Renormalization Group Approach to Einstein Equation in Cosmology

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Abstract

The renormalization group method has been adapted to the analysis of the long-time behavior of non-linear partial differential equation and has demonstrated its power in the study of critical phenomena of gravitational collapse. In the present work we apply the renormalization group to the Einstein equation in cosmology and carry out detailed analysis of renormalization group flow in the vicinity of the scale invariant fixed point in the spherically symmetric and inhomogeneous dust filled universe model.

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I. INTRODUCTION

Recently the renormalization group (RG) idea is applied to study the long-time asymptotics of the non-linear partial differential equations [1,2]. The RG transformation there is integration of the equation up to a finite time followed by a rescaling of the dependent and independent variables. The RG transformation together with the original differential equation gives a RG equation. Using the RG transformation, the problem on an infinite time is reduced to the problem on a finite time. A fixed point of the RG transformation corresponds to a scale invariant solution of the differential equation. We can obtain the long-time behavior of the equation by studying the flow around fixed points.

As an application of this RG method to the system of gravity, Koike, Hara and Adachi [3] analyzed the Einstein equation to understand the problem of critical behavior of black hole mass in gravitational collapse found by numerical study [4]. A pedagogical exposition of the RG method in the deterministic system is given by Tasaki [5] in a simple but very illustrative example of motion of a point particle in the Newtonian gravity.

Here we apply the RG to the Einstein equations in the cosmological situation. For simplicity, we shall consider only two cases. One is a homogeneous and isotropic universe filled with a perfect fluid and the other is a spherically symmetric universe filled with dust. We shall study the flow near the fixed points of the RG equations, which have self-similarity.

The astronomical observations indicate that the present universe has the hierarchical structure such as galaxies, clusters of galaxies, and super clusters, and that the two point correlation function of the galaxies and of the clusters of galaxies can be expressed roughly by a single power law [6]. The scale invariant Harrison-Zel'dovich spectrum for the primordial density perturbation has been successful in

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the study of structure formation of the universe [7]. These suggest that the present universe has some self-similarity and that the scale invariant solution plays an important role in cosmology.

In this paper we apply the renormalization group method to the Einstein equations in the cosmological context. In Sec.II, we illustrate the application of the RG method to the heat equation with the nonlinear term. In Sec.III, we apply the RG method to the Einstein equations. Section IV is devoted to the summary and discussions.

II. RENORMALIZATION GROUP TRANSFORMATION —— HEAT EQUATION WITH NONLINEAR TERM ——

In this section, we review the RG method for nonlinear partial differential equations [1]. First we consider the heat equation with the nonlinear term as a simple example:

\[
\frac{\partial u(x,t)}{\partial t} = \frac{1}{2} \left[ u''(x,t) + \lambda u^2(x,t) \right],
\]

where the prime denotes the spatial derivative and \( \lambda \) is a coupling constant. The equation (2.1) has scale invariance under the following scale transformation:

\[
x \rightarrow Lx, \\
t \rightarrow L^2 t, \\
u(x,t) \rightarrow L^2 u(Lx, L^2 t),
\]

(2.2)

where \( L \) is a parameter of the scale transformation and is taken to be larger than 1. Namely, if \( u(x,t) \) is a solution of Eq.(2.1), the scaled function,

\[
(L) u(L) = L^2 u(Lx, L^2 t),
\]

(2.3)

is also a solution of Eq.(2.1). We can thus obtain a one-parameter family of solutions, provided that \( u(x,t) \) is a solution.

Here we define the RG transformation \( R_L \) of a function of \( x \) by

\[
R_L u(x,1) = (L) u(x,1).
\]

(2.4)

In short, the \( R_L \) is a map from a set of initial data to another. It is convenient to take the initial time to be \( t = 1 \). The RG transformation \( R_L \) has a semi-group property:

\[
R_{L^n} = R_{L^{n-1}} \circ R_L.
\]

(2.5)

Letting \( t = 1 \) and then \( L^2 = t \) in Eqs.(2.3) and (2.4), we can express an arbitrary solution \( u(x,t) \) as an initial data \( u(\cdot,1) \) transformed by RG transformations:

\[
u(x,t) = t^{-1/2} u(xt^{-1/2},1).
\]

(2.6)

The large \( L \) means the late time. Repeating the RG transformation (2.4), we can see the long-time behavior of the solution \( u(x,t) \) in Eq.(2.1).

Denoting \( L = e^\tau \), we have from Eq.(2.3) that

\[
\frac{d (L)}{d \tau} = L \frac{d (L)}{dL} = 2 (L) u + x (L) u' + 2 \frac{d (L)}{dt}.
\]

(2.7)

Using the original partial differential equation (2.1) we have

\[
\frac{d (L)}{d \tau} = 2 (L) u + \lambda u^2 + x u' + u''.
\]

(2.8)
This is the equation satisfied by the scaled function \( u_L \), which we call the RG equation. We note that the equation (2.8) has no explicit scale \( L \) dependence because of the scale invariance of the original equation (2.1).

We investigate the fixed point of the RG equation (2.8). The fixed point \( u^* \) is defined by

\[
R_L u^* = u^* \tag{2.9}
\]

for any \( L > 1 \). This condition means that the field profile is unchanged after time evolution followed by suitable rescaling. In general this condition is equivalent to

\[
\frac{d}{d\tau} u^* = 0. \tag{2.10}
\]

From Eq.(2.8), \( u^* \) satisfies the following equation,

\[
2u^* + \lambda u^{*2} + xu'' + u''' = 0. \tag{2.11}
\]

In the homogeneous case, we can easily obtain the fixed points:

\[
u^* = 0 \quad \text{and} \quad -\frac{2}{\lambda}. \tag{2.12}\]

To investigate the character of the fixed point Eq.(2.12), we consider the linear perturbation around the fixed point Eq.(2.12). The perturbed quantity \( \delta u \) is defined by

\[
\delta u = \delta u^* + \delta u, \tag{2.13}
\]

where \( \delta u \) is assumed small. Substituting Eq.(2.13) into Eq.(2.8) and neglecting the second order term \( \delta u^2 \), we obtain the linearized equation for \( \delta u \):

\[
\frac{d}{d\tau} \delta u = \begin{cases} 2 \delta u + x\delta u' + \delta u'' & (u^* = 0) \\ -2 \delta u + x\delta u' + \delta u'' & (u^* = -\frac{2}{\lambda}) \end{cases}. \tag{2.14}
\]

We require the boundary condition

\[
\delta u \rightarrow 0 (|x| \rightarrow \infty). \tag{2.15}
\]

We are going to find the normal modes with the ansatz:

\[
\delta u = f(x)e^{-\frac{x^2}{2} + \omega \tau}, \tag{2.16}
\]

where \( f(x) \) is a function to be determined below and \( \omega \) is a constant. From Eqs.(2.14) and (2.16), we have

\[
\begin{align*}
f'' - xf' - (\omega - 1)f &= 0 & (u^* = 0), \\
f'' - xf' - (\omega + 3)f &= 0 & (u^* = -\frac{2}{\lambda}). \tag{2.17}
\end{align*}
\]

The regularity at \( x = 0 \) and the boundary condition at \( |x| = \infty \) imply

\[
f(x) = H_n(x), \tag{2.18}
\]

\[
\omega = \begin{cases} 1 - n & (u^* = 0) \\ -3 - n & (u^* = -\frac{2}{\lambda}) \end{cases}, \tag{2.19}
\]
where $H_n(x)$ is the Hermite polynomial and $n = 0, 1, 2, \ldots$.

