Representation properties, Racah sum rule and Biedenharn-Elliott identity for $U_q(osp(1|2))$.

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Abstract

It is shown that the universal $R$-matrix in the tensor product of two irreducible representation spaces of the quantum superalgebra $U_q(osp(1|2))$ can be expressed by Clebsch-Gordan coefficients. Racah sum rule satisfied by $U_q(osp(1|2))$ Racah coefficients and 6-$j$ symbols is derived from the properties of the universal $R$-matrix in the tensor product of three representation spaces. Considering the tensor product of four irreducible representations, it is shown that Biedenharn-Elliott identity holds for $U_q(osp(1|2))$ Racah coefficients and 6-$j$ symbols. A recursion relation for $U_q(osp(1|2))$ 6-$j$ symbols is derived from Biedenharn-Elliott identity.

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I. Introduction

In three previous papers, Ref.[1], [2], [3], we have studied the properties of irreducible representations of the quantum superalgebra $U_q(osp(1|2))$. It was shown that it is possible to construct Racah-Wigner calculus for this quantum superalgebra, in a completely similar way as in the classical algebra $su(2)$ [4] and the quantum algebra $U_q(su(2))$ [5], [6], [7] cases. In this paper, in order to complete our study of the representations of this simple quantum superalgebra, we shall consider the properties of its universal $R$-matrix in the tensor product of irreducible representations.

The universal $R$-matrix for $U_q(osp(1|2))$ has been derived and its basic properties have been exhibited in Ref.[8]. It was also considered in the construction of vertex model solutions of the graded Yang-Baxter equation in Ref.[9]. In the present paper we study the properties of the universal $R$-matrix in the tensor product of two irreducible $U_q(osp(1|2))$ representations. We give explicit formulas for the matrix representation of the universal $R$-matrix in different bases of the tensor product of representations. In particular we show that matrix elements of the universal $R$-matrix in the reduced basis of the tensor product of two representations can be expressed with $U_q(osp(1|2))$ Clebsch-Gordan coefficients (denoted $sq$-CGC). Analytic expressions for $sq$-CGC of any tensor product of irreducible representations are known [1], [10], [11], therefore the matrix representation of the universal $R$-matrix can be calculated for any tensor product.

In Ref.[2], we have defined and studied properties of Racah coefficients (denoted $sqRc$) and 6-$j$ symbols (denoted $sq6$-$j$) for the quantum superalgebra $U_q(osp(1|2))$. In particular, it was shown that, as in the classical case, $sq6$-$j$ symbols satisfy not only the usual tetrahedral symmetry but present also an additional symmetry of Regge type. Another property satisfied by $sqRc$ and $sq6$-$j$ symbols is the pseudo-orthogonality relation. This relation can be considered as an algebraic relation satisfied by $sqRc$ and $sq6$-$j$ symbols. It is known that in the classical Racah-Wigner calculus for $su(2)$ or $U_q(su(2))$, Racah coefficients and 6-$j$ symbols satisfy, besides orthogonality relations, other algebraic relations namely Racah sum rule and Biedenharn-Elliott identity. These algebraic identities have been extented to the corresponding features of the superalgebra $osp(1|2)$, Ref.[12]. In this paper we extend Racah sum rule and Biedenharn-Elliott identity for $sqRc$ and $sq6$-$j$ symbols. In both cases, the structure of these relations is completely similar to the corresponding classical ones, the only difference concerns the phases which are more complicated in the case of $U_q(osp(1|2))$. As in the classical cases, we also derive from the Biedenharn-Elliott identity, a three-term recurrence relation between $sq6$-$j$ symbols.

This paper is organized in the following way: in section II we study the properties of the universal $R$-matrix in the tensor product of irreducible representations of $U_q(osp(1|2))$. In section III, considering the tensor product of three $U_q(osp(1|2))$ representations, we derive Racah sum rule for $sqRc$ and $sq6$-$j$ symbols and we derive a simple algebraic identity following from the sum rule. Finally, in section IV, we prove Biedenharn-Elliott identity for $sq6$-$j$ symbols and the three-term recurrence relation.
II. Properties of the universal $R$-matrix in the irreducible representations of the quantum superalgebra $U_q(osp(1|2))$

A. The $U_q(osp(1|2))$ universal $R$-matrix

The quantum superalgebra $U_q(osp(1|2))$ is generated by 4 elements: 1, $H$ (even) and $v_{\pm}$ (odd) with the following (anti)commutation relations

$$[H, v_{\pm}] = \pm \frac{1}{2} v_{\pm}, \quad [v_{+}, v_{-}]_{+} = -\frac{sh(\eta H)}{sh(2\eta)},$$  \hspace{1cm} (2.1)

where the deformation parameter $\eta$ is real and $q = e^{-\frac{i\pi}{2}}$ (we choose $\eta > 0$ so that $q < 1$).

The quantum superalgebra $U_q(osp(1|2))$ is a Hopf algebra with the following coproduct $\Delta^q$ and antipode $S$

$$\Delta^q(v_{\pm}) = v_{\pm} \otimes q^H + q^{-H} \otimes v_{\pm},$$ \hspace{1cm} (2.2)

$$\Delta^q(H) = H \otimes 1 + 1 \otimes H, \quad \Delta^q(1) = 1 \otimes 1,$$ \hspace{1cm} (2.3)

$$S(H) = -H \quad S(v_{\pm}) = -q^{\mp \frac{1}{2}} v_{\pm}.$$ \hspace{1cm} (2.4)

One can consider another coalgebra structure and antipode defined by

$$\Delta'^q(v_{\pm}) = v_{\pm} \otimes q^{-H} + q^H \otimes v_{\pm},$$ \hspace{1cm} (2.5)

$$S'(H) = -H \quad S'(v_{\pm}) = -q^{\mp \frac{1}{2}} v_{\pm}.$$ \hspace{1cm} (2.6)

which, together with relations (2.1), define another Hopf algebra structure that we denote $U'_q(osp(1|2))$. Both Hopf algebra structures are related by

$$U'_q(osp(1|2)) = U_{q-1}(osp(1|2)).$$ \hspace{1cm} (2.7)

It is known [8] that it exists a canonical element $R^q \in U_q(osp(1|2)) \otimes U_q(osp(1|2))$, called the universal $R$-matrix, that defines a similarity relation between $\Delta'^q$ and $\Delta^q$:

$$R^q \Delta' = \Delta^q R^q.$$ \hspace{1cm} (2.8)

This universal $R$-matrix is:

$$R^q = q^{AH \otimes H} \sum (-1)^k q^{\frac{n(n+1)}{4}} \frac{(1+q^{-1})^k}{[k]! \gamma^k} (q^H v_{+})^k \otimes (q^H v_{-})^k,$$ \hspace{1cm} (2.9)

and it satisfies the following relations:

$$R^{13} R^{23} = (\Delta \otimes id)R^q \equiv R^{12,3},$$ \hspace{1cm} (2.10)

$$R^{13} R^{32} = (id \otimes \Delta)R^q \equiv R^{q1,23},$$ \hspace{1cm} (2.11)
where the indices $i, j = 1, 2, 3, \quad i \neq j$, in $R^{ij}$ show the embedding of $R^i = \sum r_n \otimes r_n^i$ into the tensor product $U_q(osp(1|2)) \otimes U_q(osp(1|2)) \otimes U_q(osp(1|2))$. The above relations imply Yang-Baxter equation for the universal $R$-matrix
\[
R^{12} R^{13} R^{23} = R^{23} R^{13} R^{12}. \tag{2.12}
\]
In the following we will consider the properties of the universal $R$-matrix in the tensor product of irreducible representations of the quantum superalgebra $U_q(osp(1|2))$.

