Path integral quantization of electrodynamics in dielectric media

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Abstract

In the present paper we study the Faddeev–Popov path integral quantization of electrodynamics in an inhomogenous dielectric medium. We quantize all polarizations of the photons and introduce the corresponding ghost fields. Using the heat kernel technique, we express the heat kernel coefficients in termini of the dielectricity $\epsilon(x)$ and calculate the ultra violet divergent terms in the effective action. No cancellation between ghosts and "non-physical" degrees of freedom of the photon is observed.

1 Introduction

The Casimir effect describes the forces resulting from the vacuum fluctuations (ground state energy) of the electromagnetic field in simple situations realized by conducting surfaces. These forces can be viewed as retarded Van der Waals forces between the atoms constituting the surfaces (and the bodies behind). As a generalisation of this picture one can consider some medium. It can be characterised either by atoms at positions $x_i$ with their individual polarisabilities $\alpha_i$ or by a macroscopic permittivity $\epsilon(x)$ and permeability $\mu(x)$. Again, we can calculate the resulting potential of the VanderWaals forces or the vacuum energy $E_0[\epsilon(x), \mu(x)]$ of the electromagnetic field in a background given by $\epsilon(x)$ resp. $\mu(x)$. Taking into account that real permittivity resp. permeability are functions of the photon frequency we arrive at the problem to calculate $E_0[\epsilon(x, \omega), \mu(x, \omega)]$. The dependence on $\omega$ has as a physical background, besides others the observation that any medium becomes transparent for $\omega$ sufficiently high (we do not consider inelastic effects here). Therfore $\epsilon, \mu \to 1$ for $\omega \to \infty$ should

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serve as a natural ultraviolet regularisation. This is widely believed, but not shown in a rigorous way yet.

The problem of the calculation of $E_0[\epsilon(x), \mu(x)]$, i.e. without frequency dependence, may be well posed independently. A physical justification could be that the essential contribution results after a proper renormalisation from quite low frequencies $\omega$, where $\epsilon$ and $\mu$ can be viewed as approximately independent on $\omega$. In that case we don’t have a natural regularisation and have to proceed like in the general situation with sharp boundary conditions or a general background field. For technical reasons we use the zeta-functional regularisation. Then the first step is to calculate the divergent contributions (the proper technical tool being the heat kernel expansion), the second is to formulate a model for the interpretation of the renormalisation (this is to be able to reinterpret the subtraction of the divergencies as a renormalization of classical quantities like volume, surface tension etc. as discussed in [1] or like mass and coupling constant of the background field as discussed in [2]) and, finally, in a third step to calculate the renormalised groundstate energy $E_0$. In the present paper we carry out the first step and discuss the second to some extend.

The forces resulting from the electromagnetic vacuum fluctuations in polarisable media have been given much attention to. The common features of these investigations are sharp boundaries separating regions of different values of $\epsilon(x)$ and simple geometries (planes, cylinders and so on). For instance, there have been interesting calculations (mainly with respect to the sign of the force) for a dielectric sphere [3]. Also, much attention had been spent on a possible explanation of sonoluminescence as a dynamical Casimir effect, especially in a series by Schwinger [4]. Recently, the bulk and surface energy contributions had been discussed [5, 6].

However, with respect to the renormalisation it is difficult to deal with sharp boundaries resp. non smooth background fields. It is known that additional contributions to the heat kernel expansion occur and therefor additional counter terms result for which a general theory is still missing. Therefor we restrict ourselves in the present paper to $\epsilon(x)$ which are smooth functions on $x$.

There is still another problem we have to pay attention. In the common understanding the quantisation of QED in media is done in the Coulomb gauge, i.e., the two ‘physical’ polarisations of the photon are quantised. Also there are known procedures where all polarisations of the photon are quantised and the gauge invariance (in the presence of boundaries) is restored by ghosts which have to fulfil boundary conditions too (one of the first is [7], later on it had been discussed in [8]). In most cases their contributions cancel that resulting from the ‘unphysical’ photons, but counter examples are known (e.g. for QED in curved space time, [9]). In the framework of quantum optics the canonical quantization of photons was considered in [11] without, however, analysing the ghost contributions. An alternative approach for quantization in covariant gauge without ghosts, however restricted to sharp boundaries, had been developed in [10].
In the present paper we analyse the problem of QED with a position dependent permittivity $\epsilon(x)$ from the point of view of general quantum gauge theory in an external field. We analyse the canonical path integral measure and corresponding configuration space measure. A gauge fixing term is introduced together with the ghost action. Next we analyse the ultra violet structure of the theory by means of the heat kernel expansion. No cancellation between ghosts and photon modes is obtained.

