Fixed Points and Vacuum Energy of Dynamically Broken Gauge Theories

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Abstract

We show that if a gauge theory with dynamical symmetry breaking has non-trivial fixed points, they will correspond to extrema of the vacuum energy. This relationship provides a different method to determine fixed points.

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1 Introduction.

Gauge theories without scalar bosons may undergo the process of dynamical symmetry breaking, where dynamical masses are generated, and we have the phenomenon of dimensional transmutation [1], i.e. we basically do not have arbitrary parameters once the gauge coupling constant \((g)\) is specified at some renormalization point \((\mu)\). In these theories all the physical parameters will depend on this particular coupling. Therefore, it would not be surprising if the dynamical masses follow a critical behavior totally related to the one of the coupling constant.

QED is one example of a theory that may show dynamical chiral symmetry breaking in the strong coupling regime. It has been suggested that QED in four dimensions at the same time that generates fermion masses, could develop a non-trivial ultra-violet fixed point, and this possibility is still under study as recently reviewed in Ref. [2]. This fixed point behavior could imply that four-dimensional QED is a non-trivial theory. In three dimensions QED also suffers from dynamical symmetry breaking, and recently it has been pointed out that it also may have a non-trivial infrared fixed point [3]. Such fixed points are determined as zeros of the renormalization group \(\beta\) function, and, generally speaking, they can be attractive or repulsive. According to the idea of dimensional transmutation we can think about how this critical behavior of the coupling constant is transmitted to other calculable physical quantities.

One of the quantities for which we have precise methods to compute in field theory is the vacuum energy, and we could naively think that the fixed points would appear as extrema of the vacuum energy. Such intuitive idea is not new. When Wilson developed the concepts of renormalization group and critical phenomena [4], he gave an example of the renormalization group equation making use of an analogy in classical physics of a ball rolling on a hill. In this example the equation of motion of the ball in the hill potential was related to the renormalization group equation, and the fixed point was related to a stationary point. Therefore, it seems natural to expect a deeper relation between fixed points and extrema of energy also in field theory. However, we have been unable to find a proof of this in the literature, and here we will give a simple presentation of such connection.

Our demonstration will rely heavily on two field theoretical methods: the inversion method proposed by Fukuda as a tool to compute non-perturbative
quantities in gauge theories [5], and the calculation of the vacuum energy as prescribed by Cornwall et al. [6, 7, 8]. We will discuss the case of a gauge theory without scalar bosons, with an unique coupling constant \((g)\), and we believe that a more rigorous proof can be performed including the case of a theory involving several coupling constants. This relationship provides a completely different method to determine non-perturbative fixed points, and could be tested in lattice simulations of gauge theories.

2 Vacuum energy and the inversion method.

Many years ago Cornwall and Norton [6] emphasized that the vacuum energy \((\Omega)\) in dynamically broken gauge theories could be defined as a function of the dynamical mass

\[
\text{m}_{\text{dyn}} \equiv \Sigma(p) \equiv m, \tag{1}
\]

where \(\Sigma(p)\) is the fermion self-energy, and once \(m\) obey the asymptotic behavior predicted by the operator product expansion \(\Omega = \Omega(\mu, g)\) is a completely finite quantity [6, 7, 8].

In the case of a gauge theory without bare masses, and with an unique coupling constant \((g)\), \(\Omega\) must satisfy a homogeneous renormalization group equation [9]

\[
\left( \mu \frac{\partial}{\partial \mu} + \beta(g) \frac{\partial}{\partial g} \right) \Omega = 0. \tag{2}
\]

On the other hand, the dynamically generated masses can be written as \(m = \mu f(g)\) [9], from what follows that \(\mu (\partial m / \partial \mu) = m\) and, consequently,

\[
m \frac{\partial \Omega}{\partial m} = -\beta(g) \frac{\partial \Omega}{\partial g}. \tag{3}
\]

This last and simple equation will be central to our argument, because it relates the stationary condition for the vacuum energy \((\partial \Omega / \partial m = 0)\) [6, 7, 8], to the condition of zeros of the \(\beta\) function.