## B. The $U_q(osp(1|2))$ finite dimensional representations

A representation of a quantum superalgebra $U_q(osp(1|2))$ in a finite dimensional graded space $V$ is a homomorphism $T : U_q(osp(1|2)) \rightarrow L(V, V)$ of the associative graded algebra $U_q(osp(1|2))$ into the associative graded algebra $L(V, V)$ of linear operators in $V$, such that
\[
[T(H), T(v_\pm)] = \pm \frac{1}{2} T(v_\pm), \quad [T(v_+), T(v_-)]_\pm = -{\frac{s h(\eta T(H))}{s h(2\eta)}}. \tag{2.13}
\]
Let us recall the main results concerning these representations, Ref.[1]. Any finite dimensional grade star representation of $U_q(osp(1|2))$ is characterized by four parameters: the highest weight $l$ (a non negative integer), the parity $\lambda$ of the highest weight vector in the representation space and the signature parameters $\varphi, \psi = 0, 1$ of the Hermitean form in the representation space $V$. The parity $\lambda$ and the signature $\varphi$ define the class $\varepsilon = 0, 1$ of the grade star representation by the relation $\varepsilon = \lambda + \varphi + 1 \pmod{2}$. The irreducible representation space of highest weight $l$, $V = V^l(\lambda)$ is a graded vector space of dimension $2l + 1$ with basis $e^l_m(\lambda)$, where $-l < m < l$, the parameter $\lambda = 0, 1$ is the parity of the highest weight vector $e^l_0(\lambda)$ and $\text{deg}(e^l_m(\lambda)) = l - m + \lambda \pmod{2}$. The vectors $e^l_m(\lambda)$ are constructed from the vector $e^l_0(\lambda)$ in a standard way so they depend on $q$ via the normalisation factor. The vectors $e^l_m(\lambda)$ are pseudo-orthogonal with respect to an Hermitian form in the representation space, denoted $\langle , \rangle$, and their normalization is determined by the signature parameters $\varphi, \psi$:
\[
\langle e^l_m(\lambda), e^l_n(\lambda) \rangle = (-1)^{\varepsilon(l-m)+\psi} \delta_{mn}. \tag{2.14}
\]
The operators $T(v_\pm)$ and $T(H)$ act on the basis $e^l_m(\lambda)$ in the following way:
\[
T(H)e^l_m(\lambda) = \frac{m}{2} e^l_m(\lambda),
\]
\[
T(v_+)e^l_m(\lambda) = (-1)^{l-m} \sqrt{[l-m][l+m+1]} e^l_{m+1}(\lambda), \tag{2.15}
\]
\[
T(v_-)e^l_m(\lambda) = \sqrt{[l+m][l-m+1]} e^l_{m-1}(\lambda),
\]
where the symbol $[n]$ is the graded quantum symbol defined by
\[
[n] = q^{-\frac{n}{2}} - (-1)^{n} q^{\frac{n}{2}} \frac{q^{-\frac{n}{2}}}{q^{\frac{n}{2}} + q^{\frac{1}{2}}}, \tag{2.16}
\]

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and \( \gamma = \frac{\cosh(\frac{\eta}{2})}{\sinh(2\eta)} \). Note that although the symbol \([n]\) is not invariant under the exchange \( q \leftrightarrow q^{-1} \), the relations (2.15) are invariant with respect to this substitution. Furthermore, it follows from the explicit formulas for \( e_{m}^{lq}(\lambda) \), Ref. [11] that
\[
e_{m}^{lq}(\lambda) = e_{m}^{lq^{-1}}(\lambda),
\]
thus both quantum algebras \( U_{q}(osp(1|2)) \) and \( U_{q^{-1}}(osp(1|2)) \) define the same basis in the representation space \( V^{l}(\lambda) \). In the next subsection, we shall see that this is no more true when we consider tensor product of representation spaces.

### C. Tensor product of representations and the \( T \)-matrix

In the following, the representation \( T \) of class \( \varepsilon \) which acts in the representation space \( V^{l}(\lambda) \) is denoted by \( T^{l} \). The generators \( v_{\pm} \) and \( H \) are represented in the tensor product of two irreducible representations of the same class, \( T^{l_{1}} \otimes T^{l_{2}} \) by the operators
\[
v_{\pm}^{\otimes}(1, 2) = (T^{l_{1}} \otimes T^{l_{2}})(\Delta(v_{\pm})) = T^{l_{1}}(v_{\pm}) \otimes q^{T^{l_{2}}(H)} + q^{-T^{l_{1}}(H)} \otimes T^{l_{2}}(v_{\pm}),
\]
\[
H^{\otimes}(1, 2) = (T^{l_{1}} \otimes T^{l_{2}})(\Delta(H)) = T^{l_{1}}(H) \otimes T^{l_{2}}(1) + T^{l_{1}}(1) \otimes T^{l_{2}}(H).
\]
The tensor product \( T^{l_{1}} \otimes T^{l_{2}} \) is simply reducible. The standard basis \( e_{m_{1}}^{l_{1}q}(\lambda_{1}) \otimes e_{m_{2}}^{l_{2}q}(\lambda_{2}) \) and reduced basis \( e^{l_{1}q}_{m}(l_{1}, l_{2}, \lambda) \) of the representation space \( V^{l_{1}}(\lambda_{1}) \otimes V^{l_{2}}(\lambda_{2}) \) are related by Clebsch-Gordan coefficients in the following way:
\[
e^{l_{1}q}_{m}(l_{1}, l_{2}, \lambda) = \sum_{m_{1}, m_{2}}(l_{1}m_{1}\lambda_{1}, l_{2}m_{2}\lambda_{2}|lm\lambda)_{q} e^{l_{1}q}_{m_{1}}(\lambda_{1}) \otimes e^{l_{2}q}_{m_{2}}(\lambda_{2}),
\]
or equivalently,
\[
\sum_{l, m}(-1)^{(l-m)L}(l_{1}m_{1}\lambda_{1}, l_{2}m_{2}\lambda_{2}|lm\lambda)_{q} e^{l_{1}q}_{m_{1}}(l_{1}, l_{2}, \lambda) = (-1)^{(l_{1}-m_{1})(l_{2}-m_{2})} e^{l_{1}q}_{m_{1}}(\lambda_{1}) \otimes e^{l_{2}q}_{m_{2}}(\lambda_{2}),
\]
where \( m_{1} + m_{2} = m \), \( L = l_{1} + l_{2} + l \), and \( l \) is an integer satisfying the conditions
\[
|l_{1} - l_{2}| \leq l \leq l_{1} + l_{2}, \quad \lambda = L + \lambda_{1} + \lambda_{2} \quad (\text{mod } 2).
\]
Both bases \( e^{l_{1}q}_{m_{1}}(\lambda_{1}) \otimes e^{l_{2}q}_{m_{2}}(\lambda_{2}) \) and \( e^{l_{1}q}_{m}(l_{1}, l_{2}, \lambda) \) are orthogonal but not positive definite, i.e., we have
\[
(e^{l_{1}q}_{m_{1}}(\lambda_{1}) \otimes e^{l_{2}q}_{m_{2}}(\lambda_{2}), e^{l_{1}q}_{m_{1}}(\lambda_{1}) \otimes e^{l_{2}q}_{m_{2}}(\lambda_{2})) = (-1)^{(l_{1}-m_{1}+\lambda_{1})(l_{2}-m_{2}+\lambda_{2})} \delta_{m_{1}m_{1}} \delta_{m_{2}m_{2}},
\]
\[
(e_{m}^{l_{1}q}(l_{1}, l_{2}, \lambda), e_{m'}^{l_{1}q}(l_{1}, l_{2}, \lambda)) = \delta_{ll'} \delta_{mm'}(-1)^{\varphi(l-m)+\psi},
\]
where
\[
\varphi = L + \lambda_{1} + \varphi_{2} \quad (\text{mod } 2), \quad \psi = (L + \lambda_{2})\lambda_{1} + \varphi_{2}L + \psi_{1} + \psi_{2} \quad (\text{mod } 2),
\]
We may consider also another reduced basis built with the basis $e^{l_2}_{m_2}^{-1}(\lambda)$:

\begin{equation}
    e^{l_2}_{m_2}^{-1}(l_1, l_2, \lambda) = \sum_{m_1, m_2} (l_1 m_1 \lambda_1, l_2 m_2 \lambda_2 | l m \lambda)_{q^{-1}} e^{l_1}_{m_1}^{-1}(\lambda_1) \otimes e^{l_2}_{m_2}^{-1}(\lambda_2),
\end{equation}

\begin{equation}
    = \sum_{m_1, m_2} (l_1 m_1 \lambda_1, l_2 m_2 \lambda_2 | l m \lambda)_{q^{-1}} e^{l_1}_{m_1}(\lambda_1) \otimes e^{l_2}_{m_2}(\lambda_2),
\end{equation}

where the second line is deduced from Eq.(2.17). This basis is normalized in the same way (2.24) as the basis $e^{l_2}_{m_2}(l_1, l_2, \lambda)$, but the two reduced bases are different because we have in general

\begin{equation}
    (l_1 m_1 \lambda_1, l_2 m_2 \lambda_2 | l m \lambda)_{q^{-1}} \neq (l_1 m_1 \lambda_1, l_2 m_2 \lambda_2 | l m \lambda)_{q}.
\end{equation}

Thus, there exist an automorphism $T^{q}(l_1, l_2)$ of the space $V^{l_1}(\lambda_1) \otimes V^{l_2}(\lambda_2)$ that relates both bases:

\[ T^{q}(l_1, l_2)(e^{l_2}_{m_2}(l_1, l_2, \lambda)) = e^{l_2}_{m_2}^{-1}(l_1, l_2, \lambda). \]

Using the orthogonality relations (2.20), (2.21), one can show that this operator has the following matrix form in the basis $e^{l_2}_{m_2}(l_1, l_2, \lambda)$:

\[ [T^{q}(l_1, l_2)]_{l' m', l m} = (-1)^{(l_1+l_2+l'+l'+m')(l'-m') \sum_{m_1, m_2} (-1)^{(l_1-m_1)(l_2-m_2)} \times (l_1 m_1 \lambda_1, l_2 m_2 \lambda_2 | l' m' \lambda')_{q} (l_1 m_1 \lambda_1, l_2 m_2 \lambda_2 | l m \lambda)_{q^{-1}}. \]

From this expression it follows that $[T^{q}(l_1, l_2)]^{-1} = T^{q^{-1}(l_1, l_2)}$. Note also that the matrix $T^{q}(l_1, l_2)$ is diagonal in the indices $m', m$ but it is not diagonal in the indices $l', l$. The symmetries of Clebsch-Gordan coefficients imply the following properties for the matrix elements $[T^{q}(l_1, l_2)]_{l' m', l m}$:

\begin{equation}
    [T^{q}(l_1, l_2)]_{l m, l' m'} = (-1)^{(l_1+l_2+l'+l+m)(l'-l)} [T^{q^{-1}(l_1, l_2)}]_{l' m', l m},
\end{equation}

\begin{equation}
    [T^{q^{-1}}(l_1, l_2)]_{l' m', l m} = (-1)^{m(l'+l)}(-1)^{\frac{1}{2}l(l+1)+\frac{1}{2}l'(l'-1)} [T^{q}(l_1, l_2)]_{l' m', l m},
\end{equation}

\begin{equation}
    [T^{q}(l_2, l_1)]_{l' m', l m} = (-1)^{(l_1+l_2+l_1+l_2)(l'+l)}(-1)^{\frac{1}{2}l(l+1)+\frac{1}{2}l'(l'-1)} [T^{q^{-1}(l_1, l_2)}]_{l' m', l m}. \quad (2.32)
\end{equation}

D. The permutation operator $\tau$ and the universal $R$-matrix

We shall now consider the tensor product of the previous irreducible representations in reverse order, namely $T^{l_2} \otimes T^{l_1}$. For this we introduce the permutation operator $\tau$ acting in any graded tensor product space as follows

\[ \tau(a \otimes b) = (-1)^{\deg(a) \deg(b)}(b \otimes a). \]

Therefore, for the quantum superalgebra $U_{q}(osp(1|2))$ we have

\begin{equation}
    \tau \Delta^{q}(v_{\pm}) = \Delta^{q^{-1}}(v_{\pm}) = R^{\delta} \Delta^{q}(v_{\pm})(R^{\delta})^{-1}, \quad (2.34)
\end{equation}

\begin{equation}
    \tau \Delta^{q}(H) = \Delta^{q^{-1}}(H) = R^{\delta} \Delta^{q}(H)(R^{\delta})^{-1}, \quad (2.35)
\end{equation}

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which means that the action of the operator \( \tau \) for the coproduct of generators of \( \mathcal{U}_q(\text{osp}(1|2)) \) coincides with the action of the universal \( R \)-matrix. The action of the operator \( \tau \) in the tensor product of representations \( T^{l_1} \otimes T^{l_2} \) reverse the order of the tensor product and we have

\[
\tau(e^{l_1\lambda_1}_{m_1}(\lambda_1) \otimes e^{l_2\lambda_2}_{m_2}(\lambda_2)) = (-1)^{(l_1-m_1+\lambda_1)(l_2-m_2+\lambda_2)} e^{l_2\lambda_2}_{m_2}(\lambda_2) \otimes e^{l_1\lambda_1}_{m_1}(\lambda_1),
\]

and

\[
\tau v^{\otimes q}_{\pm}(l_1, l_2) \tau^{-1} = v^{\otimes q^{-1}}_{\pm}(l_2, l_1),
\]

\[
\tau H^{\otimes q}_{\pm}(l_1, l_2) \tau^{-1} = H^{\otimes q^{-1}}_{\pm}(l_2, l_1).
\]

Using the following symmetry property of Clebsch-Gordan coefficients [1]

\[
(l_1m_1\lambda_1, l_2m_2\lambda_2|lm\lambda)_q = (-1)^{(l_1-m_1+\lambda_1)(l_2-m_2+\lambda_2)}(-1)^{(l_1+l_2+l)(\lambda_1+\lambda_2)+\lambda_1\lambda_2} \times (-1)^{\frac{1}{2}(l_1+l_2-l)(l_1+l_2-l+l)}(l_2m_2\lambda_2, l_1m_1\lambda_1|lm\lambda)_{q^{-1}},
\]

one can check that the action of \( \tau \) on the reduced basis reads

\[
\tau(e^{l\lambda}_{m}(l_1, l_2, \lambda)) = (-1)^{(l_1+l_2+l+\lambda_1)(l_1+l_2+l+\lambda_2)}(-1)^{\frac{1}{2}(l_1+l_2-l)(l_1+l_2-l-l-1)} e^{l^{-1}\lambda}_{m^{-1}}(l_2, l_1, \lambda).
\]