Our paper is organized as follows. In the next section the quantization of the theory is considered and the path integral is derived. In sec. 3 we use the heat kernel expansion to evaluate ultra violet divergencies. Concluding remarks are given in sec 4. An Appendix contains an alternative calculation to check up the results of sec. 3.

2 Canonical quantization and gauge choice

Consider the action for the electromagnetic field in a dielectric media with permittivity $\epsilon(x)$:

$$ S = \int d^4x \frac{1}{2} (\epsilon(x)E^2 - B^2) $$

To avoid technical complexities we put the permeability $\mu = 1$ and suppose that $\epsilon$ depends on spatial coordinates only.

Let us rewrite the action (1) in the canonical first order form:

$$ S_1 = \int d^4x (P^i \partial_0 A_i + A_0 \partial_i P^i - \frac{1}{\epsilon(x)} P^i P^i - \frac{1}{2} B^2) $$

Here $A_\mu$ is the vector potential. $P^i = -\epsilon(x)E_i$ is the momentum conjugate to $A_i$. Canonical Poisson brackets are

$$ \{ A_i(x, t), P^j(y, t) \} = \delta_i^j \delta(x - y) $$

The same brackets were obtained in [11]. $A_0$ plays the role of a Lagrange multiplier generating the Gauss law constraint, which in turn generates gauge transformations. According to the general method [12] of quantization of gauge theories we can write down the path integral

$$ Z = \int DA_i \epsilon A_0 DP^j J_{FP} \delta(\chi(A_i)) \exp(iS_1), $$

where $\chi(A_i)$ is a gauge fixing condition, $J_{FP}$ is the Faddeev–Popov determinant. $J_{FP} = \det\{ \chi(A), \partial_j P^j \}$. Now we can perform the integration over the momenta $P^j$. It produces the factor $\prod_x \sqrt{\epsilon(x)^3}$ which should be absorbed in the path integral measure $DA_i$. We arrive at the following expression:

$$ Z = \int D\tilde{A}_i \epsilon A_0 J_{FP} \delta(\chi(A)) \exp(iS), \quad \tilde{A}_i = \sqrt{\epsilon} A_i. $$
Our \( \tilde{A} \) variables coincide with the \( q' \) variables of Glauber and Lewenstein [11]. Note that the measure in (5) differs from the naive one \( \prod D A_\mu \). We can use the Faddeev–Popov trick to transform the path integral (5) to whatever gauge condition we prefer, introduce a gauge fixing term and ghost fields. There is nothing specific in this respect in the present model. All steps repeat those of a standard text book [12]. The result is

\[
Z = \int DA_0 \, Dc \, D\bar{c} \exp \left\{ i \int d^4x \left[ \frac{1}{4} (2\epsilon(x)(\partial_0 \epsilon^{-1/2} \tilde{A}_i - \partial_i A_0)^2 
- (\partial_i \epsilon^{-1/2} \tilde{A}_k - \partial_k \epsilon^{-1/2} \tilde{A}_i)^2 + L_{gf} + L_{ghost} \right] \right\} \tag{6}
\]

where \( L_{gf} \) and \( L_{ghost} \) are gauge fixing term and ghost action respectively. As usual, we can bring the action in (6) to the form \( \int A_\mu L_{\mu\nu} A_\nu \), where \( L_{\mu\nu} \) is a second order differential operator. In calculating the effective action and the heat kernel expansion it is much more convenient to deal with operators of Laplace type, i.e. operators with scalar leading symbol. There is a unique gauge choice which splits the \( L_{\mu\nu} \) in a direct sum of operators of Laplace type. This choice is

\[
L_{gf} = -\frac{1}{2} (\epsilon^{-1} \partial_i \epsilon^{1/2} \tilde{A}_i - \epsilon \partial_0 A_0)^2 \tag{7}
\]

\[
L_{ghost} = -\bar{c} (\epsilon^{-1} \partial_i \epsilon \partial_i + \epsilon \partial_0^2) c \tag{8}
\]

The action for the electromagnetic field \( A \) then takes the form

\[
\frac{1}{2} \int d^4x [\epsilon (\partial_i A_0)^2 - \epsilon^2 (\partial_0 A_0)^2 + (\partial_0 \tilde{A}_i)^2
+ \tilde{A}_i \epsilon^{-1/2} (\partial_j^2 \delta_{ik} - e_i \partial_k + \partial_i e_k - e_i e_k) \epsilon^{-1/2} \tilde{A}_k] , \quad e_i = \partial_i \ln \epsilon
\]

Note, that the mixing between \( A_0 \) and \( \tilde{A}_i \) is removed completely.

