Abelian-Projected Effective Gauge Theory of QCD with Asymptotic Freedom and Quark Confinement

Kei-Ichi Kondo\textsuperscript{1†}

\textsuperscript{1} Department of Physics, Faculty of Science, Chiba University, Chiba 263, Japan

\textsuperscript{†} E-mail: kondo@cuphd.nd.chiba-u.ac.jp

Abstract

Starting from SU(2) Yang-Mills theory in 3+1 dimensions, we prove that the abelian-projected effective gauge theories are written in terms of the maximal abelian gauge field and the dual abelian gauge field interacting with monopole current. This is performed by integrating out all the remaining non-Abelian gauge field belonging to SU(2)/U(1). We show that the resulting abelian gauge theory recovers exactly the same one-loop beta function as the original Yang-Mills theory. Moreover, the dual abelian gauge field becomes massive if the monopole condensation occurs. This result supports the dual superconductor scenario for quark confinement in QCD. We give a criterion of dual superconductivity and point out that the monopole condensation can be estimated from the classical instanton configuration. Therefore there can exist the effective abelian gauge theory which shows both asymptotic freedom and quark confinement based on the dual Meissner mechanism. Inclusion of arbitrary number of fermion flavors is straightforward in this approach. Some implications to lower dimensional case will also be discussed.
1 Introduction

It is one of the most important problems in particle physics to clarify the physical mechanism which realizes the quark and gluon confinement. An important question is what is the most relevant degrees of freedom to describe the confinement. In the mid-1970's, an idea of the dual Meissner vacuum of quantum chromodynamics (QCD) was proposed by Nambu [1], 't Hooft [2] and Mandelstam [3]. In this scenario, the monopole degrees of freedom plays the most important role in the confinement. This aspect can be seen explicitly through a procedure called the *abelian projection* by 't Hooft [2]. Under the abelian projection, the non-Abelian gauge theory can be regarded as an abelian gauge theory with magnetic monopole [4]. For the confinement mechanism, there are other proposals [5] which we do not discuss in this paper.

The abelian projection [2] is to fix the gauge in such a way that the maximal torus group of the gauge group $G$ remains unbroken. It goes on as follows for the gauge group SU(N),

1) One chooses a gauge-dependent local quantity $X(x) = X^A(x)T^A$ which transforms adjointly under the gauge transformation, i.e.,
   \[ X(x) \rightarrow X'(x) := U(x)X(x)U^\dagger(x). \] (1.1)

2) One performs the gauge rotation so that $X$ becomes diagonal,
   \[ X'(x) = diag(\lambda_1(x), \cdots, \lambda_N(x)), \] (1.2)
   where $\lambda_i(x) (i = 1, \cdots, N)$ are eigenvalues.

3) At the space-time point where the eigenvalues are degenerate $\lambda_i(x) = \lambda_j(x)(i \neq j, i, j = 1, \cdots, N)$, the monopole-like (hedgehog) singularity appears. The singularity does appear in the abelian gauge field $a_\mu(x)$ extracted from the non-Abelian gauge field $A'_\mu(x) = U(x)(A_\mu(x) + \frac{i}{g}\partial_\mu)U^\dagger(x)$. The monopole singularity is characterized as a topological quantity.

4) At generic point where the eigenvalues do not coincide, the gauge is not determined completely, since any diagonal gauge rotation $U$ (an element of the largest abelian subgroup, $U(1)^{N-1}$, the maximal torus group)
   \[ U(x) = diag(e^{i\theta_1(x)}, \cdots, e^{i\theta_N(x)}), \quad \sum_{i=1}^{N} \theta_i(x) = 0, \] (1.3)
   leaves $X$ invariant. Therefore, within this gauge, the theory reduces to an (N-1) fold abelian gauge-invariant theory.

The Monte Carlo studies of the abelian projection was initiated by the work [6] and the maximal abelian gauge (MAG) was adopted in the simulation on the lattice [7]. Recent extensive studies of abelian projection (see [8] for a review) have confirmed the *abelian dominance* proposed in Ref.[9]. This states that the non-Abelian gauge field
$A_\mu$ in $SU(N)/U(1)^{N-1}$, behaving as a charged field under the residual $U(1)^{N-1}$ gauge rotation, is not important in the low energy physics and the maximal abelian part $U(1)^{N-1}$ plays the dominant role in the quark and gluon confinement. In analytical studies, the abelian dominance was assumed from the beginning to construct the effective low energy theory of QCD [9, 10]. Assuming the abelian dominance, one can show that, if the monopole condensation occurs, charged quarks and gluons are confined due to dual Meissner effect. The monopole condensation is expected to bring the mass for the dual gauge field. An effective theory of monopole currents was investigated also on the lattice [11]. In fact, recent Monte Carlo simulations [12] support the abelian dominance and furthermore monopole dominance. However, there seems to be no analytical proof of abelian dominance.

An deficit of the abelian projection is the gauge-dependence of the procedure of abelian projection. The quantity $X$ is a gauge-dependent quantity and the field variable in which the monopole appears is not a gauge invariant quantity. Therefore the result seems to depend crucially on the gauge selected in the abelian projection. However, this would not be a real problem, since it is possible to put the abelian projection in a gauge-invariant form, if we desire to do so, see [13, 14].

The real problem is another in our view. In the abelian-projected theory, the magnetic monopole degrees of freedom appear as the singularity in the abelian gauge field. The magnetic current $k_\mu$ is obtained as the divergence of the dual abelian field strength $\tilde{f}_{\mu\nu}$,

$$\partial^\nu \tilde{f}_{\mu\nu} = k_\mu, \quad \tilde{f}_{\mu\nu} := \frac{1}{2} \epsilon_{\mu\nu\rho\sigma} f^{\rho\sigma},$$  \hspace{1cm} (1.4)

in the similar way that the equation of motion relates the field strength $f_{\mu\nu}$ to the electric current $j_\mu$,

$$\partial^\nu f_{\mu\nu} = j_\mu.$$  \hspace{1cm} (1.5)

If the U(1) potential $a_\mu$ is non-singular, the abelian field strength $f_{\mu\nu} := \partial_\mu a_\nu - \partial_\nu a_\mu$ leads to vanishing magnetic current, $k_\mu = 0$, which is nothing but the Bianchi identity for the U(1) field, $\partial^\nu \tilde{f}_{\mu\nu} \equiv 0$. So, if one needs the non-zero magnetic current, the abelian field must include a singularity. However, we do not think that it is sound as a quantum field theory to treat the singularity of the field variable as the essential ingredient from the very outset. In the lattice gauge theory, such a singularity does not appear due to the lattice regularization [15] and the monopole contribution is extracted from the gauge-invariant magnetic flux, although the monopole dominance is supported in the Monte Carlo simulation on the lattice. Moreover, it should be noted that the magnetic monopole does not exists in the original non-Abelian gauge theory. Magnetic monopole appears only after the abelian projection (see Appendix C).

The purpose of this paper is to derive the abelian-projected effective gauge theory (APEGT) of QCD as a quantum field theory, from which we should start the analysis. For simplicity, we restrict the following argument to the $G = SU(2)$ case. $SU(3)$ case is more involved and will be presented in a subsequent paper. In this paper, without
using various assumptions (actually with no assumptions), we derive the APEGT
of Yang-Mills (YM) theory and QCD. This is done by integrating out off-diagonal
fields belonging to the SU(2)/U(1) based on the functional integral formalism. We
use the word "effective" in the sense of the Wilson renormalization group [16], since
the abelian-projected theory is obtained after integrating out the degrees of freedom
corresponding to non-abelian gauge fields \( A^\pm_\mu := (A^1_\mu \pm iA^2_\mu)/\sqrt{2} \) which behave as massive charged matter fields and don’t play the important role in the low energy
physics of confinement. Such a strategy can be exactly performed in the \( N = 2 \)
supersymmetric YM and QCD [17].

We show that the off-diagonal field gives rise to the non-trivial magnetic monopole
current for the abelian part,
\[
K^\mu = \frac{1}{2} \partial_\nu (\epsilon^{\mu\nu\rho\sigma} \epsilon^{ab3} A^a_\rho A^b_\sigma), \quad a, b = 1, 2.
\] (1.6)
In other words, the charged off-diagonal gluon field plays the role of the source of the
monopole. Although the definition (1.6) of monopole current seems to be different
from the usual definition based on the singularity of the abelian field, we show that
both are equivalent to each other (apart from the Dirac string singularity). In the
APEGT, the singularity does not appear apparently, although we can always include
the singularity if necessary.

The effective dual Ginzburg-Landau (GL) theory derived assuming the abelian
dominance does not have sufficient predictive power, since it contains undetermined
free parameters. On the contrary, all the quantities in APEGT are calculable and
all the effects of the non-Abelian gauge field are included in the APEGT. In fact, we
show that the APEGT recovers exactly the same one-loop beta function as that of
the original non-Abelian gauge theory. The dual abelian gauge field follows naturally
in the course of the derivation of the theory and has a coupling with the monopole
current. This interaction leads to the dual Meissner effect due to monopole con-
densation. The resulting non-zero mass of the dual gauge field gives the non-zero
string tension, i.e. linear potential for static quarks. Thus the string tension is
determined by the monopole loop condensate, \( \langle K^\mu(x)K^\nu(x)\rangle/\delta^{(4)}(0) \), (see section 4 for
precise definition). The monopole condensate plays the role of the order parameter
for confinement.

Moreover, we discuss a possibility that the non-zero monopole condensation is de-
olvine from the instanton configuration. Hence instanton may lead to the confinement
against the conventional wisdom [18].

In our approach, the inclusion of fermions is straightforward. Hence APEGT is
also a starting point to study the relationship between the confinement and the chiral
symmetry breaking (or restoration) [19, 20].

This paper is organized as follows. In section 2, we derive the APEGT for the
maximal abelian part by integrating out the remaining non-Abelian gauge field. In
this step, we introduce the auxiliary tensor field which is converted to the dual gauge
field. The dual gauge field is essential to discuss the dual Meissner effect in section
4. APEGT is first obtained in the form including the logarithmic determinant. The
logarithmic determinant is explicitly calculated. It generates the gauge invariant

\[
K^\mu = \frac{1}{2} \partial_\nu (\epsilon^{\mu\nu\rho\sigma} \epsilon^{ab3} A^a_\rho A^b_\sigma), \quad a, b = 1, 2.
\] (1.6)
form due to U(1) gauge invariance. An effect of this term is the renormalization of the abelian gauge field. In section 3, we calculate the one-loop beta function without using the Feynman diagram. It is shown to agree with the original non-Abelian gauge theory. In this sense, the effective theory recovers the asymptotic freedom. In section 4, we discuss the dual Meissner effect. If the monopole loop condensation occurs, the dual vector field becomes massive. In section 5, we include the fermion into the APEGT. In section 6, we discuss the lower dimensional case. In the final section we give conclusion and discussion.

2 Abelian-projected effective gauge theory

2.1 Separation of the abelian part and introduction of the dual field

First, we decompose the field $A_\mu$ into the diagonal (maximal abelian U(1)) and the off-diagonal part SU(2)/U(1),

$$A_\mu(x) = \sum_{A=1}^{3} A^A_\mu(x) T^A := a_\mu(x) T^3 + \sum_{a=1}^{2} A^a_\mu(x) T^a. \quad (2.1)$$

We adopt the following convention. The generators of the Lie algebra $T^A (A = 1, \cdots, N^2 - 1)$ for the gauge group $G = SU(N)$ are taken to be hermitian satisfying $[T^A, T^B] = i f^{ABC} T^C$ and normalized as $\text{tr}(T^A T^B) = \frac{1}{2} \delta^{AB}$. The generators in the adjoint representation is given by $T^A_{\mu\nu} = -\frac{1}{4} \epsilon^{ABC}$. For SU(2), $T^A (A = 1, 2, 3)$ with Pauli matrices $\sigma^A$, the structure constant $f^{ABC} = \epsilon^{ABC}$. The indices $a, b, \cdots$ denote the off-diagonal parts.