We will suppose that we have a gauge theory with a critical coupling \(g_c\) separating the symmetric and asymmetric phases. For \(g < g_c\) we do not have dynamically generated masses \((m\) and \(\Omega\) are equal to zero), and we do not expect any non-trivial fixed point. For \(g > g_c\) we are in the asymmetric phase, \(m\) and \(\Omega\) are different from zero, and as long as \(\partial \Omega / \partial g \neq 0\), what
we will show that is true when $g > g_c$, the extrema of energy happens for the fixed points of the theory. Therefore, in the following we will show that in a gauge theory with dynamically generated masses (or condensation), the condition for an extrema of the vacuum energy:

$$
\beta(g) \frac{\partial \Omega}{\partial g} \bigg|_{\partial \Omega / \partial m = 0} = 0 ,
$$

always imply $\beta(g) = 0$.

Before discussing the behavior of the theory for $g > g_c$, it is interesting to see what are the consequences if the theory has a fixed point at $g_\star = g_c$. Note that the notion of critical point for the chiral transition and fixed point should not necessarily coincide, although this is exactly what is expected in QED. A bifurcation of the self-energy, or dynamical mass, will happens between the symmetric and asymmetric regions, and we can perform a very simple analysis to know the behavior of the mass (and the vacuum energy) in the neighborhood of the fixed point $g_\star$. The dynamical mass is given by

$$
m = \mu \exp \left[ - \int^g dg \frac{\beta(g)}{\beta(g)} \right] ,
$$

and assuming that near $g_\star$ the $\beta$ function can be approximated by

$$
\beta(g) = b(g - g_\star) + ... ,
$$

where $b$ is a constant, we obtain

$$
m = \mu (g - g_\star)^{-1/b} .
$$

On dimensional grounds the vacuum energy will be given by $\Omega \propto cm^4$, where $c$ is a calculable negative number [9]. For $g < g_\star$ the dynamical mass is zero (as well as the vacuum energy), and for $g > g_\star$ it deviates from zero according to Eq.(5). The dynamically broken phase exist for $g > g_\star$, and in this case $g_\star$ coincides with the critical coupling for mass generation $g_c$. For $b > 0$ we note that the vacuum energy as a function of the coupling constant will behave as displayed in Fig.(1a). In Fig.(1a) $\Omega = 0$ for $g < g_\star$, it has a bifurcation at $g_\star$, and its absolute value diminishes as $g$ is increased. For $b < 0$ the vacuum energy $\Omega$ is depicted in Fig.(1b). According to the sign of $b$ we have a local
minimum or a maximum of the vacuum energy at the point \( g = g_* \). The discontinuity in Fig.(1a) is artificial, and it will depends on our ability to compute the dynamical mass behavior at the transition point. Actually this will be a measure of the phase transition order. The above argument is valid only on the vicinity of the fixed point, where Eq.(6) is reliable.

We can now proceed to the case when \( g_* > g_c \) (the possibility that \( g_* = g_c \) as was discussed above, is nothing else than a limit of this case). Our demonstration will not depend on the specific form of the \( \beta \) function. Considering that in the broken phase \( m \neq 0 \) and that the functional derivative of \( \Omega \) with respect to \( m(\Sigma) \) vanishes at the extrema of the vacuum energy (or when \( m \) is a solution of the Schwinger-Dyson equations) [6, 7, 8], we can assert that the extrema of energy will occur at the fixed points just showing that in the right-hand side of Eq.(3), \( \partial \Omega / \partial g \neq 0 \) for \( g > g_c \).

To compute the vacuum energy in dynamically broken gauge theories we need to introduce a bilocal field source \( J(x, y) \), since we are interested in theories which admit condensation of composite operators as, for instance, \( \langle \bar{\psi} \psi \rangle \) [7]. \( \Omega \) will be calculated after a series of steps starting from the generating functional \( Z(J) \) [7]:

\[
Z(J) = \exp[iW(J)] = \int d\phi \exp \left[ i \left( \int d^4x L(x) + \int d^4x d^4y \phi(x) J(x,y) \phi(y) \right) \right], \tag{8}
\]

where \( \phi \) can be a fermion or gauge boson field. From the generating functional we determine the effective action \( \Gamma(G) \) which is a Legendre transform of \( W(J) \):

\[
\Gamma(G) = W(J) - \int d^4x d^4y G(x,y) J(x,y), \tag{9}
\]

where \( G \) is a complete propagator, and from Eq.(9) we obtain

\[
\delta \Gamma / \delta G(x,y) = -J(x,y). \tag{10}
\]

