Hamiltonian Thermodynamics of Black Holes in Generic 2-D Dilaton Gravity

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Abstract

We consider the Hamiltonian mechanics and thermodynamics of an eternal black hole in a box of fixed radius and temperature in generic 2-D dilaton gravity. Imposing boundary conditions analogous to those used by Louko and Whiting for spherically symmetric gravity, we find that the reduced Hamiltonian generically takes the form:

\[ H(M, \phi_+) = \sigma_0 E(M, \phi_+) - \frac{N_0}{2\pi} S(M) \]

where \( E(M, \phi_+) \) is the quasilocal energy of a black hole of mass \( M \) inside a static box (surface of fixed dilaton field \( \phi_+ \)) and \( S(M) \) is the associated classical thermodynamical entropy. \( \sigma_0 \) and \( N_0 \) determine time evolution along the world line of the box and boosts at the bifurcation point, respectively. An ansatz for the quantum partition function is obtained by fixing \( \sigma_0 \) and \( N_0 \) and then tracing the operator \( e^{-\beta H} \) over mass eigenstates. We analyze this partition function in some detail both generically and for the class of dilaton gravity theories that is obtained by dimensional reduction of Einstein gravity in \( n+2 \) dimensions with \( S^n \) spherical symmetry.

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I. INTRODUCTION

Although it has been more than twenty years since its discovery, the thermodynamic behaviour of black holes [1,2] is still not well understood. One of the most compelling mysteries is the microscopic source of the Bekenstein-Hawking entropy. Some progress has been made recently in specific contexts. String theory [3] has successfully accounted for the microscopic states for certain extremal and near extremal black holes. A completely different and more geometrical approach has been proposed by Carlip [4] and by Balachandran and collaborators [5]. According to this proposal black hole boundary conditions give rise to surface terms at the horizon that break the diffeomorphism invariance of the theory. Diffeomorphism invariance can only be restored by assuming the existence of new, physical modes that live on the horizon\(^1\). Carlip [4] has counted the resulting edge states in the gauge theory formulation of 2+1 anti-deSitter gravity to obtain the correct entropy of BTZ black holes [6]\(^2\). Another interesting proposal, due to Jacobson [8], invokes Sakharov’s theory of induced gravity [9]. In this approach gravity emerges as a macroscopic, bulk theory obtained by integrating out quantum fields in the effective action. Frolov, Fursaev and Zelnikov [10] have shown that induced gravity can in at least some cases successfully account for black hole entropy in terms of microscopic states of the underlying quantized fields.

The fact that these very diverse approaches can all be made to work, suggests that the correct explanation for black hole entropy may in some ways be universal [10]: it should not depend on the form of the macroscopic gravitation theory, nor on the details of the underlying microscopic quantum theory. This is also emphasized by recent work of Wald [11], who showed that black hole thermodynamics and entropy are generic features of diffeomorphism invariant theories with a curvature term in the action. It is therefore important to examine black hole thermodynamics and statistical mechanics using a variety of methods in as many different theories as possible. Identifying model independent features might provide clues about the geometrical source of black hole entropy.

The purpose of the present paper is to examine the thermodynamics of black holes in a large class of theories in 1+1 spacetime dimensions. Most derivations of the canonical partition function start from the Euclidean action [12]. Recently, however, Louko and Whiting (LW) [13] applied the canonical formalism of Kuchar [14] to derive the Hamiltonian boundary terms for an eternal black hole inside a box of fixed radius and temperature. They restricted consideration to spacelike hypersurfaces that ended on the bifurcation point on the interior of an eternal black hole along a static slice. These boundary conditions are well suited to the study of thermodynamics since they allow for analytic continuation to Euclidean spacetime. Remarkably, LW found that these boundary conditions lead to a surface contribution to the Hamiltonian that was proportional to the entropy. They then used the resulting reduced Hamiltonian to derive the partition function for a canonical ensemble of

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\(^1\) An alternative, and effectively equivalent description states that as a result of the black hole boundary conditions, some diffeomorphism modes on the horizon become physical.

\(^2\) This proposal however seems to need some modification when applied to dilaton gravity theories in 1+1 spacetime dimensions [7].
black holes. The same techniques have since been applied to string inspired gravity [15] by Bose et al [16], to Reissner-Nordstrom-anti de Sitter black holes [17] and to Lovelock black holes [18]. In the following, we will show that the results of Louko and Whiting generalize easily to the case of generic vacuum 2-D dilaton gravity [19]. In particular, the form of the reduced Hamiltonian and resulting partition function is essentially model independent. The classical thermodynamic entropy arises generically as a surface term at the bifurcation point and the resulting Hamiltonian looks more like a free energy [20] than a Hamiltonian for a pure quantum system. This result appears to lend support to Jacobson’s conjecture [8] that the gravitational action is best thought of as an effective theory describing the bulk properties of an as yet unknown underlying microscopic theory.