Then the field strength

$$F_{\mu\nu}(x) := \sum_{A=1}^{3} F^A_{\mu\nu}(x) T^A := \partial_\mu A_\nu(x) - \partial_\nu A_\mu(x) - i [A_\mu(x), A_\nu(x)] \quad (2.2)$$

is decomposed as

$$F_{\mu\nu}(x) = [f_{\mu\nu}(x) + C_{\mu\nu}(x)] T^3 + S^a_{\mu\nu}(x) T^a,$$

$$f_{\mu\nu}(x) := \partial_\mu a_\nu(x) - \partial_\nu a_\mu(x),$$

$$S^a_{\mu\nu}(x) := D_\mu [a]^{ab} A_\nu^b - D_\nu [a]^{ab} A_\mu^b,$$

$$C_{\mu\nu}(x) T^3 := -i [A_\mu(x), A_\nu(x)], \quad (2.3)$$

where the derivative $D_\mu [a]$ is defined by

$$D_\mu [a] = \partial_\mu + i [a, T^3, \cdot], \quad D_\mu [a]^{ab} := \partial_\mu \delta^{ab} - \epsilon^{abc} a_\mu. \quad (2.4)$$

Hence the diagonal part $F^3_{\mu\nu}$ of the field strength is given by

$$F^3_{\mu\nu} = f_{\mu\nu} + C_{\mu\nu}, \quad C_{\mu\nu} := \epsilon^{abc} A_\mu^a A_\nu^b. \quad (2.5)$$
Next, we rewrite the Yang-Mills (YM) action

\[ S_{YM}[A] = -\frac{1}{2g^2} \int d^4x \, \text{tr}(F_{\mu\nu} F^{\mu\nu}). \]  

(2.6)

By using

\[ \text{tr}(f_{\mu\nu} S^{\mu\nu}) = 0 = \text{tr}(C_{\mu\nu} S^{\mu\nu}), \]  

(2.7)

the YM action is rewritten as

\[ S_{YM}[A] = -\frac{1}{4g^2} \int d^4x \left[ (f_{\mu\nu} + C_{\mu\nu})^2 + (S_{\mu\nu}^a)^2 \right]. \]  

(2.8)

Here we introduce an antisymmetric auxiliary tensor field $B_{\mu\nu}$ in order to linearize the $(C_{\mu\nu})^2$ term. This procedure enables us to perform the Gaussian integration over the off-diagonal gluon fields $A_a^{\mu}(a = 1, 2)$. It turns out that the tensor field $B_{\mu\nu}$ plays the role of the "dual" field to the abelian gluon field $a_\mu$. We find that there are two ways to introduce the "dual" tensor field.

One way is to introduce the tensor field $B_{\mu\nu}$ such that the tensor $B_{\mu\nu}$ is the dual of the diagonal field strength $\mathcal{F}^3_{\rho\sigma}$,

\[ B_{\mu\nu} \leftrightarrow \frac{1}{2} \epsilon^{\mu\nu\rho\sigma} F^3_{\rho\sigma} = \frac{1}{2} \epsilon^{\mu\nu\rho\sigma} (f_{\rho\sigma} + C_{\rho\sigma}). \]  

(2.9)

This is achieved in the tree level by the following action

\[ S_{apBFYM}[A, B] = \int d^4x \left[ \frac{1}{4} \epsilon^{\mu\nu\rho\sigma} B_{\rho\sigma} (f_{\mu\nu} + C_{\mu\nu}) - \frac{1}{4} g^2 B_{\mu\nu} B^{\mu\nu} - \frac{1}{4} g^2 (S_{\mu\nu}^a)^2 \right]. \]  

(2.10)

This theory is equivalent to the BF-YM theory,

\[ S_{BFYM}[A, B] = \int d^4x \left[ \frac{1}{4} \epsilon^{\mu\nu\rho\sigma} B_{\rho\sigma} A_{\mu\nu} - \frac{1}{4} g^2 B_{\mu\nu} B^{\mu\nu} A \right]. \]  

(2.11)

Actually, by identifying $B_{\mu\nu} = B^3_{\mu\nu}$, the action (2.10) is obtained from (2.11) by separating the diagonal part from the off-diagonal part and integrating out the off-diagonal auxiliary tensor field $B^a_{\mu\nu}(a = 1, 2)$. Quite recently, equivalence of the BF-YM theory with the YM theory has been proved in the quantum level, see [22]. This theory is interesting from the topological point of view.

Another way is to introduce the tensor field as a dual to $C_{\rho\sigma}$ at the tree level,

\[ B_{\mu\nu} \leftrightarrow \frac{1}{2} \epsilon^{\mu\nu\rho\sigma} C_{\rho\sigma}. \]  

(2.12)

Thus we are lead to the action,

\[ S_{apYM}[A, B] = \int d^4x \left[ -\frac{1}{4g^2} (f_{\mu\nu} f_{\mu\nu} + 2 f_{\mu\nu} C_{\mu\nu}) + \frac{1}{4} \epsilon^{\mu\nu\rho\sigma} B_{\rho\sigma} C_{\mu\nu} - \frac{1}{4} g^2 B_{\mu\nu} B^{\mu\nu} \right. \]

\[ \left. - \frac{1}{4g^2} (S_{\mu\nu}^a)^2 \right]. \]  

(2.13)

\[ ^1 \text{This procedure is similar to the field strength approach for non-Abelian gauge theory [21].} \]
In this case, \( \frac{1}{2} \epsilon_{\mu\nu\rho\sigma} f_{\rho\sigma} \) is generated through the radiative correction as shown in section 2.4. In either case, Gaussian integration over \( B_{\mu\nu} \) recovers the action (2.8) and hence the original YM action. This model (2.13) is simpler than the model (2.10) in the actual treatment, since the topological theory need some delicate treatment [22]. (Equivalence of two formulations is shown in Appendix A.) In what follows, we focus on the action (2.13) which is essentially equivalent to that derived by Quandt and Reinhardt [23].

2.2 Gauge-fixing

We discuss the gauge-fixing term. This is independent from the choice of the action. The gauge-fixing term is constructed based on the BRST formalism. We consider the gauge given by

\[
F^{\pm}[A, a] := (\partial^\mu \pm i \xi a^\mu) A^{\pm}_\mu = 0, \quad (2.14)
\]

\[
F^3[a] := \partial^\mu a_\mu = 0, \quad (2.15)
\]

where we have used the \((\pm, 3)\) basis,

\[
\mathcal{O}^{\pm} := (\mathcal{O}^1 \pm i \mathcal{O}^2) / \sqrt{2}. \quad (2.16)
\]

The gauge fixing with \( \xi = 0 \) is the Lorentz gauge, \( \partial^\mu A^\mu = 0 \). In particular, \( \xi = 1 \) corresponds to the differential form of the maximal abelian gauge (MAG) which is expressed as the minimization of the functional

\[
\mathcal{R}[A] := \frac{1}{2} \int d^4x [(A^1_\mu(x))^2 + (A^2_\mu(x))^2] = \int d^4x A^{\pm}_\mu(x) A^{\mp}_\mu(x). \quad (2.17)
\]

The differential MAG condition (2.14) corresponds to a local minimum of the gauge fixing functional \( \mathcal{R}[A] \), while the MAG condition (2.17) requires the global (absolute) minimum. The differential MAG condition (2.14) fixes gauge degrees of freedom in SU(2)/U(1) and is invariant under the residual U(1) gauge transformation. An additional condition (2.15) fixes the residual U(1) invariance. Both conditions (2.14) and (2.15) then completely fix the gauge except possibly for the Gribov problem. It is known that the differential MAG (2.14) does not spoil renormalizability of YM theory [24]. An implication of this fact is shown in Appendix B.

From physical point of view, we expect that MAG introduces the non-zero mass \( m_A \) for the off-diagonal gluons, \( A^1_\mu, A^2_\mu \). This is suggested from the form (2.17) which is equal to the mass term for \( A^1_\mu, A^2_\mu \), although we need an independent proof of this statement. This motivates us to integrate out the off-diagonal gluons in the sense of Wilsonian renormalization group and allows us to regard the resulting theory as the low-energy effective gauge theory written in terms of massless fields alone which describes the physics in the length scale \( R > m_A^{-1} \). The abelian dominance will be realized in the physical phenomena occurring in the scale \( R > m_A^{-1} \). In this sense

\[
\sum_\pm P^\pm Q^\mp = P^+ Q^- + P^- Q^+ = P^a Q^a, \quad \sum_\pm (\pm) P^\pm Q^\mp = -P^+ Q^- + P^- Q^+ = i e^{a b c} P^a Q^b(a, b = 1, 2).
\]
the choice of MAG is not unique in realizing abelian dominance. We can equally take the gauge so that the off-diagonal gluon fields acquire non-zero masses. Then the abelian-projected effective gauge theory obtained by integrating out the massive off-diagonal gluons will be valid in the low energy region below the energy scale given by the off-diagonal gluon mass.

We introduce the Lagrange multiplier field $\phi^\pm$ and $\phi^3$ for the gauge-fixing function $F^\pm[A]$ and $F^3[A]$, respectively. It is well known that the gauge fixing term in the BRST quantization is given by \[ L_{GF} = -i\delta_B G, \] (2.18)

where $G$ carries the ghost number $-1$ and is a hermitian function of Lagrange multiplier field $\phi^\pm, \phi^3$, ghost field $c^A$, antighost field $\bar{c}^A$, and the remaining field variables of the original lagrangian. In this paper we consider a simple gauge given by \[ G = \sum_{\pm} \bar{c}^\pm (F^\pm[A,a] + \frac{\alpha}{2} \phi^\pm) + \bar{c}^3 (F^3[a] + \frac{\beta}{2} \phi^3). \] (2.19)

For the most general gauge fixing, see [26].

The BRST transformation in the usual basis is

\[
\begin{align*}
\delta_B A^\mu_\pm &= (\partial_\mu \pm ia_\mu)c^\pm \mp iA^\pm_\mu c^3, \\
\delta_B a_\mu &= \partial_\mu c^3 + i(A^+_\mu c^- - A^-_\mu c^+), \\
\delta_B c^\pm &= \mp ic^\pm c^3, \\
\delta_B c^3 &= -ic^+ c^-, \\
\delta_B \phi^\pm &= i\phi^\pm c^3, \\
\delta_B \phi^3 &= 0, \\
\delta_B B^\mu_\pm &= -i[c, B^\mu_\pm].
\end{align*}
\] (2.20)

Then the BRST transformation in the $(\pm, 3)$ basis is given by

\[
\begin{align*}
\delta_B A^\pm_\mu &= (\partial_\mu \pm ia_\mu)c^\pm \mp iA^\pm_\mu c^3, \\
\delta_B a_\mu &= \partial_\mu c^3 + i(A^+_\mu c^- - A^-_\mu c^+), \\
\delta_B c^\pm &= \mp ic^\pm c^3, \\
\delta_B c^3 &= -ic^+ c^-, \\
\delta_B \phi^\pm &= i\phi^\pm c^3, \\
\delta_B \phi^3 &= 0, \\
\delta_B B^\pm_\mu &= \mp ic^\pm B^3_\mu \pm ic^3 B^\pm_\mu, \\
\delta_B B^3_\mu &= i(c^+ B^-_\mu - c^- B^+_\mu).
\end{align*}
\] (2.21)

Under the local U(1) gauge transformation,

\[
a_\mu \rightarrow a_\mu + \partial_\mu \omega, \quad \mathcal{O}^\pm \rightarrow e^{\mp i\omega} \mathcal{O}^\pm \quad \mathcal{O}^3 \rightarrow \mathcal{O}^3.
\] (2.22)

Hence $a_\mu$ transforms as a U(1) gauge field, while $A^\pm_\mu$ and $B^\pm_\mu$ behave as charged matter fields under the U(1) gauge transformation. It turns out that $B^3_\mu$ and

\[
\begin{align*}
\mathcal{C}_{\mu\nu} &= i \sum_{\pm} (\pm) A^\pm_\mu A^\pm_\nu.
\end{align*}
\] (2.23)
are U(1) gauge invariant as expected.