From Eq. (2.19), $u^* = -2/\lambda$ is an attractor because all $\omega$’s are negative. On the other hand, $u^* = 0$ has only one relevant mode ($n = 0$).

We can discuss the long-time behavior of a solution of the nonlinear diffusion equation (2.1), if $u(x, 1)$ is sufficiently close to the self-similar profile, $u^*$. From Eq. (2.12), we obtain the two self-similar profiles.

Suppose the initial spatial profile of $\delta u$ is expressed as a superposition of the normal modes $H_n e^{-x^2/2}$. As we have seen from Eq. (2.19), if $u(x, 1)$ is sufficiently close to the fixed point $u^* = -2/\lambda$, the solution approaches to $-2t^{-1}/\lambda$ in course of time because all modes of perturbation are irrelevant. On the other hand, there is the only one growing mode ($n = 0$) of perturbation around the fixed point $u^* = 0$. As time goes on, the behavior of the solution near this fixed point is dominated by the relevant mode $n = 0$.

This relevant mode corresponds a Gaussian distribution $t^{-1/2} e^{-x^2/(2t)}$.

From this instructive example, we see that if the perturbation around the fixed point has a finite number of relevant modes or no relevant modes, we have some prediction power for the long-time behavior of nonlinear partial differential equation.

### III. Renormalization Group for Einstein Equation

In this section, we apply the RG method, which is explained in the previous section, to the Einstein equations.

Take a synchronous reference frame where the line element is

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = -dt^2 + \gamma_{ij}(x^k, t)dx^i dx^j.$$  \hspace{1cm} (3.1)

Throughout this paper Latin letters will denote spatial indices and Greek letters spacetime indices. The matter is taken to be a perfect fluid characterized by the energy-momentum tensor

$$T_{\mu\nu} = (\rho + p)u_\mu u_\nu + pg_{\mu\nu},$$  \hspace{1cm} (3.2)

where $p$, $\rho$, and $u_\mu$ are pressure, energy density, and four-velocity, respectively. We assume that the equation of state of the fluid is

$$p = (\Gamma - 1)\rho,$$  \hspace{1cm} (3.3)

where $\Gamma$ is a constant. The Einstein equations are

$$\dot{\gamma}_{ij} = 2K_{ij},$$  \hspace{1cm} (3.4)

$$K_{ij} = -\frac{3}{2}R_{ij} - KK_{ij} + 2K^l_i K_{lj} + \frac{\kappa\rho}{2} \{2\Gamma u_i u_j + (2 - \Gamma)\gamma_{ij}\},$$  \hspace{1cm} (3.5)

$$\kappa\rho = \frac{3R + K^2 - K^m_i K^l_m}{2(1 + \Gamma u_i u^i)},$$  \hspace{1cm} (3.6)

$$\kappa\Gamma u_i = \frac{1}{\sqrt{1 + u_i u^i}}(K^l_i K_{lj} - K_{lj}),$$  \hspace{1cm} (3.7)

where $K_{ij}$ is the extrinsic curvature, $3R_{ij}$ is the Ricci tensor associated with $\gamma_{ij}$, and $\kappa$ $\equiv 8\pi G$. A dot denotes the derivative with respect to $t$, a semicolon denotes the covariant derivative with respect to $\gamma_{ij}$.

Hereafter we consider the RG transformation for the dynamical variables $\gamma_{ij}$ and $K_{ij}$. In the following subsections, we investigate the two cases; the one is a homogeneous and isotropic universe, the other is a spherically symmetric inhomogeneous dust universe.

#### A. Homogeneous and Isotropic case

We consider the homogeneous and isotropic universe as a simple case. This case is rather trivial because the field equation becomes an ordinary differential equation. Nonetheless, this gives a nice warming up model to familiarize us to the RG approach to the universe. In this case, the spatial metric is written by
\[ \gamma_{ij}(x^k, t) = \frac{a^2(t)}{1 + \frac{k(t)r^2}{4}} \delta_{ij}, \]  
(3.8)

where \( r^2 \equiv \delta_{ij}x^ix^j \), and \( a(t) \) and \( k(t) \) are the functions of time \( t \) to be studied. Substituting Eq.(3.8) into Eqs.(3.4) – (3.7), we get

\[ \dot{a} = aH, \]  
(3.9)

\[ \dot{H} = -3H^2 - \frac{2k}{a^2} + \frac{2 - \Gamma}{2}\kappa\rho, \]  
(3.10)

\[ \dot{k} = 0, \]  
(3.11)

\[ \kappa\rho = 3 \left[ H^2 + \frac{k}{a^2} \right], \]  
(3.12)

where \( H \) is the Hubble parameter. From the conservation law of the energy density, we have

\[ \kappa\rho = Ma^{-3\Gamma}, \]  
(3.13)

where \( M \) is an arbitrary constant.

First, we consider the following scale transformation.

\[ t \rightarrow Lt, \]
\[ a(t) \rightarrow (L)^{\frac{1}{a}} a(t) \equiv L^{-2/(3\Gamma)}a(Lt), \]  
(3.14)

\[ k(t) \rightarrow (L)^{\frac{1}{a}} k(t) \equiv L^{2(3\Gamma - 2)/(3\Gamma)}k(Lt), \]  
(3.15)

where \( L \) is a parameter of scale transformation and larger than 1. From Eqs.(3.10) and (3.13), under this scale transformation the variables \( H \) and \( \rho \) are scaled in the following way,

\[ H(t) \rightarrow (L)^{\frac{1}{a}} H(t) = LH(Lt), \]  
(3.16)

\[ \rho(t) \rightarrow (L)^{\frac{1}{a}} \rho(t) = L^2\rho(Lt). \]  
(3.17)

The equations of motion (3.9)–(3.12) are invariant under the scale transformation Eqs.(3.15)–(3.17) and equivalently the scaled variables satisfy the original equation.

Second, we define the RG transformation \( R_L \):

\[ R_L a(1) = \frac{d}{d\tau}, \quad R_L H(1) = \frac{d}{d\tau} H(1), \quad R_L k(1) = \frac{d}{d\tau} k(1). \]  
(3.18)

Letting \( t = L \), we have formulas

\[ a(t) = \frac{a(t)}{(L)^{\frac{1}{a}}(1)}, \]  
(3.19)

\[ H(t) = \frac{H(t)}{(L)^{-\frac{1}{a}}(1)}, \]  
(3.20)

\[ k(t) = \frac{k(t)}{(L)^{-\frac{1}{a}}(1)}, \]  
(3.21)

which we shall use later to see the long-time behavior of \( a, H, \) and \( k \).