Thus the operator \( \tau \) that relates the bases \( e^{l\lambda}_{m}(l_1, l_2, \lambda) \) and \( e^{l^{-1}\lambda}_{m^{-1}}(l_2, l_1, \lambda) \) is a representation isomorphism:

\[
\tau : (T^{l_1} \otimes T^{l_2}) \mathcal{U}_q(\text{osp}(1|2)) \rightarrow (T^{l_2} \otimes T^{l_1}) \mathcal{U}_{q^{-1}}(\text{osp}(1|2)).
\]

The operator \( R^q \) in the representation \( T^{l_1} \otimes T^{l_2} \) will be denoted \( R^q(l_1, l_2) \equiv R^q(1, 2) \). In such a representation its action is

\[
R^q(1, 2) v^{\otimes q}_{\pm}(l_1, l_2) R^q(1, 2)^{-1} = v^{\otimes q^{-1}}_{\pm}(l_1, l_2),
\]

\[
R^q(1, 2) H^{\otimes q}_{\pm}(l_1, l_2) R^q(1, 2)^{-1} = H^{\otimes q^{-1}}_{\pm}(l_1, l_2).
\]

Since \( R^q(1, 2) \) does not exchange the two spaces, in this case the action of the operator \( \tau \) does not coincide with the action of \( R^q(1, 2) \).

### D.1. Matrix elements of \( R^q(1, 2) \) in the tensor product basis

Using relations (2.15) one can show that, in the standard basis \( e^{l\lambda}_{m}(\lambda_1) \otimes e^{l\lambda}_{m}(\lambda_2) \), the operator \( R^q(1, 2) \) has the following matrix form

\[
[R^q(1, 2)]_{m_1m_2n_1n_2} = (-1)^{k(\lambda_1+1)+\frac{1}{2}k(k-1)} q^{\frac{1}{2}k(3k+1)} q^{\frac{1}{2}k(n_1-n_2)+(n_1+k)(n_2-k)} \frac{(1+q^{-1})^k}{|k|!} \\
\times \left[ \frac{l_1-n_1!}{l_1+n_1+k!} \frac{l_2+n_2!}{l_2-n_2+k!} \right]^{\frac{1}{2}} \delta_{m_1, n_1+k} \delta_{m_2, n_2-k}.
\]
For the simplest case when \( l_1 = l_2 = 1 \), \( R^q(1,2)_{m_1m_2,n_1n_2} \) has the following form

\[
[R^q(1,1)]_{m_1m_2,n_1n_2} = \begin{pmatrix}
q & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & a & 0 & 0 & 0 & 0 \\
0 & 0 & q^{-1} & 0 & b & 0 & e & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & c & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & d \\
0 & 0 & 0 & 0 & 0 & 0 & q^{-1} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & q
\end{pmatrix},
\] (2.45)

where

\[
a = d = (-1)^{l_1}(q - q^{-1}), \quad c = b = q^{-\frac{1}{2}}a, \quad e = (-1)^{l_1}(1 + q^{-1})a.
\] (2.46)

When \( \lambda_1 = 0 \), this matrix coincides with the corresponding matrix given in Ref.[8].

**D.2. Matrix elements of \( R^q(1,2) \) and \( R^{-1,q}(1,2) \) in the reduced bases**

Relations (2.42), (2.43) suggest that the vector \( e^{l,q-1}_{m}(l_1,l_2,\lambda) \) is proportional to the vector \( R^q(1,2)e^{l,q}_{m}(l_1,l_2,\lambda) \). In the appendix we prove that indeed we have

\[
R^q(1,2)e^{l,q}_{m}(l_1,l_2,\lambda) = q^{\frac{1}{2}(l(l+1)-l_1(l_1+1)-l_2(l_2+1))}e^{l,q-1}_{m}(l_1,l_2,\lambda).
\] (2.47)

From this relation it follows that

\[
[R^q(1,2)]_{-1}^{-1} = [R^{-1,q}(1,2)].
\] (2.48)

Furthermore, formula (2.47) allows one to calculate the matrix elements of the operators \( R^q(1,2) \) and \( R^{-1,q}(1,2) \) in the reduced bases \( e^{l,q}_{m}(l_1,l_2,\lambda) \) and \( e^{l,q-1}_{m}(l_1,l_2,\lambda) \). Let us introduce the following notation:

- \( \mathbb{R}_q \) (\( \mathbb{R}_q^{-1} \)) is the matrix of the operator \( R^q(1,2) \) (\( R^{-1,q}(1,2) \)) in the basis \( e^{l,q}_{m}(l_1,l_2,\lambda) \),
- \( \mathbb{R}_e \) (\( \mathbb{R}_e^{-1} \)) is the matrix of the operator \( R^q(1,2) \) (\( R^{-1,q}(1,2) \)) in the basis \( e^{l,q-1}_{m}(l_1,l_2,\lambda) \),

and let \( D^q \) be a diagonal matrix of the form

\[
[D^q]_{l,m,l'm'} = q^{\frac{1}{2}(l(l+1)-l_1(l_1+1)-l_2(l_2+1))}\delta_{l,l'}\delta_{m,m'}.
\] (2.49)

Using relation (2.47) and the orthogonality relations (2.20), (2.21) for \( sq \)-CGc, one can derive the following matrix relations:

\[
\mathbb{R}_q = T^q D^q, \quad \mathbb{R}_e^{-1} = D^{-1,q} T^{-1,q},
\] (2.50)

\[
\mathbb{R}_e = D^q T^q, \quad \mathbb{R}_e^{-1} = T^{-1,q} D^{-1,q}.
\] (2.51)
where $T^q$ is the matrix given by the formula (2.29). In particular, for the first relation we have

$$[\mathcal{R}^q_{\ell,l';m',lm}] = q^{\frac{1}{2}(l(l+1)-l_l(l_h+1)-l_l(l_h+1)-1)}(l_1 + l_2 + l')^{(l'-m')} \times \sum_{m_1 m_2} (-1)^{(l_1 - m_1)(l_2 - m_2)}(l_1 m_1 \lambda_1, l_2 m_2 \lambda_2 | l' m' \lambda') q(l_1 m_1 \lambda_1, l_2 m_2 \lambda_2 | l m \lambda)_{q^{-1}}. \quad (2.52)$$

i.e., the matrix elements of the operator $R^q(1,2)$ can be expressed in terms of $sq$-CGc. The values of the $sq$-CGc for arbitrary tensor product of irreducible representations can be calculated from the analytical expression given in Ref. [2], [10], [11], therefore the matrix elements of $R^q(1,2)$ in any tensor product of irreducible representations can be deduced from formula (2.52). For example, calculating the values of corresponding $sq$-CGc for $l_1 = l_2 = 1$, we get from formula (2.52)

$$[\mathcal{R}^q_{l,l';m',lm}] = \begin{pmatrix}
q^{-2} a & 0 & 0 & 0 & 0 & 0 & qc & 0 & 0 \\
0 & q^{-1} b & 0 & 0 & 0 & qd & 0 & 0 & 0 \\
0 & 0 & q^{-1} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & q^{-1} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & q & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & q & 0 & 0 & 0 \\
0 & -q^{-1} d & 0 & 0 & 0 & 0 & q & 0 & 0 \\
0 & 0 & 0 & q^{-1} d & 0 & 0 & 0 & q & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & q
\end{pmatrix}, \quad (2.53)$$

where

$$a = \frac{q}{q^2 - q + 1}, \quad b = \frac{2q}{q^2 + 1}, \quad c = \frac{q^3 - q^2 + q - 1}{\sqrt{q^2 + 1(q^2 - q + 1)}}, \quad d = (-1)^{l_1} q^2 - 1 \frac{1}{q^2 + 1}. \quad (2.54)$$