In the usual basis, we can write
\[ G = \sum_{a=1,2} \bar{c}^a (F^a[A,a] + \frac{\alpha}{2} \phi^a) + \bar{c}^3 (F^3[a] + \frac{\beta}{2} \phi^3), \]  
(2.24)

where
\[ F^a[A,a] := (\partial^\mu \delta^{ab} - \xi \epsilon^{a\beta\alpha} A^\beta) A^b_{\mu} := D^{\mu a}[a] \xi A^b_{\mu}. \]  
(2.25)

For the gauge fixing function (2.19) with the BRST transformation (2.21), or (2.24) with (2.20), straightforward calculation leads to the gauge fixing lagrangian
(2.18),
\[ L_{GF} = \phi^a F^a[A,a] + \frac{\alpha}{2} (\phi^a)^2 + i \epsilon^a D^{\mu a}[a] \xi D^{\mu b}[a] \phi^b + i \epsilon^3 \partial^\mu \partial_\phi \phi^3 
- i \xi \epsilon^a A^\mu_{\alpha} A^a_{\beta} A^b_{\mu} - A^c_{\mu} A^\mu_{\epsilon} \epsilon^{a\beta\epsilon} \phi^b 
+ i \epsilon^a \epsilon^{a\beta\epsilon} [(1 - \xi) A^b_{\mu} \phi^b + F^b[A,a] \phi^3] 
+ \phi^3 F^3[a] + \frac{\beta}{2} (\phi^3)^2 - i \epsilon^3 \partial^\mu (\epsilon^{a\beta\epsilon} A^a_{\mu} \phi^b). \]  
(2.26)

This reduces to the usual form in the Lorentz gauge, \( \xi = 0 \).

Finally we introduce the source term,
\[ L_J = A^a_{\mu} J^a_{\mu} + \phi^a J^3_{\phi}, \]  
(2.27)

which will be necessary to calculate the correlation functions.

### 2.3 Integration over SU(2)/U(1)

Our strategy is to integrate out the off-diagonal fields, \( \phi^a, A^a_{\mu}, c^a, \bar{c}^a \) (and \( B^a_{\mu\nu} \) for BF-YM case) belonging to the Lie algebra of SU(2)/U(1) and to obtain the effective abelian gauge theory written in terms of the diagonal fields \( a_{\mu}, B_{\mu\nu} \) (and ghost fields \( c^3, \bar{c}^3 \) if we need a completely gauge-fixed theory also for the residual U(1) gauge invariance).

First of all, when \( \alpha \neq 0 \), \(^3\) the Lagrange multiplier field \( \phi^a \) can be easily integrated out. The result is
\[ \phi^a F^a[A,a] + \frac{\alpha}{2} (\phi^a)^2 + \phi^a J^a_{\phi} \rightarrow - \frac{1}{2\alpha} (F^a[A,a])^2 - \frac{1}{\alpha} F^a[A,a] J^a_{\phi}. \]  
(2.28)

Next, as a preliminary procedure to integrate out \( A^a_{\mu} \), we rewrite the last term in the action (2.13) into
\[ (S^a_{\mu\nu})^2 = -2 A^a_{\mu} W_{\mu\nu} A^b_{\nu} + 2 \partial_\mu (A^a_{\nu} S^a_{\mu\nu}), \]
\[ W_{\mu\nu} := (D^\rho [a] D_{\rho} [a])^{ab} \phi^b - \epsilon^{ab\epsilon} f_{\mu\nu} - D_{\mu}[a]^{ac} D_{\nu}[a]^{cb}, \]  
(2.29)

\(^3\)The case of \( \alpha = 0 \) should be treated separately. Since \( F^a[A,a] = DA \) is linear in \( A^a_{\mu} \), the \( \phi^a \) integration can be performed finally after integrating out the \( A^a_{\mu} \) field. However, it generates the additional complicated logarithmic determinant, \( \ln \det [DQ^{-1} D] \). Such a case was treated in [23]. The choice of the gauge-fixing parameter should not change the physics, since it appears due to a gauge choice. Therefore we don’t treat this case in this paper.
where we have used

$$[D_\mu[a]^{ac}, D_\nu[a]^{cb}] = -\epsilon^{abc} f_{\mu\nu}. \quad (2.30)$$

Discarding the surface term, \(^4\) we arrive at

$$S_{YM} = S_{YM}[a, A, B, c, \bar{c}, J] = S_1[a, B] + S_2[a, c, \bar{c}] + S_3[a, A, B, c, \bar{c}, J], \quad (2.31)$$

$$S_1 = \int d^4x \left[ -\frac{1}{4g^2} f_{\mu\nu} f^{\mu\nu} - \frac{1}{4g^2} B_{\mu\nu} B^{\mu\nu} \right], \quad (2.32)$$

$$S_2 = \int d^4x \left[ i\bar{c}^\alpha D^{\mu\nu}[a]^{\xi} D^{\nu\xi}[a]^{\alpha} + i\bar{c}^\alpha \partial^\mu \partial_\nu c^\beta + \partial^\nu (\partial^\mu a^\alpha) + \frac{\beta}{2} (\phi^3)^2 \right], \quad (2.33)$$

$$S_3 = \int d^4x \left[ \frac{1}{2g^2} A^a_\mu Q^{ab}_\mu A^b_\nu + A^a_\mu \left( G^a_\mu + \frac{1}{\alpha} D^{ab}[a]^{\xi} J^{\mu a} + J^{\mu a} \right) \right], \quad (2.34)$$

$$Q^{ab}_{\mu\nu} := (D_\rho[a] D_\nu[a])^{ab} \delta_{\mu\nu} - 2\epsilon^{abc} f_{\mu\nu} + \frac{1}{2} g^2 \epsilon^{abc} \epsilon_{\mu\nu\rho\sigma} B^{\rho\sigma}$$

$$- 2ig^2 (\bar{c}^\alpha c^\beta - \bar{c}^\beta c^\alpha) \phi^3 \phi + D_\mu[a]^{ac} D_\nu[a]^{cb} + \frac{1}{\alpha} D_\mu[a]^{ac} D_\nu[a]^{cb}, \quad (2.35)$$

$$G^c_\mu := i(\partial_\mu \phi^3) \epsilon^{c\beta\gamma} + i\bar{c}^\alpha \epsilon^{abc} \left( (1 - \xi) (\partial_\mu c^\beta) \delta^{bc} - \xi \epsilon^{b\gamma\delta} a^\delta c^3 \right) - i\partial_\mu (\bar{c}^\alpha \epsilon^{abc} c^3) \quad (2.36)$$

where we have rescaled the parameter \(\alpha\) to absorb the \(g\) dependence.

All the terms appearing in the resulting YM action are at most quadratic in \(A^a_\mu\). The field \(A^\mu_\nu(a = 1, 2)\) in \(S_3\) can be eliminated using the Gaussian integration and we obtain

$$iS_0 = \ln \int [dA^a_\mu] \exp \left\{ i \int d^4x \left[ \frac{1}{2g^2} A^a_\mu Q^{ab}_\mu A^b_\nu + A^a_\mu \left( G^a_\mu + \frac{1}{\alpha} D^{ab}[a]^{\xi} J^{\mu a} + J^{\mu a} \right) \right] \right\} = -\frac{1}{2} \ln \det(Q^{ab}_{\mu\nu}) + \frac{g^2}{2} G^{a_1}_{\mu_1} (Q^{-1})^{ab}_{\mu_\nu} G^{b_1}_{\nu_1} + g^2 \left( \frac{1}{\alpha} D^{ac}[a]^{\xi} J^{\nu a}_c + J^{\mu a} \right) (Q^{-1})^{ab}_{\mu_\nu} G^{b_1}_{\nu_1}$$

$$- \frac{g^2}{2\alpha} J^{\mu a} (Q^{-1})^{ab}_{\mu_\nu} J^{\nu b}. \quad (2.37)$$

Thus we can derive the effective abelian gauge theory

$$S_E = S_0[a, A, B, c, \bar{c} ; J] + S_1[a, B] + S_2[a, c, \bar{c}],$$

$$S_0 = -\frac{1}{2} \ln \det(Q^{ab}_{\mu\nu}) + \frac{g^2}{2} G^{a_1}_{\mu_1} (Q^{-1})^{ab}_{\mu_\nu} G^{b_1}_{\nu_1} + g^2 \left( \frac{1}{\alpha} D^{ac}[a]^{\xi} J^{\nu a}_c + J^{\mu a} \right) (Q^{-1})^{ab}_{\mu_\nu} G^{b_1}_{\nu_1}$$

$$- \frac{g^2}{2\alpha} J^{\mu a} (Q^{-1})^{ab}_{\mu_\nu} J^{\nu b} + \frac{g^2}{\alpha} J^{\mu a} (Q^{-1})^{ab}_{\mu_\nu} J^{\nu b}. \quad (2.38)$$

As will be shown in the next subsection, \(\ln \det Q\) gives the renormalization of the fields \(a_\mu, B_{\mu\nu}\) and \(c^a\). The residual U(1) invariant theory is obtained by putting \(\phi^3 = 0\) and \(\bar{c}^3 = c^3 = 0\) (hence \(G^a_\mu = 0\)). Therefore, the resulting APEGT is greatly simplified.

\(^4\)This will be justified, since the off-diagonal gluons become massive due to MAG.
On the other hand, the effective abelian BF-YM theory is obtained if $S_1$ and $Q_{ab}^{\mu\nu}$ in $S_3$ are replaced by

\begin{align}
S_1 &= \int d^4x \left[ \frac{1}{4} \epsilon^{\mu\nu\rho\sigma} B_{\rho\sigma} f_{\mu\nu} - \frac{1}{4} g^2 B_{\mu\nu} B^{\mu\nu} \right], \\
Q_{ab}^{\mu\nu} &= (D_\rho [a] D_\rho [a])^{ab} \delta_{\mu\nu} - \epsilon^{ab\kappa\lambda} f_{\mu\nu} + \frac{1}{2} g^2 \epsilon^{ab\kappa\lambda} \epsilon_{\mu\nu\rho\sigma} B^{\rho\sigma} \\
&\quad - 2 i g^2 (\bar{c}^a c^b - \bar{c}^a c^a) \delta_{\mu\nu} - 2 i g^2 (\bar{c}^a c^b - \bar{c}^a c^a) \delta_{\mu\nu} - D_\mu [a]^{ac} D_\nu [a]^{cb} + \frac{1}{\alpha} D_\mu [a]^{ac} D_\nu [a]^{cb},
\end{align}

where the $G$ is the same as (2.36). This case is discussed in Appendix A.

### 2.4 Calculation of logarithmic determinant

In MAG ($\xi = 1$), the last two terms in $Q$ cancel by taking $\alpha = 1$ (they disappear also for $\alpha = 0$ [23]),

\begin{align}
Q_{ab}^{\mu\nu} &= (D_\rho [a] D_\rho [a])^{ab} \delta_{\mu\nu} - 2 \epsilon^{ab\kappa\lambda} f_{\mu\nu} + \frac{1}{2} g^2 \epsilon^{ab\kappa\lambda} \epsilon_{\mu\nu\rho\sigma} B^{\rho\sigma} \\
&\quad - 2 i g^2 (\bar{c}^a c^b - \bar{c}^a c^a) \delta_{\mu\nu}. \tag{2.40}
\end{align}

In order to calculate the $\ln \det Q$, we use the $\zeta$ function regularization or heat kernel method (see e.g. [27]),

\begin{align}
\ln \det Q &= - \lim_{s \to -1} \frac{d}{ds} \frac{\mu^{2s}}{\Gamma(s)} \int_0^\infty dt \ t^{s-1} \text{Tr}(e^{-tQ}), \tag{2.41}
\end{align}

where $\text{Tr}$ is understood in the functional sense. In this subsection the calculations are performed in Euclidean formulation.