The physical solutions will correspond to \( J(x,y) = 0 \), which will reproduce the Schwinger-Dyson equations (SDE) of the theory [7]. In general, if \( J \) is the source of the operator \( \mathcal{O} \), we have [10]

\[
\frac{\delta \Gamma}{\delta J}
\bigg|_{J=0} = \langle 0 | \mathcal{O} | 0 \rangle. \tag{11}
\]
For translationally invariant (t.i.) field configurations we can work with the effective potential given by

\[ V(G) \int d^4x = -\Gamma(G)_{t.i.} \].

(12)

Finally, from the above equations we can define the vacuum energy as [7, 8]

\[ \Omega = V(G) - V_{pert}(G) \],

(13)

where we are subtracting from \( V(G) \) its perturbative counterpart, and \( \Omega \) is computed as a function of the nonperturbative propagators \( G \), i.e. its self-energies, \( \Sigma \) or \( \Pi \), whether we are working with fermions or gauge bosons, and is zero in the absence of mass generation. Ultimately, \( \Omega \) is a function of the dynamical masses of the theory. Note that in the cases when we do not have bare masses \( V_{pert}(G) = 0 \), and \( \Omega = V(G) \) [7, 8].

There is a long discussion in the literature if the vacuum energy \( \Omega \) can be identified with the effective potential as described above, if the effective potential is single-valued, gauge-invariant, etc... [11]. However, we stress that all these problems are absent at the stationary points of the vacuum energy [12], where even different formulations of the effective potential for composite operators lead to the same stationary point [13, 14], and it is exactly for these points that we must compute Eq.(4). We can now write Eq.(4) in the following form:

\[ -\beta(g) \left[ \frac{\partial \Omega}{\partial J} \frac{\partial J}{\partial g} \right]_{J=0} = 0. \]

(14)

However, \( \partial \Omega/\partial J = -\partial \Gamma/\partial J \), and as a consequence of Eq.(11) we have

\[ \beta(g) \langle 0 | O | 0 \rangle \frac{\partial J}{\partial g} \bigg|_{J=0} = 0. \]

(15)

The vacuum condensate \( \langle 0 | O | 0 \rangle \) is different from zero for \( g > g_c \). Therefore, it remains to show that \( \partial J/\partial g \big|_{J=0} \) is also different from zero in the same condition, what can be accomplished through the so called “inversion method” [5].

Fukuda has devised a very ingenious method to determine nonperturbative quantities [5]. He noticed that to compute a nonperturbative quantity
like $\langle 0 | \mathcal{O} | 0 \rangle \equiv \vartheta$, the usual procedure is to introduce a source $J$ and to calculate the series:

$$\vartheta = \sum_{n=0}^{\infty} g^n h_n(J).$$

(16)

In practice we have to truncate Eq.(16) at some finite order, and it gives us only the perturbative solution $\vartheta = 0$ when we set $J = 0$. The right-hand side of Eq.(16) should be double valued at $J = 0$ for another solution to exist, which is not the present case. The alternative method proposed by Fukuda is to invert Eq.(16), solving it in favor of $J$ and regarding $\vartheta$ as a quantity of the order of unity. We obtain the following series:

$$J = \sum_{n=0}^{\infty} g^n k_n(\vartheta),$$

(17)

where the $k_n$'s satisfying $n \leq m$ ($m$ being some finite integer) are calculable from $h_n$ also satisfying $n \leq m$. We can find a nonperturbative solution of $\vartheta$ by setting $J = 0$ through a truncated version of Eq.(17). The details of the method can be found in Ref. [5]. The important point for us is that by construction of Eq.(17) we verify that when $J = 0$ and $\vartheta \neq 0$ (i.e. $g > g_c$), the same value of $\vartheta$ that satisfy Eq.(17) leads trivially to

$$\frac{\partial J}{\partial g}|_{J=0} \neq 0.$$ 

(18)

Therefore, the two terms, $\partial J/\partial g$ and $\langle 0 | \mathcal{O} | 0 \rangle$, of Eq.(15) never can be equal to zero in the broken phase! According to this and looking at Eq.(15), the only possibility to obtain $\partial \Omega/\partial m = 0$ (for $g > g_c$) is when we have a fixed point ($\beta(g) = 0$), from where comes our main assertion that fixed points are extrema of the vacuum energy. It is also clear that if $\beta(g) \neq 0$ in the condensed phase we are forced to say that the critical and fixed point coincide ($g_c = g_*$), and, of course, it corresponds to an extreme of energy.

