Ectoplasm Has No Topology: The Prelude$^{1,2}$

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ABSTRACT

Preliminary evidence is presented that a long overlooked and critical element in the fundamental definition of a general theory of integration over curved Wess-Zumino superspace lies with the imposition of “the Ethereal Conjecture” which states the necessity of the superspace to be topologically “close” to its purely bosonic sub-manifold. As a step in proving this, a new theory of integration of closed super p-forms is proposed.

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Presently ‘Salam-Strathdee superspace [?]' is almost universally accepted as the requisite mathematical setting for describing supersymmetric field theories. Even so, there remain a fairly large number of open questions about superspace, particularly with regard to those with large values of \( N (\equiv N_F) \) or \( D (\equiv N_B) \). There are also particularly pointed questions that remain largely unanswer about a general theory of integration on the curved versions of these spaces also known as ‘Wess-Zumino superspace [?].’ To answer some of these questions, extensions such as ‘harmonic superspace’ have been developed especially by the late Dr. Ogievetsky and collaborators. Although my discussion today will only tangentially touch on such constructions, I wish to dedicate this talk to Victor Isaakovich’s memory.

Near the beginning of research using superspace, more mathematically motivated investigators such as Rogers [?] asked a question we may paraphrase as,

“Is it possible to construct a superspace whose topological properties are significantly different from those of its purely bosonic subspace?”

In all cases of interest to physicists to date the answer appears to be, “No!” The emphasis on this negation is mine own because I believe that there is a hidden message in this answer.

In establishing a nomenclature appropriate to researching these issues, one often finds the ‘spiritualist’ denotations (see for example [?])

monomials in \( x \equiv \text{body of the superspace}, \)

monomials in \( x \) and \( \theta \) or purely \( \theta \equiv \text{soul of the superspace.} \)

In deference to this convention, I may call the ‘basic substance’ of which the soul is composed, the “ectoplasm” of superspace.

There is a peculiar sense in which the question of how to construct integration measures over curved superspaces is unanswered. Arnowitt, Nath and Zumino [?] first suggested such integration measures should be written as

\[
\int d\mu \equiv \int d^{N_B+N_F} E^{-1} = \int d^{N_B+N_F} \left[ sdet \left( E_{AB} M(\theta, x) \right) \right]^{-1} .
\]

for a superspace of \( N_B \) bosonic coordinate and \( N_F \) fermionic coordinates.

In principle this is perfectly consistent. In practice, however, for any theory with large \( N_B \) or \( N_F \) (\( N_F = 4 \) is large), this becomes an impractical way to obtain component results in a supergravity theory of ‘physical’ interest. The impracticality arises because the complete \( \theta \)-expansion of the superdeterminant of the inverse vielbein
\[ \text{sdet} \left( \mathbf{\mathcal{M}} (\theta, x) \right)^{-1} \] is complicated to calculate\(^4\). For practical calculations an alternative to the method of Arnowitt, Nath and Zumino is required. To my knowledge, only two such alternatives exist in the literature. They have been discussed in three books listed by authors below.

a. “Covariant Theta Expansion” - Wess & Bagger, [?]  
b. “Density Projectors” - Gates, Grisaru, Roček & Siegel, [?]  
  - Buchbinder & Kuzenko. [?]

I will obviously speak on the second of these because I have recently found increasing and unexpected indications that it is directly connected to more general issues of the calculus and topology of curved supermanifolds with torsion.

I begin by writing the “Ectoplastic Integration Theorem” (or E.I.T.). There should exist an operator \( \mathcal{D}^{NF} \) such that

\[
\int d^{N_B+N_F} z \ E^{-1} \mathcal{L} = \int d^{N_B} z \ e^{-1} \left| \mathcal{D}^{NF} \mathcal{L} \right| ,
\]

independent of the superfield \( \mathcal{L} \) that appears in this equation and where

\[
e^{-1} \equiv \left[ \text{det} \left( e_{\underline{m}}^{\underline{m}}(x) \right) \right]^{-1} , \quad \mathcal{D}^{NF} \mathcal{L} \equiv \lim_{\theta \to 0} \left( \mathcal{D}^{NF} \mathcal{L} \right) .
\]

This theorem is of a similar form to that of the standard Gauss’, Green’s or Stoke’s Theorems of multi-variable calculus. It is different, however, because the operator \( \mathcal{D}^{NF} \) appears on the “wrong” side of the equation from the standard multi-variable calculus analogs. The E.I.T. is also the natural extension of the Berezinian definition of integrating over Grassmann numbers [?].

