SUPERSYMMETRIC CONSTRUCTION OF
EXACTLY SOLVABLE POTENTIALS
AND NON-LINEAR ALGEBRAS∗

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Abstract

Using algebraic tools of supersymmetric quantum mechanics we construct
classes of conditionally exactly solvable potentials being the supersymmetric
partners of the linear or radial harmonic oscillator. With the help of the
raising and lowering operators of these harmonic oscillators and the SUSY
operators we construct ladder operators for these new conditionally solvable
systems. It is found that these ladder operators together with the Hamilton
operator form a non-linear algebra which is of quadratic and cubic type for
the SUSY partners of the linear and radial harmonic oscillator, respectively.

1 Introduction, summary and outlook

During the last decade supersymmetric (SUSY) quantum mechanics has become an
important tool in various branches of theoretical physics. In particular, in quantum
mechanical problems SUSY has been found to be a very useful algebraic tool [1].
For example, the class of exactly solvable quantum systems has been enlarged by
such methods [2]. Quite recently these methods have even been extended to the
construction of conditionally exactly solvable problems [3], where, in addition, it
has been shown that these systems have a non-linear algebraic structure.

It is the aim of this paper to generalize the approach given in [3] to a much
wider class of conditionally exactly solvable systems being the SUSY partners of the
linear or radial harmonic oscillator. In doing so we will first review the basic tools
of SUSY quantum mechanics [1] which we are going to use. In Section 3 we will
present in some detail the general construction principle previously suggested by us
[3]. In Section 4 we present the results for the linear harmonic oscillator. Section
5 and 6 contain our results on the radial harmonic oscillator with unbroken and
broken SUSY, respectively.

Besides the construction of conditionally exactly solvable problems we also anal-
yse their algebraic structure, which turns out to be uniquely characterized be their
SUSY partner. That is, for the SUSY partners of the linear oscillator we obtain a
quadratic algebra and for the radial oscillator (unbroken as well as broken SUSY)
we find a cubic algebra.

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As becomes clear from our general method in Section 3, the present approach can also be applied to other shape-invariant SUSY systems such as the radial hydrogen atom, Morse or Pöschl-Teller oscillator. Another application is in the construction of exactly solvable drift potentials associated with the Fokker-Planck equation. In fact, for the harmonic oscillator case this has, in essence, already been done by Hongler and Zheng [4].

2 Supersymmetric quantum mechanics

Witten’s model of supersymmetric quantum mechanics consists of a pair of standard Schrödinger Hamiltonians

\[ H_{\pm} = -\frac{1}{2} \frac{d^2}{dx^2} + V_{\pm}(x) \]  

acting on the Hilbert space \( \mathcal{H} \) of square integrable functions on the configuration space \( M \), which we will assume to be the real line in the case of the linear harmonic oscillator, \( \mathcal{H} = L^2(\mathbb{R}) \), or the positive half line in the case of the radial harmonic oscillator, \( \mathcal{H} = \{ \psi \in L^2(\mathbb{R}^+) | \psi(0) = 0 \} \). The so-called SUSY partner potentials

\[ V_{\pm}(x) = W^2(x) \pm W'(x) \]  

are given by the SUSY potential \( W : M \to \mathbb{R} \) and its derivative \( W' = dW/dx \). In terms of the SUSY operators

\[ A = \frac{1}{\sqrt{2}} \left( \frac{d}{dx} + W(x) \right) , \quad A^\dagger = \frac{1}{\sqrt{2}} \left( -\frac{d}{dx} + W(x) \right) \]  

the SUSY partner Hamiltonians read \( H_+ = AA^\dagger \geq 0 \) and \( H_- = A^\dagger A \geq 0 \).

