right)^2 + \left( \frac{z - z_0}{c} \right)^2 = 1$$  \hspace{1cm} (IV.9)

$$z_0 = -\frac{c^2 - a^2}{c}, \quad 0 < a \leq b \leq c.$$  \hspace{1cm} (IV.10)

For a \( z - z_0 > 0 \) branch, the square of the distance between \((x, y, z)\) and an arbitrary point \((X, Y, Z)\) is given by

$$f_{XYZ}(x) = (X - x)^2 + (Y - y)^2 + (Z - z)^2$$  \hspace{1cm} (IV.11)

$$= (X - x)^2 + (Y - y)^2 + \left( Z - (c\sqrt{1 - (x/a)^2 - (y/b)^2 + z_0}) \right)^2.$$  \hspace{1cm} (IV.12)

The light path through \((X, Y, Z)\) is given by

$$\frac{\partial f_{XYZ}(x)}{\partial x} = 0, \quad \frac{\partial f_{XYZ}(x)}{\partial y} = 0$$  \hspace{1cm} (IV.13)

$$\Rightarrow X = A(x, y)Z + B(x, y), \quad \text{and} \quad Y = C(x, y)Z + D(x, y),$$  \hspace{1cm} (IV.14)

$$A = -\left( \frac{\partial z(x, y)}{\partial x} \right)_{E_3}, \quad B = z\left( \frac{\partial z(x, y)}{\partial x} \right)_{E_3} + x,$$  \hspace{1cm} (IV.15)

$$C = -\left( \frac{\partial z(x, y)}{\partial y} \right)_{E_3}, \quad D = z\left( \frac{\partial z(x, y)}{\partial y} \right)_{E_3} + y.$$  \hspace{1cm} (IV.16)

From Fig.5 showing the light paths, it is known that a caustic is formed around the origin. Only when \( a, b \) and \( c \) are equal to each other, the SOEP becomes zero-dimensional (at the origin). \( a = b \neq c \) implies the endpoints form a one-dimensional set which is an interval on the \( z \)-axis, \([2z_0, 0]\). Otherwise, the SOEP is two-dimensional (Fig.4).

The Taylor series of the potential \( f \) at the origin is given by

$$\tilde{f}_{x=0}(x) = \frac{a^4}{c^2} + \frac{-a^2 + c^2}{4a^4} x^4 + \frac{-a^2 + c^2}{8a^6} x^6 + \frac{b^2 - a^2}{b^2} y^2 + \frac{-a^2 + c^2}{2a^2b^2} x^2 y^2$$  \hspace{1cm} (IV.17)

$$+ \frac{3(-a^2 + c^2)}{8a^4b^2} x^4 y^2 + \frac{-a^2 + c^2}{4b^4} y^4 + \frac{3(-a^2 + c^2)}{8a^2b^4} x^2 y^4 + \frac{-a^2 + c^2}{8b^6} y^6$$  \hspace{1cm} (IV.18)

$$+ O \left( x^7 \right).$$  \hspace{1cm} (IV.19)
The structure of the caustic is controlled by the leading term of \( \tilde{f} \) about \( x, y \). When \( a < b \leq c \), \( \tilde{f} \sim ax^4 + \beta y^2 + \gamma \) produces a cusp (type \( A_3 \)). Then, the two-dimensional SOEP is structurally stable. On the other hand, \( \tilde{f} \) becomes \( \alpha(y^4 + 2x^2y^2 + x^4) + \gamma \) with \( a = b \neq c \). This case corresponds to the line SOEP and it is not structurally stable. Incidentally, if \( a < b < c \), there is also another cusp at \((0, 0, -(b^2 - a^2)/c)\) (carefully see Fig.5). The Taylor expansion of \( f \) around this cusp tells that it is also stable as long as \( a \neq b \) and \( b \neq c \). With \( b \to a \), this cusp approaches the cusp at the origin and degenerate into the unstable structure. On the contrary, when \( b \) is equal to \( c \), this cusp disappears at the center of the ellipsoid. The example given in [4] corresponds to this case. Of course, the zero-dimensional SOEP \((a = b = c)\) is not structurally stable.

