Fixed point actions in SU(3) gauge theory: surface tension and topology *

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This work is organized in two independent parts. In the first part are presented some results concerning the surface tension in SU(3) obtained with a parametrized fixed point action. In the second part, a new, approximately scale-invariant, parametrized fixed point action is proposed which is suitable to study the topology in SU(3).

1. INTRODUCTION

Lattice actions living on the direction of the space of couplings which originates at the fixed point of a renormalization group (RG) transformation and leaves orthogonally the critical surface, are called fixed point (FP) actions. They are classical perfect actions and have many interesting properties [1]. In particular, FP actions possess scale-invariant instanton solutions.

Given a RG transformation, the FP action of any lattice configuration is well defined and can be determined numerically by multigrid minimization [2]. In Monte Carlo simulations, however, only simple enough parametrizations of the FP action can be used.

In Sect. 2 the so called “type III” parametrized FP action proposed in Ref. [3] is used to study some typical observables in SU(3) thermodynamics at high temperatures (free energy density) and at criticality (surface tension). In Sect. 3 a new, approximately scale-invariant parametrization of the FP action is proposed to be used in SU(3) topology.

2. SU(3) THERMODYNAMICS

In lattice numerical simulations, the temperature \( T \) and the volume \( V \) are determined by the lattice size \( N^3 \times N_{\tau}, (N_{\tau} < N_{\sigma}) \) through \( T = 1/(N_{\tau}a) \) and \( V = (N_{\sigma}a)^3 \).

At high temperatures, high momentum modes give relevant contributions to the free energy density. In this regime, thermodynamic quantities are strongly influenced by the finite lattice cut-off \( a^{-1} \), i.e. they show \( 1/N_{\tau}^n \) corrections from the continuum at fixed \( T \). For the Wilson action these cut-off effects are very large: in the case of the energy density of the ideal gluon gas at \( N_{\tau} = 4 \) they are as large as 50% [4]. Therefore, the continuum can be extrapolated only from lattice results at relatively large values of \( N_{\tau} \).

In Ref. [5] the free energy density of SU(3) at \( T/T_c = 4/3, 3/2, 2 \) has been determined using the type III parametrized FP action. Simulations were performed on lattices as small as \( 8^3 \times 2 \) and \( 12^3 \times 3 \). Results at \( N_{\tau} = 3 \) showed already a good agreement with the continuum extrapolated from the Wilson action determinations on lattices with \( N_{\tau} = 6 \) and 8 (see also Ref. [6]).

At the critical temperature of the first-order deconfinement transition of SU(3), there can be mixed states where the confined and the deconfined phases coexist, separated by an interface. These mixed states have an additional free energy \( F = \sigma A \) (\( \sigma \) is the surface tension, \( A \) the area of the interface). The frequency distribution of any order parameter \( \Omega \) at the transition shows a typical double-peak structure, where the two peaks correspond to the pure phase configurations, while the region in-between corresponds to configurations containing an interface. The peaks become more pronounced when the volume is increased. The leading volume dependence of the surface tension can be determined by [7]

\[
\left( \frac{\sigma}{T_c^3} \right) = \frac{1}{2} \left( \frac{N_{\tau}}{N_{\sigma}} \right)^2 \ln \frac{P_{\min}}{P_{\max,1} P_{\max,2}},
\]

with \( \Omega = \gamma_1 \Omega_1 + \gamma_2 \Omega_2, \ \gamma_1 + \gamma_2 = 1 \). Here \( P_{\min} \) is the minimum of the distribution \( P(\Omega) \), \( P_{\max,1} \) and \( P_{\max,2} \) are the two maxima corresponding to

*Presented by A. Papa who is the author of Sect. 2. Sect. 3 was done in collaboration with F. Farchioni.
Table 1
Parameters of the runs and results for $\beta_c$ and $\chi_L/N_\sigma^3$. One iteration is the combination of a 20-hit Metropolis and 4 over-relaxation updatings. Errors have been estimated by the jackknife method.