The total action with gauge fixing and ghost term is invariant under the BRST transformations with the parameter \( \sigma(x) \):

\[
\begin{align*}
\delta A_0 &= \partial_0 \sigma c \\
\delta \tilde{A}_i &= \epsilon^{1/2} \partial_i \sigma c \\
\delta c &= 0 \\
\delta \bar{c} &= (\epsilon^{-1} \partial_i \epsilon^{1/2} \tilde{A}_i + \epsilon \partial_0 A_0) \sigma
\end{align*} \tag{10}
\]

which are given here to complete the picture.

### 3 Effective action and heat kernel expansion

Now we are able to integrate over \( A_0 \), \( \tilde{A} \) and the ghosts. The resulting path integral reads after Wick rotation to the Euclidean domain:

\[
Z = Z[A_0] Z[\tilde{A}] Z[\bar{c}, c] , \tag{11}
\]
where the separate contributions are of the form:

\[
\begin{align*}
Z[A_0] &= \det^{-1/2}(-\partial_i\epsilon\partial_i - \epsilon^2\partial_0^2) \\
Z[\tilde{A}] &= \det^{-1/2}\left(-\frac{1}{\epsilon}\partial_0^2\delta_{ij} - \partial_0^2\delta_{ij} - G_i\partial_j + G_j\partial_i - M_{ij}\right) \\
Z[\tilde{c}, c] &= \det(-\epsilon^{-1}\partial_i\epsilon\partial_i - \epsilon\partial_0^2)
\end{align*}
\]

(12)

We introduced the notations:

\[
\begin{align*}
G_i &= \frac{e_i}{\epsilon} \\
M_{ij} &= \frac{1}{\epsilon}(e_{ij} - e_i e_j) \\
e_{ij} &= \partial_i e_j
\end{align*}
\]

(13)

For the functional determinants we use the integral representation

\[
\log \det(L) = \int_0^\infty \frac{dt}{t} K(L; t)
\]

(14)

where the heat kernel \(K(L; t)\) for a second order elliptic operator \(L\) is

\[
K(L; t) = \text{Tr}\exp(-tL)
\]

(15)

The ultraviolet behavior of functional determinants is given by the asymptotic expansion of the heat kernel (15) as \(t \to +0\). Since all the operators are of Laplace type, we can use the general theory [13]. Each of the operators has the structure

\[
L = -(g^{\mu\nu}\partial_\mu\partial_\nu + a^\sigma\partial_\sigma + b)
\]

(16)

where \(g^{\mu\nu}\) plays the role of a metric. \(a^\sigma\) and \(b\) are local sections of endomorphism \(\text{End}(V)\) of certain vector bundle. By introducing a connection \(\omega_\mu\) in the vector bundle \(V\), one can bring \(L\) to the form:

\[
L = -(g^{\mu\nu}\nabla_\mu \nabla_\nu + E)
\]

(17)

where \(\nabla\) is a sum of the Riemannian covariant derivative with respect to the metric \(g\) and the connection \(\omega\). The explicit form of \(\omega\) and \(E\) is

\[
\begin{align*}
\omega_\delta &= \frac{1}{2}g_{\delta\phi}(a^\nu + g^{\mu\sigma}\Gamma_{\mu\sigma}^\nu) \\
E &= b - g^{\mu\nu}(\partial_\mu \omega_\nu + \omega_\mu \omega_\nu - \omega_\sigma \Gamma_{\mu\nu}^\sigma)
\end{align*}
\]

(18)

As usual, \(\Gamma\) denotes the Christoffel connection.