To see why this is a practical improvement in calculational matters, let me consider the case of flat 4D, \( N = 1 \) superspace where the E.I.T. becomes

\[
\int d^4 x \ d^2 \theta \ d^2 \bar{\theta} \ \mathcal{L} = \frac{1}{2} \left\{ \int d^4 x \left[ \mathcal{D}^2 \mathcal{L} \right] + \text{h.c.} \right\} ,
\]

where

\[
D_\alpha \equiv \partial_\alpha + i \frac{1}{2} \bar{\theta}^\alpha \partial_{\underline{\alpha}} , \quad \bar{D}_{\underline{\alpha}} \equiv \bar{\partial}_{\underline{\alpha}} + i \frac{1}{2} \theta^\alpha \partial_\alpha .
\]

Anyone familiar with rigid supersymmetry can attest to the practical utility of the above equation. For example, if I define \( \mathcal{L} \equiv \mathcal{F} \Phi \) where \( \bar{D}_{\underline{\alpha}} \Phi = 0 \) use the

\(^4\)To my knowledge, this calculation has only been done explicitly by no more than six physicists to this date for 4D, \( N = 1 \) supergravity.
component field definitions \( A(x) \equiv \Phi \), \( \psi_\alpha(x) \equiv D_\alpha \Phi \) and \( F(x) \equiv D^2 \Phi \), apply the E.I.T. and use of the Leibnitz rule for differentiation, it is simple to show

\[
\int d^4 x \ d^2 \theta \ d^2 \bar{\theta} \ \Phi \Phi = \int d^4 x \left[ -\frac{1}{2} (\mathcal{D}^2 A) (\partial_\alpha A) - i \bar{\psi} \gamma^\alpha \partial_\alpha \psi + F \bar{F} \right].
\]  

(6)

No explicit \( \theta \)-expansion was required at any point to derive this component result. Thus, it should be obvious why it is calculationally superior to use the E.I.T. By using techniques that are essentially the same as above, we simple by-pass the need to know the explicit structure of the \( \theta \)-expansion of \( \left[ sdet \left( E_\Delta \mathcal{M}(\theta, x) \right) \right]^{-1} \).

From this viewpoint, the whole problem becomes how to develop a theory for the calculation of the operator \( D^{NF} \) that appears in equation (2). The expression \( e^{-1}[D^{NF} \mathcal{L}] \) is called “the density projection operator” or “density projector” (see ‘Superspace’ [?] or ‘Ideas’ [?] ). It should be clear that this operator, in the general case, can be written as

\[
\int d^{NB} z \ e^{-1} \left[ D^{NF} \mathcal{L} \right] = \int d^{NB} z \ e^{-1} \left[ \sum_{i=0}^{NF} c_{(NF-i)} (\nabla \cdots \nabla)^{NF-i} \mathcal{L} \right],
\]  

(7)

in terms of some field-dependent coefficients \( c_{(NF-i)} \) and powers of the spinorial super-space supergravity covariant derivative \( \nabla_\alpha \). How are these coefficients to be found?

In ‘Superspace’ [?] it was shown that given the local supersymmetry variations of some matter superfield, it is possible to re-construct these coefficients. In ‘Ideas’ [?], it was shown that the density projector follows after solving the constraints to find the basic supergravity pre-potentials. Neither of these approaches is a theory\footnote{We may think of the approach in [?] as a ‘handicraft’ method for summarizing component results. The fact that it was required to go component at all, was equivalent to an admission that we did not have an \textit{a priori} theoretical basis for this result.} for \( D^{NF} \). In the early to middle eighties, Zumino was the first to raise the question of a purely \textit{theoretical} basis for this operator. This bring us to the point of my presentation.

In the rest of my presentation, I will attempt to convince the reader that the answer can be found in the study of super topology similar to the investigations by Rogers. I will argue that local supergravity theories (as a principle) obey what I call “The Ethereal Conjecture” which largely determines the form of \( D^{NF} \).