<table>
<thead>
<tr>
<th>lattice</th>
<th># iterations</th>
<th>$\beta$ reweighting</th>
<th>$\beta_c$</th>
<th>$\chi_L/N_\sigma^3$</th>
</tr>
</thead>
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<tr>
<td>$12^3 \times 3$</td>
<td>3</td>
<td>645099</td>
<td>3.58995</td>
<td>3.58982(9)</td>
</tr>
<tr>
<td>$12^3 \times 4$</td>
<td>1</td>
<td>171209</td>
<td>3.69915</td>
<td>3.70046(20)</td>
</tr>
<tr>
<td>$16^3 \times 4$</td>
<td>3</td>
<td>135662</td>
<td>3.70025</td>
<td>3.7009(5)</td>
</tr>
</tbody>
</table>

the values $\Omega_1$ and $\Omega_2$ of $\Omega$ in the pure states of the two phases at infinite volume.

A convenient choice for the order parameter in SU(3) is the absolute value of the Polyakov loop $L = 1/N_\sigma^3 \sum_\tilde{\nu} T_\nu \prod_{n=1}^{N_\tau} U_{\tilde{\nu}, n_\lambda}$. Numerical simulations with the type III parametrized FP action have been performed on three lattices for values of $\beta$ close to criticality (Table 1 and Fig. 1). Ferrenberg-Swendsen reweighting has been applied in order to make the peaks in the Polyakov loop distribution have the same height. The critical couplings have been determined through the location of the peak in the Polyakov loop susceptibility $\chi_L = N_\sigma^3 (\langle |L|^2 \rangle - \langle |L| \rangle^2)$. They are in agreement within errors with the results of Ref. [3] where the so called Columbia definition was adopted. In Table 2 the results for $\sigma/T_c^3$ are compared with other determinations using the tree-level Symanzik improved action (plaquette + rectangle) and the tadpole improved action (with the same loops) [8].

$$\left( \frac{\sigma}{T_c^3} \right)_V = \left( \frac{\sigma}{T_c^3} \right)_{\infty} - \left( \frac{N_\tau}{N_\sigma} \right)^2 \left[ c + \frac{1}{4} \ln N_\sigma \right].$$

Table 2 shows that the tadpole improved action has no cut-off dependence from $N_\tau = 3$ to $N_\tau = 4$, thus indicating for $\sigma/T_c^3$ a continuum value equal to 0.0155(16) (corresponding to $\sigma \sim 7$ Mev/fm$^2$). The type III FP action results indicate a significant cut-off dependence from $N_\tau = 3$ to $N_\tau = 4$. Moreover, the result on lattice $16^3 \times 4$ is consistent with the other two determinations, but the infinite volume extrapolation at $N_\tau = 4$ is not. This evident cut-off dependence is surprising in consideration of the results obtained with the same action in the case of the free energy density. The surface tension is, however, a difficult quantity to determine for the high statistics required and for the complicated volume dependence. Further checks are necessary before drawing a definite conclusion.

For comparison with the Wilson action see Ref. [7].

Table 2
$\sigma/T_c^3$ for various lattice actions on several lattices. Errors estimated by the jackknife method.

<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>$12^3 \times 3$</td>
<td>0.0234(24)</td>
<td>0.0158(11)</td>
<td>0.0307(8)</td>
</tr>
<tr>
<td>$12^3 \times 4$</td>
<td>0.0196(11)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$16^3 \times 4$</td>
<td>0.0148(16)</td>
<td>0.0147(14)</td>
<td>0.0180(21)</td>
</tr>
<tr>
<td>$24^3 \times 4$</td>
<td>0.0136(25)</td>
<td>0.0119(21)</td>
<td></td>
</tr>
<tr>
<td>$32^3 \times 4$</td>
<td>0.0116(23)</td>
<td>0.0125(17)</td>
<td></td>
</tr>
<tr>
<td>$\infty^3 \times 4$</td>
<td>0.0152(26)</td>
<td>0.0152(20)</td>
<td>0.026(6)</td>
</tr>
</tbody>
</table>

3. TOPOLOGY

It is well known that FP actions possess scale-invariant instanton solutions [2]: the dependence of the FP action on the instanton size $\hat{\rho}$ is flat up to $\hat{\rho}$ as small as $\sim 1$ lattice spacing [9]. FP actions combined with a suitable definition of the topological charge [9–11] allow a consistent determination of the topological susceptibility.