With the help of the operators (3) it is easy to show that \( H_+ \) and \( H_- \) are essentially isospectral. To be more explicit, let us denote the eigenfunctions and eigenvalues of \( H_\pm \) by \( \psi_{\pm n} \) and \( E_{\pm n} \), respectively. That is,

\[ H_{\pm} \psi_{\pm n}(x) = E_{\pm n} \psi_{\pm n}(x) , \quad n = 0, 1, 2, \ldots . \]  

In the case of unbroken SUSY (here we will use the convention [1] that the zero-energy eigenstate of the SUSY system belongs to \( H_- \)) we have for the ground state of \( H_- \) the relations

\[ E_0^- = 0 , \quad \psi_0^- (x) = C \exp \left\{ - \int dx W(x) \right\} \in \mathcal{H} \]  

with \( C \) being a proper normalization constant. The remaining spectrum of \( H_- \) coincides with the complete spectrum of \( H_+ \) and the corresponding eigenfunctions are related by SUSY transformations:

\[ E_{n+1}^- = E_n^+ > 0 , \quad \psi_{n+1}^- (x) = (E_n^+)^{-1/2} A^\dagger \psi_n^+(x) , \quad \psi_n^+(x) = (E_{n+1}^-)^{-1/2} A \psi_{n+1}^-(x) . \]  

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In the case of broken SUSY $H_+$ and $H_-$ are strictly isospectral and the eigenfunctions are also related by SUSY transformations:

\[ E_n^- = E_n^+ > 0 , \quad \psi_n^-(x) = (E_n^+)^{-1/2} A^\dagger \psi_n^+(x) , \]
\[ \psi_n^+(x) = (E_n^-)^{-1/2} A \psi_n^-(x) . \]  (7)

Though above relations (6) and (7) are also valid in the cases of continuous spectra, we consider in this paper only systems having a purely discrete spectrum.

With the help of the relations (5) and (6) or (7) it is obvious that knowing the spectral properties of, say, $H_+$ one immediately obtains the complete spectral properties of the SUSY partner Hamiltonian $H_-$. It is this fact which is our basis for the construction of (conditionally) exactly solvable potentials, by which we mean that the eigenvalues and eigenfunctions of the corresponding Schrödinger Hamiltonian can be given in an explicit closed form (under certain conditions obeyed by the potential parameters). Furthermore, the SUSY operators (3) also allow to construct from known ladder operators of $H_+$ the corresponding ladder operators for $H_-$ which turn out to closed a non-linear algebra.

3 Construction of exactly solvable potentials

In this section we present our basic idea for the construction of (conditionally) exactly solvable potentials. As already anticipated in the last section, the basic idea is to choose the SUSY potential $W$ such that the partner potential $V_+$ becomes one of the well-known exactly solvable ones, that is, the eigenvalue problem for the corresponding Hamiltonian $H_+$ is exactly solvable. In this way we can eventually find (through a proper choice of $W$) new partner potentials which are also exactly solvable. That is, the spectral properties for $H_-$ are obtainable via the SUSY transformations (6) or (7).

In order to find an appropriate class of SUSY potentials we make the ansatz [3]

\[ W(x) = \Phi(x) + f(x) \]  (8)

where $\Phi$ is a so-called shape-invariant SUSY potential [1], that is, for $f \equiv 0$ the corresponding partner potentials $V_\pm$ belong to a known class of exactly solvable ones. For a non-vanishing $f$ we have

\[ V_+(x) = \frac{1}{2} [\Phi^2(x) + \Phi'(x) + f^2(x) + 2\Phi(x)f(x) + f'(x)] . \]  (9)

If we now choose $f$ such that it obeys the following generalized Riccati equation

\[ f^2(x) + 2\Phi(x)f(x) + f'(x) = 2(\varepsilon - 1) , \]  (10)

at least under certain conditions on the parameters contained in $\Phi$ and certain values of $\varepsilon \in \mathbb{R}$, than the two partner potentials read

\[ V_+(x) = \frac{1}{2} \Phi^2(x) + \frac{1}{2} \Phi'(x) + \varepsilon - 1 , \]  (11)
\[ V_-(x) = \frac{1}{2} \Phi^2(x) - \frac{1}{2} \Phi'(x) - f'(x) + \varepsilon - 1 . \]  (12)
Clearly, the potential $V_+$ is by construction shape-invariant and therefore exactly solvable. Via the SUSY transformation we can now also solve the eigenvalue problem for $H_-$ associated with the above $V_-$ which, due to our assumption that the potential parameters had to take certain values, is sometimes called a conditionally exactly solvable potential [5]. A first and obvious condition on the parameter $\varepsilon$ is that it has to be large enough in order to give rise to a strictly positive Hamiltonian $H_+ > 0$. If this would not be the case, than the SUSY transformations would lead to “wavefunctions” which are not in the Hilbert space $\mathcal{H}$. This, for example, may happen if the solution $f$ of (10) contains a singularity in the configuration space $M$.