From expressions (2.42), (2.43), (2.47), it follows that the operator $R^q(1,2)$ yields an isomorphism of two representations

$$R^q(1,2) : (T^{i_1} \otimes T^{i_2})(U(q(osp(1|2)))) \rightarrow (T^{i_1} \otimes T^{i_2})(U_{q^{-1}}(osp(1|2))). \quad (2.55)$$

E. The operator $R^q$

Finally, in the following we will use the operator $R^q$ that is a composition of the operators $\tau$ and $R^q$

$$R^q = \tau \circ R^q. \quad (2.56)$$

Properties of the operators $\tau$ and $R^q$, in particular Yang-Baxter equation (2.12) satisfied by $R^q$, implies that $R^q$ satisfies the following relation

$$([R^q \otimes id] (id \otimes R^q)) ([R^q \otimes id] (R^q \otimes id)) = (id \otimes [R^q \otimes id]) (id \otimes [R^q \otimes id]). \quad (2.57)$$
Further, from relations (2.10), we get the relations
\[ (\mathcal{R}^q \otimes \text{id}) (\text{id} \otimes \mathcal{R}^q) = \tau(12, 3) R^{12,3} = \mathcal{R}^{q12,3}, \] (2.58)
\[ (\text{id} \otimes \mathcal{R}^q) (\mathcal{R}^q \otimes \text{id}) = \tau(1, 23) R^{12,23} = \mathcal{R}^{q1,23}, \] (2.59)
where
\[ \tau(12, 3) = (\tau \otimes \text{id})(\text{id} \otimes \tau), \quad \tau(1, 23) = (\text{id} \otimes \tau)(\tau \otimes \text{id}). \] (2.60)
In the tensor product \( T^1 \otimes T^2 \) the operator \( \mathcal{R}^q \) is represented by the operator
\[ \mathcal{R}^q(1, 2) = \tau \circ \mathcal{R}^q(1, 2) \]
and formulas (2.58), (2.59) take the form
\[ (\mathcal{R}^q(1, 3) \otimes \text{id}(2)) (\text{id}(1) \otimes \mathcal{R}^q(2, 3)) = \mathcal{R}^{q12,3}(12, 3), \] (2.61)
\[ (\text{id}(2) \otimes \mathcal{R}^q(1, 3)) (\mathcal{R}^q(1, 2) \otimes \text{id}(3)) = \mathcal{R}^{q1,23}(1, 23). \] (2.62)
Using relations (2.40), (2.47) we derive immediately the action of \( \mathcal{R}^q(1, 2) \) on the reduced basis
\[ \mathcal{R}^q(1, 2) e_{m}^{l}(l_1, l_2, \lambda) = (-1)^{(l_1+l_2+(\lambda_1+\lambda_2))(l_1+l_2+(\lambda_1+\lambda_2))} \times (-1)^{(l_1+l_2-1)(l_1+l_2-l-1)} \times q^{l_2(l_1+1)-l_1(l_1+1)-l_2(l_2+1)} e_{m}^{l}(l_2, l_1, \lambda), \] (2.63)
and its action on the operators \( v^q_{\pm}(l_1, l_2), H^q(l_1, l_2) \) reads
\[ \mathcal{R}^q(1, 2) v^q_{\pm}(l_1, l_2) \mathcal{R}^q(1, 2)^{-1} = v^q_{\pm}(l_2, l_1), \] (2.64)
\[ \mathcal{R}^q(1, 2) H^q(l_1, l_2) \mathcal{R}^q(1, 2)^{-1} = H^q(l_2, l_1), \] (2.65)
where
\[ \mathcal{R}^q(1, 2)^{-1} = R^{-1}(1, 2) \circ \tau. \] (2.66)
The operator \( \mathcal{R}^q(1, 2) \) defines then an isomorphism of representations
\[ \mathcal{R}^q(1, 2) : (T^1 \otimes T^2) (U_q(osp(1|2))) \to (T^2 \otimes T^1) (U_q(osp(1|2))). \] (2.67)
The universal \( R \)-matrix is the appropriate tool for obtaining numerous relations between Racah coefficients and \( sg6-j \) symbols. In the following section we shall first use it to derive Racah sum rule for \( sg6-j \) symbols.
III. Racah sum rule for the $sq - 6j$ symbols.

In order to derive Racah sum rule, we will consider the action of the operator $R^3$ on the reduced bases of the tensor product of three irreducible representations of the same class $T^{i_1} \otimes T^{i_2} \otimes T^{i_3}$. The operators $H, v_\pm$ are represented in this tensor product by

$$v_\pm^\otimes (1, 2, 3) = (T^{i_1} \otimes T^{i_2} \otimes T^{i_3})((\Delta \otimes id)\Delta(v_\pm)),$$

$$H^\otimes (1, 2, 3) = (T^{i_1} \otimes T^{i_2} \otimes T^{i_3})((\Delta \otimes id)\Delta(H)).$$

The reduction of the tensor product $V^{i_1}(\lambda_1) \otimes V^{i_2}(\lambda_2) \otimes V^{i_3}(\lambda_3)$ of representation spaces can be done, as in the classical case, in two different schemes:

$$T^l \subset ((T^{i_1} \otimes T^{i_2}) \otimes T^{i_3}), \quad T^l \subset (T^{i_1} \otimes (T^{i_2} \otimes T^{i_3})).$$

The bases $e^l_m(l_1, l_2, 3, \lambda)$ and $e^l_m(l_1, l_3, \lambda)$ corresponding to these reduction schemes are orthogonal and normalized in the following way

$$(e^l_m(l_1, l_2, 3, \lambda), e^{l'}_{m'}(l_1', l_2, 3, \lambda)) = (-1)^{\varphi_{123}(l-\lambda_1)+\psi_{123}} \delta_{ll'} \delta_{mm'} \delta_{l_1 l_1'} \delta_{l_2 l_2'} = g_{l_1 l_2 m_1 m_2},$$

$$(e^l_m(l_1, l_2, 3, \lambda), e^{l'}_{m'}(l_1, l_3, \lambda)) = (-1)^{\varphi_{123}(l-\lambda_1)+\psi_{123}} \delta_{ll'} \delta_{mm'} \delta_{l_2 l_3} \delta_{l_3 l_3'} = g_{l_2 l_3 m_2 m_3'},$$

where

$$\varphi_{123} = \varphi_{123} = \lambda_1 + \lambda_2 + \varphi_3 \pmod{2},$$

$$\psi_{123} = (l_1 + l_2 + l_1)(\lambda + 1) + (\lambda_1 + \lambda_3 + \varphi_2)\lambda + \sum_{i<j} \lambda_i \lambda_j + \sum_{i=1}^{3} \psi_i \pmod{2},$$

$$\psi_{123} = (l_2 + l_3 + l_2)(\lambda + 1) + (\lambda_1 + \lambda_3 + \varphi_2)\lambda + \sum_{i<j} \lambda_i \lambda_j + \sum_{i=1}^{3} \psi_i \pmod{2},$$

with

$$\lambda = \sum_{i=1}^{3} (\lambda_i + l_i) + l.$$

The $sqRacah$ Coefficients $U^s(l_1, l_2, l_3, l, l_12, l_23, q)$ of the quantum superalgebra $U_q(osp(1|2))$ are defined, in the standard way, as the coefficients that relate reduced bases in two different reduction schemes [2]