Instead of using the powerful, but technically involved formalism of Kuchar [14], we apply the simpler techniques applied extensively to generic dilaton gravity in a variety of recent papers [21]-[27]. The key steps in this method are to reparametrize the theory so as to make the generic action take a very simple form, and then to rewrite the Hamiltonian constraint as the spatial divergence of the mass observable [14,28,21]. Our analysis shows in a clear and completely general way how the thermodynamical entropy emerges from the canonical analysis. Moreover we are able to analyse the semi-classical limit of the resulting thermodynamic partition function generically to show the emergence of the usual Bekenstein Hawking entropy/temperature relations. Finally we analyze in detail an interesting sub-class of models obtained by dimensional reduction of Einstein gravity in n+2 dimensions. We will show that the qualitative features of the partition function are more or less independent of n: the specific heat is always positive and there is for all n a phase transition from a high temperature, semi-classical phase to a low temperature quantum phase dominated by zero mass (Planckian) black holes. This agrees with previous work [12,13] for spherically symmetric gravity (which corresponds to the n = 2 case). For n = 2 the transition is first order, but as n increases the transition “weakens”, i.e. resembles more closely a second order transition. In the n → ∞ limit, for which the theory resembles string inspired dilaton gravity [15], the transition appears to be strictly second order.

The paper is organized as follows: In Section 2, we review some important features of generic dilaton gravity, including the action, space of solutions and classical black hole thermodynamics. Section 3 reviews the Hamiltonian analysis for the given boundary conditions, and derives the necessary surface terms. The generic partition function for a canonical ensemble of black holes in a box at fixed temperature is derived and analyzed in the semi-classical approximation in Section 4. Section 5 contains results specific to dimensionally reduced Einstein Gravity in n+2 dimensions with n-dimensional spherical symmetry (SnG). Section 6 closes with conclusions and prospects for future work.

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3A preliminary form of this result, for slightly different boundary conditions, was first reported in [25].
The most general action functional depending on the metric tensor $g_{\mu\nu}$ and scalar field $\psi$ in two spacetime dimensions that contains at most second derivatives of the fields can be written [19]:

$$I[g, \psi] = \frac{1}{2G} \int d^2x \sqrt{-g} \left( \frac{1}{2} g^{\alpha\beta} \partial_{\alpha} \psi \partial_{\beta} \psi + \frac{1}{l^2} V(\psi) + D(\psi) R(g) \right).$$

(1)

where $R(g)$ is the Ricci curvature scalar. The dilaton potential, $V(\psi)$, is an arbitrary function of the dilaton field. In the above, the fields $\psi$ and $g_{\mu\nu}$ are taken to be dimensionless, as is the 2-D Newton constant, $G$. This requires the inclusion of a coupling constant, $l$, of dimension length in the potential term.

If $D(\psi)$ is a differentiable function of $\psi$ such that $D(\psi) \neq 0$ and $\frac{dD(\psi)}{d\psi} \neq 0$ for any admissible value of $\psi$ then the kinetic term for the scalar field can be eliminated by means of the (invertible) field redefinition [19,21]:

$$g_{\mu\nu} = \Omega^2(\psi) g_{\mu\nu}$$

(2)

$$\phi = D(\psi)$$

(3)

where

$$\Omega^2(\psi) = \exp \left( \frac{1}{2} \int \frac{d\psi}{(dD/d\psi)} \right)$$

(4)

In terms of the new fields, the action Eq.(1) takes the form:

$$I = \frac{1}{2G} \int d^2x \sqrt{-g} \left( \phi R(g) + \frac{1}{l^2} V(\phi) \right).$$

(5)

where $V$ is defined as:

$$V(\phi) = \frac{\nabla(\psi(\phi))}{\Omega^2(\psi(\phi))}$$

(6)

This reparametrization accomplishes two things: it makes the action and resulting Hamiltonian simpler by eliminating the kinetic term for the scalar and it allows us to classify all possible theories in terms of a single function of the dilaton field, namely the dilaton potential. For example, spherically symmetric 4-D Einstein gravity [29] is described by a dilaton gravity theory with $V \propto 1/\sqrt{\phi}$, while string inspired dilaton gravity [15] corresponds to $V = \text{constant}$.

In the following we consider the action in the form Eq.(5). The field equations are:

$$R + \frac{1}{l^2} \frac{dV}{d\phi} = 0$$

(7)

$$\nabla_{\mu} \nabla_{\nu} \phi - \frac{1}{2l^2} g_{\mu\nu} V(\phi) = 0$$

(8)
It follows directly from the above field equations that all solutions have at least one Killing vector given by [22]

\[ k^\mu = l \epsilon^{\mu\nu} \partial_\nu \phi / \sqrt{-g} \]  

(9)

where the constant \( l \) has been included to ensure that the vector components are dimensionless. Note that the dilaton field is also constant along this Killing vector.