Third, we derive the RG equation. Letting \( L = e^\tau \), the infinitesimal transformation of \( \frac{d}{d\tau} a, \frac{d}{d\tau} H, \) and \( \frac{d}{d\tau} k \) with respect to \( \tau \) is

\[ \frac{d}{d\tau} a = -\frac{2}{3\Gamma} \frac{a}{(L)^{\frac{1}{a}}} + \frac{\partial a}{\partial t}, \]
\[ \frac{d}{d\tau} H = \frac{(L)}{H} + \frac{\partial H}{\partial t}, \]
\[ \frac{d}{d\tau} k = \frac{2(3\Gamma - 2)(L)}{3\Gamma} k + \frac{\partial k}{\partial t}. \]  
(3.22)
Using the equations of motion (3.9), (3.10), and (3.11), Eqs.(3.22) can be rewritten as

\[
\frac{d}{d\tau} \left( L a^2 \right) = -\frac{2}{3}\Gamma a + \frac{(L)}{a H},
\]

\[
\frac{d}{d\tau} \left( L H \right) = \frac{(L)}{H} + \left[ \frac{(L)}{-H^2 + \frac{2 - 3\Gamma}{6}\kappa \rho} \right],
\]

\[
\frac{d}{d\tau} \left( L k \right) = \frac{2(3\Gamma - 2)(L)}{3\Gamma} k.
\]

These equations (3.23) are the RG equations.

Here we investigate the fixed point of the RG equations. The fixed point \((a^*, H^*, k^*)\) is defined by

\[
R_L a^* = a^*, \quad R_L H^* = H^*, \quad R_L k^* = k^*.
\]

(3.24)

The above conditions can be rewritten as

\[
\frac{d}{d\tau} \left( L a^* \right) = 0, \quad \frac{d}{d\tau} \left( L H^* \right) = 0, \quad \frac{d}{d\tau} \left( L k^* \right) = 0.
\]

(3.25)

From Eqs.(3.12) and (3.23), the fixed point is

\[
a^* = \left( \frac{3\Gamma^2}{4} \right)^{1/3}, \quad H^* = \frac{2}{3\Gamma}, \quad k^* = 0, \quad \kappa \rho^* = 3H^*2.
\]

(3.26)

This fixed point corresponds to a flat Friedmann universe.

Note that if \(\Gamma\) is taken to be \(2/3\), there is another fixed point where \(k^*\) is non-zero. For the non-zero \(k^*\) case, the term of the spatial curvature can be absorbed into the term of the energy density of matter because the dependence of the scale factor on each term is the same. Thus the \(k^* = 0\) case includes the non-zero \(k^*\) case. Hereafter we concentrate on only the \(k^* = 0\) case.

In order to study the flow in the RG around the fixed point, we consider the perturbation around the fixed point. The perturbed quantities \(\delta a, \delta H, \) and \(\delta k\) are defined by

\[
\frac{(L)}{a} = a^* + \delta a, \quad \frac{(L)}{H} = H^* + \delta H, \quad \frac{(L)}{k} = k^* + \delta k,
\]

(3.27)

where \(\delta a, \delta H, \) and \(\delta k\) are assumed small qualities. From Eq.(3.23), the perturbed quantities satisfy the linearized equations,

\[
\frac{d}{d\tau} \left( L \delta a \right) = a^* \delta H,
\]

(3.28)

\[
\frac{d}{d\tau} \left( L \delta H \right) = -\delta H - \frac{3\Gamma}{2a^*} \delta k,
\]

(3.29)

\[
\frac{d}{d\tau} \left( L \delta k \right) = \frac{2(3\Gamma - 2)}{3\Gamma} \delta k,
\]

(3.30)

where we neglect the second order term \(\delta a^2\) and \(\delta H^2\) and use the linearized equation of Eq.(3.12):

\[
\kappa \delta \rho = \frac{4}{\Gamma} \delta H + \frac{3}{a^*} \frac{(L)}{\delta k}.
\]

(3.31)

Substituting Eq.(3.28) into Eq.(3.29), \(\delta a\) satisfies
\[
\frac{d^2(\delta a)}{d\tau^2} + \frac{d\delta a}{d\tau} + \frac{3\Gamma (L)}{2a^*} \delta k = 0. \tag{3.32}
\]

We solve Eq.(3.32);
\[
\delta a = f_1 e^{-\tau} + f_2 e^{2\sin^2 - 1/3t}, \tag{3.33}
\]
where \(f_1\) and \(f_2\) are arbitrary constants. From the solution (3.33), we can see the flow in RG around the fixed point. If \(3\Gamma - 2 < 0\), this fixed point is an attractor. On the other hand if \(3\Gamma - 2 > 0\), there is a single relevant mode. Note that in the case \(3\Gamma - 2 > 0\), the matter we consider satisfies the strong energy condition.

From the flow in the RG around the fixed point, we can see the long-time behavior of the homogeneous and isotropic universe. If \(3\Gamma - 2 < 0\) and setting the initial profile in the vicinity of the fixed point \((a^*, H^*, k^*)\), the spacetime will approach to the flat Friedmann universe \(a(t) = a^* t^{2/3\Gamma}\). On the other hand, if \(3\Gamma - 2 > 0\), the spacetime will deviate from the flat Friedmann universe because there is a relevant mode \(\delta a(t) = f_2 t^{2(3\Gamma - 2)/3\Gamma}\).

In the context of the usual cosmological perturbation around a flat Friedmann universe, \(f_1\) mode corresponds to the decaying mode and \(f_2\) mode corresponds to the growing mode, which implies the gravitational instability, because the matter should satisfy the strong energy condition.