$$e^l_m(l_1, l_3, \lambda) = \sum_{l_23} (-1)^{(l_1 + l_3 + l_23)(\lambda + 1)} U^s(l_1, l_2, l_3, l, l_12, l_23, q) e^l_m(l_1, l_23, \lambda),$$

or equivalently

$$e^l_m(l_1, l_23, \lambda) = \sum_{l_12} (-1)^{(l_1 + l_2 + l_12)(\lambda + 1)} U^s(l_1, l_2, l_3, l, l_12, l_23, q) e^l_m(l_1, l_2, \lambda).$$
We have the following relation between $sqRc$ and $sq6-j$ symbols [2]

\[
U^q(l_1, l_2, l_3, l_1, l_{12}, l_{23}, q) = (-1)^{(l_2-1)(l_2+l_2+12)}(1)_{(l_1+l_2+12)}(l_1-1)(l_1+l_2+l_3)(l_1-1)(l_1+l_2+12) \\
\times (-1)^{(l_2+l_2+12+l_2)}(1)_{(l_2+l_2+12+1)} q^{\frac{l(l+1)}{2}} \sqrt{2l_{12}+1}\sqrt{2l_{23}+1} \left\{ \begin{array}{ccc}
  l_1 & l_2 & l_{12} \\
  l_3 & l & l_{23} \\
\end{array} \right\}_q.
\]

(3.12)

Let us consider the matrix $W_{lm'}$ of the operator $id(1) \otimes R^q(3,2)$ in the bases $e^q_m(l_{12}, l_2, \lambda)$ and $e^q_m(l_{13}, l_2, \lambda)$

\[
W_{lm',lm} = g_{ll'}_m(l_{12}, l_2, \lambda, e^q_m(l_{13}, l_2, \lambda), id(1) \otimes R^q(3,2), e^q_m(l_{13}, l_2, \lambda))
\]

(3.13)

Using relations (2.20), (3.10), (2.63), one obtains

\[
\begin{align*}
& id(1) \otimes R^q(3,2), e^q_m(l_{13}, l_2, \lambda) \\
& = \sum_{l_{13}, l_{23}} (-1)^{(l_1+l_3+l_{12}+l_{23})}(-1)^{(l_2+l_3+l_{23}+\lambda_2)}(1)_{(l_2+l_3+l_{23}+\lambda_2)} \\
& \times (-1)^{(l_2+l_3-l_{23})(l_2+l_3-l_{23})} q^{\frac{l(l+1)}{2}} U^q(l_1, l_2, l_3, l_1, l_{12}, l_{23}, q) U^q(l_1, l_2, l_3, l_1, l_{12}, l_{23}, q) e^q_m(l_{13}, l_2, \lambda)
\end{align*}
\]

(3.14)

Therefore, the matrix $W_{lm',lm}$ has the form

\[
W_{lm',lm} = \sum_{l_{23}} (-1)^{(l_1+l_3+l_{12}+l_{23})}(-1)^{(l_2+l_3+l_{23}+\lambda_2)}(1)_{(l_2+l_3+l_{23}+\lambda_2)} \\
\times (-1)^{(l_2+l_3-l_{23})(l_2+l_3-l_{23})} q^{\frac{l(l+1)}{2}} U^q(l_1, l_2, l_3, l_1, l_{12}, l_{23}, q) U^q(l_1, l_2, l_3, l_1, l_{12}, l_{23}, q) \delta_{ll'} \delta_{mm'}
\]

(3.15)

It is diagonal and it does not depend on $m$.

On the other hand we have

\[
\begin{align*}
& id(1) \otimes R^q(3,2), e^q_m(l_{13}, l_2, \lambda) \\
& = ([R^q(1,2)]^{-1} \otimes id(3)) (R^q(1,2) \otimes id(3)) (id(1) \otimes R^q(3,2)) e^q_m(l_{13}, l_2, \lambda),
\end{align*}
\]

(3.16)

\[
\begin{align*}
& = (-1)^{(l_2+l_3+l_{12}+l_{13})}(-1)^{(l_2+l_3+l_{12}+l_{13})} q^{\frac{l(l+1)}{2}} U^q(l_2, l_1, l_3, l_1, l_{12}, l_{13}, q) e^q_m(l_{13}, l_2, \lambda),
\end{align*}
\]

(3.17)

where we have used relations (2.61), (2.63). Applying further relations (2.20), (3.10), (2.63), we find that

\[
([R^q(1,2)]^{-1} \otimes id(3)) e^q_m(l_2, l_{13}, \lambda) = \sum_{l_2} (-1)^{(l_1+l_2+l_{12})}(-1)^{(l_1+l_2+l_{12}+\lambda_2)}(1)_{(l_1+l_2+l_{12}+\lambda_2)} \\
\times (-1)^{(l_1+l_2+l_{12})} q^{\frac{l(l+1)}{2}} U^q(l_2, l_1, l_3, l_1, l_{12}, l_{13}, q) e^q_m(l_{12}, l_3, \lambda).
\]

(3.18)
Substitution of this relation into (3.16) and of the resulting formula into relation (3.13) yields another expression for the matrix $W_{l'm',lm}$

$$W_{l'm',lm} = (-1)^{\frac{1}{2}(l_2+l_3+l_2+\lambda_2)}(-1)^{\frac{1}{2}(l_2+l_3-\lambda_2)}(-1)^{\frac{1}{2}(l_1+l_2+l_1+\lambda_2)}(-1)^{\frac{1}{2}(l_1+l_2-l_1+\lambda_2)}(-1)^{\frac{1}{2}(l_1+l_2-l_2)}(-1)^{\frac{1}{2}(l_1+l_2-l_2-\lambda_2)}$$

$$\times (-1)^{\frac{1}{2}(l_2+l_3-\lambda_2)-l_2(l_2+1)}(-1)^{\frac{1}{2}(l_1+l_2-l_1+\lambda_2)-l_1(l_1+1)}q^{\frac{1}{2}[l(l+l+1)-l_2(l_2+1)-l_1(l_1+1)]}U^*(l_2,l_1,l_3,l_1,l_2,l_1^2,l_1^3,q)\delta_{l'1,m',m}.$$  

(3.19)

Comparison of this expression with relation (3.15) furnishes Racah sum rule for $U_q(osp(1|2))$

Racah coefficients :

$$\sum_{l_23}(-1)^{(l_2+l_3+l_2)}(-1)^{(l_2+l_3+\lambda_2)}(-1)^{(l_1+l_2+l_2)}(-1)^{\frac{1}{2}(l_2+l_3-l_2)}(-1)^{\frac{1}{2}(l_2+l_3-\lambda_2)}$$

$$\times q^{\frac{1}{2}(l_2+l_3-\lambda_2)-l_2(l_2+1)}(-1)^{\frac{1}{2}(l_1+l_2-l_1+\lambda_2)}U^*(l_1,l_3,l_2,l_1,l_2,l_3)U^*(l_1,l_2,l_3,l_1,l_2,l_3)$$

$$= (-1)^{(l_2+l_3+l_2+\lambda_2)}(-1)^{(l_1+l_2+l_2+\lambda_2)}(-1)^{(l_2+l_3+\lambda_2)}q^{\frac{1}{2}[l(l+l+1)-l_2(l_2+1)-l_1(l_1+1)]}U^*(l_2,l_1,l_3,l_1,l_2,l_1^2,l_1^3,q)$$

(3.20)