The most general solution to the field equations in the generic theory depends on one coordinate invariant parameter. In a coordinate system adapted to the Killing vector (the analogue of Schwarzschild coordinates), the solution takes the form [22,23]:

\[ ds^2 = -(j(\phi) - 2GlM)dt^2 + \frac{1}{(j(\phi) - 2GlM)}dx^2, \]

\[ \phi = x/l, \]  

(10)

where \( j(\phi) = \int_0^\phi d\tilde{\phi}V(\tilde{\phi}) \) and the parameter \( M \) will turn out to be the ADM mass. In these coordinates, the Killing vector points along the time axis, from which one can easily deduce that its norm is

\[ |k|^2 = -l^2|\nabla \phi|^2 = (2GlM - j(\phi)). \]  

(11)

Given the above equation, it is clear that the general solution has an apparent horizon at the surface \( \phi = \phi_0 = \text{constant} \) for \( \phi_0 \) given by

\[ 2GlM = j(\phi_0) \]  

(12)

Whether or not this is an event horizon depends on the global properties of the solution, which in turn depends on the form of the function \( j(\phi) \). We assume that \( j(0) = 0 \) and that \( j(\phi) \) goes to infinity monotonically as \( \phi \to \infty \) so that \( \phi_0 \) is unique. If in addition the surface \( \phi = 0 \) is excluded from the manifold\(^4\) the resulting spacetime has precisely the same global structure as the radial part of a Schwarzschild black hole.

It is worth emphasizing that the global spacetime structure in these models derives not only from the spacetime metric, but from the structure of the dilaton field as well. This is not unreasonable given that one cannot say \textit{a priori} which metric is physical\(^5\), \( g_{\mu\nu} \) or for example \( f(\phi)g_{\mu\nu} \). In fact in the present parametrization, the metric corresponding to the vacuum or matter free solution does not have all the properties normally expected for an isolated Schwarzschild black hole. The Ricci scalar in Eq.(7), which is the only independent curvature invariant in two spacetime dimensions, does not vanish in the asymptotic region \( \phi \to \infty \), in general. More importantly, the curvature does not depend directly on the mass parameter, and is non-vanishing when \( M = 0 \). Thus the vacuum metric is not Minkowskian.

\(^4\)The effective Newton constant is infinite at \( \phi = 0 \), and the field redefinitions that we used to get to the simplified action in general are singular there.

\(^5\)i.e. to which metric ordinary matter is minimally coupled.
It is therefore reassuring that we can do a conformal reparametrization to a “physical metric” which does have most of the expected properties. In particular, define $\tilde{g}$ by

$$\tilde{g}_{\mu\nu} = \frac{1}{j(\phi)}g_{\mu\nu}$$  \hspace{1cm} (13)$$

By transforming coordinates to $\tilde{x}$ such that

$$d\tilde{x} = \frac{d\phi}{j(\phi)}$$  \hspace{1cm} (14)$$

one can see that the physical metric takes the usual Schwarzschild form:

$$d\tilde{s}^2 = \left(1 - 2GM/\tilde{j}(x)\right)dt^2 - \left(1 - 2GM/\tilde{j}(x)\right)^{-1}d\tilde{x}^2$$  \hspace{1cm} (15)$$

where $\tilde{j}(x) \equiv j(\phi(\tilde{x}))$. Note that this metric approaches the Minkowski metric as $\phi \to \infty$ providing $j(\phi) \to \infty$ in this limit. Moreover, the Ricci scalar

$$R(\tilde{g}) = \left[\frac{GM}{l}(V' - \frac{2V^2}{j})\right]$$  \hspace{1cm} (16)$$

does vanish when $M = 0$. This metric is asymptotically flat and has a curvature singularity at $\phi = 0$ for potentials of the form $V = k\phi^a$, where $a < 1$.

The thermodynamic properties of these black hole solutions are derived in [22]. Here we merely quote the results. The Bekenstein-Hawking entropy for black holes in generic dilaton gravity is:

$$S(M) = \frac{2\pi}{G}\phi_0$$  \hspace{1cm} (17)$$

where $\phi_0$ is the value of the dilaton at the horizon:

$$\phi_0 = j^{-1}(2GMl)$$  \hspace{1cm} (18)$$

The corresponding Hawking temperature is:

$$T_H(M) = \frac{V(\phi_0)}{4\pi l}$$  \hspace{1cm} (19)$$

It is important to note that the conformal reparametrizations expressed in Eq.(3) and Eq.(13) do not affect the classical thermodynamics.

III. HAMILTONIAN ANALYSIS AND BOUNDARY TERMS

The Hamiltonian analysis for generic dilaton gravity has been presented in many works. Here we summarize the results, using the notation and conventions of [24]. We start by decomposing the metric as follows:

$$ds^2 = e^{2\rho} \left[-u^2 dt^2 + (dx + vdt)^2\right]$$  \hspace{1cm} (20)$$
where $x$ is a local coordinate for the spatial section $\Sigma$ and $\rho$, $u$ and $v$ are functions of spacetime coordinates $(x,t)$. For convenience we work with the form of the action in Eq.(5). The transition to the physical metric can be done by a simple point canonical transformation that mixes $\rho$ and $\phi$, but leaves the lapse and shift functions unchanged. In terms of the parametrization Eq.(20), the action Eq.(5) takes the form (up to surface terms):

$$I = \int dt \int_{\sigma_-}^{\sigma_+} dx \left[ \frac{1}{G} \left( \dot{\phi} (vp' + v' - \dot{\rho}) + \frac{\phi'}{u} (uu' - vv' + v\dot{\rho} + u^2 \rho' - v^2 \rho') \right) \right.$$ 

$$+ \frac{1}{2} u e^{2\rho} V(\phi) \frac{l^2}{l^2} \right]$$

(21)