Note that the SUSY operators (3) are given by

$$A = \frac{1}{\sqrt{2}} \left( \frac{d}{dx} + \Phi(x) + f(x) \right), \quad A^\dagger = \frac{1}{\sqrt{2}} \left( -\frac{d}{dx} + \Phi(x) + f(x) \right)$$

(13)

and should leave the Hilbert space invariant, $A : \mathcal{H} \to \mathcal{H}$, $A^\dagger : \mathcal{H} \to \mathcal{H}$.

In order to search for such regular solutions of (10) we linearize it by setting $f(x) = u'(x)/u(x)$, which leads to an ordinary, homogeneous and linear second-order differential equation:

$$u''(x) + 2\Phi(x)u'(x) + 2(1 - \varepsilon)u(x) = 0.$$  

(14)

In terms of $u$ the conditionally exactly solvable potential than reads

$$V_-(x) = \frac{1}{2} \Phi^2(x) - \frac{1}{2} \Phi'(x) + \frac{u'(x)}{u(x)} \left( 2\Phi(x) + \frac{u'(x)}{u(x)} \right) - \varepsilon + 1.$$  

(15)

The regularity condition on $f$ now amounts to obtain the most general solutions of (14) which is (without loss of generality) strictly positive on $M$. This is equivalent to require that $V_-$ does not have any additional singularities besides that of $V_+$. The latter may only exist at $x = 0$ for the case $M = \mathbb{R}^+$. In the following we will consider three examples corresponding to the linear harmonic oscillator with unbroken SUSY and the radial harmonic oscillator with unbroken as well as broken SUSY. For these systems we also know how to construct ladder operators which close a linear algebra. With the help of the SUSY operators (13) we are then able to obtain ladder operators for the conditionally exactly solvable system $H_-$ which turn out to close a non-linear algebra.

## 4 The linear harmonic oscillator

As a first example we will consider the SUSY potential of the linear harmonic oscillator on the real line $M = \mathbb{R}$:

$$\Phi(x) = x$$  

(16)

It is straightforward to verify that in this case the potential (11) is indeed that of the linear harmonic oscillator

$$V_+(x) = \frac{1}{2} x^2 + \varepsilon - \frac{1}{2}$$  

(17)
whose energy eigenvalues and eigenfunctions are
\[ E_n^+ = n + \varepsilon, \quad \psi_n^+(x) = \left[ \sqrt{\pi} 2^n n! \right]^{-1/2} H_n(x) \exp\{-x^2/2\}, \quad (18) \]
where \( H_n \) denotes the Hermite polynomial of order \( n = 0, 1, 2, \ldots \). As we require a strictly positive spectrum for \( H_+ \) we arrive at a first condition on the parameter \( \varepsilon \) which reads \( \varepsilon > 0 \).

Let us now turn to the solution of (14) with the linear SUSY potential (16). With the substitution \( z = -x^2 \) this differential equation can be transformed into that of the confluent hypergeometric function and thus the most general solution reads (\( \alpha, \beta \in \mathbb{R} \) are two additional system parameters)
\[ u(x) = \alpha F_1 \left( \frac{1-\varepsilon}{2}, \frac{1}{2}, -x^2 \right) + \beta x F_1 \left( \frac{1+\varepsilon}{2}, \frac{3}{2}, -x^2 \right) \]
\[ = e^{-x^2} \left[ \alpha F_1 \left( \frac{1}{2}, \frac{1}{2}, x^2 \right) + \beta x F_1 \left( \frac{1+\varepsilon}{2}, \frac{3}{2}, x^2 \right) \right]. \quad (19) \]