Given the geometric quantities \(g, \omega\) and \(E\), we are able to calculate the coefficients \(a_n\) of the asymptotic expansion

\[
\text{Tr}(f \exp(-tL)) = t^{-2} \sum_{n=0}^\infty t^n a_n(f, L)
\]

(19)

for a function \(f\). The coefficients \(a_n(f, L)\) contain information on the asymptotics of the heat kernel diagonal \(<x|\exp(-tL)|x>\). The analytical
expressions for the first coefficients are known [13]:

\[ a_0 = \frac{1}{(4\pi)^2} \text{tr}_V \int d^4x g^{1/2} f \]
\[ a_1 = \frac{1}{(4\pi)^2} \text{tr}_V \int d^4x g^{1/2} f(E + \frac{\tau}{6}) \]
\[ a_2 = \frac{1}{(4\pi)^2} \text{tr}_V \int d^4x g^{1/2} \left( \frac{1}{360} f(60E_{i\mu}^\mu + 60\tau E + 180E^2 + 30\Omega_{\mu\nu}\Omega^{\mu\nu} + 12\tau\mu \mu + 5\tau^2 - 2\rho^2 + 2R^2) \right) \]  

(20)

Here \( R, \rho \) and \( \tau \) are Riemann tensor, Ricci tensor and scalar curvature of the metric \( g \) respectively. Semicolon denotes covariant differentiation, \( E_{i\mu} = \nabla_\mu E \). All indices are lowered and raised with the metric tensor, \( \text{tr}_V \) is the bundle (matrix) trace, \( \Omega \) is the field strength of the connection \( \omega \):

\[ \Omega_{\mu\nu} = \partial_\mu \omega_\nu - \partial_\nu \omega_\mu + \omega_\nu \omega_\mu - \omega_\mu \omega_\nu \]  

(21)

The three coefficients (20) are enough to describe the one–loop ultra violet divergencies in four dimensional quantum field theory in an infinite space–time.

Now our problem is reduced to the calculation of the geometric quantities appearing in (20). For the ghost operator we have

\[ g_{ij} = \delta_{ij}, \quad g_{00} = \epsilon^{-1}(x) \]
\[ \Gamma^i_{00} = \frac{1}{2\epsilon} e_i, \quad \Gamma^0_{0i} = -\frac{1}{2} e_i \]
\[ \omega_0 = 0, \quad \omega_i = \frac{3}{4} e_i \]
\[ E = -\frac{3}{4} e_{ii} - \frac{3}{16} e_i e_i \]
\[ R^i_{jkl} = 0 \]
\[ R^0_{i0j} = -\frac{1}{2} e_{ij} + \frac{1}{4} e_i e_j \]
\[ \rho_{ij} = \frac{1}{2} e_{ij} - \frac{1}{4} e_i e_j \]
\[ \rho_{00} = \frac{1}{2\epsilon} (e_{ii} - \frac{1}{2} e_j e_j) \]
\[ \tau = e_{ii} - \frac{1}{2} e_i e_i \]  

(22)

For the operator acting on \( A_0 \) the relevant quantities are:

\[ g_{ij} = \epsilon^{-1}\delta_{ij}, \quad g_{00} = \epsilon^{-2}(x) \]
\[ \Gamma^i_{00} = \frac{1}{\epsilon} e_i, \quad \Gamma^0_{0i} = -e_i \]
\[ \Gamma^k_{ij} = -\frac{1}{2} (e_i \delta_{jk} + e_j \delta_{ik} - e_k \delta_{ij}) \]
\[ \omega_0 = 0, \quad \omega_i = \frac{5}{4} e_i \]
$E = \epsilon \left(-\frac{5}{4}e_{ii} + \frac{5}{16}e_{i}e_{i}\right)$

$R^{i}_{jkl} = \frac{1}{2}(-e_{ij}\delta_{ik} + e_{jk}\delta_{il} + e_{il}\delta_{kj} - e_{ik}\delta_{lj})$

$+\frac{1}{4}(e_{p}e_{p}(\delta_{il}\delta_{ki} - \delta_{jk}\delta_{li}) + e_{k}e_{j}\delta_{il} - e_k e_i \delta_{jl} - e_l e_j \delta_{ki} + e_l e_i \delta_{jk})$

$R^{0}_{0ij} = -e_{ij} + \frac{1}{2}e_{i}e_{j}\delta_{ij}$

$\rho_{ij} = \frac{3}{2}e_{ij} + \frac{1}{2}\delta_{ij}e_{kk} + \frac{1}{4}e_{i}e_{j} - \frac{3}{4}\delta_{ij}e_{k}e_{k}$