First, we calculate the trace of $e^{-tQ}$. To estimate this quantity, we use the plane wave basis,

\begin{align}
\text{Tr}(e^{-tQ}) &= \int d^4x \text{tr}\langle x | e^{-tQ} | x \rangle = \int d^4x \text{tr} \int d^4k \frac{(2\pi)^4}{(2\pi)^4} e^{-ikx} e^{-tQ} e^{ikx}. \tag{2.42}
\end{align}

By making use of the relation,

\begin{align}
[D_\mu^{ab}, e^{\pm ikx}] &= \pm ik_\mu e^{\pm ikx} \delta^{ab}, \tag{2.43}
\end{align}

we find

\begin{align}
e^{-ikx} e^{-t(D_\rho [a]^{ab} \delta_{\mu\nu}) e^{ikx}} &= \exp[-t(D_\rho [a]^{ac} + ik_\rho \delta^{ac})(D_\rho [a]^{cb} + ik_\rho \delta^{cb}) \delta_{\mu\nu}]. \tag{2.44}
\end{align}

Furthermore the rescaling of $k_\mu$, $k_\mu \to k_\mu / \sqrt{t}$, leads to

\begin{align}
\text{Tr}(e^{-tQ}) &= \int d^4x \frac{1}{t^2} \text{tr} \int \frac{d^4k}{(2\pi)^4} e^{k_\mu k_\nu} \exp[-(2i\sqrt{t} k_\mu D_\mu + tQ)] \\
&= \int d^4x \frac{1}{t^2} \sum_{n=0}^\infty \frac{(-1)^n}{n!} \text{tr} \int \frac{d^4k}{(2\pi)^4} e^{k_\mu k_\nu} (2i\sqrt{t} k_\mu D_\mu + tQ)^n. \tag{2.45}
\end{align}
where we have omitted the unit operator, \( \delta_{ab} \delta_{\mu\nu} \). It is obvious that all terms odd w.r.t. \( k_{\mu} \) in the expansion go to zero in the integration. Thus we obtain

\[
\text{Tr}(e^{-tQ}) - \text{Tr}(e^{-tQ_0}) = \int \frac{d^4x}{16\pi^2} \text{tr} \left[ \frac{1}{2} Q^2 - D^2 Q + \frac{1}{6} (2D^2 D^2 + D_\mu D_\nu D_\mu D_\nu) \right] + O(t),
\]

where we have used the cyclicity of trace and the replacement

\[
k_{\mu}k_{\nu} \rightarrow \frac{1}{4} k^2 \delta_{\mu\nu},
\]

\[
k_{\mu}k_{\nu}k_{\alpha}k_{\beta} \rightarrow \frac{1}{24} (k^2)^2 (g_{\mu\nu}g_{\alpha\beta} + g_{\mu\alpha}g_{\nu\beta} + g_{\mu\beta}g_{\nu\alpha}),
\]

which is applied in the integrand of the integration formula

\[
\int \frac{d^4k}{(2\pi)^4} e^{k^2} (k^2)^m = \frac{(-1)^m}{16\pi^2} (m + 1)! (m = 0, 1, 2, \cdots).
\]

Separating the first term from the other terms in \( Q \),

\[
Q_{\mu\nu} := (D_\rho[a] D_\rho[a])^{ab} \delta_{\mu\nu} + \tilde{Q}_{\mu\nu}^{ab},
\]

we see

\[
\text{Tr}(e^{-tQ}) - \text{Tr}(e^{-tQ_0}) = \frac{1}{16\pi^2} \int d^4x \text{tr} \left[ \frac{1}{2} \tilde{Q}^2 + \frac{1}{12} [D_\mu, D_\nu][D_\mu, D_\nu] \right] + O(t),
\]

where any cross term between \( D \) and \( \tilde{Q} \) does not appear.

The first term is obtained as

\[
\text{tr} \left( \frac{1}{2} \tilde{Q}^2 \right) = 2\kappa f_{\mu\nu} f^{\mu\nu} - \frac{1}{2} g^4 \kappa B_{\mu\nu} B^{\mu\nu} - \kappa g^2 \epsilon^{\mu\nu\rho\sigma} B_{\rho\sigma} f_{\mu\nu} - 8g^4 (\tilde{c}^a \tilde{c}^b - \tilde{c}^a \delta^{ab} \tilde{c}^d \delta^{ba}),
\]

and the second term is

\[
\text{tr} \left( \frac{1}{12} [D_\mu, D_\nu][D_\mu, D_\nu] \right) = -\frac{1}{3} \kappa f_{\mu\nu} f^{\mu\nu},
\]

where

\[
\kappa := C_2(G) := f^{3cd} f^{3cd} = 2.
\]

The zero-order term of the expansion with respect to \( t \) is equal to the free term

\[
\text{Tr}(\exp[-tQ_0]) := \text{Tr}(\exp[-t\partial^2 \delta^{ab} \delta_{\mu\nu}]) = \frac{4N(N - 1) \int d^4x}{16\pi^2 t^2}.
\]
Thus we obtain (apart from the 4-body ghost interaction terms, see Appendix B) the U(1) invariant result,

\[ \frac{1}{2} \ln \det Q_{\mu\nu}^b = \int d^4x \left[ \frac{1}{4g^2} z_a f_{\mu\nu} f^{\mu\nu} + \frac{1}{4} z_b g^2 B_{\mu\nu} B^{\mu\nu} + \frac{1}{2} z_c B_{\mu\nu} \tilde{f}_{\mu\nu} + \cdots \right], \quad (2.55) \]

where

\[ z_a = -\frac{20}{3} \kappa \frac{g^2}{16\pi^2} \ln \mu, \quad z_b = +2\kappa \frac{g^2}{16\pi^2} \ln \mu, \quad z_c = +4\kappa \frac{g^2}{16\pi^2} \ln \mu. \quad (2.56) \]

Therefore, in the absence of the source \( J_\mu^a = 0 = J_\phi^a \),

\[ S_0 + S_1 = \int d^4x \left[ -\frac{1}{4g^2} z_a f_{\mu\nu} f^{\mu\nu} - \frac{1}{4} z_b g^2 B_{\mu\nu} B^{\mu\nu} + \frac{1}{2} z_c B_{\mu\nu} \tilde{f}_{\mu\nu} + \cdots \right]. \quad (2.57) \]

Integrating out the \( B_{\mu\nu} \) field, we will obtain an additional contribution,

\[ -\frac{1}{4g^2} z_c^2 (1 + z_b)^{-1} f_{\mu\nu} f^{\mu\nu}. \quad (2.58) \]

However, in the one-loop level, this term is irrelevant. Therefore, the cross term does not contribute in the one-loop level.

For later convenience, we calculate another determinant coming from the integration over ghost fields. For the action,

\[ S_F = \int d^4x \ i \bar{c}^a D_\mu^{ac}[a] D_\mu^{cb}[a] c^b, \quad (2.59) \]

we obtain up to one-loop

\[ S_c = \ln \int [d\bar{c}][dc] \exp \left\{ -\int d^4x \ i \bar{c}^a D_\mu^{ac}[a] D_\mu^{cb}[a] c^b \right\} \]

\[ = \ln \det(D_\mu^{ac}[a] D_\mu^{cb}[a]) \]

\[ = \int d^4x \frac{1}{4g^2} z'_a f_{\mu\nu} f^{\mu\nu} + \cdots, \quad z'_a := \frac{2}{3} \kappa \frac{g^2}{16\pi^2} \ln \mu. \quad (2.60) \]

For the abelian-projected effective BF-YM theory, see Appendix A.

### 2.5 APEGT with monopole

The antisymmetric (abelian) tensor \( B_{\mu\nu} \) has the Hodge decomposition in 3+1 dimensions (see section 6 for other dimensions),

\[ B_{\mu\nu} = b_{\mu\nu} + \tilde{\chi}_{\mu\nu}, \quad b_{\mu\nu} := \partial_\mu b_\nu - \partial_\nu b_\mu. \quad \tilde{\chi}_{\mu\nu} = \frac{1}{2} \epsilon_{\mu\nu\alpha\beta} (\partial^\alpha \chi^\beta - \partial^\alpha \chi^\beta). \quad (2.61) \]

The tensor \( B_{\mu\nu} \) has six degrees of freedom, while the fields \( b_\mu \) and \( \chi_\mu \) have eight. This mismatch is not a problem, since two degrees are redundant; the gauge transformation

\[ b_\mu(x) \to b'_\mu(x) = b_\mu(x) - \partial_\mu \theta, \quad \chi_\mu(x) \to \chi'_\mu(x) = \chi_\mu(x) - \partial_\mu \varphi, \quad (2.62) \]
leave $B_{\mu\nu}$ invariant. In the function integral, the integration over $B_{\mu\nu}$ is replaced by an integration over $b_{\mu}$ and $\chi_{\mu}$, provided that the gauge degrees of freedom are fixed in (2.62). These gauge fixing are not explicitly presented in the following, since they can be easily implemented.

In this case, we obtain

$$S_0 + S_1 = \int d^4x \left[ \frac{1 + z_a}{4g^2} f_{\mu\nu} f^{\mu\nu} - \frac{1 + z_b}{4} g^2 (b_{\mu\nu} b^{\mu\nu} + \bar{\chi}_{\mu\nu} \bar{\chi}^{\mu\nu}) + \frac{1}{2} z_c b_{\mu\nu} \bar{f}_{\mu\nu} + \frac{1}{2} z_c \chi_{\mu\nu} f_{\mu\nu} + \cdots \right],$$

(2.63)

At one-loop level, integration over $\chi$ leads to

$$S_E = \int d^4x \left[ -\frac{1 + z_a}{4g^2} f_{\mu\nu} f^{\mu\nu} + i\bar{c}^a D_\mu [a] D_\nu^{\mu\nu} c^b - \frac{1 + z_b}{4} g^2 b_{\mu\nu} b^{\mu\nu} - z_c b_\mu k^\mu \right],$$

(2.64)

where we have defined the magnetic current,

$$k^\mu := \partial^\nu \tilde{f}_{\mu\nu}, \quad \tilde{f}_{\mu\nu} := \frac{1}{2} \epsilon_{\mu\nu\rho\sigma} f^{\rho\sigma}.$$  

(2.65)

Here we have neglected the ghost self-interaction terms (see Appendix B) and higher derivative terms coming from the logarithmic determinant of $Q$. This is the APEGT written in terms of the abelian gauge field $a_{\mu}$ and the dual gauge field $b_{\mu}$ (an effect of the off-diagonal ghost field is studied in the next section). This theory has $U(1)_e \times U(1)_m$ symmetry where the abelian gauge field $a_{\mu}$ has $U(1)_e$ symmetry and the dual abelian gauge field $b_{\mu}$ has $U(1)_m$ symmetry which is guaranteed by the conservation $\partial_\mu k^\mu = 0$. If the field $a_{\mu}$ is singular, the magnetic current $k_\mu$ is non-zero and couples with the dual field $b_{\mu}$. This interaction leads to the dual Meissner effect, see section 4. In the absence of magnetic current, the dual field $b_{\mu}$ decouples from the theory. Note that the renormalizations of the fields $a_{\mu}, b_{\mu}$ are different each other.

APEGT can be considered as an interpolating theory which reduces to a theory with an action $S[a]$ by integrating out the $b_{\mu}$ field or to another theory with $S[b]$ by integrating out $a_{\mu}$ field. The theory $S[a]$ is suitable for describing the weak coupling region, while $S[b]$ is more suitable for the strong coupling region. However, both theories give the dual description of the same physics. In the next section, we see an aspect of this picture.

### 3 One-loop beta function and asymptotic freedom

Neglecting the contribution from dual gauge field, the APEGT is reduced to the $U(1)$ gauge theory,

$$S_E = \int d^4x \left[ -\frac{1 + z_a}{4g^2} f_{\mu\nu} f^{\mu\nu} + i\bar{c}^a D_\mu [a] D_\nu^{\mu\nu} c^b \right].$$  

(3.1)

This APEGT is similar to the scalar quantum electrodynamics. But the scalar field is replaced with the ghost field. We can show that the running coupling $g$ exhibits
asymptotic freedom, i.e. the beta function has negative coefficient. The beta function is obtained from the calculation of the logarithmic determinant in the previous section.