In the above dots and primes denote differentiation with respect to time and space, respectively, while $\sigma_+$ and $\sigma_-$ are the outer and inner spatial boundaries. The canonical momenta for the fields $\{\phi, \rho\}$ are:

$$\Pi_\phi = \frac{1}{Gu} (vp' + v' - \dot{\rho})$$

(22)

$$\Pi_\rho = \frac{1}{Gu} (-\dot{\phi} + v\phi')$$

(23)

The momenta conjugate to $u$ and $v$ vanish: these fields play the role of Lagrange multipliers that are needed to enforce the first class constraints associated with diffeomorphism of the classical action. A straightforward calculation leads to the canonical Hamiltonian (up to surface terms which will be discussed below):

$$H_c = \int dx \left( vF + \frac{u}{2G}G \right)$$

(24)

where

$$P = \rho' \Pi_\rho + \phi' \Pi_\phi - \Pi'_\rho \sim 0$$

(25)

$$G = 2\phi'' - 2\phi' \rho' - 2G^2 \Pi_\phi \Pi_\rho - e^{2\rho} V(\phi) \frac{l^2}{l^2} \sim 0$$

(26)

are secondary constraints.

Since 2-D dilaton gravity obeys a generalized Birkhoff theorem [23] we expect there to be only one independent, diffeomorphism invariant physical observable, namely the mass of the black hole. The phase space, however, must have even dimension, and it turns out that there is one physical pair of canonical variables. The first of these is most easily derived by defining the following linear combination of constraints:

$$\tilde{\mathcal{E}} := l e^{-2\rho} (-\phi'G + GP_\rho P)$$

$$= \frac{\partial M}{\partial x}$$

(27)

where:

$$M := \frac{l}{2G} \left( e^{-2\rho} (G^2 \pi_\rho^2 - (\phi')^2) + \frac{j(\phi)}{l^2} \right) = \frac{1}{2Gl} (|k|^2 + j(\phi))$$

(28)
Clearly $\mathcal{M}$ is a constant on the constraint surface. One can verify that it commutes weakly with the constraints. As discussed in [22], the constant mode of $\mathcal{M}$ is a physical observable, corresponding to the ADM mass of the solution\(^6\). We henceforth call $\mathcal{M}$ the mass observable, to distinguish it from the total Hamiltonian. Its conjugate momentum [24]

$$P_{\mathcal{M}} = -\frac{G}{l} \int dx \frac{e^{2\rho} \pi_{\rho}}{[(G \pi_{\rho})^2 - (\phi')^2]}$$

is only invariant under diffeomorphisms that vanish at the boundaries of the system. Thus, the Hamiltonian analysis is consistent with the generalized Birkhoff theorem. It is possible, following Kuchar [14] to do a canonical transformation to variables such that $\{\mathcal{M}, P_{\mathcal{M}}\}$ are used as one pair of the phase space variables but this will not be necessary for our purposes.

In terms of the new constraint, the canonical Hamiltonian is:

$$H_c = \int_{\sigma_-}^{\sigma_+} dx (-\tilde{u} \mathcal{M}' + \tilde{v} \mathcal{P}) + H_+ - H_-$$

where

$$\tilde{u} = \frac{ue^{2\rho}}{l\phi'}$$

$$\tilde{v} = v + \frac{uG \Pi_{\rho}}{\phi'}$$

$H_+$ and $H_-$ are boundary terms determined by the requirement that the surface terms in the variation of $H_c$ vanish for a given set of boundary conditions.

For concreteness we will choose black hole boundary conditions considered recently by Louko and Whiting [13] in the case of spherically symmetric gravity. In particular we consider the analogue of an eternal black hole in a box of fixed constant radius. This requires keeping the value of the dilaton fixed and time independent at $u_+$: $\phi(u_+) = \phi_+$, $\phi' = 0$. The latter condition implies that $\tilde{v}_+ = 0$, so that the only contribution to $H_+$ will come from the first term in the Hamiltonian:

$$\delta H_+ (\mathcal{M}) = \tilde{u} \delta \mathcal{M}|_{\sigma_+}$$

From $\tilde{v}_+ = 0$ and Eq.(32) it follows that $\Pi_{\phi}|_{\sigma_+} = -v \phi'/uG|_{\sigma_+}$. After substituting this equation into the definition of the mass observable, a bit of algebra yields:

$$\tilde{u}_+^2 = \frac{l^2 g_{tt}}{2G \mathcal{M}l - j(\phi)}|_{\sigma_+}$$

Following [13] we fix the metric along the outer boundary, $g_{tt}|_{\sigma_+} = constant = g_{tt}^+$, in which case Eq.(33) can be integrated to yield:

\(^6\)It also corresponds to the Casimir invariant that characterizes solutions in the Poisson sigma model approach [30].
\[
H_+(\mathcal{M}) = \frac{Q^+(\phi_+)}{l G} \left(1 - \sqrt{1 - \frac{2GMl}{j(\phi_+)}}\right)
\]
fact find exact physical eigenstates of the mass observable \[21,25,26\]. Alternatively, one can reduce the phase space to the physical degrees of freedom at the classical level and then quantize. Here we will follow the program of Louko and Whiting \[13\] and do the latter. On the constraint surface \(\mathcal{M}\) is independent of the spatial coordinates. The physical phase space consists of the mass observable \(\mathcal{M}|_{\text{phys}}\) and its canonical conjugate. It is convenient to introduce a dimensionless mass parameter:

\[
M := \mathcal{M} l
\]  

(42)

and dimensionless box size

\[
B := j(\phi_+) / G
\]  

(43)

In terms of these, the reduced action is simply:

\[
I = \int dt (P_M \dot{M} - H(M; B, N_0))
\]  

(44)

with Hamiltonian:

\[
H(M; B, N_0)) = E(M, B) - \frac{N_0}{2\pi} S(M).
\]  

(45)

\(E(M, B)\) is the quasi-local energy given by:

\[
E(M, B) = \frac{j(\phi_+)}{l G} \left(1 - \sqrt{1 - \frac{2GM}{j(\phi_+)}} \right) = \frac{B}{l} \left(1 - \sqrt{1 - \frac{2M}{B}} \right)
\]  

(46)

while \(S(M)\) classical entropy defined in the previous section:

\[
S(M) = \frac{2\pi}{G} j^{-1}(2GM) .
\]  

(47)

Note that we have set \(Q_+ = 1\) without loss of generality. In addition to the dynamical variable \(M\), the Hamiltonian \(H(M; B, N_0)\) depends on the size, \(B\), of the box in which the black hole is placed and the rate of change of the unit normal to constant \(t\) surfaces at the interior point, as given by \(N_0\).

We would like to compute the quantum partition function:

\[
Z(\beta; B, N_0) = \text{Tr} \left[ e^{-\beta \hat{H}(M; B, N_0)} \right]
\]  

(48)

where the trace is over all physical mass eigenstates. This requires knowledge of the mass spectrum for the theory. However, since we are effectively working in action-angle variables, we cannot gain any information about the mass spectrum without making further assumptions. For example, in reference \[27\] it was shown that by assuming periodicity of the angle variable \(P_M\), it is possible to derive a discrete mass spectrum for Euclidean black holes. If the period corresponded to the inverse Hawking temperature, as required to make the Euclidean soliton solutions regular, the spectrum was given by:

\[
S(M) = 2\pi n
\]  

(49)
for any positive integer \( n \). Remarkably, this is the same spectrum as obtained via Dirac quantization of Euclidean black holes in the generic theory \([25]\). Arguments also exist for the quantization of mass in the Lorentzian sector as well \([32]-[36]\)\(^7\).

In the following, we assume that the spectrum of the mass operator is bounded below by zero and above by \( M_+ = j(\phi_+)/2Gl = B/2 \), so that the exterior boundary remains outside the horizon of the corresponding black hole. In this case the partition function can formally be written as a sum over mass eigenstates \( |M> \):

\[
Z(\beta; B, N_0) = \int_0^{B/2} dM \mu(M) <M|M|e^{-\beta H}|M>
\]

\[
= \int_0^{B/2} dM \mu(M) <M|M|e^{-\beta E(M;B)+\frac{\beta N_0}{2}\pi S(M)}
\]

One important question concerns the choice of measure \( \mu(M) \). It should be a smooth function of positive weight. The specific form of \( \mu(M) \) should not affect the qualitative features of the partition function, which we expect to be dominated by the exponentials. For simplicity, we therefore choose

\[
\mu(M) = 1
\]

A more rigorous derivation of the measure should be possible within the Dirac quantization procedure of \([25]\), for example. This is currently under investigation.

The expression Eq.(51) for the partition function is divergent because the states \(|M>\) are not normalizable. As first argued in \([13]\) it is reasonable to regulate this expression by replacing

\[
<M|M> = \delta(0)
\]

by the inverse of the volume of the configuration space so that the regularized partition function is:

\[
Z(\beta; B, N_0) = \frac{2}{B} \int_0^{B/2} dM e^{-\beta E(M;B)+\frac{\beta N_0}{2}\pi S(M)}
\]

The only remaining ambiguity in this expression is the choice of \( N_0 \). Although our calculation is Lorentzian, we allow ourselves to be motivated by consistency with the Euclidean path integral, and choose:

\[
N_0 = \frac{2\pi}{\beta}
\]

which is the value needed to avoid the conical singularity in the Euclidean sector for solutions periodic in time with period equal to the inverse temperature \( \beta \). It is worth remarking that in the present context this choice seems somewhat artificial: it leads to a Hamiltonian in

\(^7\)An extensive list of references is given in \([35]\).
the Lorentzian framework that is explicitly temperature dependent. We will nonetheless follow the “traditional path” and see that it gives rise to potentially interesting physics.