$\rho_{00} = \frac{1}{\epsilon}(e_{ii} - \frac{3}{2}e_{j}e_{j})$

$\tau = \epsilon(4e_{ii} - \frac{7}{2}e_{i}e_{i})$

For the operator acting on $\hat{A}$ we obtain:

$g_{ij} = \epsilon\delta_{ij}, \quad g_{00} = 1$

$\Gamma^{k}_{ij} = \frac{1}{2}(e_{i}\delta_{jk} + e_{j}\delta_{ik} - e_{k}\delta_{ij})$

$\omega_{ab}^{i} = \frac{1}{2}(-e_{a}\delta_{bl} + e_{b}\delta_{al} - \frac{1}{2}e_{i}\delta_{ab})$

$E_{ab} = \frac{1}{4\epsilon}(e_{kk}\delta_{ab} + e_{a}e_{b} + \frac{5}{4}e_{p}e_{p}\delta_{ab})$

$R^{i}_{jkl} = \frac{1}{2}(e_{ij}\delta_{ik} - e_{jk}\delta_{il} - e_{il}\delta_{kj} + e_{ik}\delta_{lj})$

$+\frac{1}{4}(e_{p}e_{p}(\delta_{il}\delta_{ki} - \delta_{jk}\delta_{li}) + e_{k}e_{j}\delta_{il} - e_k e_i \delta_{jl} - e_l e_j \delta_{ki} + e_l e_i \delta_{jk})$

$\rho_{jk} = -\frac{1}{\epsilon}(e_{jk} + e_{pp}\delta_{jk}) + \frac{1}{4}(e_{k}e_{j} - e_{p}e_{p}\delta_{kj})$

$\tau = \frac{1}{\epsilon}(-2e_{pp} - \frac{1}{2}e_{p}e_{p})$

Here for convenience we prefer to keep the distinction between coordinate indices $\{i, j, k, l\}$ and bundle indices $\{a, b\}$, though they all run from 1 to 3. In the equations (22) - (24) repeated indices are contracted with the flat space metric $\delta_{ij}$.

It is instructive to express the heat kernel coefficients in terms of $\epsilon$ and its derivatives:

$K_{gh}(f, t) = \frac{1}{(4\pi t)^{2}} \int d^{4}x\epsilon^{-1/2}f\{1 + t\left(-\frac{7}{12}e_{ii} - \frac{13}{48}e_{i}e_{i}\right)$

$+\frac{t^{2}}{360}\left(-33e_{i}e_{jj} - 18e_{i}e_{ij} - 8e_{i}e_{ij} + \frac{237}{4}e_{ii}e_{jj} + \frac{531}{8}e_{ij}e_{i}e_{j} + \frac{33}{4}e_{i}e_{ij}e_{j} + \frac{837}{64}e_{i}e_{i}e_{j}e_{j} + O(t^{3})\right)\}$

$K_{[A_{0}]}(f, t) = \frac{1}{(4\pi t)^{2}} \int d^{4}x\epsilon^{-5/2}f\{1 + te\left(-\frac{7}{12}e_{ii} - \frac{13}{48}e_{i}e_{i}\right)$

$+\frac{t^{2}\epsilon^{2}}{360}\left(-27e_{i}e_{jj} - 60e_{i}e_{ij} - 41e_{i}e_{ij} + \frac{119}{4}e_{ii}e_{jj}\right)\}$
Here $e_{i...j} = \partial_i \ldots \partial_j \ln \epsilon$. This completes the calculation of the UV divergent terms.

We can define a "total" heat kernel as $K_{[A_0]} + K_{[\tilde{A}]} - 2K_{gh}$. We see, that the contribution of ghosts is not cancelled by that of $A_0$ and of the "non–physical" components of $\tilde{A}$.

As a check, in the Appendix we derive (25) by an alternative method.

The asymptotic expansion constructed above gives $2n$ spatial derivatives of $\epsilon$ in any $a_0$. Hence it is clear that certain smoothness of $\epsilon(x)$ is needed. Our expansion is not valid if $\epsilon$ changes abruptly, as e.g. for a bubble in water. For the configurations of latter type boundary terms in the heat kernel expansion should be taken into account.