As we are searching for strictly positive solutions the real parameter \( \alpha \) must not vanish and thus can be set to unity without loss of generality. In addition the real parameter \( \beta \) has to obey the inequality \( |\beta| < 2 \Gamma \left( 1+\frac{\varepsilon}{2} \right) / \Gamma \left( 1+\varepsilon \right) \) which follows from the positivity condition \( u > 0 \) via the asymptotic form
\[ u(x) = x^{\varepsilon-1} \left( \frac{\Gamma(1/2)}{\Gamma(\varepsilon/2)} + \beta \frac{\Gamma(3/2)}{\Gamma(1+\varepsilon)} \right) \left[ 1 + O(1/x) \right]. \quad (20) \]

Note that for \( \beta = 0 \) the positivity requirement on \( u \) leads to \( \varepsilon > 0 \), a condition already obtained above from the positivity of \( H_+ \). Under these conditions the potential (15) is given by
\[ V_-(x) = \frac{1}{2} x^2 - \varepsilon + \frac{1}{2} + \frac{u'(x)}{u(x)} \left( 2 x + \frac{u'(x)}{u(x)} \right) \]
\[ = \frac{1}{2} x^2 - \frac{1}{2}, \quad (21) \]
which is now a conditionally exactly solvable potential. A plot of this potential for \( 0 < \varepsilon \leq 3 \) and \( \beta = 0 \) is given in Figure 1. For small \( \varepsilon \) and \( \beta = 0 \) the potential \( V_- \) exhibits two deep and one shallow minimum which is located at the origin. In fact, the parameter \( \varepsilon \) is the tunneling splitting due to the tunnel effect between the two deep minima. For large values of \( \varepsilon \) the shallow minimum at the center \( x = 0 \) becomes deeper and the other two minima, which are symmetrically located about the origin, disappear. For non-vanishing \( \beta \) the basic structure of \( V_- \) is the same but now it is no longer symmetric about \( x = 0 \).

The groundstate energy eigenvalue and eigenfunction of \( H_- \) for the above potential (21) are given by
\[ E_0^- = 0, \quad \psi_0^-(x) = C \frac{u(x)}{u(x)} \exp\{-x^2/2\}. \quad (22) \]

Note that because of (20) the above groundstate wavefunction is square integrable and therefore SUSY is unbroken. The remaining spectral properties of \( H_- \) follow
Figure 1: The family (21) of SUSY partner potentials corresponding to the linear harmonic oscillator potential (17). Here we have shown only the symmetric case $\beta = 0$.

from those of $H_+$ via the SUSY transformation (6):

$$E_{n+1}^- = n + \varepsilon, \quad \psi_{n+1}^-(x) = \frac{\exp\{-x^2/2\}}{\sqrt{\pi} 2^{n+1} n! (n + \varepsilon)^{1/2}} \left( H_{n+1}(x) + H_n(x) \frac{u'(x)}{u(x)} \right). \quad (23)$$

Let us also remark that for $\beta = 0$ and an odd integer $\varepsilon = 2N + 1 > 0$ the solution (19) becomes a polynomial in $x^2$ of degree $N$ with no real zeros, that is, $u(x) = (1 + g_1 x^2) \cdots (1 + g_N x^2)$ with $g_i > 0$. These cases, in particular for $N = 1$ and 2, have been discussed in [3]. See, however, also [6] for a different approach to such cases and their connection to non-linear superalgebras.

Let us now turn to the construction of ladder operators for $H_-$. In doing this we first recall the well-known ladder operators for the linear harmonic oscillator $H_+$

$$a = \frac{1}{\sqrt{2}} \left( \frac{d}{dx} + x \right), \quad a^\dagger = \frac{1}{\sqrt{2}} \left( -\frac{d}{dx} + x \right), \quad (24)$$

which obey the linear algebra

$$[H_+, a] = -a, \quad [H_+, a^\dagger] = a^\dagger, \quad [a, a^\dagger] = 1 \quad (25)$$

and act on the eigenstates of $H_+$ as follows

$$a \psi_n^+(x) = \sqrt{n} \psi_{n-1}^+(x), \quad a^\dagger \psi_n^+(x) = \sqrt{n+1} \psi_{n+1}^+(x). \quad (26)$$
With the help of the SUSY operators (13) we can now construct similar ladder operators [3] for the SUSY partner $H_-$:

$$B = A^\dagger a A, \quad B^\dagger = A^\dagger a^\dagger A.$$  \hfill (27)