### 3 Examples and summary.

We would like to show some specific calculations of the above expressions to see how they can be computed in practice. As an example that the terms $\partial \Omega/\partial J$ and $\partial J/\partial g$ of Eq.(14) are different from zero in the condensed phase,
we can compute them in the case of four-dimensional quantum electrodynamics \((QED_4)\), which is supposed to have a non-trivial ultra-violet fixed point \([2]\). In \(QED_4\) the term \(\partial \Omega / \partial J\) will be given by \(\langle \bar{\psi} \psi \rangle\), i.e. we have dynamical mass generation and a condensate is formed when the gauge coupling constant \(\alpha \equiv e^2 / 4\pi\) is larger than a certain critical value \(\alpha_c\). The term \(\partial J / \partial g\) is also different from zero for \(\alpha > \alpha_c\), and these terms can be computed with the help of the inversion formula determined in Ref. [15]. We will not repeat here all the steps of Ref. [15], where the inversion method was applied to \(QED_4\), and Eq.(17) was obtained up to two-loop level in a gauge invariant way, whose result is:

\[
J = \frac{4\pi^2}{N_f \Lambda_f^2} \left[ 1 - \frac{\alpha}{\alpha_c} \right] \langle \bar{\psi} \psi \rangle + \frac{64\pi^6}{\eta N_f^3 \Lambda_f^8} \frac{\alpha}{\alpha_c} \left( \frac{16\pi^4}{N_f^2 \Lambda_f^4 \Lambda_p^2} \langle \bar{\psi} \psi \rangle^2 \right) ^3 \ln \left( \frac{1}{\alpha_c / \alpha} \right) + \mathcal{O} \left( \langle \bar{\psi} \psi \rangle^3 \ln \langle \bar{\psi} \psi \rangle^2 \right),
\]

where \(\alpha_c = 2\pi / 3\eta\), \(N_f\) is the number of flavors, \(\eta = \Lambda_p^2 / \Lambda_f^2\), the \(\Lambda\)'s are ultraviolet cutoffs associated to the photon and fermion self-energies \([15]\), and in the following we will assume that \(\Lambda_f = \Lambda_p = \Lambda\). The solution \(J = 0\) exists only for \(\alpha > \alpha_c\), and the expression of \(\langle \bar{\psi} \psi \rangle\) for this range of coupling constant can be easily obtained. For instance, near \(\alpha = \alpha_c\) it behaves as

\[
\langle \bar{\psi} \psi \rangle \approx - \frac{\Lambda^3 N_f}{8\pi^2} \frac{1}{\ln((1 - \alpha_c / \alpha)^{1/2})}.
\]

Substituting the result of \(\langle \bar{\psi} \psi \rangle\) into \(\partial J / \partial g\) we verify that this term is different from zero for \(\alpha > \alpha_c\). In the case of \(QED_4\) Eq.(14) will have a zero only if there is a fixed point for \(\alpha \geq \alpha_c\). Note that for \(\alpha < \alpha_c\) there is not mass generation and \(\Omega \equiv 0\), and at \(\alpha = \alpha_c\) we may expect a bifurcation in the vacuum energy due to the phase transition. It is obvious in this case that the critical and fixed points are the same.