### B. Spherically symmetric case

We consider the spherically symmetric inhomogeneous case. In this case, the spatial metric is written by
\[
\gamma_{ij}(x^k, t)dx^i dx^j = A^2(r, t)dr^2 + B^2(r, t)(d\theta^2 + \sin^2 \theta d\phi^2). \tag{3.34}
\]
Namely, \(\gamma_{rr} = A^2(r, t), \gamma_{\theta\theta} = B^2(r, t),\) and \(\gamma_{\phi\phi} = B^2 \sin^2 \theta\) while the other components of the spatial metric vanish. As a simple case, we investigate the universe filled with dust, \(i.e. \Gamma = 1\) and we can set \(u^i = 0\) in Eqs.(3.4) – (3.7). The Einstein equations are
\[
\gamma_{ij} = 2K_{ij} \tag{3.35}
\]
\[
K_{ij} = -3R_{ij} + 2K_iK_j - KK_{ij} + \frac{1}{2} \kappa \rho \gamma_{ij}, \tag{3.36}
\]
\[
\kappa \rho = \frac{1}{2} [3R + K^2 - K^m_{l} K^l_{m}], \tag{3.37}
\]
\[
K^j_{i;j} - K_{,i} = 0. \tag{3.38}
\]

Here we consider the following scale transformation.
\[
r \rightarrow Lr,
\]
\[
t \rightarrow L^\alpha t,
\]
\[
\gamma_{rr}(r, t) \rightarrow \gamma_{rr}(r, t) \equiv L^{2-2\alpha} \gamma_{rr}(Lr, L^\alpha t), \tag{3.39}
\]
\[
\gamma_{\theta\theta}(r, t) \rightarrow \gamma_{\theta\theta}(r, t) \equiv L^{-2\alpha} \gamma_{\theta\theta}(Lr, L^\alpha t), \tag{3.40}
\]
\[
K_{rr}(r, t) \rightarrow K_{rr}(r, t) \equiv L^{2-\alpha} K_{rr}(Lr, L^\alpha t), \tag{3.41}
\]
\[
K_{\theta\theta}(r, t) \rightarrow K_{\theta\theta}(r, t) \equiv L^{-\alpha} K_{\theta\theta}(Lr, L^\alpha t), \tag{3.42}
\]
\[
\rho(r, t) \rightarrow \rho(L, r, t) \equiv L^{2\alpha} \rho(Lr, L^\alpha t), \tag{3.43}
\]
where \(\alpha\) is an arbitrary constant because the coordinate transformation, \(r \rightarrow r^\beta\), yields substitution of \(\alpha/\beta\) for \(\alpha\) in Eqs.(3.39)–(3.43). Without loss of generality, we take \(\alpha\) to be positive. Because of the scale
invariance of the Einstein equations Eqs.(3.35) – (3.38), the scaled variables \((\gamma_{rr})^{(L)}, (\gamma_{\theta\theta})^{(L)}, (K_{rr})^{(L)}, (K_{\theta\theta})^{(L)}\), and \((\rho)^{(L)}\) also satisfy Eqs.(3.35) – (3.38).

We derive the RG equation. Letting \(L = e^\tau\), the infinitesimal transformation of \((\gamma_{ij})^{(L)}\) and \((K_{ij})^{(L)}\) with respect to \(\tau\) is

\[
\frac{d\gamma_{rr}^{(L)}}{d\tau} = 2(1 - \alpha)\gamma_{rr}^{(L)} + r\partial_\tau \gamma_{rr}^{(L)} + 2\alpha K_{rr}^{(L)},
\]

(3.44)

\[
\frac{d\gamma_{\theta\theta}^{(L)}}{d\tau} = -2\alpha \gamma_{\theta\theta}^{(L)} + r\partial_\tau \gamma_{\theta\theta}^{(L)} + 2\alpha K_{\theta\theta}^{(L)},
\]

(3.45)

\[
\frac{dK_{rr}^{(L)}}{d\tau} = (2 - \alpha)K_{rr}^{(L)} + r\partial_\tau K_{rr}^{(L)} + \alpha \left[ \frac{1}{4} (\gamma_{rr}^{(L)})^2 - \frac{1}{3} R_{ij}^{\gamma_{rr}} - 3R_{rr}^{(L)} + 2K_{i\ell}K_{\ell}^{(L)} - K K_{rr}^{(L)} + \frac{1}{4} \left( \frac{L}{K} \text{L}^{(L)}(L) - K_{l}^{(L)}(L) K_{l}^{(L)}(L) \right) \right],
\]

(3.46)

\[
\frac{dK_{\theta\theta}^{(L)}}{d\tau} = -\alpha K_{\theta\theta}^{(L)} + r\partial_\tau K_{\theta\theta}^{(L)} + \alpha \left[ \frac{1}{4} (\gamma_{\theta\theta}^{(L)})^2 - \frac{1}{3} R_{ij}^{\gamma_{\theta\theta}} - 3R_{\theta\theta}^{(L)} + 2K_{i\ell}K_{\ell}^{(L)} - K K_{\theta\theta}^{(L)} + \frac{1}{4} \left( \frac{L}{K} \text{L}^{(L)}(L) - K_{l}^{(L)}(L) K_{l}^{(L)}(L) \right) \right],
\]

(3.47)

where \(\gamma_{ij}^{(L)}\) is the Ricci tensor associated with \((\gamma_{ij})^{(L)}\). In the derivation of Eqs.(3.44) – (3.47), the equations of motion (3.35) and (3.36) are used. These equations (3.44) – (3.47) are the RG equations.

In terms of \(A^{(L)}\) and \(B^{(L)} (A^2 = \gamma_{rr}^{(L)} \text{ and } B^2 = \gamma_{\theta\theta}^{(L)})\), the RG equations (3.44)–(3.47) read

\[
\frac{d^2 A^{(L)}}{d\tau^2} = \left[ 2(1 - \alpha) + \frac{1}{B} \frac{dB}{d\tau} - \frac{r B'}{B} \right] \frac{dA^{(L)}}{d\tau} - 2r \frac{dA'}{d\tau} = \frac{1}{A^2} \left[ \frac{1}{2} \frac{dB}{d\tau} + \frac{A}{B} \frac{dA'}{d\tau} \right],
\]

\[
\frac{d^2 B^{(L)}}{d\tau^2} = \left[ 2\alpha + \frac{1}{2B} \frac{dB}{d\tau} - \frac{r B'}{B} \right] \frac{dB^{(L)}}{d\tau} - 2r \frac{dB'}{d\tau} = \frac{1}{2B} \left[ \frac{1}{2} \frac{dB}{d\tau} + \frac{A}{B} \frac{dA'}{d\tau} \right],
\]

(3.48)

(3.49)

where the prime denotes the derivative with respect to \(r\). From Eqs.(3.37) and (3.38), we obtain

\[
\kappa^{(L)} = \frac{2}{\alpha^2 A} \left[ \alpha + \frac{dA}{B} \frac{dB}{d\tau} - \frac{r B'}{B} \right] \frac{dA}{d\tau} + \frac{1}{\alpha^2 A} \left[ 2(2\alpha - 1) - \frac{r A'}{A} - \frac{r A'}{A} + \frac{dB}{B} \frac{dB}{d\tau} \right] = \frac{1}{A^3 B^2} \left[ \frac{L}{A^3} + 2 \frac{L}{A^3 B} - \frac{L}{A^3 B} - 2 \frac{L}{A^3 B} \right],
\]

(3.50)
\[
\frac{dA}{dr} - \frac{c}{A} d\frac{dA}{dr} - r A B' + r A B'' = 0. 
\]

(3.51)

Letting \( t = L^\alpha \), the original variables \( A(r, t) \) and \( B(r, t) \) are expressed by the scaled variables \( A^{(L)} \) and \( B^{(L)} \):

\[
A(r, t) = t^{(\alpha - 1)/\alpha} A^{(L)}(rt^{-1/\alpha}, 1), \quad (3.52)
\]

\[
B(r, t) = t^{(1/\alpha)} B^{(L)}(rt^{-1/\alpha}, 1). \quad (3.53)
\]

Here we investigate the fixed point of the RG equations (3.48) and (3.49) defined by

\[
\frac{dA^*}{d\tau} = 0, \quad \frac{dB^*}{d\tau} = 0. \quad (3.54)
\]

At the fixed point,

\[
A(r, t) = t^{(\alpha - 1)/\alpha} A^* (rt^{-1/\alpha}, 1) = t^{(\alpha - 1)/\alpha} \times \text{(function of } rt^{-1/\alpha} \text{ only)} \quad (3.55)
\]

and

\[
B(r, t) = t^{(1/\alpha)} B^* (rt^{-1/\alpha}, 1) = t \times \text{(function of } rt^{-1/\alpha} \text{ only)} \quad (3.56)
\]

is a self-similar solution.