Finally, our assumptions have lead us to the following quantum partition function for generic dilaton gravity:

\[ Z(\beta; B) = \frac{2}{B} \int_0^{B/2} dM e^{S(M)} e^{-\beta E(M; B)} \]  

(56)

It is interesting that the classical thermodynamical entropy generically enters the integral as the logarithm of an apparent degeneracy of states with mass \( M \). However it must be remembered that this derivation does not explain the degeneracy via microscopic states, and therefore does not solve the black hole entropy problem. It merely re-expresses it in a Hamiltonian context, and more importantly, shows that it is a completely generic feature of the class of theories considered.

We close this Section by examining the partition function in the semi-classical approximation. To this end we write the partition function as:

\[ Z(\beta, B) = \frac{2}{B} \int_0^{B/2} dM e^{-I(M; \beta, B)} \]  

(57)

where

\[ I \equiv \beta E(M, B) - S(M) \]  

(58)

It is clear that if a stable minimum \( \overline{M}(\beta, B) \) of \( I \) exists, then the partition function can be approximated to first order in the semi-classical approximation by:

\[ Z(\beta, B) \sim e^{-I(\overline{M}, B)} \]  

(59)

so that the free energy obtained from the partition function is:

\[ \mathcal{F} = -T \ln(Z) = I(\overline{M}, B)/\beta = E(\overline{M}, B) - T S(\overline{M}) \]  

(60)

The minimum \( \overline{M} \) is found as usual at the extremum of the free energy:

\[ \left. \frac{\partial \mathcal{F}}{\partial M} \right|_{\overline{M}} = \left. \frac{\partial E(M, B)}{\partial M} \right|_{\overline{M}} - T \left. \frac{\partial S(M)}{\partial M} \right|_{\overline{M}} = 0 \]  

(61)

Using the fact that

\[ \left. \frac{\partial E(M, B)}{\partial M} \right|_{\overline{M}} = \frac{1}{l \sqrt{1 - \frac{2\overline{M}}{B}}} \]  

(62)

and that

\[ \text{GK is grateful to Valeri Frolov for useful discussions on this issue.} \]

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is the asymptotic inverse Hawking temperature associated with a black hole of mass $M$, we find:

$$\beta = \sqrt{1 - \frac{2M}{B}} \beta_H(M)$$

Thus, in the semi-classical approximation, the inverse temperature $\beta$ of the box is related to the Hawking temperature by the usual blue shift factor. Note that the blue shift is calculated relative to the physical metric introduced in Section 2. This is because the Hamiltonian was normalized with respect to this metric.

One can also verify that in the semi-classical approximation, the entropy $S_{GF}$ of the gravitational field obtained from the partition function is equal to the classical thermodynamic entropy $S(M)$:

$$S_{GF} = \left(1 - \beta \frac{\partial}{\partial \beta}\right) \ln Z = S(M)$$

V. S$^N$ SPHERICAL GRAVITY

We now consider the case of dimensionally reduced Einstein gravity in $n+2$ dimensions with $S^n$ spherical symmetry (SnG). Higher dimensional black holes of this form were analyzed in detail at the classical level in [33]. For $n = 2$ this theory corresponds precisely to spherically symmetric gravity, while in the $n = \infty$ limit it goes over to the String Inspired Dilaton gravity model [SIG]. We start with the Einstein-Hilbert action in $n+2$ spacetime dimensions:

$$I_{EH}^{(n+2)} = \frac{1}{16\pi G^{(n+2)}} \int d^{n+2}x \sqrt{-g^{(n+2)}} R^{(n+2)}$$

and impose spherical symmetry via the ansatz:

$$ds^2_{(n+2)} = \bar{g}_{\alpha\beta}(x,t) dx^\alpha dx^\beta + r^2(x,t) d\Omega^{(n)}$$

In the above, $x$ is a radial coordinate, $\{x^\alpha = x, t\}$ and $\Omega^{(n)}$ is the volume form on the unit $n$-sphere and $r(x,t)$ is the invariant radius of an $n$-sphere at $(x,t)$.

The dimensionally reduced action takes the form:

$$I_{EH}^{(n+2)} = \frac{\mathcal{V}^{(n)}}{16\pi G^{(n+2)}} \int d^2x \sqrt{-\bar{g}} r^n \left( R(\bar{g}) + \frac{n(n-1)}{r^2} + n(n-1) \left| \nabla_\psi \right|^2 \right)$$

where the volume of the unit $n$-sphere is:

$$\mathcal{V}^{(n)} = \int \Omega^{(n)} = \frac{2\pi^{(n+1)/2}}{\Gamma(\frac{1}{2}(n+1))}$$
Eq. (68) is of the form of a generic dilaton gravity theory in 1+1 dimensions. We now define a new, dimensionless scalar $\bar{\psi}$:

$$
\bar{\psi}(x, t) = \left(\frac{r}{l}\right)^{\frac{n}{2}}
$$

(70)

$l$ is an arbitrary constant with dimension length, which we take without loss of generality to be the Planck length in $n + 2$ dimensions:

$$
l^n \equiv G^{(n+2)}
$$

(71)