As a final example we would like to present a calculation of the vacuum energy showing its behavior with respect to the coupling constant. Actually, a few years ago one of us \([16]\) computed \(\langle \Omega \rangle\), which denotes the values of \(\Omega\) at the stationary points \([17]\), in the case of quenched \(QED_4\). \(\langle \Omega \rangle\) was
computed using approximate solutions of the Schwinger-Dyson equations for
the fermion propagator, and the minimum of energy was obtained for each
value of the coupling constant ($\alpha$). It was observed that the deepest minimum
occurs exactly for the critical value of the coupling constant expected to
be a fixed point. This calculation was also extended to include the effect
of four-fermion interactions [18]. Here we will compute $\langle \Omega \rangle$ in the case of
QED taking into account the vacuum polarization effects as discussed by the
authors of Ref. [19, 20]. Note that this is not going to be a proof that QED in
this approximation has a fixed point, because we will not include the effect
of the four-fermion interactions (which can change completely the result),
and we will make use of very rough approximations to the Schwinger-Dyson
equations for the dynamical mass. A complete calculation of $\langle \Omega \rangle$ without
the approximations that we are going to make it is not an easy task, and
could only be performed numerically. However, we do find a behavior similar
to the one presented in Fig.(1), indicating that such studies have still to be
pursued. $\langle \Omega \rangle$ is given by the following expression [17]:

$$
\langle \Omega \rangle = 2\nu N_f \int \frac{d^4p}{(2\pi)^4} \left[ \ln[1 - \Sigma^2(p)/p^2] + \Sigma^2(p)/[p^2 - \Sigma^2(p)] \right], \quad (21)
$$

where $\Sigma(p)$ is the fermion self-energy of QED. Considering the photon po-
larization $\Pi(p^2) = (\alpha N_f/3\pi) \ln(\Lambda^2/p^2)$ and defining

$$
z = \frac{3\pi}{\alpha N_f} + \ln \frac{\Lambda^2}{p^2}, \quad (22)
$$

the fermion self-energy in the large $z$ limit has the form (in the Landau
gauge) [19, 20]

$$
\Sigma(p^2) = z^\gamma [C_1 \Phi(a, c; z) + C_2 \Psi(a, c; z)], \quad (23)
$$

where $\gamma = 3/(2\sqrt{N_f})$, $a = \gamma(1 - \gamma)$, $c = 1 + 2\gamma$, and $\Phi(a, c; z)$, $\Psi(a, c; z)$ are
the confluent hypergeometric functions.

Eq.(23), for a given number of flavors, exist only above a certain critical
coupling constant. These critical couplings have been determined numeri-
cally, and up to four fermion flavors they are [19, 20, 21]: $\alpha_c = 2.00(N_f = 1)$, 2.75($N_f = 2$), 3.51($N_f = 3$), 4.31($N_f = 4$). These couplings would be
candidates for fixed points as in quenched QED. Eq.(21) can be computed
numerically with the use of Eq.(23). A quite reasonable approximation to the full result can also be obtained noticing that the infrared behavior of Eq.(23) at leading order is weakly dependent on the coupling constant [20]. Therefore we can expand Eq.(21) for small values of $\Sigma/p$, obtaining

$$\frac{8\pi^2}{m^4N_f} \langle \Omega \rangle \approx -\frac{1}{2} \int_1^{(\Lambda/m)^2} dx \frac{\Sigma^4}{x} + O\left(\frac{\Sigma^6}{x^2}\right),$$

(24)

where $\Sigma = \Sigma/m$ and $x = p^2/m^4$. In agreement with the approximation leading to Eq.(24), $\langle \Omega \rangle$ can be calculated with the help of the asymptotic forms of Eq.(23), for which (when $\Lambda/m \gg 1$) we have [19, 20]

$$C_1 \approx -\frac{m^3}{\Lambda^2} e^{-\frac{3\pi}{\alpha N_f}} \left[ \frac{3\pi}{\alpha N_f} + \ln \frac{\Lambda^2}{m^2} \right]^{\frac{1}{\gamma^2}},$$

$$C_2 \approx m \left[ \frac{3\pi}{\alpha N_f} + \ln \frac{\Lambda^2}{m^2} \right]^{\frac{1}{\gamma^2}},$$

(25)

and

$$\Phi(a, c; z) \sim z \rightarrow +\infty \frac{\Gamma(c)}{\Gamma(a)} e^{z(a-c)} e^{z(a-c)} \Phi(a, c; z) \sim z \rightarrow +\infty z^{-a}. $$

(26)