In the spherically symmetric spacetime filled with dust, the general solution of the Einstein equations is the Tolman-Bondi solution (A1) – (A2) in the Appendix. Therefore we can obtain the fixed point from the Tolman-Bondi solution with self-similarity rather than solving the equations (3.54) directly. The precise form of these are

For \( c = 0 \) :

\[
A^*(r, 1) = \frac{\alpha r^{\alpha/3 - 1}(1 - 3pr^\alpha)}{3(1 - pr^\alpha)^{\frac{3}{2}}}, \quad (3.57)
\]

\[
B^*(r, 1) = r^{\alpha/3}(1 - pr^\alpha)^{\frac{5}{2}}, \quad (3.58)
\]

\[
\kappa \rho^*(r, 1) = \frac{4}{3(1 - pr^\alpha)(1 - 3pr^\alpha)}; \quad (3.59)
\]

For \( c > 0 \) :

\[
A^*(r, 1) = \frac{\alpha}{(1 + c)^{1/2}r} \left[ \frac{2}{9c} (\cosh \eta - 1)r^\alpha - c^{1/2} \frac{\sinh \eta}{\cosh \eta - 1} \right], \quad (3.60)
\]

\[
B^*(r, 1) = \frac{2}{9c} r^\alpha (\cosh \eta - 1), \quad (3.61)
\]

\[
\sinh \eta - \eta = \frac{2c^{3/2}}{2} (r^{-\alpha} - p), \quad (3.62)
\]

\[
\kappa \rho^*(r, 1) = \frac{9c^2}{2^{1/2} c (\cosh \eta - 1)} \left[ \frac{2^{1/2} c^{1/2}}{2c} (\cosh \eta - 1) - c^{1/2} \frac{\sinh \eta}{\cosh \eta - 1} \right]; \quad (3.63)
\]

For \( c < 0 \) :

\[
A^*(r, 1) = \frac{\alpha}{(1 - |c|)^{1/2}r} \left[ \frac{2}{9|c|} (1 - \cos \eta)r^\alpha - |c|^{1/2} \frac{\sin \eta}{1 - \cos \eta} \right], \quad (3.64)
\]

\[
B^*(r, 1) = \frac{2}{9|c|} r^\alpha (1 - \cos \eta), \quad (3.65)
\]

\[
\eta - \sin \eta = \frac{9|c|^{3/2}}{2} (r^{-\alpha} - p), \quad (3.66)
\]

\[
\kappa \rho^*(r, 1) = \frac{9c^2}{r^\alpha (1 - \cos \eta)^2} \left[ \frac{2^{1/2} c^{1/2}}{9|c|} (1 - \cos \eta) - |c|^{1/2} \frac{\sin \eta}{1 - \cos \eta} \right]; \quad (3.67)
\]
where \( c \) and \( p \) are constants.

The constant \( c \) can be interpreted as the total energy of the universe in the analogy of the Newtonian mechanics. By the signature of the constant \( c \), these fixed points are classified into the following three. The universe with \( c = 0 \) is similar to the flat Friedmann universe. The universes with a positive \( c \) or a negative \( c \) are similar to the open and closed Friedmann universes, respectively. Especially when \( c = p = 0 \), the above fixed point coincides with the flat Friedmann universe and the spacetime becomes homogeneous.

In the context of the RG, we can treat the time evolution of the field variables as the map from a set of initial data to another. If the initial data is taken to be the above fixed point \( A^*, B^*, \) and \( \rho^* \), the spacetime will evolve into the Tolman-Bondi solution with self-similarity. In the cases of \( c = 0 \) and \( c > 0 \), the fixed point is not regular if \( p > 0 \). For \( c < 0 \), the fixed point has singularities irrespective of the signature of \( p \) because the spacetime is similar to the closed Friedmann universe and will recollapse.

Since we should set a regular initial data in the physical situation, we investigate only the case of \( c = p = 0 \) because we have already studied it in the previous subsection.

To study the behavior of the flow in the RG around the fixed point, we consider the linear perturbation around the self-similar Tolman-Bondi solution. We assume the spatial metric variables for \( \delta A \) and \( \delta B \) in the following form,

\[
\begin{align*}
\delta A &= a(r)e^{\omega \tau}, \\
\delta B &= b(r)e^{\omega \tau}.
\end{align*}
\]  

The perturbed quantities \( a(r) \) and \( \delta \rho^* \) are expressed by \( b(r) \);

\[
\begin{align*}
\frac{a}{A^*} &= \frac{b'}{B^*} - \frac{c_1}{2(1+c)} r^\omega, \\
\frac{\delta \rho^*}{\rho^*} &= e^{\omega \tau} \left[ \frac{9(\omega + \alpha)}{4\alpha} c_2 r^\omega - 2 \frac{b}{B^*} - \frac{b'}{B^*} \right],
\end{align*}
\]  

where \( c_1 \) and \( c_2 \) are arbitrary constants (see Appendix).

As for the spherical modes of the perturbation, we can easily obtain the solutions.