In terms of $\bar{\psi}$, the action Eq. (68) takes the form Eq. (1) providing we make the following identifications:

$$
\frac{1}{2G} = \frac{8(n - 1)V^{(n)}}{16\pi n}
$$

(72)

$$
D(\bar{\psi}) = \frac{n}{8(n - 1)}\bar{\psi}^2
$$

(73)

$$
\nabla(\bar{\psi}) = \frac{n^2}{8}\bar{\psi}^{(2n-4)/n}
$$

(74)

We can now put the action in the canonical form Eq. (5) by the conformal reparametrization Eq. (3) with

$$
\phi = D(\bar{\psi}) = \frac{n}{8(n - 1)}\bar{\psi}^2
$$

(75)

and

$$
\Omega^2(\bar{\psi}) = C\bar{\psi}^{\frac{2(n-1)}{n}}
$$

(76)

where $C$ is a constant of integration. The transformed dilaton potential is:

$$
V(\phi) = \frac{n(n - 1)}{C} \left(\frac{n}{8(n - 1)}\right)^{\frac{n+1}{n}} \phi^{-1/n}
$$

(77)

Note that the dilaton field can be expressed in terms of the invariant radius $r$ as follows:

$$
\phi = \frac{n}{8(n - 1)} \left(\frac{r}{l}\right)^{n}
$$

(78)

In order to determine the constant $C$ in Eq. (76), recall that the ADM mass Eq. (36) was derived by normalizing the time component of the physical metric $\tilde{g}_{\mu\nu} = g_{\mu\nu}/j(\phi)$ to be unity at spatial infinity. In the present context, we would like the physical metric to coincide with the projection $\tilde{g}_{\mu\nu} = \Omega^{-2}g_{\mu\nu}$ of the higher dimensional metric. Clearly, this will be true if $\Omega^2 = j(\phi)$, which requires:

$$
C = \frac{n^2}{8(n - 1)}
$$

(79)
With this choice, the dilaton potential is:

\[
V(\phi) = (n - 1) \left( \frac{n}{8(n - 1)} \right)^{1/n} \phi^{-1/n} \tag{80}
\]

We can now apply the results of generic dilaton gravity to investigate the thermodynamics of black holes in SnG. In terms of the dimensionless mass parameter \( M = \mathcal{M} l \), the entropy is:

\[
S(M) = \frac{2\pi}{G} j^{-1}(2GM) \tag{81}
\]

\[
= \frac{2\pi}{G} \left( \frac{8(n - 1)}{n} \right)^{1/(n-1)} \left( \frac{2GM}{n} \right)^{n/(n-1)}
\]

\[
= 4\pi \left( \frac{16\pi}{n^n V(n)} \right)^{1/(n-1)} (M)^{n/(n-1)}
\]

\[
= c(n) M^{\frac{n}{n-1}}
\]

where we have used Eq.(72) and defined

\[
c(n) = 4\pi \left( \frac{16\pi}{n^n V(n)} \right)^{1/(n-1)} \tag{82}
\]

We note for future reference that \( c(n) \to 4\pi/\sqrt{\pi n} \to 0 \) as \( n \to \infty \).

If we express Eq.(82) in terms of the invariant radius of the horizon(Eq.(78)), we get the expected result that the entropy is one quarter of the area of the horizon expressed in Planck units [33]:

\[
S(M) = \frac{1}{4} \frac{r_0^n V(n)}{G(n+2)} \tag{83}
\]

A straightforward calculation reveals that the Hawking temperature for black holes in SnG is:

\[
T_H(M) = \frac{(n - 1)}{4nc(n)l} M^{\frac{1}{n-1}} \tag{84}
\]

For \( n = 2 \) the above expressions give \( S = 4\pi G^{(4)} M^2 \) and \( T_H(\mathcal{M}) = 1/8\pi G^{(4)} \mathcal{M} \) which are the correct entropy and temperature for a Schwarzschild black hole. As \( n \to \infty \), \( S \to 4\pi \mathcal{M} l/\sqrt{\pi n} \), while \( T_H \to \sqrt{\pi n}/4\pi l \). These are the entropy and temperature of black holes in String Inspired Gravity, up to a factor of \( \sqrt{\pi n} \).

We will now examine in detail the partition function Eq.(57). In the present case, it takes the form:

\[
Z(\beta, B) = \frac{2}{B} \int_0^B dM e^{-I(M, \beta, B)} \tag{85}
\]

where the “action” \( I \) is:
\[ I = \tilde{\beta} B \left( 1 - \sqrt{1 - \frac{2M}{B}} \right) - c(n)(M)^{n/n-1} \]  