We define the wave function renormalization for $a_\mu$ and $c^a$ by

$$ a_\mu^R = Z_a^{-1/2} a_\mu, \quad c^c_R = Z_c^{-1/2} c. \quad (3.2) $$

For the 3-point $a_\mu c \bar{c}$ vertex, the renormalized coupling constant is defined by

$$ g^R = Z_a^{1/2} Z_c Z_g^{-1} g. \quad (3.3) $$

It should be remarked that the effective abelian gauge theory (3.1) has U(1) gauge invariance and we can derive the Ward-Takahashi (WT) identity for this symmetry. For example, the 3-point vertex function and the ghost propagator obeys the well known WT identity which is similar to that in scalar QED. This implies that $Z_g = Z_c$ (independently on the order of the perturbation). Therefore the coupling constant for the $a_\mu c \bar{c}$ vertex is determined by $Z_a$ alone,

$$ g^R = Z_a^{1/2} g. \quad (3.4) $$

Note that $Z_a$ is obtained by integrating the ghost field, i.e. ln det $D^2$, if we remember (2.60). Adding this contribution to (3.1), we obtain

$$ Z_a = 1 - z_a + z'_a = 1 + \frac{g^2}{16\pi^2} \frac{22C_2(G)}{3} \ln\mu, \quad C_2(G) := f^{3ad} f^{3cd} = 2. \quad (3.5) $$

Thus the beta function is easily calculated:

$$ \beta(g) := \mu \frac{dg^R}{d\mu} = -\frac{b_0}{16\pi^2} g^3, \quad b_0 = \frac{11C_2(G)}{3} > 0. \quad (3.6) $$

Thus the APEGT exhibits asymptotic freedom as the original YM theory. 6

In order to obtain the RG beta function, we could have used the Feynman graph technique. By the perturbation expansion in the coupling constant, we can ascertain the Ward relation $Z_g = Z_c$. 7 An origin of asymptotic freedom ($z'_a$) is understood as follows. By the Ward relation, asymptotic freedom is explained by the vacuum polarization of the abelian gauge field alone. This diagram up to order $g^2$ is quite similar to those of scalar QED by replacing the complex scalar field $\phi, \phi^*$ with the ghost, antighost field $c^a, \bar{c}^a$;

$$ c^a D^a_{\mu} \{a\} D^b_{\mu} \{a\} c^c \leftrightarrow |(\partial_{\mu} - ie a_{\mu})\phi|^2 = -\phi^*(\partial_{\mu} - ie a_{\mu})^2 \phi. \quad (3.8) $$

An essential difference is the signature due to ghost loop. This minus sign changes the non asymptotic freedom of scalar QED into asymptotic freedom in the effective

6 This fact was first obtained in the gauge $\alpha = 0$ based on quite complicated calculations [23].

7 Explicit calculation based on the perturbation theory shows that

$$ Z_g = Z_c = 1 - \frac{g^2}{16\pi^2} (\beta - 3) \ln \mu, \quad (3.7) $$

where $\beta$ is the gauge-fixing parameter.
abelian gauge theory in question. The additional dominant contribution \((z_a)\) comes from the gluon self-interaction which is already included in the action of APEGT through the calculation of \(-\frac{1}{2} \ln \det Q\). Summation of two contributions gives exactly the same beta function as the original YM theory.

In other words, the APEGT is the abelian gauge theory with QCD-like running coupling constant \(g(\mu)\),

\[
S_E[a] = \int d^4x \left[ -\frac{1}{4g(\mu)^2} f_{\mu\nu} f^{\mu\nu} \right], \quad \frac{1}{g(\mu)^2} = \frac{1}{g(\mu_0)^2} + \frac{b_0}{8\pi^2} \ln \frac{\mu}{\mu_0}.
\]  

(3.9)

4 Monopole condensation and dual Meissner effect

In section 2, we have obtained the APEGT with magnetic current (after the ghost integration),

\[
S_E[a, b, k] = \int d^4x \left[ -\frac{1}{4g^2} f_R^{\mu\nu} f_R^{\mu\nu} - \frac{1}{4} b_R^{\mu\nu} b_R^{\mu\nu} - \frac{1}{g} z_c / Z^{1/2}_b b_R^{\mu} k^\mu \right].
\]  

(4.1)

The interaction term between the dual gauge field \(b_\mu\) and the magnetic current \(k_\mu\) is generated by the radiative correction through the gluon self-interaction. The action leads to the field equation for the renormalized field,

\[
\partial_\mu f_R^{\mu\nu} = j_R^\nu, \quad \partial_\mu b_R^{\mu\nu} = k_R^\nu,
\]  

(4.2)

where we have defined

\[
k_R^\mu := \frac{1}{g} (z_c / Z_b^{1/2}) k^\mu, \quad Z_b^{1/2} = 1 - z_b / 2.
\]  

(4.3)

Integrating out the dual field \(b_\mu\), we obtain the effective action for the monopole loop,

\[
S_E[a, k] \cong \int d^4x \left[ -\frac{Z^{-1}_a}{4g^2} f_{\mu\nu} f^{\mu\nu} + \frac{1}{g^2} k^\mu D_{\mu\nu} k^{\nu} \right],
\]  

(4.4)

where \(D_{\mu\nu}\) is the massless vector propagator. Such a monopole action was predicted on a lattice in [11].

For our purposes, it is more convenient to use the local lagrangian formalism invented by Zwanziger [28] for the system having both electric and magnetic currents.

Before that, we will give a different treatment which is helpful to discuss the relationship between the monopole condensation and the instanton. We show how the magnetic monopole current is calculated in the original YM theory.

4.1 Definition of the monopole current

We show that the current \(K_\mu\) defined by

\[
K_\mu = \frac{1}{2} \epsilon^{\mu\nu\rho\sigma} \partial_\nu (\epsilon^{ab3} A_\rho^a A_\sigma^b) = \frac{1}{2} \epsilon^{\mu\nu\rho\sigma} \partial_\nu C_{\rho\sigma}
\]  

(4.5)
is interpreted as the magnetic monopole current. This current is topologically conserved, i.e., \( \partial_\mu K^\mu = 0 \). For a while, we use a different normalization of the field \( \mathcal{A} \rightarrow g \mathcal{A} \). Usually the abelian gauge field \( a_\mu \) defined by \( a_\mu(x) := \text{tr}[T^3 A_\mu(x)] \) can have singularities if the field \( \mathcal{A} \) is gauge transformed by the rotation matrix \( U(x) \) as \( \mathcal{A}_\mu(x) \rightarrow \mathcal{A}^U_\mu(x) \),

\[
\mathcal{A}^U_\mu(x) := U(x) \mathcal{A}_\mu(x) U^\dagger(x) + \frac{i}{g} U(x) \partial_\mu U^\dagger(x),
\]

such that the gauge transformed field \( \mathcal{A}^U_\mu(x) \) satisfies the abelian gauge fixing condition, e.g. MAG. It is this singularity that leads to a non-zero magnetic current. Under the gauge transformation (4.6), the field strength is transformed as \( F_{\mu\nu} \rightarrow F^U_{\mu\nu} \),

\[
F^U_{\mu\nu}(x) = U(x) F_{\mu\nu}(x) U^\dagger(x)
= \partial_\mu \mathcal{A}^U_\nu(x) - \partial_\nu \mathcal{A}^U_\mu(x) - i g [\mathcal{A}^U_\mu(x), \mathcal{A}^U_\nu(x)],
\]

see Appendix C. The abelian gauge field strength is extracted as

\[
f_{\mu\nu} := \partial_\mu a^U_\nu - \partial_\nu a^U_\mu = \text{tr}[T^3 (\partial_\nu \mathcal{A}^U_\rho - \partial_\rho \mathcal{A}^U_\nu)]
= \text{tr} [T^3 (U F_{\mu\nu} U^\dagger + i g [\mathcal{A}^U_\mu(x), \mathcal{A}^U_\nu(x)])].
\]

The definition of the magnetic current is

\[
k_\mu := \frac{1}{2} \epsilon_{\mu\nu\rho\sigma} \partial^\rho f^{\nu\sigma}.
\]

The first term in (4.6) is non-singular. Hence (4.8) shows that the first term gives vanishing contribution in the magnetic current. Only the second term

\[
\tilde{\mathcal{A}}_\mu(x) := \frac{i}{g} U(x) \partial_\mu U^\dagger(x)
\]

give a non-vanishing magnetic current. If \( U(x) \) is not singular, \( \tilde{\mathcal{A}}_\mu \) is a pure gauge and hence the field strength constructed from \( \tilde{\mathcal{A}}_\mu \) is zero, \( \tilde{F}_{\mu\nu}(x) := \partial_\mu \tilde{\mathcal{A}}_\nu(x) - \partial_\nu \tilde{\mathcal{A}}_\mu(x) - i g [\tilde{\mathcal{A}}_\mu(x), \tilde{\mathcal{A}}_\nu(x)] \equiv 0 \). For the singular \( U(x) \), this is modified as

\[
\tilde{F}_{\mu\nu}(x) := \partial_\mu \tilde{\mathcal{A}}_\nu(x) - \partial_\nu \tilde{\mathcal{A}}_\mu(x) - i g [\tilde{\mathcal{A}}_\mu(x), \tilde{\mathcal{A}}_\nu(x)] = \frac{i}{g} U(x) [\partial_\mu, \partial_\nu] U^\dagger(x).
\]

Thus we obtain the expression of the magnetic current,

\[
k_\mu = \frac{1}{2} \epsilon_{\mu\nu\rho\sigma} \partial^\rho \text{tr}[T^3 (\partial_\rho \tilde{\mathcal{A}}_\sigma - \partial_\sigma \tilde{\mathcal{A}}_\rho)]
= \frac{1}{2} \epsilon_{\mu\nu\rho\sigma} \partial^\rho \text{tr}(T^3 i g [\tilde{\mathcal{A}}_\rho, \tilde{\mathcal{A}}_\sigma]) + \frac{1}{2} \epsilon_{\mu\nu\rho\sigma} \partial^\rho \text{tr}(T^3 \frac{i}{g} U [\partial_\rho, \partial_\sigma] U^\dagger).
\]

The magnetic current is composed of two parts. The second part corresponds to the contribution from the Dirac string. Therefore the first part is the contribution from
the magnetic monopole which agrees with (4.5) in the original normalization of the field $\mathcal{A}$. This can be seen also from (4.9), since

\[
k_{\mu} = \frac{1}{2} \epsilon_{\mu\nu\rho\sigma} \partial^\nu \text{tr}(T^3 i g [\mathcal{A}_\rho, \mathcal{A}_\sigma]) + \frac{1}{2} \epsilon_{\mu\nu\rho\sigma} \partial^\nu \text{tr}(T^3 F^{U}_{\mu\nu})
\]

\[
= \frac{1}{2} \epsilon_{\mu\nu\rho\sigma} \partial^\nu \text{tr}(T^3 i g [\mathcal{A}_\rho, \mathcal{A}_\sigma]) + \frac{1}{2} \epsilon_{\mu\nu\rho\sigma} \partial^\nu \text{tr}(T^3 \frac{i}{g} U[\partial_\rho, \partial_\sigma] U^\dagger). \tag{4.14}
\]

For details, see Appendix C.

### 4.2 Dual effective abelian theory

In the following we present somewhat different picture of monopole condensation leading to the dual Meissner effect. By extracting the $b_{\mu}$ dependent pieces from the action (2.31), and inserting the identity

\[
1 = \int [dK^\mu] \delta(K^\mu - \frac{1}{2} \epsilon_{\mu\nu\rho\sigma} \partial^\nu (\epsilon^{ab} A^a_\rho A^b_\sigma)), \tag{4.15}
\]

the partition function $Z_{YM}$ is written as

\[
Z_{YM}[J] := \int d\mu e^{-S_{YM}} = \int d\mu \int [dK^\mu] \delta(K^\mu - \frac{1}{2} \epsilon_{\mu\nu\rho\sigma} \partial^\nu (\epsilon^{ab} A^a_\rho A^b_\sigma))
\]

\[
\times \exp \left\{ -S_{YM}[a, A, \chi, c, \bar{c}; J] - \int d^4x \left[ -\frac{1}{4} g^2 b_{\mu\nu} b^{\mu\nu} + b^\mu K_\mu \right] \right\}, \tag{4.16}
\]

where the measure $d\mu$ denotes the integration over all the fields.