For \( c = 0 \):

\[
b(r) = r^\omega \left[ \frac{9}{20} c_1 r^{-\alpha/3} (1 - pr^\alpha)^{4/3} + \frac{3}{4} c_2 r^{\alpha/3} (1 - pr^\alpha)^{2/3} - \frac{2}{3} c_3 r^{4\alpha/3} (1 - pr^\alpha)^{-1/3} \right],
\]  

where \( c_3 \) is another arbitrary constant. The density contrast is

\[
\begin{align*}
\frac{\delta \rho^*}{\rho^*} &= e^{\omega \tau} r^{\omega} \left\{ -\frac{9}{20\alpha} c_1 r^{-2\alpha/3} (1 - pr^\alpha)^{2/3} (1 - 3pr^\alpha)^{-1} [3\omega + \alpha - 3(\omega + \alpha)pr^\alpha] \\
&\quad - \frac{9\omega}{2\alpha} c_2 pr^\alpha (1 - 3pr^\alpha)^{-1} + \frac{2\omega}{\alpha} c_3 r^{\alpha} (1 - pr^\alpha)^{-1} (1 - 3pr^\alpha)^{-1} [\omega + 2\alpha - (\omega + 3\alpha)pr^\alpha] \right\}.
\end{align*}
\]  

In the expression for the linear perturbation Eq.(3.73) there are three terms corresponding to \( c_1, c_2, \) and \( c_3 \) so that there should be a gauge mode hidden in Eq.(3.73) because the number of physical modes has
the regularity condition that \( \delta \approx \omega \tau \), where \( \tau \) is given by
\[
\Delta (A8). \quad \text{The gauge mode is given by}
\]
with \( f \) being an arbitrary constant. Because we should fix the freedom of gauge we choose \( f \) so that
\[
(\delta \rho + \delta \rho_g)/\rho^* \text{ behave as nicely as possible at } r = \infty \text{ because we are interested in the perturbation modes which are finite at } r = \infty.
\]
We use the following condition as a convenient gauge condition.
\[
f = -\frac{9(\omega + 3\alpha)}{40\alpha^2} p^{2/3} c_1 + \frac{3\omega}{4\alpha^2} c_2 - \frac{\omega + 3\alpha}{3\alpha^2 p} c_3.
\]
We fix the gauge mode by the above condition and obtain the physical perturbation \( \delta \hat{\rho} = \delta \hat{\rho} + (L) \) as follows.
\[
\frac{\delta (L)}{\rho^*} = e^{\omega \tau} \frac{\Delta_1 (1 - 3\rho^*)^{-1}}{\rho^*} \left[ p^{-2/3} r^{-2\alpha/3} (1 - \rho^*)^{2/3} [3\omega + \alpha - 3(\omega + 3\alpha)\rho^*] \right] + (\omega + 3\alpha) p^r \hat{\rho}^r (2 - 3\rho^r) (1 - \rho^r)^{-1} + \Delta_2 p^r \hat{\rho}^r (1 - \rho^r)^{-1} (1 - 3\rho^r)^{-1},
\]
where
\[
\Delta_1 = -\frac{9p^{2/3}}{20\alpha} c_1,
\]
\[
\Delta_2 = -\frac{3\omega}{2\alpha} (c_2 - \frac{4}{9p} c_3).
\]
Since we consider only the case of \( p < 0 \), the coordinate \( r \) can be taken from 0 to \( \infty \). We demand the regularity condition that \( \delta \hat{\rho} / \rho^* \) should be finite at the boundary, \( r = 0 \) and \( r = \infty \). This condition implies that \( \Delta_1 \) with \( 2\alpha/3 < \omega < \alpha \) and \( \Delta_2 \) modes with \( -\alpha < \omega < \alpha \) are allowed. The mode with \( \omega = 0 \) corresponds to change of \( p \) in the self-similar solution Eqs.(3.59) and \( \rho^* \) remains constant independent of \( r \) in the direction. Although this fixed point is not a repeller, it has many relevant modes, \( \Delta_1 \) with \( 0 < \omega \) \( \leq \) \( \alpha \) and \( \Delta_2 \) with \( 0 < \omega \) \( \leq \alpha \). Note that a suitable linear combination of the \( \Delta_1 \) and \( \Delta_2 \) modes will have an asymptotic behavior \( \approx T^{2\alpha - 3\alpha/2} \) at \( r = \infty \). For such modes, \( 2\alpha/3 < \omega \leq 2\alpha \) is allowed. These modes which satisfy the regularity condition are not discrete but continuous. We suppose that this special feature arises because our matter is assumed to be dust. To summarize, the possible value of \( \omega \) ranges from \( -\alpha \) to \( 2\alpha \). The flow of RG in the vicinity of the fixed point is shown in Fig.(1).
In the case of \( c = p = 0 \) where the fixed point corresponds to the flat Friedmann universe, the \( c_2 \) modes in the density contrast (3.73) are gauge modes. The regularity condition implies that \( c_1 \) mode with \( \omega = 2\alpha/3 \) and \( c_3 \) mode with \( \omega = -\alpha \) are allowed. This result corresponds to the homogeneous and isotropic case in the previous section. Compared with the usual cosmological perturbation in the synchronous comoving reference frame, this result appears to be strange because the only spherical modes of the linear perturbation allowed are constant in space. However, the time coordinate used in this RG method is different from the usual cosmic time coordinate, the solution allowed by the regularity condition in each case does not coincide in general. Moreover, in the homogeneous universe, there is no non-trivial characteristic profile of field variables. If the fixed point is a homogeneous universe, the RG method may have no advantage since the RG approach respects the self-similar profile. But if the fixed point is an inhomogeneous universe, we believe that the RG method may be useful.
Compared with \( c = 0 (p < 0) \) case, the value of \( \omega \) allowed in the case \( (p = 0) \) is the lower limit in the case \( (p < 0) \). The effect of nonlinearity of gravity makes the growth rate of the density contrast large.
For \( c > 0 \):
The gauge mode is given by
\begin{equation}
\begin{aligned}
b(r) &= r^{\omega} \left\{ -\frac{1}{9c^2} c_1 \left[ 2(\cosh \eta - 1) - \frac{3 \sinh \eta (\sinh \eta - \eta)}{\cosh \eta - 1} \right] \\
&\quad + \frac{1}{2c} \left[ \cosh \eta - 1 - \frac{\sinh \eta (\sinh \eta - \eta)}{\cosh \eta - 1} \right] - \frac{e^{1/2} \sinh \eta}{\cosh \eta - 1} c_3 \right\}.
\end{aligned}
\end{equation}

The density contrast is
\begin{equation}
\begin{aligned}
\frac{\delta \rho}{\rho^*} &= \frac{e^{\omega \tau} r^{\omega}}{2 \alpha \left[ \frac{9c}{9c^2} (\cosh \eta - 1) - \frac{e^{1/2} \sinh \eta}{\cosh \eta - 1} \right]}
\left\{ \frac{1}{9c^2} c_1 \left[ (\omega + 3\alpha) r^{\alpha} \left[ 2(\cosh \eta - 1) - \frac{3 \sinh \eta (\sinh \eta - \eta)}{\cosh \eta - 1} \right] + \frac{27c^{3/2} \alpha}{2} \left[ \frac{\sinh \eta}{\cosh \eta - 1} + \frac{(2 \cosh \eta + 1)(\sinh \eta - \eta)}{(\cosh \eta - 1)^2} \right] \right] \\
&\quad - \frac{1}{2c} c_2 \left[ \frac{2\alpha (\cosh \eta - 1) - (\omega + 3\alpha) \sinh \eta (\sinh \eta - \eta)}{\cosh \eta - 1} \right] + \frac{9c^{2/3}}{2} \left[ (\omega - \alpha) \sinh \eta \cosh \eta - 1 + \frac{\alpha (2 \cosh \eta + 1)(\sinh \eta - \eta)}{(\cosh \eta - 1)^2} \right] \\
&\quad + c^{1/2} c_3 \left[ (\omega + 3\alpha) r^{\alpha} \frac{\sinh \eta}{\cosh \eta - 1} - \frac{9c^{2/3} (2 \cosh \eta + 1)}{2(\cosh \eta - 1)^2} \right] \right\}.
\end{aligned}
\end{equation}