Obviously, these operators act as lowering and raising operators:

$$B \psi_n^- (x) = \sqrt{E_{n+1}^+ n E_{n+1}^-} \psi_n^- (x), \quad B^\dagger \psi_{n+1}^- (x) = \sqrt{E_{n+1}^+ (n+1) E_{n+1}^-} \psi_{n+2}^- (x).$$  \hfill (28)

However, the ground state remains isolated, that is, $B \psi_0^- (x) = 0 = B^\dagger \psi_0^- (x)$. With these relations one can easily verify that the ladder operators $B$ and $B^\dagger$ close together with the Hamiltonian $H_-$ the non-linear algebra

$$[H_-, B] = -B, \quad [H_-, B^\dagger] = B^\dagger, \quad [B, B^\dagger] = 3H_2^2 - (2\varepsilon - 1)H_-, \quad \hfill (29)$$

which is of quadratic type. Due to unbroken SUSY, i.e. $H_- \psi_0^- (x) = 0$, this algebra is defined on the full Hilbert space $\mathcal{H} = L^2(\mathbb{R})$.

5 The radial harmonic oscillator with unbroken SUSY

As a second example we consider the SUSY potential

$$\Phi(x) = x - \frac{\gamma + 1}{x}, \quad \gamma \geq 0,$$  \hfill (30)

which in turn gives rise to the radial harmonic oscillator potential

$$V_+ (x) = \frac{x^2}{2} + \frac{(\gamma + 1)(\gamma + 2)}{2x^2} + \varepsilon - \gamma - \frac{3}{2}.$$  \hfill (31)

The energy eigenvalues and eigenfunction of the corresponding Hamiltonian $H_+$ are

$$E_n^+ = 2n + 1 + \varepsilon, \quad \psi_n^+ (x) = \left[ \frac{2n!}{\Gamma(n + \gamma + 5/2)} \right]^{1/2} x^{\gamma+2} L_n^{\gamma+3/2}(x^2)e^{-x^2/2},$$  \hfill (32)

where $L_n^\nu$ denotes a Laguerre polynomial of degree $n$ [7]. Positivity of $H_+$ leads to the first restriction, \( \varepsilon > -1 \).

Let us now consider the positive solutions of (14), which reads

$$u(x) = {_1F_1}(\frac{1-\varepsilon}{2}, -\gamma - \frac{1}{2}, -x^2) + \beta x^{2\gamma+3}{_1F_1}(2 + \gamma - \frac{\varepsilon}{2}, \frac{5}{2} - \gamma, -x^2).$$  \hfill (33)

Here we have already set the parameter \( \alpha = 1 \) without loss of generality. Positivity of the above solution amounts in requiring the following conditions on the parameters \( \beta, \gamma \) and \( \varepsilon \):

$$0 < \frac{\Gamma(-\gamma - \frac{1}{2})}{\Gamma(\varepsilon/2 - \gamma - 1)}, \quad |\beta| < \frac{\Gamma(-\gamma - \frac{1}{2})}{\Gamma(\varepsilon/2 - \gamma - 1)} \frac{\Gamma(\frac{1+\varepsilon}{2})}{\Gamma(5/2 + \gamma)}. \hfill (34)$$
Figure 2: A family of SUSY partner potentials (35) corresponding to the radial harmonic oscillator class (31). Here we have only shown the cases $\beta = 0$ and $\gamma = 1$. Note that because of condition (34) the allowed ranges of $\varepsilon$ are $0 < \varepsilon < 2$ and $4 < \varepsilon < \infty$. For the forbidden regions the figure clearly shows singularities in $V_-$. 