In order to see that the APEGT can exhibit dual Meissner effect, we consider the effective action $S[b]$ written in terms of $b_{\mu}$ which is obtained by integrating out all the fields except for $b_{\mu}$,

\[
Z_{YM}[J] := \int [db_{\mu}] \exp \{-S[b]\}. \tag{4.17}
\]

Then $S[b]$ is obtained as

\[
S[b] = -\frac{1}{4} g^2 \int d^4x b_{\mu\nu} b^{\mu\nu} + \ln(\exp[\int d^4x b_{\mu}(x) K_\mu(x)])_0, \tag{4.18}
\]

where the expectation value for a function $f$ of the field is defined by

\[
\langle f(A) \rangle_0 := \int d\bar{\mu} \int [dK^\mu] \delta(K^\mu - \frac{1}{2} \epsilon_{\mu\nu\rho\sigma} \partial^\nu (\epsilon^{ab} A^a_\rho A^b_\sigma))
\]

\[
\times \exp \left\{ -S_{YM}[a, A, \chi, c, \bar{c}; J] \right\} f(A), \tag{4.19}
\]

where $d\bar{\mu}$ denotes the normalized measure without $[db_{\mu}]$ so that $\langle 1 \rangle_0 = 1$. It turns out that

\[
S[b] = -\frac{1}{4} g^2 \int d^4x b_{\mu\nu}(x) b^{\mu\nu}(x) + \int d^4x \langle K_\mu(x) \rangle_0 b^\mu(x)
\]

\[
+ \frac{1}{2} \int d^4x \int d^4y \langle K_\mu(x) K_\nu(y) \rangle_0 c b^\mu(x) b^\nu(y) + O(b^3), \tag{4.20}
\]
where \( \langle K_\mu(x)K_\nu(y) \rangle_c \) is the connected correlation function obtained from the normalized expectation value, \( \langle f(A) \rangle := \langle f(A) \rangle_0 / \langle 1 \rangle_0 \), e.g., \( \langle f(A)g(A) \rangle_c = \langle f(A)g(A) \rangle - \langle f(A) \rangle \langle g(A) \rangle \).

We can obtain a similar expression for the APEGT using the action (2.64). Hence the argument in the next subsection can be extended also to the APEGT.

### 4.3 Dual Meissner effect due to monopole condensation

The effective dual abelian theory \( S[b] \) has \( U(1) \) symmetry, \( b_\mu \to b_\mu + \partial_\mu \theta \), which is different from the \( U(1) \) symmetry for the abelian field \( a_\mu \) and is called the magnetic \( U(1)_m \) symmetry hereafter. The magnetic current satisfies the conservation \( \partial_\mu K^\mu = 0 \), consistently with the \( U(1)_m \) symmetry. This implies that the correlation function of the magnetic monopole current is transverse,

\[
\langle K_\mu(x)K_\nu(y) \rangle_c = \left( \delta_{\mu\nu} \partial^2 - \partial_\mu \partial_\nu \right) M(x - y). \tag{4.21}
\]

As long as the magnetic \( U(1)_m \) symmetry is not broken, the dual gauge field \( b_\mu \) is always massless as can be seen from (4.20) and (4.21). Therefore non-zero mass for the dual gauge field implies breakdown of the \( U(1)_m \) symmetry.

If \( U(1)_m \) symmetry is broken in such a way that

\[
\langle K_\mu(x)K_\nu(y) \rangle_c = g^2 \delta_{\mu\nu} \delta^{(4)}(x - y)f(x) + \cdots, \tag{4.22}
\]

the mass term is generated,

\[
S[b] = \int d^4x \left[ -\frac{1}{4} g^2 b_{\mu\nu}(x)b^{\mu\nu}(x) + \frac{1}{2} g^2 m_b^2 b_\mu(x)b_\mu(x) + \cdots \right], \tag{4.23}
\]

if we write \( f(x) = m_b^2 \). This can be called the dual Meissner effect; the dual gauge field acquires a mass given by

\[
m_b^2 = \frac{1}{4g^2} \Phi(0), \tag{4.24}
\]

if the monopole loop condensation occurs in the sense that,

\[
\Phi(x) := \lim_{y \to x} \frac{\langle K_\mu(x)K_\mu(y) \rangle_c}{\delta^{(4)}(x - y)} \neq 0. \tag{4.25}
\]

This is a criterion of dual superconductivity of QCD.\(^8\) It is consistent with the picture of dual superconductor scenario for quark confinement proposed by Nambu, 't Hooft and Mandelstam [1, 2, 3]. In the translation invariant theory, \( \Phi(x) \) is an \( x \)-independent constant which depends only on the gauge coupling constant \( g \). If we take a specific classical configuration to estimate them, \( x \)-dependence may appear, see the effective dual GL theory in the latter half of this subsection.

It should be remarked that \( \Phi \) is not the local order parameter in the usual sense. In order to find the non-zero value of \( m_b \), we must extract, from the magnetic monopole

\(^8\)For other proposals, see [29] and references therein.
current correlation function $\langle K_\mu(x)K_\nu(y) \rangle_c$, a piece which is proportional to the Dirac delta function $\delta^{(4)}(x - y)$ diverging as $y \to x$. Therefore, if such type of strong short-range correlation between two magnetic monopole loops does not exist, $\Phi$ is obviously zero. This observation seems to be consistent with the result of lattice simulations. The monopole loops exist both in the confinement and the deconfinement phases. However, in the deconfinement phase the monopole currents are dilute and the vacuum contains only short monopole loops with some non-zero density. In the confinement phase, on the other hand, the monopole trajectories form the infinite long loops and the monopole currents form a dense cluster, although there is a number of small mutually disjoint clusters [30].

It should be remarked that APEGT doesn’t need any scalar field. In this sense, the mechanism in which the dual gauge field acquires a mass is different from the dual Higgs mechanism. Nevertheless, we can always introduce the scalar field into APEGT so as to recover the spontaneously broken $U(1)_m$ symmetry,

$$\frac{1}{2} m_b^2 b_\mu(x)b_\mu(x) \to \frac{1}{2} m_b^2 (b_\mu(x) - \partial_\mu \theta(x))^2 = |(\partial_\mu - ib_\mu(x))\phi(x)|^2,$$

where we identify

$$\phi(x) = \frac{m_b}{\sqrt{2}} e^{i\theta(x)}. \quad (4.27)$$

Indeed, the result is invariant under $b_\mu \to b_\mu + \partial_\mu \alpha$ and $\theta \to \theta + \alpha (\phi \to e^{i\alpha} \phi)$. Such a scalar field is called the Stückelberg field or Batalin-Fradkin field [31]. The case (4.27) is obtained as an extreme type II limit (London limit),

$$\lim_{\lambda \to \infty} V(\phi), \quad V(\phi) := \lambda (|\phi(x)|^2 - m_b^2/2)^2,$$

or non-linear $\sigma$ model with a constraint, $\delta(|\phi(x)|^2 - m_b^2/2)$. The value $\phi_0$ at which the potential $V(\phi)$ has a minimum is proportional to the mass $m_b$ of dual gauge field,

$$m_b = \sqrt{2}\phi_0 = \frac{\sqrt{\Phi}}{2g}. \quad (4.29)$$

In the deconfinement phase, the minimum is given by $\phi_0 = 0 (m_b = 0)$, while in the confinement phase the minimum is shifted from zero $\phi_0 \neq 0 (m_b \neq 0)$ which corresponds to monopole condensation. Thus the dual abelian gauge theory with an action $S[b]$ is equivalent to (the London limit of) the dual GL theory (or the dual Abelian Higgs model with radial part of the Higgs field being frozen),

$$S[b] = \int d^4x \left[ -\frac{1}{4} b_\mu b^{\mu\nu} + |(\partial_\mu - ig^{-1} b_\mu)\phi|^2 + \lambda (|\phi|^2 - \phi_0^2)^2 + \cdots \right], \quad (4.30)$$

where we have rescaled the field $b_\mu \to b_\mu/g$. Note that the inverse coupling $g^{-1}$ has appeared as a coupling constant. This implies that the dual theory is suitable for describing the strong coupling region.

19
Now we compare our approach with the previous approach \cite{32, 33} where the summation over the monopole trajectories are performed. The monopole trajectories is expressed by the four-vector, \( x^\mu = \bar{x}_l^\mu(\tau_l) \), \( l = 1, 2, \cdots, N \) where \( \tau_l \) is an arbitrary parameter characterizing the trajectory and \( N \) is the total number of loops. Then the monopole current is written as

\[
K^\mu(x) = \frac{4\pi}{g} n_l \int d\tau_l \, \dot{x}^\mu_l(\tau_l) \delta^{(4)}(x - \bar{x}_l(\tau_l)), \quad \dot{x}^\mu := \frac{\partial \bar{x}^\mu}{\partial \tau},
\]

with \( n_l \) being the winding number. Then the interaction \( b^\mu K^\mu \) between the dual field and the monopole current is written as

\[
\int d^4x b^\mu(x) K^\mu(x) = \frac{4\pi}{g} \sum_{l=1}^N n_l \int d\tau_l \, b^\mu(\bar{x}_l(\tau_l)) \dot{x}^\mu_l(\tau_l).
\]

Summation over all configurations containing arbitrary number of monopole loops with all possible winding number and trajectories is performed based on the identity \cite{33},

\[
\sum_{N=0}^\infty \frac{1}{N!} \prod_{l=1}^N \int [d\bar{x}_l] \exp \left\{ i \sum_{l=1}^N \int d\tau_l \left[ M \dot{x}^2_l(\tau_l) + Q_{\mu_l}(\bar{x}_l(\tau_l)) \dot{x}^\mu_l(\tau_l) \right] \right\} = \exp \text{Tr} \ln H = \det(H)
\]

\[
= \int [d\phi] \exp \left\{ i \int d^4x \left[ (\partial^\mu + ig_{\mu}(x)) \phi(x) \right]^2 - M^2 |\phi(x)|^2 \right\},
\]

\[
H := \frac{1}{2} (p^\mu - Q^\mu)^2 - \frac{1}{2} M^2, \quad (4.33)
\]

where \( \phi \) is a complex scalar field. Both side are equal to the vacuum-to-vacuum transition amplitude of the theory consisting of charged scalar particles of mass \( M \) in the presence of an external electromagnetic field \( Q^\mu \).

If \( n_l \) is restricted to \( n_l = \pm 1 \), the field theoretical quantity is obtained,

\[
| (\partial^\mu + ig_{\mu}(x)) \phi(x) |^2, \quad g_{\mu} := \frac{4\pi}{g} n,
\]

where \( Q_{\mu} = g_{\mu} b^\mu \) and \( \phi \) plays the role of the monopole. Assuming a mass term of the monopole field and the repulsive self-interaction among the monopoles, the low energy (infrared) effective theory of the GL type, the effective dual GL theory, was proposed \cite{10},

\[
-\frac{1}{4} b_{\mu\nu}(x) b^{\mu\nu}(x) + |(\partial^\mu + ig_{\mu}(x)) \phi(x) |^2 - \lambda (|\phi(x)|^2 - v^2)^2.
\]

(4.34)

If the monopole condensation occurs in the sense that \( |\phi(x)| = v \neq 0 \), the mass term of the dual gauge field is generated, and the GL theory reduces to (note that the normalization for the field \( b_{\mu} \) is different from (4.23). )

\[
-\frac{1}{4} b_{\mu\nu}(x) b^{\mu\nu}(x) + \frac{1}{2} m_b^2 b_{\mu}(x)^2, \quad m_b \equiv g_{\mu} v,
\]

20
This is the so-called dual Meissner effect. Precisely speaking, the classical solution
\[ \phi(x) \text{ is not a constant and is a function of } x \text{ such that } \phi(x) \to v \text{ as } |x| \to \infty \text{ and } \phi(x) \to 0 \text{ as } |x| \to 0. \]
The characteristic length separating both behaviors is the coherence length \( \xi := \sqrt{2/m_\phi} \). The ratio

\[ \kappa_{GL} := \delta/\xi = m_\phi/(\sqrt{2}m_b) \]  \hspace{1cm} (4.35)

is called the GL parameter where \( \delta := 1/m_b \) is the penetration depth. The constant \( |\phi(x)| \equiv v \neq 0 \) corresponds to \( m_\phi = \infty \) or \( \xi = 0 \) and \( \kappa_{GL} = \infty \), a special case of type II superconductor \( \kappa_{GL} > 1/\sqrt{2} \). In the APEGT, this effect is expressed by the \( x \)-dependent mass \( m_b(x) \).