As noticed in Ref. [20] only at the momentum scale of $O(m)$ the solution $C_1 \Phi(a, c; z)$ becomes significant compared to $C_2 \Psi(a, c; z)$. The use of these asymptotic expressions is reliable in a considerably large region of momenta, since the full numerical solution of the Schwinger-Dyson equation can be fitted by $C_2 \Psi$ in almost all the interval of the integral of Eq.(24) in the case of $N_f = 1$ [20]. However, the solution given by Eq.(23) becomes a poor approximation for the numerical solution of the full self-energy equation when $N_f \geq 2$ [19]. To compute the vacuum energy we used the asymptotic expressions given by Eq.(26), and the values of $\langle \Omega \rangle$ are not expected to match the ones that would be obtained with the complete solution of the self-energy as we increase $N_f$, but they will keep roughly the same behavior, due to the fact that the integral in Eq.(24) depends essentially on the ultraviolet behavior of $\Sigma(p)$. We assumed $(\Lambda/m)^2 = 10^{15}$, and kept the infrared cutoff equal to 100 in order to be consistent with the approximation of small values of $\Sigma/p$ that lead us to Eq.(24).

Our results for the calculation of the vacuum energy at stationary points are shown in Fig.(2), where $\langle \Omega \rangle$ is plotted as a function of $\alpha$ for $N_f = 1, 2$, and 3. As in the case of quenched QED (without [16] and with four-fermion interaction [18]), the points of minimum energy occur exactly at the critical
couplings $\alpha_c = 2.00, 2.75$ and $3.51$ respectively, which are the ones expected to be fixed points. Again, we stress that the discontinuity reflects only the poor handling of the dynamical mass at the phase transition point. With this calculation we could say that QED when considering vacuum polarization effects may have non-trivial fixed points. Unfortunately, there are indications that the introduction of four-fermion interactions leads to a non-interacting theory [22]. It would be interesting to compute $\langle \Omega \rangle$ taking into account the effect of four-fermion interactions (with coupling constant $G$), and the full solution of the fermionic self-energy with the vacuum polarization effects. In this case $\langle \Omega \rangle$ would appear as a surface in the space of the coupling constants $(g,G)$. We can predict the following possibilities: a) A non-trivial fixed point would be indicated by a minimum of the vacuum energy for finite values of the coupling constants. b) A trivial theory would come out with a minimum located at infinity for an infinite value of the four-fermion interaction as observed in Ref. [22]. Although this calculation cannot be performed analytically it may provide useful information on the strong coupling regime of QED.

The connection between fixed points and vacuum energy provides a totally different way to find fixed points. As $\langle \Omega \rangle$ is a gauge invariant physical quantity it can be computed in numerical lattice simulations of gauge theories for different values of the coupling constant. Therefore, it is possible to obtain the curve of minima of energy in the regions of small and strong coupling, approaching the region of the phase transition, and according to our previous discussion the point of minimum energy connecting these different regions will indicate the fixed point.

In conclusion, we have shown that the extrema of the vacuum energy are associated to the fixed points of dynamically broken gauge theories. The vacuum energy is identically zero for $g < g_c$, where $g_c$ is the coupling separating the symmetric and broken phases. For $g > g_c$ we are in the broken phase, and, using the “inversion method”, it is possible to verify explicitly that the extrema condition (Eq.(14)) can have a zero, only if the $\beta$ function has a zero. In the expected case that $g_* = g_c$, i.e. critical and fixed points coincide, we use a very simple argument to show that it is an extremum of energy. This case can be seen as a limiting one when these points necessarily do not coincide. Our demonstration can be extended to the cases with several coupling constants, and possibly may be established on more formal grounds. We also presented an example of vacuum energy calculation which
shows the expected behavior discussed here. It would be interesting if such
relationship could be investigated by other methods, such as direct lattice
calculations.

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Figure Captions

Fig.(1) Expected behavior of the vacuum energy in a theory with a fixed point at $g_\star = g_c$. a) For positive values of $b$; b) For negative values of $b$.

Fig.(2) The vacuum energy at stationary points $\langle \Omega \rangle$ calculated as a function of the coupling constant $\alpha$, in the case of QED with $N_f$ fermions and considering effects of vacuum polarization. The curves are for $N_f = 1, 2, 3$, and the minima of energy are respectively at $\alpha = 2.00, 2.75$ and $3.51$. 
Figure 1.a
Figure 1.b
Figure 2