Substituting in this expression relation (3.12) that relates $sq$Racah coefficients and $sq6$-$j$ symbols, we derive Racah sum rule satisfied by the latter :

$$\sum_{l_23}(-1)^{(l_1+l_2+l_2+\lambda_1)}(-1)^{(l_1+l_2+l_2)}(-1)^{(l_1+l_2+\lambda_2)}(-1)^{(l_1+l_2+\lambda_2+\lambda_3)}(-1)^{(l_1+l_2+\lambda_2+\lambda_3)+\lambda_3}$$

$$\times (-1)^{\frac{1}{2}(l_2+l_3-\lambda_2)-l_2(l_2+1)}(-1)^{\frac{1}{2}(l_1+l_2-\lambda_2)-l_1(l_1+1)}$$

$$\times q^{\frac{1}{2}(l_2+l_3+\lambda_2+\lambda_3)+l_2(l_2+1)+l_1(l_1+1)}[2l_23+1] \left\{ \begin{array}{ccc} 1 & 2 & \beta_23 \hline l_1 & l_2 & l_3 \end{array} \right\} \left\{ \begin{array}{ccc} 1 & 2 & \beta_23 \hline l_2 & l_1 & l_3 \end{array} \right\}$$

$$= q^{\frac{1}{2}[l(l+l+1)+l_2(l_2+1)+l_1(l_1+1)]} \left\{ \begin{array}{ccc} 1 & 2 & \beta_23 \hline l_1 & l_2 & l_3 \end{array} \right\} \left\{ \begin{array}{ccc} 1 & 2 & \beta_23 \hline l_2 & l_1 & l_3 \end{array} \right\}$$

(3.21)

Using the orthogonality relation for the $sq6$-$j$ symbols

$$\sum_{l_23}(-1)^{(l_1+l_2+l_2+\lambda_1)}[2l_23+1] \left\{ \begin{array}{ccc} 1 & 2 & \beta_23 \hline l_1 & l_2 & l_3 \end{array} \right\} \left\{ \begin{array}{ccc} 1 & 2 & \beta_23 \hline l_2 & l_1 & l_3 \end{array} \right\}$$

$$= (-1)^{(l_2+l_3+l_2+\lambda_1)}\delta_{\beta_23,\beta_23},$$

(3.22)

one can rewrite the Racah sum rule (3.21) in the following form.
\[
\sum_{l_{13}} (-1)^{\Theta} q^{-\frac{1}{2} \left[ l_{23}(l_{23}+1) + l_{12}(l_{12}+1) + l_{13}(l_{13}+1) \right]} \left[ 2l_{13} + 1 \right] \left\{ \begin{array}{ccc}
l_2 & l_1 & l_{12} \\
l_3 & l & l_{13}
\end{array} \right\}_q \left\{ \begin{array}{ccc}
l_1 & l_3 & l_{13} \\
l_2 & l & l_{23}
\end{array} \right\}_q
\]

= \ q^{-\frac{1}{2} \left[ l_1(l_1+1) + l_2(l_2+1) + l_3(l_3+1) \right]} \left\{ \begin{array}{ccc}
l_1 & l_2 & l_{12} \\
l_3 & l & l_{23}
\end{array} \right\}_q,
(3.23)

where the phase argument \( \Theta \) is

\[
\Theta = \frac{1}{2} (l_2 + l_3 - l_{23})(l_2 + l_3 - l_{23} - 1) + \frac{1}{2} (l_1 + l_2 - l_{12})(l_1 + l_2 - l_{12} - 1)
+ \frac{1}{2} (l_2 + l_{13} - l)(l_2 + l_{13} - l - 1) + \frac{1}{2} \mathcal{L}(\mathcal{L} + 1) + \mathcal{L}l + (l_2 + l_3 + l_{23})l_{23}
+ (l_1 + l_2 + l_{12})l_{12} + (l_1 + l_3 + l_{13})l_{13}.
(3.24)
\]

In the limit \( q \to 1 \) this formula becomes Racah sum rule for the 6-\( j \) symbols for the superalgebra \( osp(1|2) \) and it takes the form given in Ref.[12].

If \( l_{12} = 0 \) and \( l_1 = l_2, \ l_3 = l \), then Racah sum rule (3.23) leads to the summation formula

\[
\sum_{l_{13}} (-1)^{(l_1+l_3)(l_{13}-l_{13}')} q^{-\frac{1}{2} \left[ l_{13}(l_{13}+1) + l_{13}'(l_{13}'+1) \right]} \left[ 2l_{13} + 1 \right] \left\{ \begin{array}{ccc}
l_1 & l_3 & l_{13} \\
l_1' & l_3' & l_{13}'
\end{array} \right\}_q = q^{-\frac{1}{2} l_1(l_1+1) + l_3(l_3+1)}.
(3.25)
\]

### IV. Biedenharn-Elliott identity for the quantum superalgebra \( U_q(osp(1|2)) \)

The tensor product \( T^{l_1} \otimes T^{l_2} \otimes T^{l_3} \otimes T^{l_4} \) of four irreducible representations of the same class can be reduced into irreducible representations according to one of the following five reduction schemes:

\[
\begin{align*}
\left[ (T^{l_1} \otimes T^{l_2}) \otimes T^{l_3} \right] \otimes T^{l_4}, & \\
\left[ (T^{l_1} \otimes (T^{l_2} \otimes T^{l_3})) \right] \otimes T^{l_4}, & \\
T^{l_1} \otimes [(T^{l_2} \otimes T^{l_3}) \otimes T^{l_4}], & \\
T^{l_1} \otimes [T^{l_2} \otimes (T^{l_3} \otimes T^{l_4})], & \\
(T^{l_1} \otimes T^{l_2}) \otimes (T^{l_3} \otimes T^{l_4}) & 
\end{align*}
(4.1)
(4.2)
(4.3)
(4.4)
(4.5)
\]

Biedenharn-Elliott identity is concerned with the relation between the schemes (4.1) and (4.4). One can go from one to the other either by the chain

\[
(4.1) \to (4.2) \to (4.3) \to (4.4),
(4.6)
\]

or by the chain

\[
(4.1) \to (4.5) \to (4.4).
(4.7)
\]

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each step involving \( sqRacah \) coefficients. Since the overall matrices that connect the two reduction schemes, (4.1) and (4.4), must be independent of the chain, there must exist a relation between combinations of products of three \( sqRacah \) coefficients for the first chain and combinations of products of two \( sqRacah \) coefficients for the second chain. This is precisely the Biedenharn-Elliott identity. Writing down the equality of corresponding matrix elements, Biedenharn-Elliott identity for \( sqRacah \) coefficients reads

\[
\sum_{l_{23}} (-1)^{l_2 + l_3 + l_{23} + l_4 + l_{234} + 1} (-1)^{l_1 + l_2 + l_{12}} (-1)^{l_1 + l_2 + l_3 + l_{123} + 1} (-1)^{l_1 + l_2 + l_3 + l_{123} + 1 + l + l_{12}} \nonumber \]

\[
U^s(l_1, l_2, l_3, l_{123}, l_{12}, l_{23}, q) U^s(l_1, l_{23}, l_4, l, l_{123}, l_{234}, q) U^s(l_2, l_3, l_4, l_{234}, l_{23}, l_{34}, q) \nonumber \]

\[
= (-1)^{(l_1 + l_2 + l_{12}) + (l_1 + l_2 + l_{3} + l_{123} + l + l_{124})} (-1)^{(l_1 + l_2 + l_{12}) + (l_1 + l_2 + l_3 + l_{123} + l_{234} + l_{23} + l_{34} + l + 1)} \nonumber \]

\[
U^s(l_{12}, l_3, l_4, l, l_{123}, l_{234}, q) U^s(l_1, l_2, l_{3}, l, l_{12}, l_{234}, q). \quad (4.8) \nonumber \]