The corresponding partner potential reads

$$V_-(x) = \frac{x^2}{2} + \frac{\gamma(\gamma + 1)}{2x^2} - \gamma - \varepsilon + \frac{1}{2} + \frac{u'(x)}{u(x)} \left(2x - 2\frac{\gamma + 1}{x} + \frac{u'(x)}{u(x)} \right)$$

which, because of the above conditions, is now also a conditionally exactly solvable potential. In Figure 2 we have shown this potential for $\beta = 0$, $\gamma = 1$ and $-1 < \varepsilon \leq 8$. Note that for $\varepsilon \leq 0$ and $2 \leq \varepsilon \leq 4$ the potential (35) exhibits singularities as expected because these values of $\varepsilon$ are not allowed for $\gamma = 1$. As SUSY remains unbroken for all the allowed values of the parameters the groundstate energy of the SUSY partner Hamiltonian $H_-$ vanishes and the corresponding eigenstate is obtained from (5). The remaining spectral properties of $H_-$ are found via the SUSY transformations (6):

$$E_0^- = 0, \quad \psi_0^-(x) = \frac{C}{u(x)} x^{\gamma+1} e^{-x^2/2},$$

$$E_{n+1}^- = 2n + 1 + \varepsilon, \quad \psi_{n+1}^-(x) = \frac{1}{\sqrt{4n+2+2\varepsilon}} \left(- \frac{d}{dx} + x - \frac{\gamma + 1}{x} + \frac{u'(x)}{u(x)} \right) \psi_n^+(x).$$

In order to construct the ladder operators for $H_-$ we recall the corresponding operators for the radial harmonic oscillator [7] which in essence are build up from
those given in (24):
\[ c = a^2 - \frac{(\gamma + 1)(\gamma + 2)}{2x^2}, \quad c^\dagger = (a^\dagger)^2 - \frac{(\gamma + 1)(\gamma + 2)}{2x^2}. \] (37)

These operators act on the eigenstates of \( H_+ \) as follows
\[
\begin{align*}
c \psi_n(x) &= -2\sqrt{n(n + \gamma + 3/2)} \psi_{n-1}^+(x), \\
c^\dagger \psi_n^+(x) &= -2\sqrt{(n + 1)(n + \gamma + 5/2)} \psi_{n+1}^+(x).
\end{align*}
\] (38)

and, as in the previous example, close a linear Lie algebra
\[
[H_+, c] = -2c, \quad [H_+, c^\dagger] = 2c^\dagger, \quad [c, c^\dagger] = 4(H_+ + \gamma - \varepsilon + 3/2). \] (39)

Furthermore, they also allow to construct ladder operators for the quantum system characterized by \( H_- \):
\[ D = A^\dagger c A, \quad D^\dagger = A^\dagger c^\dagger A. \] (40)

These operators act on eigenstates of \( H_- \) in the following way:
\[
\begin{align*}
D \psi_{n-1}^-(x) &= -2\sqrt{E_{n-1}^- n(n + \gamma + 3/2) E_{n+1}^-} \psi_n^-(x), \\
D^\dagger \psi_{n+1}^-(x) &= -2\sqrt{E_{n+1}^- (n + 1)(n + \gamma + 5/2) E_{n+2}^-} \psi_{n+2}^-(x), \\
D \psi_0^-(x) &= 0 = D^\dagger \psi_0^-(x).
\end{align*}
\] (41)

The last line shows that the ground state is again isolated, a fact due to unbroken SUSY. From the above relations one verifies that these operators together with the Hamiltonian close the non-linear algebra
\[
\begin{align*}
[H_-, D] &= -2D, \quad [H_-, D^\dagger] = 2D^\dagger, \\
[D, D^\dagger] &= 8H_-^3 + 12(\gamma - \varepsilon + 3/2) H_-^2 - 4(2\varepsilon\gamma - \varepsilon^2 + 3\varepsilon - 1) H_-,
\end{align*}
\] (42)

which is of cubic type.