(86)

where \( \tilde{\beta} = \beta/l \) is a dimensionless inverse temperature. First of all we note that this partition function describes a thermodynamically stable system (i.e positive specific heat) for all finite \( n \). Moreover there exists an interesting phase structure. Numerical plots of the logarithm of the partition function as a function of \( B \) and \( \tilde{\beta} \) for \( n = 2, n = 3 \) and \( n = 9 \) are presented in Figs 1a), 1b) and 1c), respectively. For \( n = 2 \) there is a “kink” in the partition function that signals a phase transition from the semi-classical region at high temperature (low \( \tilde{\beta} \)) to a quantum phase consisting of a gas of microscopic black holes at low temperature\(^9\). This was first described by York and Whiting for spherically symmetric gravity [12]. The transition appears to be strongly first order for large box size, but weakens as \( B \) decreases. In fact, at very low \( B \) there appears to be a vapour phase, which is neither pure semi-classical nor pure quantum. As the dimension \( n \) is increased, the phase transition appears to weaken for all \( B \). These qualitative features can also be deduced by examining \( I \). This is plotted in Figs 2a), 2b) and 2c) for \( n = 2, 3, 9 \) and fixed box size. At high temperatures the partition function is clearly dominated by a non-zero minimum, \( \overline{M} \) of \( I \), but the system goes through a first order phase transition as the temperature is decreased. However, as \( n \) increases, the transition is weakened: at the critical temperature the value of the mass at the minimum moves towards zero, and the height of the potential barrier decreases. Remarkably, it is possible to solve analytically for the critical temperature as a function of the mass \( \overline{M} \) at the minimum. In the semi-classical phase, \( \overline{M} \) approximates the average black hole mass, and is obtained from:

\[ \frac{\partial I}{\partial \overline{M}} \bigg|_{\overline{M}} = 0 \]  

(87)

which yields:

\[ \tilde{\beta} = \sqrt{1 - \frac{2\overline{M}}{B}} \frac{nc(n)}{n-1} \overline{M}^{\frac{1}{n-1}} \]  

(88)

so that, as indicated in the previous section, the temperature of the box is the red-shifted Hawking temperature for a black hole of mass \( \overline{M} \). At the critical point, we also have that

\[ I(\overline{M}) = \tilde{\beta} B\left( 1 - \sqrt{1 - \frac{2\overline{M}}{B}} \right) - c(n)\overline{M}^{\frac{n}{n-1}} = 0 \]  

(89)

Solving Eq.(88) for \( 2\overline{M}/B \) and substituting into Eq.(89), we find that the (dimensionful) critical temperature is related to the mean mass at the transition point by:

\[ \beta_c = \frac{nc(n)l}{n+1} \overline{M}^{\frac{1}{n-1}} \]  

(90)

This formula can be explicitly verified by examining Fig. 3, where we plot the free energy at the critical point for \( n = 3 \) for a variety of box sizes. Note that as \( n \to \infty \), the critical temperature goes to the constant \( c(n)l \) and is independent of the box size and mean mass.

---

\(^9\) The flattened region at the top of each figure is a consequence of the graphic presentation.
By substituting Eq.(90) into Eq.(89) it is possible to solve for the mean mass at the critical point as a function of the box size. It is remarkably simple:

\[ \overline{M} = \frac{2n}{(n+1)^2} B \]  

(91)

This expression supports our claim that as \( n \) increases, the first order transition becomes weaker: the mean mass at the critical point goes to zero.

Finally we note that, for all finite \( n \) the partition function diverges as the box size is taken to infinity. That is, as \( B \to \infty \),

\[ I \to \tilde{\beta} M - c(n) M^{n/n-1} \]  

(92)

Clearly, the second term dominates for all finite \( n \) for any temperature as \( M \to \infty \), so that the integral will diverge, as claimed.

**VI. CONCLUSIONS**

We have examined in some detail the Hamiltonian thermodynamics of black holes in generic 2-D dilaton gravity. As the SnG example shows, these models can in some cases be thought of as truncated higher dimensional theories, and are therefore not simply “toy models”. One important result of our analysis is that the classical thermodynamic entropy generally contributes to the boundary term at the bifurcation point in such a way as to make the Hamiltonian look like a free energy [20]. The consequences of this were examined in detail for the partition function of SnG.

The main puzzle in black hole thermodynamics is how the most simple gravitational system (i.e. a spherically symmetric black hole) can have sufficient complexity to account for the Beckenstein-Hawking entropy. This puzzle is highlighted by our analysis, which shows explicitly in a Hamiltonian context how an apparent “degeneracy” of states generically arises in the partition function, despite the fact that the mass eigenstates are non-degenerate. This interesting behaviour can be interpreted as support for the conjecture by Jacobson [8,10] that the gravitational action should be thought of as a bulk action induced by the interactions of microscopic quantum fields. In the context of such an interpretation, it is natural that the Hamiltonian for the gravitational field encodes the degeneracy of the states in the fundamental microscopic theory.

**VII. ACKNOWLEDGEMENTS**

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REFERENCES


Fig 1a: Partition Fn for n=2
Fig 1b Partition Fn for $n=3$
Fig 1c. Partition Fn for n=9
Fig 2a) Action, \( I \), Near Critical Temperature \( n=2, B=5 \)
Fig 2b) Action, I, Near Critical Temperature n=3, B=5
Fig 2c) Action, I, Near Critical Temperature n=9, B=5
Fig 3. Critical Action n=3