4 Conclusions and discussion

In the present paper we performed the path integral quantization of electromagnetic fields in a dielectric medium. As a first step, we considered the first order action and derived the canonical Poisson brackets. Next, we constructed the canonical (simplectic) measure in the phase space. We built up a measure in the configuration space by means of an integration over the canonical momenta. This measure appeared to be different from the naive one. By choosing a suitable gauge fixing condition (7) we reduced the path integral to a product of three determinants of operators of Laplace type. For the evaluation of the ultra violet divergent parts of this determinants the standard heat kernel technique [13] is available. Our results are re-checked by another technique (see Appendix). We observed no cancellation of ultra violet divergencies between ghosts and any "non–physical" components of the vector potential. Thus it is highly unlikely that the full quantized electrodynamics in dielectric media is equivalent to a theory where only two polarizations of photons are quantized.

The next step to do is to work out a suitable cut–off procedure for the path integral. This problem is very non–trivial in the present case. Since $\epsilon \rightarrow 1$ at high frequencies, the cut–off is physical, it will not be removed after a renormalization. Therefore, we must be sure that the basic properties of the quantum field theory, as unitarity and absence of gauge anomaly, are valid at finite cut–off. After having solved this problem, it will be possible to consider the vacuum energy densities and other physical quantities of interest.
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Appendix

In this Appendix we describe briefly an alternative method for the evaluation of the heat kernel expansion which we used to control our results.

We can represent the functional trace in the r.h.s. of (15) as an integral over \( x \) of diagonal matrix elements between \( <x| \) and \( |x> \) and insert ”unity” expressed via an integral of momentum eigenstates:

\[
\text{Tr} \exp(-tL) = \int \frac{d^4x d^4k}{(2 \pi)^4} <x| \exp(-tL)|k> <k|x>
\]  

(26)

The generic form of the matrix element in (26) is \( <x| F_1(\epsilon, \partial \epsilon) F_2(\partial) |k> \), where \( F_1 \) and \( F_2 \) are some polynomials of \( \epsilon \) and its derivatives and of \( \partial \) respectively. Acting on the left \( F_1 \) is replaced by its value in the point \( x \). Acting on the right, \( F_2 \) is replaced by \( F_2(i k) \). It is easy to see that the result is

\[
\int \frac{d^4x d^4k}{(2 \pi)^4} \exp(-tL(\epsilon(x), \partial_{\mu} \rightarrow \partial_{\mu} + ik_{\mu}))
\]

(27)

where we should take all external fields in the point \( x \), shift all derivatives by \( ik \), and drive derivatives to the right. It is understood, that \( \partial \) standing at the very right position vanishes.

Consider the heat kernel for the ghost operator:

\[
K_{gh}(t) = \int \frac{d^4x d^4k}{(2 \pi)^4} \exp(t(\partial_j^2 + 2ik_j \partial_j + (\partial_j \log \epsilon) \partial_j + ik_j(\partial_j \log \epsilon) - k^2 - \epsilon \omega^2))
\]

(28)

where \( \{k_\mu\} = \{\omega, k_j\} \). Time derivatives are dropped out because \( \epsilon(x) \) is static.

To obtain a small \( t \) asymptotic expansion of (28), one should isolate the factor \( \exp(-t(k^2 + \epsilon \omega^2)) \) and expand the rest of the expression in a power series of operators and functions involved. Next one should integrate over momenta and collect all terms with the same powers of proper time \( t \). Denote the exponential in (28) as \( \exp(A + B) \) , where \( A = -t(k^2 + \epsilon \omega^2) \).

Note, that \( A \) does not commute with \( B \). However, the repeated commutator \( [[[B, A], A], A] \) vanishes. This allows us to present the exponential as follows (see e.g. [14])

\[
\exp(A + B) = \exp A(1 + B + \frac{1}{2}[B, A] + \frac{1}{6}[[B, A], A] + \frac{1}{2} B^2 + \frac{1}{2}[B, A] B + \frac{1}{6}[B, [B, A]] + \frac{1}{8}[B, A]^2 + \ldots
\]

(29)
We retained all the terms which contribute to the two leading terms of the asymptotic expansion proportional to $t^{-2}$ and $t^{-1}$.

Acting as explained above we obtain the asymptotic expansions for the heat kernels $K_{gh}$, $K_{[A_0]}$ and $K_{[\tilde{A}]}$. The first two terms are in complete agreement with (25). Calculations of the third terms are too complicated to be done just for a control.

References