The gauge mode is given by
\begin{equation}
\begin{aligned}
\frac{\delta \rho_g}{\rho^*} &= f r^{\omega + 1} e^{\omega \tau} \frac{\delta \rho}{\rho^*} \\
&= - \frac{\alpha f e^{\omega \tau} r^{\omega}}{2 \alpha \left[ \frac{9c}{9c^2} (\cosh \eta - 1) - \frac{e^{1/2} \sinh \eta}{\cosh \eta - 1} \right]}
\left\{ \frac{4}{9c} r^{\alpha} (\cosh \eta - 1) - \frac{4c^{1/2} \sinh \eta}{\cosh \eta - 1} + \frac{9c^{2/3} (\omega + 3\alpha \sinh \eta)}{(\cosh \eta - 1)^2} c_3 \right\}.
\end{aligned}
\end{equation}

Similarly to the case of $c = 0$, we demand the regularity condition at $r = 0$ and $r = \infty$.

(i) case $1 \ (\rho < 0)$:

We use the following convenient gauge condition,
\begin{equation}
\begin{aligned}
f &= \frac{1}{4\alpha^2} \left\{ \frac{\omega + 3\alpha}{c} c_1 \left[ 2 - \frac{3 \sinh \eta_0 (\sinh \eta_0 - \eta_0)}{(\cosh \eta_0 - 1)^2} \right] \\
&\quad - \frac{9}{2c} c_2 \left[ 2\alpha - (\omega + 3\alpha) \frac{\sinh \eta_0 (\sinh \eta_0 - \eta_0)}{(\cosh \eta_0 - 1)^2} \right] + \frac{9c^{2/3} (\omega + 3\alpha \sinh \eta_0)}{(\cosh \eta_0 - 1)^2} c_3 \right\},
\end{aligned}
\end{equation}

where $\eta = \eta_0$ corresponds to $r \to \infty$ and $\eta_0$ is thus given by $\sinh \eta_0 - \eta_0 = -9pc^{3/2}/2$.

By using the above condition Eq.(3.82), we obtain the physical perturbation \( \delta \rho = \delta \rho_g + (L) \delta \rho_g \) as follows.

\begin{equation}
\begin{aligned}
\frac{\delta^{(L)} \rho}{\rho^*} &= \frac{e^{\omega \tau} r^{\omega}}{\rho^*} \left\{ \alpha \left[ \frac{9c}{9c^2} (\cosh \eta - 1) - \frac{e^{1/2} \sinh \eta}{\cosh \eta - 1} \right] \right\}
\left\{ 3(\omega + 3\alpha) r^{\alpha} (\cosh \eta - 1) \left[ \frac{\sinh \eta_0 (\sinh \eta_0 - \eta_0)}{(\cosh \eta_0 - 1)^2} - \frac{\sinh \eta (\sinh \eta - \eta)}{(\cosh \eta - 1)^2} \right] \\
&\quad + \frac{9c^{3/2}}{2} \left[ 4\omega + 9\alpha - 6(\omega + 3\alpha) \frac{\sinh \eta_0 (\sinh \eta_0 - \eta_0)}{(\cosh \eta_0 - 1)^2} \right] \frac{\sinh \eta}{\cosh \eta - 1} + 3\alpha \frac{(2 \cosh \eta + 1)(\sinh \eta - \eta)}{(\cosh \eta - 1)^2} \right\] \\
&\quad + \frac{9c^{2/3}(\omega + 3\alpha) r^{\alpha} (\cosh \eta - 1)}{2(\cosh \eta - 1)^2} \left[ \frac{\sinh \eta_0}{(\cosh \eta - 1)^2} - \frac{\sinh \eta_0}{(\cosh \eta_0 - 1)^2} \right] - \frac{9c^{3/2}\alpha (2 \cosh \eta + 1)}{2(\cosh \eta - 1)^2} \\
&\quad + \frac{9(\omega + 3\alpha) c^{2/3} \sinh \eta_0}{4(\cosh \eta_0 - 1)^2} \left[ \frac{4 \sinh \eta + 9c^{3/2} (2 \cosh \eta + 1)}{2(\cosh \eta - 1)^2} \right] \right\}.
\end{aligned}
\end{equation}
\[ \Delta_1^+ = \frac{1}{9c^2} c_1, \]  
(3.84)  
\[ \Delta_2^+ = -\frac{9pc^{1/2}}{4}(c_2 - \frac{4}{9p} c_3), \]  
(3.85)  
\[ \sinh \eta - \eta = 9c^{3/2}(r^{-\alpha} - p)/2. \]

The regularity condition at \( r = 0 \) and \( r = \infty \) implies that \( \Delta_1^+ \) modes with \( \omega = \alpha \) and \( \Delta_2^+ \) modes with \( 0 \leq \omega \leq \alpha \) are allowed. In this case, this fixed point is a repeller up to the zero mode because all other modes which satisfy the regularity condition have a positive \( \omega \). Note that a suitable linear combination of the \( \Delta_1^+ \) and \( \Delta_2^+ \) modes will have an asymptotic behavior \( \approx r^{\omega - 2\alpha} \). Therefore the possible value of \( \omega \) ranges from 0 to 2\( \alpha \). The flow of RG in the vicinity of the fixed point is shown in Fig. (2).

(ii) case 2 (\( p = 0 \)):

The gauge condition which we use is,

\[ f = -\frac{9}{4\alpha} c_2. \]

(3.87)

By using the above condition Eq.(3.87), we obtain the physical perturbation,

\[ \frac{\delta (L)}{\delta \rho^*} = \frac{e^{\omega \tau} r^\omega}{\alpha} \left\{ \Delta_1^+ \left[ (\omega + 3\alpha)r^\alpha \left[ 2(\cosh \eta - 1) - \frac{3 \sinh \eta(\sinh \eta - \eta)}{\cosh \eta - 1} \right] \right] + \Delta_2^+ \left[ (\omega + 3\alpha)r^\alpha \frac{\sinh \eta}{\cosh \eta - 1} - \frac{9\alpha c^{2/3}(2 \cosh \eta + 1)}{2(\cosh \eta - 1)^2} \right] \right\}, \]

(3.88)

where

\[ \Delta_1^+ = \frac{1}{9c^2} c_1, \]

(3.89)  
\[ \Delta_2^+ = c^{1/2} c_3. \]

(3.90)

The regularity condition at \( r = 0 \) and \( r = \infty \) implies that no modes are allowed.