Figure 1. Polyakov loop distribution on lattices $12^3 \times 4$ and $16^3 \times 4$ at $\beta = 3.69915$. Data have been normalized to the total statistics.
In this Section we present a new, approximately scale-invariant parametrization of the type III FP action for SU(3), determined by a fit procedure on the (numerically estimated) “exact” values of the FP action of both typical Monte Carlo configurations and “hand-made” instanton solutions with \( \hat{\rho} \gtrsim 1 \). The procedure to build such instantons was first described in Ref. \[9\] for the O(3) \( \sigma \) model and applied to SU(2) in Ref. \[12\]. Differently from Ref. \[12\], we bypassed the problem of non-existence of single instantons on lattices with periodic b.c. by working with open boundaries. We performed the following steps:

- we discretized smooth continuum instantons on a \( 6^4 \) lattice, in order to have configurations on which Wilson action and typical Symanzik actions are the same within a few per mill;
- we blocked down three times by the type III RG averaging, in order to make the final instanton size \( \hat{\rho} \sim 1 \);
- we estimated by numerical minimization the “exact” FP action of the blocked configurations.

The blocked configurations were included finally in the parametrization procedure together with \( \sim 500 \) typical Monte Carlo configurations in the range \( \beta_{\text{Wilson}} = 5.1 - 50.3 \). We searched for parametrizations of the form

\[
A(U) = \frac{1}{N} \sum C \geq 1 c_i(C) [N - \text{ReTr}(U_C)]^i, \tag{3}
\]

where \( C \) denotes any closed path, \( U_C \) stands for \( \prod_{C} U_{\mu}(n) \). Skipping the many technical subtleties induced by the open b.c. in the full procedure and the details of the fit, we just mention that we defined the action on a finite lattice \( \Lambda \), \( A_\Lambda = \sum_{x \in \Lambda} A(x) \), through

\[
A(x) = \frac{1}{N} \sum C \geq x, i \geq 1 c_i(C) [N - \text{ReTr}(U_C)]^i \text{perimeter}(C). \tag{4}
\]

We found a nice parametrization involving the plaquette, the bent rectangle and the twisted perimeter-8 loop \((x,y,z,t,-x,-y,-z,-t)\), with the couplings given in Table 3. In Fig. 2 parametrized and minimized FP action are compared with the finite volume instanton action on the continuum.

![Figure 2. Minimized and parametrized FP action of the blocked instanton configurations versus the instanton size \( \hat{\rho} \). The solid line is the finite volume instanton action on the continuum.](image)

### Table 3: Couplings of the parametrized FP action.

<table>
<thead>
<tr>
<th>loop</th>
<th>( c_1 )</th>
<th>( c_2 )</th>
<th>( c_3 )</th>
<th>( c_4 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>plaquette</td>
<td>-0.5441</td>
<td>1.9405</td>
<td>-0.4663</td>
<td>0.0162</td>
</tr>
<tr>
<td>bent rect.</td>
<td>0.1099</td>
<td>-0.1150</td>
<td>-0.0087</td>
<td>0.0073</td>
</tr>
<tr>
<td>twisted-8</td>
<td>-0.0089</td>
<td>0.0048</td>
<td>0.0154</td>
<td>-0.0014</td>
</tr>
</tbody>
</table>

REFERENCES

1. P. Hasenfratz, these proceedings.
6. E. Laermann, these proceedings.

\(^3\)These Monte Carlo configurations were obtained in Ref. \[3\].