V. SUMMARY AND DISCUSSIONS

We have investigated the stability of the topology of the EH (TOEH). First the stability of a spherical topology is investigated under linear perturbation in a spherically symmetric background. In linear order, \( L = 2 \) even parity mode changes the structure of the SOEP and the TOEH, and odd parity mode and \( L = 1 \) mode do not. In higher order, however, mode coupling between the modes with different \( L \) or parity will cause the instability of the TOEH even for these modes not changing the TOEH in the linear order. For \( L > 2 \) even parity mode, more detailed investigation will be required. Anyway, we have seen that the trivial TOEH is generally unstable under the linear perturbation.

In this discussion of the linear perturbation, we consider Oppenheimer-Snyder spacetime as the example of an always spherical EH. Nevertheless, the result will be same to other non-eternal EHS with spherical symmetry since we have never used the concrete geometry of the spacetime other than the spherical symmetry.

How can we interpret the fact that the TOEH is insensitive to some modes of the perturbation? In a sense, when we give odd parity or \( L = 1 \) perturbation to the spherical EH, the change of the TOEH in the higher order would not be able to be detected (though it is not
trivial how one can observe it). For, while a local geometry around an observer is perturbed with the same strength as the given perturbation, the TOEH is not so. The change of the TOEH in the higher order would be prevented being observed.

Second, by a simple discussion in catastrophe theory, the structural stability of the SOEP is studied in a more general situation. Assuming the ellipsoidal wave front, the stability of zero-, one-, and two-dimensional SOEP is investigated. We see that the two-dimensional SOEP is stable, and the one- and zero-dimensional is not. Therefore, the TOEH with handles (a torus, a double torus, ...) is generic.

Though in the present article we meet only the SOEP with a cusp catastrophe, as discussed in [9] there will be the possibilities of some further types, the ‘swallowtail’, the ‘pyramid’, and so on. They will form other structurally stable SOEPs. The TOEH with them will be revealed in our forthcoming work.

One may expect to give some restriction to the TOEH introducing certain conditions about matter field. The present results imply, however, that it is likely hopeless. It seems that the symmetry of the spacetime affect the structure of the SOEP of the EH and the TOEH, and is easily disturbed by perturbation. If one does not concern the scale of the topological structure of the EH, the TOEH can generally become complicated.

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REFERENCES


FIGURES

the zero-dimensional SOEP

the one-dimensional SOEP

the two-dimensional SOEP

FIG. 1. EHs with the zero-, one- and two-dimensional SOEP are shown. We see that the one-dimensional SOEP becomes coalescence of arbitrary number of spherical EHs. For the two-dimensional SOEP, only sections of the EH and the SOEP are drawn. It can become the EH with arbitrary number of handles. It is also possible to change the EH into the trivial creation of a spherical EH.
FIG. 2. The latest light paths with maximal $\delta\eta_{\text{crit}}$ ($x, y$ direction in this figure) form an endpoint at the origin with $\eta = \eta_{\text{crit}} + \delta\eta_{\text{crit}}(\theta = \pi/2)$. On the other hand, a light path on the other axis ($z$-axis in this figure) crosses light paths from other directions and form an endpoint there. Thus the SOEP gets a dimension in this ($z$-) direction.
FIG. 3. The light paths for the elliptic wave front with $a = 2, b = 1$ are drawn. There are the crossing points of the light paths which are the endpoints of a null surface corresponding to the wave front, on the $x$-axis, $[2x_0, 0]$. A cusp is formed at the origin.
FIG. 4. The SOEP becomes zero-dimensional for the spherical wave front. In the prolate-spheroidal wave front, the one-dimensional SOEP appears. Otherwise, the ellipsoidal wave front produces the two-dimensional SOEP.
FIG. 5. The light paths for the ellipsoidal wave front with $a = 1, b = 1.3, c = 1.5$ are drawn. A cusp is formed at the origin. Watching this figure carefully, one will see that also another cusp exists at $(0, 0, (b^2 - a^2)/c)$