From the linear perturbation analysis around the fixed point, we see that the long-time behavior of the spherically symmetric dust universe is separated into two types. One is the case that the fixed point is a repeller. In this case, the Tolman-Bondi solution with self-similarity does not play an important role in expanding universe because this fixed point is unstable and the spacetime will diverge from this fixed point. In the other case, the fixed point has both relevant and irrelevant modes. Although this fixed point is not a repeller, it has continuously many relevant modes. Thus it is not as straightforward as in the case of gravitational collapse [3] to extract the long-time behavior of the universe, because it is sensitive to the initial condition and therefore we cannot uniquely predict the outcome. In the final section, we briefly discuss how to treat the fixed point which has many relevant modes of the perturbation.

IV. SUMMARY AND DISCUSSIONS

We considered the spherically symmetric but inhomogeneous universe filled with dust, where the Einstein equations have scale invariance Eqs.(3.39)–(3.42) and applied the renormalization group method to...
study its long-time asymptotics. The fixed point of the RG transformation is a self-similar solution with scale invariance of the Einstein equations. In order to study the flow of the RG around this fixed point, the linear perturbation analysis is used. We impose the perturbation on the regularity at the boundary where the radial coordinate \( r \) equals zero or infinity. This boundary means that the area radius equals zero or infinity in the case of \( c = 0 \), on the other hand, in the case of \( c > 0 \), it equals a finite or infinity. The fixed point is the Tolman-Bondi solution with self-similarity, which includes the flat Friedmann universe. The behavior of the fixed point is separated into two types. Both types have many relevant modes of the perturbation.

The Tolman-Bondi solution with self-similarity is unstable against almost all spherical modes of linear perturbation. The spacetime will deviate from this fixed point. It is necessary to study the non-spherical mode of perturbation to say something more definite. In the cosmological problem, only the statistical quantities are meaningful if we think of comparison with observations. There are some works in the RG approach \[8–10\] on the universe which has a hierarchical structure. We may contemplate further development of the RG approach to cosmology formulated in the present work by introducing some kind of volume or statistical average for observables like energy density and the Hubble constant of the universe. The statistical concept is needed not only for comparison with observations but also for us to proceed further in the analysis of the RG equation because we have continuously many relevant (growing) modes around the fixed points. That is, the long-time behavior of the universe is sensitive to the initial configuration which we have no a priori control and we have to consider statistical likelihood of the initial values.

We remark that the introduction of volume average in a finite region of the universe potentially introduces the scale invariance violation by hand because the exact scale invariance holds only for an infinite space. Note that in quantum field theories and statistical physics of the second order phase transition the scale invariance violations are hidden in the form of cut-off of the spectrum of physical modes. We shall elaborate our present observation in our future work.

The self-similar solution given by Eqs.(3.52) and (3.53) through the fixed point of the RG equation is essentially a function of \( t^{-1/\alpha_F} = t^{1-1/\alpha_F}/t \), which is roughly the fraction of physical distance to the horizon scale of the Friedmann universe. Also note that in the case of non-linear diffusion equation Eq.(2.6) implies the self-similar solution is a function of the ratio of the distance \( r \) to the diffusion length \( \sqrt{t} \). In the both cases, the self-similar solution is a function of the distance in the unit of physically relevant time dependent scale. We believe this is a general phenomenon and the physical background of the RG equation which governs how dynamical variables deviate from the self-similar solution.

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APPENDIX A: TOLMAN-BONDI SOLUTION WITH SELF-SIMILARITY

In the spherically symmetric universe filled with dust, the most general solution of the Einstein equations is the Tolman-Bondi solution \[11\]:

\[
A(r, t) = \frac{B'(r, t)}{\sqrt{1 + C_1(r)}}, \tag{A1}
\]

\[
B(r, t) = \begin{cases} 
\left(\frac{9C_2(r)}{4}\right)^{1/3} \left[t - C_3(r)\right]^{2/3} & \text{for } C_1(r) = 0 \\
\frac{C_2(r)}{2C_1(r)}(\cosh \eta - 1), & \begin{cases} 
t - C_3(r) = \frac{C_2(r)}{2C_1(r)}(\sinh \eta - \eta) & \text{for } C_1(r) > 0 \\
C_3(r) = \frac{C_2(r)}{2C_1(r)}(\eta - \sin \eta) & \text{for } C_1(r) < 0 
\end{cases}
\end{cases} \tag{A2}
\]
\[ \kappa \rho(r, t) = \frac{C_2'(r)}{B^2 B'}, \tag{A3} \]

where \( C_1(r), C_2(r), \) and \( C_3(r) \) are arbitrary functions of \( r \) and a prime denotes the derivative with respect to \( r \). By taking \( C_1(r) = c, C_2(r) = 4r^\alpha/9, \) and \( C_3(r) = pr^\alpha, \) we obtain the Tolman-Bondi solution with self-similarity Eqs. (3.59)–(3.67).

As for the calculation of linear perturbation, since we concentrate on the spherical modes of perturbation around a self-similar solutions it is enough to consider the linear perturbation of arbitrary functions, \( C_1(r), C_2(r), \) and \( C_3(r) \). The perturbed quantities, \( \delta C_1(r), \delta C_2(r), \) and \( \delta C_3(r) \), can be expressed by a superposition of modes with different \( \omega \) and taken in the following form,

\[ \delta C_1(r) = c_1 r^{\omega'}, \tag{A4} \]
\[ \delta C_2(r) = c_2 r^{\omega'+\alpha}, \tag{A5} \]
\[ \delta C_3(r) = c_3 r^{\omega'+\alpha}. \tag{A6} \]

By coordinate transformation of \( r, \)

\[ r \longrightarrow r + F(r), \tag{A7} \]

where \( F(r) \) is an arbitrary function of \( r \), we obtain the gauge mode of linear perturbation. This function \( F(r) \) also can be expressed by the superposition of modes with different \( \omega \) in the form,

\[ F(r) = f r^{\omega+1}. \tag{A8} \]

\[ \begin{align*}
[13] & The RG equations also can be seen as the Einstein equations in the new coordinate system \((\tau, \xi, \theta, \phi)\), where 
& \( \tau = (\ln t)/\alpha, \) \( \xi = r t^{1/\alpha}, \) \( A(\xi, \tau) = t^{(1-\alpha)/\alpha} A(r, t), \) and \( B(\xi, \tau) = t^{-1} B(r, t). \) One might be puzzled by 
& observing that in this new coordinate system, the lapse function can be zero at \( \xi^2 A = \alpha^2. \) However, there 
& are no physical meaning of this point because the Einstein equations are found to be regular at this point.
\]
FIG. 1. TBSS represents the Tolman-Bondi solution with self-similarity in the case of $c = 0$ and $p < 0$. The axes correspond to the modes of linear perturbation Eq.(3.76).

FIG. 2. TBSS represents the Tolman-Bondi solution with self-similarity in the case of $c > 0$ and $p < 0$. The axes correspond to the modes of linear perturbation Eq.(3.83).