For \( sq6-j \) symbols it is rewritten as

\[
\sum_{l_{123}} (-1)^{2l_{23} + 1} \left\{ \begin{array}{ccc} l_1 & l_2 & l_{12} \\ l_3 & l_{123} & l_{23} \end{array} \right\}_q^s \left\{ \begin{array}{ccc} l_1 & l_{23} & l_{123} \\ l_4 & l & l_{234} \end{array} \right\}_q^s \left\{ \begin{array}{ccc} l_2 & l_3 & l_{23} \\ l_{234} & l_{34} \end{array} \right\}_q^s \nonumber \]

\[
= \left\{ \begin{array}{ccc} l_{12} & l_3 & l_{123} \\ l_4 & l & l_{34} \end{array} \right\}_q^s \left\{ \begin{array}{ccc} l_1 & l_2 & l_{12} \\ l_3 & l & l_{234} \end{array} \right\}_q^s, \quad (4.9) \nonumber \]

where the phase argument \( \Theta \) is given by

\[
\Theta = \frac{1}{2} \left( \sum_{all} l_i \right) \left( \sum_{all} l_i^2 - 1 \right) + l_{23} (l + l_{12} + l_{34}) + (l_4 + l_{234}) (l_3 + l + l_{123} + l_{34}) + (l_{12} + l_{123}) (l_{23} + l + l_{123} + l_{34}) + (l_1 + l_2 + l_{12}) (l_1 + l_3 + l_4 + l_{34} + l + l_{234}) + l_2 + l_4 + l + l_{34} + l_{123} + l_{234}. \quad (4.10) \nonumber \]

Similarly as in the cases of \( su(2) \) and \( u_q(su(2)) \) three-term recurrence relation for \( sq6-j \) symbols follows from the Biedenharn-Elliott identity (4.9). Putting \( l = 1 \) into this identity and using the known analytical expressions of the \( sq6-j \) symbols with one argument equal to 1 (see the table in Ref.[3]) we derive a recurrence relation that can be written, after a change in the notation,

\[
A \left\{ \begin{array}{ccc} a & b & e \\ c & d & f - 1 \end{array} \right\}_q^s + B \left\{ \begin{array}{ccc} a & b & e \\ c & d & f \end{array} \right\}_q^s + C \left\{ \begin{array}{ccc} a & b & e \\ c & d & f + 1 \end{array} \right\}_q^s = 0, \quad (4.11) \nonumber \]

where
\[ A = (-1)^{a+d+f+1}[2f+2][a+d+f+1][a-d+f][a+d-f+1][-a+d+f] \times [b+c+f+1][b-c+f][b+c-f+1][-b+c+f])^{\frac{1}{2}}, \]  
\hspace{1cm} (4.12)  

\[ B = (-1)^{a+c+f+1}[2f+1][a-d+f][a+d-f] + (-1)^{a+b+f}[-a+d+f][a+d+f+2] \times [(b+c-f)[b-c+f] + (-1)^{b+c+f}[-b+c+f][b+c+f+2)]  
+ (-1)^{2f}[2f+1][2f+2][e+c-d][e-c+d]  
+ (-1)^{e+c+d}[-e+c+d][e+c+d+2]), \]  
\hspace{1cm} (4.13)  

\[ C = (-1)^{b+c+f}[2f][a+d+f+2][a-d+f+1][a+d-f][-a+d+f+1] \times [b+c+f+2][b-c+f+1][b+c-f][-b+c+f+1])^{\frac{1}{2}}, \]  
\hspace{1cm} (4.14)  

and  
\[ \Psi = (a+b+e)(a+b+c+d) + (d+e+c)(b+d+e+f) \]  
\[ + (a+d+f)(a+c+e+f) + c+d. \]  
\hspace{1cm} (4.15)
Appendix

A  Action of the universal $R$-matrix on the reduced basis $e_m^{l_2}(l_1, l_2, \lambda)$

In this appendix we prove the relation (2.47). For this, we recall that the vector $e_m^{l_2}(l_1, l_2, \lambda)$ can be expressed in the form (ref.[1])

$$e_m^{l_2}(l_1, l_2, \lambda) = \frac{1}{(N(l, m, q))^\frac{1}{2}} P_m^{l_2}(1, 2) e_l^{l_2}(\lambda_1) \otimes e_{l_1}^{l_2}(\lambda_2),$$  \hspace{1cm} (A.1)

where

$$P_m^{l_2}(1, 2) = \left( \frac{[l + m]!}{[2l]! [l - m]!} \right)^\frac{1}{2} v_0^{l_2}(l_1, l_2)^l_m P_m^{l_2}(1, 2),$$  \hspace{1cm} (A.2)

$$P_m^{l_2}(1, 2) = (T_1 \otimes T_2)(\Delta(P^{l_2})) = \sum_{r=0}^{\infty} c_r(l)(v_0^{l_2}(1, 2))^r (v_0^{l_2}(1, 2))^r,$$  \hspace{1cm} (A.3)

and $P^{l_2}$ is the projection operator on the highest weight vector of weight $l$. The coefficients $c_r(l)$ are of the form

$$c_r(l) = \frac{[2l + 1]!}{[2l + r + 1]! [r]! \gamma_r}.$$  \hspace{1cm} (A.4)

The value of the normalisation factor $N(l, m, q)$ is given by the formula

$$N(l, m, q) = q^{\frac{1}{2}(l_1 + l_2 - 1)(l_1 + l_2 - l - 1)} \frac{[2l + 1]! [2l_1]!}{[l_1 + l_2 + l]! [l_1 + l_2 + l + 1]!}.$$  \hspace{1cm} (A.5)

From relations (2.42) and (A.2, A.3) it follows that

$$R^{l_2}(1, 2) P_m^{l_2}(1, 2) R^{-1}(1, 2) = P_m^{l_2}(1, 2),$$

and one deduces successively the relations

$$R^{l_2}(1, 2) e_m^{l_2}(l_1, l_2, \lambda) = (N(l, m, q))^{-\frac{1}{2}} R^{l_2}(1, 2) P_m^{l_2}(1, 2) R^{-1}(1, 2)$$

$$\times R^{l_2}(1, 2) e_l^{l_2}(\lambda_1) \otimes e_{l_1}^{l_2}(\lambda_2),$$  \hspace{1cm} (A.6)

$$= (N(l, m, q))^{-\frac{1}{2}} q^{l_1(l_1-1)} P_m^{l_2}(1, 2) e_l^{l_2}(\lambda_1) \otimes e_{l_1}^{l_2}(\lambda_2),$$  \hspace{1cm} (A.7)

$$= \left( \frac{N(l, m, q^{-1})}{N(l, m, q)} \right)^\frac{1}{2} q^{l_1(l_1-1)} e_l^{l_2}(1, l_1, l_2, \lambda),$$  \hspace{1cm} (A.8)

The norm ratio $\frac{N(l, m, q^{-1})}{N(l, m, q)}$ is calculated from the definition (A.5), and one gets finally the action of the universal $R$-matrix on the reduced basis $e_m^{l_2}(l_1, l_2, \lambda)$:

$$R^{l_2}(l_1, l_2) e_m^{l_2}(l_1, l_2, \lambda) = q^{\frac{1}{2}(l_1(l_1-1)+l_2(l_2-1)-l_1-l_2)} e_m^{l_2}(l_1, l_2, \lambda).$$  \hspace{1cm} (A.9)

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References