6 The radial harmonic oscillator with broken SUSY

So far we have considered only examples with unbroken SUSY. However, the radial harmonic oscillator also allows for a broken SUSY. Here in essence the second term in (30) is opposite in sign. Hence, we consider the SUSY potential [1]
\[ \Phi(x) = x + \frac{\gamma + 1}{x}, \quad \gamma \geq 0, \] (43)

which yields the radial harmonic oscillator potential
\[ V_+(x) = \frac{x^2}{2} + \frac{\gamma(\gamma + 1)}{2x^2} + \varepsilon + \gamma + \frac{1}{2} \] (44)
Figure 3: A family of SUSY partner potentials (47) corresponding to the radial harmonic oscillator class (44) with broken SUSY. Here we have only shown the case $\gamma = 1$. Note that the allowed range of $\varepsilon$ is $-4 < \varepsilon$.

and the following spectral properties of the corresponding Hamiltonian $H_+$

$$E^+_n = 2n + 2\gamma + 2 + \varepsilon, \quad \psi^+_n(x) = \left[\frac{2n!}{\Gamma(n + \gamma + 3/2)}\right]^{1/2} x^{\gamma+1} L_n^{\gamma+1/2} (x^2) e^{-x^2/2}. \quad (45)$$

Clearly, we have the condition $-2 - 2\gamma < \varepsilon$. This condition is identical with the one obtained from positivity of the solution of (14)

$$u(x) = 1F_1(\frac{1-\varepsilon}{2}, \gamma + \frac{3}{2}, -x^2). \quad (46)$$

Note that the second linearly independent solution of (14) is not allowed ($\beta = 0$) in order for SUSY to remain broken. The corresponding partner potential reads

$$V_-(x) = \frac{x^2}{2} + \frac{(\gamma + 1)(\gamma + 2)}{2x^2} + \gamma - \varepsilon + \frac{3}{2} + \frac{u'(x)}{u(x)} \left(2x + 2\gamma + 1 + \frac{u'(x)}{u(x)}\right) \quad (47)$$

and the spectral properties of the associated $H_-$ are immediately obtained from (7)

$$E^-_n = 2n + 2\gamma + 2 + \varepsilon, \quad \psi^-_n(x) = \frac{1}{\sqrt{4n + 4\gamma + 4 + 2\varepsilon}} \left(-\frac{d}{dx} + x + \frac{\gamma+1}{x} + \frac{u'(x)}{u(x)}\right) \psi^+_n(x). \quad (48)$$

In Figure 3 we show the potential (47) for $\gamma = 1$ and $-5 \leq \varepsilon \leq 2$. 

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As before, we can introduce ladder operators $D = A^\dagger c A$ and $D^\dagger = A^\dagger c^\dagger A$ which obey the non-linear algebra

\[
[H_-, D] = -2D, \quad [H_-, D^\dagger] = 2D^\dagger, \\
[D, D^\dagger] = 8H^2_3 - 12(\gamma + \varepsilon + 1/2)H^2_2 + 4(2\varepsilon\gamma + \varepsilon^2 + \varepsilon + 1)H_- .
\] (49)

This algebra can also be obtained from the unbroken SUSY case (42) by replacing $\gamma$ by $-\gamma - 2$. However, in contrast to the unbroken case, here the ladder operators act on all eigenstates of $H_-$. In other words, the ground state is not isolated. In fact we have the relation

\[
\psi_n^- (x) = \left( -\frac{1}{4} \right)^n [n! (\gamma + \frac{3}{2})_n (\gamma + 1 + \frac{\varepsilon}{2})_n (\gamma + 2 + \frac{\varepsilon}{2})_n]^{-1/2} (D^\dagger)^n \psi_0^- (x) 
\] (50)

with groundstate wavefunction

\[
\psi_0^- (x) = \frac{x^{\gamma+1} \exp\left\{-x^2/2\right\}}{\sqrt{(2\gamma + \varepsilon + 2)\Gamma(\gamma + \frac{3}{2})}} \left( 2x - \gamma - 1 + \frac{\gamma + 1}{x} + \frac{u'(x)}{u(x)} \right). 
\] (51)

A discussion for the special case $\varepsilon = 3$ and arbitrary $\gamma$ is given in [3].

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