### 4.4 Monopole action

It is easy to show that monopole condensation actually occurs, if we use the lattice version \([11, 30]\) of the monopole action \(^9\) (4.4),

\[ S_m = -\frac{1}{4g^2} \sum_x f_\mu(x)f^{\mu\nu}(x) + \sum_{x,y} \frac{1}{g^2} k^\mu(x)D_{\mu\nu}(x-y)k^\nu(y). \]  \hspace{1cm} (4.36)

The monopole condensate (4.25) is calculated as follows. From (4.36), we can extract the self-mass term,

\[ S_{ma} = \frac{D(0)}{g^2} \sum_x k^\mu(x)k^\mu(x), \quad D(0) < \infty. \]  \hspace{1cm} (4.37)

The self-mass term with constant \( |k^\mu(x)| = 1 \) (see \([11]\)) is proportional to length of monopole loops. The probability that the monopole loop with length \( L \) will appear somewhere is

\[ P_L = 7^L \exp(-S_{ma}) = \exp \left[ (C - D(0)/g^2)L \right], \]  \hspace{1cm} (4.38)

where \( C = \ln 7 \) for non backtracking walk on the 4-dimensional hypercubic lattice. For sufficiently large \( g^2 \) (\( g^2 > D(0)/C \)), \( P_L \uparrow \infty \) as \( L \uparrow \infty \) and long loops give dominant contribution to the functional integral. On the other hand, \( P_L \downarrow 0 \) as \( L \uparrow \infty \), if \( g^2 \) is small (\( g^2 < D(0)/C \)). This indicates that in the infinite volume limit long monopole loops make no finite contribution. This is a simple energy-entropy (action-entropy) argument. Taking into account the entropy contribution is equivalent to adding an action,

\[ S_{en} = -C \sum_x k^\mu(x)k^\mu(x), \quad C < \infty. \]  \hspace{1cm} (4.39)

Therefore we obtain

\[ \Phi = \left( C - \frac{D(0)}{g^2} \right)^{-1}. \]  \hspace{1cm} (4.40)

---

\(^9\)On the lattice, the monopole action is obtained from the radially fixed Abelian Higgs model (of Villain type) by lattice duality transformation \([35]\).
This shows that, if the coupling $g$ is sufficiently strong, we have positive $\Phi$ and non-zero $m_b$. In other words, if the entropy of a monopole loop exceeds the energy, monopole condensation occurs. The region exhibiting monopole condensation extends to smaller and smaller values of $g$ for longer loops due to recent studies [30]. The above argument is valid for long loops. For more details, see [30]. The monopole action in the continuum needs more careful treatment as in three-dimensional case [36] which will be treated in a subsequent paper.

In the usual language of field theory, the term $k^\mu(x)D_{\mu\nu}(x-y)k^\nu(y)$ corresponds to the quartic self-interaction, especially, the self-mass term $k^\mu(x)k^\nu(x)$ to the contact quartic self-interaction. Therefore, it is assumed that the self-interaction among monopole loops does not essentially change the above picture. It should be remarked that higher order expansion generates interactions between monopole loops. For example, the self-interaction among the monopoles,

$$\langle K_\mu(x)K_\nu(y)K_\rho(z)K_\sigma(w) \rangle = \lambda(g)\left[ \delta_{\mu\nu}\delta_{\rho\sigma}\delta^4(x-y)\delta^4(z-w)\delta^4(x-z) \right] + \cdots$$

induces quartic self-interactions for $b_\mu$,

$$\int d^4x d^4yd^4zd^4w b_\mu(x)b_\nu(y)b_\rho(z)b_\sigma(w)\langle K_\mu(x)K_\nu(y)K_\rho(z)K_\sigma(w) \rangle$$

This renormalizes the mass term in (4.23) through radiative corrections. In this sense the criterion (4.25) is the tree-level criterion. The monopole interaction is expected to be weak repulsive.

### 4.5 Confinement and instanton

The effective abelian theory $S[a]$ written in terms of $a_\mu$ is obtained by integrating out the dual gauge field. This theory with an action $S[a]$ gives a dual description of the same physics as that given by $S[b]$. Following the Zwanziger formalism [28], (we don’t repeat the details, see [10] and [19]), if the dual gauge field acquires non-zero mass $m_b$ (due to monopole condensation), we obtain

$$S_{EFF}[a] = \int d^4x \left[ -\frac{1}{4g(\mu)^2}f_{\mu\nu}(x)f^{\mu\nu}(x) + \frac{1}{2}a^\mu(x)\frac{n^2m_b^2(x)}{(n \cdot \partial)^2 + n^2m_b^2(x)}X_{\mu\nu}(\partial)a^\nu(x) \right],$$

$$X_{\mu\nu}(\partial) := \frac{1}{n^2}\epsilon^{\lambda\mu\alpha\beta}\epsilon^{\lambda\nu\gamma\delta}n_\alpha n_\gamma \partial_\beta \partial_\delta,$$

where $n$ is an arbitrary fixed four-vector appearing in the Zwanziger formalism. The coupling constant $g(\mu)$ is the running coupling constant obeying the same $\beta$ function.

---

10We remember that quartic self-interaction in the scalar $\lambda \phi^4$ theory can be understood as the intersection probability of two random walks with repulsive interaction.
as the YM theory. In the limit $m_b \to 0$, (4.46) reduces to (3.9). The effective theory (4.46) leads to the linear potential $V(r) = \sigma r$ between static color charges and the string tension $\sigma$ is given by

$$
\sigma = \frac{Q^2}{4\pi} m_b^2 f(\kappa_{GL}),
$$

(4.47)

where $f(x)$ is a function depending on the method of calculations [10, 19]. The essential part $m_b^2$ in the string tension follows simply due to the dimensional analysis, irrespective of the details of the calculation.

The monopole condensation can be estimated based on the classical configuration of $A(x)$ satisfying the gauge fixing condition $F^a[A, a] = 0$,

$$
\langle K_\mu(x)K_\mu(y) \rangle = Z_{YM}^{-1} \int [dA(x)] e^{-S_{YM}[A]} \delta(F[A, a])
\times \int [dK_\mu] \delta(K_\mu - \frac{1}{2} g^2 \epsilon_{\mu\nu\rho\sigma} \partial^\nu (\epsilon^{ab\delta} A^{a}_\rho A^{b}_\sigma)) K_\mu(x)K_\mu(y). (4.48)
$$

Note that the MAG condition $F^a[A, a] = 0$ is satisfied by the classical multi-instanton solution [37, 38] of 'tHooft type,

$$
A^a_\mu(x) = \bar{\eta}_{\mu\nu} \partial_\nu f(x), \quad \bar{\eta}_{\mu\nu} := \epsilon_{\mu\nu} + \delta_{\mu\alpha} \delta_{\nu4} - \delta_{\mu4} \delta_{\nu\alpha} = -\bar{\eta}_{v\mu}.
$$

(4.49)

Therefore the classical instanton configuration may have a possibility to generate the monopole condensation. Actually, it has been shown that monopole loop formation and its condensation are intimately correlated with the instanton configuration [13, 39, 40, 41, 42, 43, 44, 45]. Therefore it is quite interesting to clarify whether the instanton configuration gives sufficient monopole loop condensation for quark confinement.

The one-instanton solution has the form,

$$
A^a_\mu(x) = 2\bar{\eta}_{\mu\nu} x_\nu g(x^2),
$$

(4.50)

where the prime denotes the differentiation with respect to the squared Euclidean distance $x^2 = |x|^2 = \sum_{\mu=1}^4 (x_\mu)^2$. Then the monopole current is written as

$$
K^\mu(x) = \epsilon_{\mu\alpha3} \bar{\eta}_{\alpha\beta} x^\nu (x^2 g^2(x^2))',
$$

(4.51)

where we have used the property,

$$
\epsilon_{ABC} \bar{\eta}^B \bar{\eta}^C = \delta_{\mu\alpha} \bar{\eta}_{\mu3} - \delta_{\mu3} \bar{\eta}_{\mu\alpha} + \delta_{\mu\beta} \bar{\eta}_{\mu\alpha} - \delta_{\mu\alpha} \bar{\eta}_{\mu3}.
$$

(4.52)

One-instanton solution with center at $x = z$ in the singular gauge is given by

$$
g((x - z)^2) = \frac{\rho^2}{|x - z|^2(|x - z|^2 + \rho^2)},
$$

(4.53)

while in the non-singular gauge

$$
g((x - z)^2) = \frac{1}{|x - z|^2 + \rho^2}.
$$

(4.54)
The expectation value (4.48) is replaced by the integration over the collective coordinates, $\rho, z_{\mu}$. Our preliminary calculation using the one-instanton solution in the singular gauge leads to non-zero monopole loop condensation. According to [13], however, one needs interpolating gauge between the singular and non-singular to derive monopole loop around the instanton. Moreover, in order to incorporate the interaction between instanton and anti-instanton and the resulting large monopole loop formation [13, 41, 42, 43, 44], we need more hard works. The details of this problem will be given in a subsequent paper [46].

5 Inclusion of fermion

In order to discuss the QCD, we add the fermionic action,

$$S_F = \int d^4 x \, \bar{\psi} [i \gamma^{\mu} D_\mu [A] - m] \psi, \quad D_\mu [A] := \partial_\mu - i A_\mu. \quad (5.1)$$

The contribution from the fermionic action is evaluated as

$$\int [d \bar{\psi}] [d \psi] \exp \left\{ - \int d^4 x \bar{\psi} [i \gamma^{\mu} D_\mu [A] - m] \psi \right\}
\begin{align*}
&= (\det [i \gamma^{\mu} D_\mu [A] - m])^{N_f} \\
&= \exp [N_f \ln \det [i \gamma^{\mu} D_\mu [A] - m]] \\
&= \exp \left[ \frac{N_f}{2} \ln \det [i \gamma^{\mu} D_\mu [A] - m]^2 \right]. \quad (5.2)
\end{align*}$$

In a similar way as in section 2, we can calculate the logarithmic determinant,

$$\text{Tr}(\exp[-t(i \gamma^{\mu} D_\mu[A])^2]) - \text{Tr}(\exp[-t(i \gamma^{\mu} \partial_\mu)^2])
= \int d^4 x \frac{g^2}{16 \pi^2} \frac{2}{3} r(F) \langle F^{a}_\mu \rangle^2 + O(t), \quad (5.3)$$

and

$$\ln \frac{\det(i \gamma^{\mu} D_\mu [A])^2}{\det(i \gamma^{\mu} \partial_\mu)^2} = \int d^4 x \frac{g^2}{16 \pi^2} \frac{2}{3} r(F) \ln \mu^2 \langle F^{a}_\mu \rangle^2, \quad (5.4)$$

where $r(F)$ is the dimension of fermion representation. In this calculation, we have used the commutator,

$$[D_\mu [A], D_\nu [A]] = -i \mathcal{F}_{\mu \nu}. \quad (5.5)$$

At one-loop level, it is easy to see that we can replace $(\mathcal{F}_{\mu \nu})^2$ in this contribution by $(f_{\mu \nu})^2$. If we add this contribution to the APEGT obtained in section 2, the APEGT of QCD is obtained (apart from the gauge-fixing term and the abelian ghost term),

$$S = \int d^4 x \left[ -\frac{1 + z_a}{4g^2} f_{\mu \nu} f^{\mu \nu} - \frac{1 + z_b}{4} g^2 b_{\mu \nu} b^{\mu \nu} - z_c b_\mu K^\mu + \bar{\psi} (i \gamma^{\mu} D_\mu [a] - m) \psi + i \bar{c}^c D_\mu [a] D^{bc} [a] c^c \right],
\quad D_\mu [a] := \partial_\mu - ia_\mu T^3. \quad (5.6)$$
In the region $K_\mu \approx 0$, it is clear that this theory recovers the one-loop beta function of QCD,

$$b_0 = \frac{11}{3} C_2(G) - \frac{4}{3} N_f r(F).$$

(5.7)

The monopole condensation and resulting dual Meissner effect can be treated in a similar way to section 4. We can discuss the chiral symmetry breaking based on APEGT of QCD (5.6), see e.g. [19].

6 Lower dimensional case

In the 2+1 dimensional case, we introduce the auxiliary vector field $B_\mu$ (instead of the tensor field $B_{\mu \nu}$ in 3+1 dimensional case). Then, corresponding to (2.10) or (2.13), the action is rewritten as

$$S_{apBFYM}[A, B] = \int d^3x \left[ \frac{1}{4} \epsilon^{\mu \rho \sigma} B_\rho (f_{\mu \sigma} + C_{\mu \sigma}) - \frac{1}{4} g^2 B_\mu B^\mu - \frac{1}{4 g^2} (S^a_{\mu \nu})^2 \right],$$

(6.1)

or

$$S_{apYM}[A, B] = \int d^3x \left[ - \frac{1}{4 g^2} (f_{\mu \nu} f_{\mu \nu} + 2 f_{\mu \nu} C_{\mu \nu}) + \frac{1}{4} \epsilon^{\mu \rho \sigma} B_\rho C_{\mu \nu} - \frac{1}{4 g^2} B_\mu B^\mu - \frac{1}{4 g^2} (S^a_{\mu \nu})^2 \right].$$

(6.2)

At the tree level, the dual vector field has the respective correspondence,

$$B_\mu \leftrightarrow \frac{1}{2} \epsilon^{\mu \rho \sigma} (f_{\rho \sigma} + C_{\rho \sigma}), \quad \frac{1}{2} \epsilon^{\mu \rho \sigma} C_{\rho \sigma}.$$

(6.3)

In order to discuss the monopole contribution, we use the decomposition,

$$B_\mu = \partial_\mu \phi + \frac{1}{2} \epsilon_{\mu \alpha \beta} \chi^{\alpha \beta}, \quad \chi_{\mu \nu} := \partial_\mu \chi_\nu - \partial_\nu \chi_\mu.$$

(6.4)

Hence APEGT of the 2+1 dimensional YM theory is given by

$$S_1[a, \phi, \chi] = \int d^3x \left[ - \frac{1}{4 g^2} f_{\mu \nu} f_{\mu \nu} - \frac{1}{4} g^2 \left( (\partial_\mu \phi)^2 + \chi_{\mu \nu}^2 \right) \right],$$

(6.5)

and

$$Q^{ab}_{\mu \nu} := (D_\rho [a] D_\rho [a])^{ab} \delta_{\mu \nu} - 2 \epsilon^{ab3} f_{\mu \nu} + \frac{1}{2} g^2 \epsilon^{ab3} (\epsilon_{\mu \rho \sigma} \delta_\rho \phi + \chi_{\mu \nu})$$

$$- 2 g^2 (\bar{c}^a c^b - \bar{c}^b c^a) \delta_{\mu \nu} - D_\mu [a]^{ac} D_\nu [a]^{cb} + \frac{1}{\alpha} D_\mu [a]^{ac} D_\nu [a]^{cb}. \quad (6.6)$$

[11] The vector $B_\mu$ has has three degrees of freedom, while the real scalar $\phi$ has one and the vector $\chi_\mu$ has three. One redundant degrees of freedom corresponds to that of the gauge transformation of $\chi_\mu$. 25
In 2+1 dimensional case, instead of interaction $b_\mu K^\mu$ between the dual gauge field and the magnetic current, we obtain the interaction term between the dual scalar $\phi$ and the monopole density $\rho$,

$$\rho(x)\phi(x), \quad \rho(x) := e^{\mu\nu}\partial_\mu C_{\mu\nu}(x),$$  \hspace{1cm} (6.7)

since

$$\int d^3x e^{\mu\nu} B_\mu C_{\mu\nu} = \int d^3x \left[ -\phi e^{\mu\nu}\partial_\mu C_{\mu\nu} + \chi_{\mu\nu} C_{\mu\nu} \right].$$  \hspace{1cm} (6.8)

The effective dual theory is the scalar theory with

$$S[\phi] \overset{\text{def}}{=} \int d^3x (\partial_\mu \phi(x))^2 + \int d^3x (\rho(x))\phi(x) + \frac{1}{2} \int d^3x \int d^3y \langle \rho(x)\rho(y) \rangle_c \phi(x)\phi(y) + \cdots.$$  \hspace{1cm} (6.9)

In 1+1 dimensional case, the dual tensor reduces to a one-component scalar $B$.

$$S_{apBFYM}[A, \phi] = \int d^2x \left[ \frac{1}{4} e^{\mu\nu}(f_{\mu\nu} + 2\epsilon_{\mu\nu\rho} C_{\rho\sigma}) \phi - \frac{1}{4} g^2 \phi^2 - \frac{1}{4} g^2 (S^a_{\mu\nu})^2 \right],$$  \hspace{1cm} (6.10)

or

$$S_{apYM}[A, \phi] = \int d^2x \left[ -\frac{1}{4} g^2 (f_{\mu\nu} f_{\mu\nu} + 2f_{\mu\nu} C_{\mu\nu}) + \frac{1}{4} e^{\mu\nu}\phi C_{\mu\nu} - \frac{1}{4} g^2 \phi^2 - \frac{1}{4} g^2 (S^a_{\mu\nu})^2 \right].$$  \hspace{1cm} (6.11)

The tree-level correspondence is given by

$$\phi \leftrightarrow \frac{1}{2} e^{\rho\sigma}(f_{\rho\sigma} + 2\epsilon_{\rho\sigma\nu} C_{\nu\mu}), \quad \frac{1}{2} e^{\rho\sigma} C_{\nu\mu}.$$  \hspace{1cm} (6.12)

Thus 1+1 dimensional YM theory is reduced to an effective abelian gauge theory with

$$S_1[a, \phi] = \int d^2x \left[ -\frac{1}{4} g^2 f_{\mu\nu} f_{\mu\nu} - \frac{1}{4} g^2 \phi^2 \right],$$  \hspace{1cm} (6.13)

and

$$Q^{ab}_{\mu\nu} := (D_\rho[a]D_\rho[a])^{ab} \delta_{\mu\nu} - 2\epsilon^{ab3} f_{\mu\nu} + \frac{1}{2} g^2 \epsilon^{ab3} \epsilon_{\mu\nu} \phi$$

$$-2i g^2 \xi (\bar{c}^a c^b - \bar{c}^b c^a) \delta_{\mu\nu} - D_\mu[a]^{ac} D_{\nu[a]}^{cb} + \frac{1}{\alpha} D_\mu[a]^{ac} D_{\nu[a]}^{cb}.$$  \hspace{1cm} (6.14)

In this case, the interaction term is induced,

$$\phi(x)\epsilon_{\mu\nu} f_{\mu\nu}(x).$$  \hspace{1cm} (6.15)

It is interesting to compare these formulations with the previous approaches [36, 47, 48, 49]. Detailed analyses of the lower dimensional case will be given in a forthcoming paper.  

26
7 Conclusion and discussion

We have derived abelian-projected effective gauge theories (APEGT) of YM theory and QCD. This has been performed by integrating out all off-diagonal non-Abelian gauge fields belonging to $SU(2)/U(1)$. The obtained APEGT is written in terms of the maximal abelian gauge field $a_\mu$ and the dual abelian gauge field $b_\mu$ which couples to the magnetic monopole current $K_\mu$. First, we have shown that the APEGT has the same one-loop beta function as the original non-Abelian gauge theories. Hence the APEGT exhibits asymptotic freedom (at one-loop level).

Next, we have shown that the dual vector field introduced to linearize the gluon self-interaction has an interaction with the magnetic current. Due to this interaction, the dual gauge field can become massive if the monopole loop condensation occurs. This is interpreted as the dual Meissner effect. We have shown that the mass of the dual gauge field is given by the monopole loop condensation $\langle K_\mu(x)K^\mu(x) \rangle/\delta^{(4)}(0) \neq 0$. This is our criterion of dual superconductivity. A method of showing monopole condensation is to consider the monopole action. The lattice monopole action [11, 30] gives a simple proof of monopole condensation.

If we apply the Zwanziger formalism to the APEGT with magnetic monopole, we can show that the static quark potential contains a linear part proportional to the quark separation. APEGT with monopole is sufficient to show quark confinement. This supports the abelian dominance. The monopole dominance will be confirmed by evaluating the monopole condensate, since the string tension is determined from the mass $m_b$ of dual gauge field. We have pointed out that this condensation can be estimated by the classical instanton configuration. Intimate relationship between confinement and instanton will be understood from the viewpoint of topological field theory of Schwarz type, BF-YM theory.

This work justifies some aspects of the pioneering works by Ezawa and Iwazaki [9] and Suzuki [10] based on the effective dual GL model. However, the APEGT has no free parameter and is of predictive power in sharp contrast with the previous works where the abelian dominance was assumed from the beginning. The APEGT has the complete correspondence to the original YM theory.

We have chosen the gauge group $SU(2)$ for mathematical simplicity. To discuss the confinement in the real world, we must discuss the $SU(3)$ case. This case will be treated in a subsequent paper [46].

Acknowledgments

This work was inspired by a series of lectures given by Tsuneo Suzuki at Chiba University in January 1997. After submitting this paper for publication, I have enjoyed fruitful discussions with Tadahiko Kimura, Taro Kashiwa, Shoichi Sasaki, Hideo Suganuma and Hiroshi Toki. I would like to thank all of them. Especially, many suggestions by H. Suganuma greatly helped me to improve the paper. I also thanks Maxim Chernodub for sending valuable comments. This work is supported in part by the Grant-in-Aid for Scientific Research from the Ministry of Education, Science
A  APEGT of BF-YM theory

In the similar way as in section 2, the APEGT of BF-YM is obtained as

\[ S_0 + S_1 + S_2 = \int d^4x \left[ -z_a \frac{1}{4g^2} f_{\mu\nu} f^{\mu\nu} - \frac{1}{4} (1 + z_b) g^2 B_{\mu\nu} B^{\mu\nu} + \frac{1}{2} (1 - z_c) B_{\mu\nu} F_{\mu\nu}^a + i c^a D^{\mu\nu [a] \xi} D_\mu^{\xi b} [a] c^b \right], \tag{A.1} \]

where

\[ z_a = -\frac{2}{3} \frac{g^2}{16\pi^2} \ln \mu, \quad z_b = +2 \frac{g^2}{16\pi^2} \ln \mu, \quad z_c = +2 \frac{g^2}{16\pi^2} \ln \mu. \tag{A.2} \]

Integrating out the tensor field \( B \), we obtain

\[ S_E = \int d^4x \left[ -z_a \frac{1}{4g^2} f_{\mu\nu} f^{\mu\nu} - \frac{1}{4} (1 + z_b) (1 + z_b)^{-1} (1 - z_c) f_{\mu\nu} f^{\mu\nu} + i c^a D^{\mu\nu [a] \xi} D_\mu^{\xi b} [a] c^b \right]. \tag{A.3} \]

Hence, at one-loop level, this reduces to

\[ S_E = \int d^4x \left[ -(1 + h) \frac{1}{4g^2} f_{\mu\nu} f^{\mu\nu} + i c^a D^{\mu\nu [a] \xi} D_\mu^{\xi b} [a] c^b \right], \tag{A.4} \]

where

\[ h = z_a - z_b - 2z_c = -\frac{20}{3} \frac{g^2}{16\pi^2} \ln \mu. \tag{A.5} \]

This agrees with the APEGT of YM theory given in section 2. Therefore, two types of APEGT are equivalent to each other.

B  Ghost interaction and gauge fixing

If we adopt more general gauge fixing functional,

\[ G = \sum_{\pm} \bar{c}^\mp (F^\pm [A, a] + \frac{\alpha}{2} \phi^\pm) + \bar{c}^3 (F^3 [a] + \frac{\beta}{2} \phi^3) + \alpha \eta \sum_{\pm} (\pm) \bar{c}^\pm c^\mp + \alpha \zeta \bar{c}^3 c^3, \tag{B.1} \]

the gauge fixing part \( \mathcal{L}_{GF} = -i \delta_\beta G \) has the additional contribution,

\[ \mathcal{L}^{\prime}_{GF} = -\sum_{\pm} (\pm) \bar{c}^\mp \frac{\alpha}{\beta} \eta F^3 [a] c^\pm - \sum_{\pm} (\pm) \bar{c}^3 \eta F^\pm [A, a] c^\mp + \sum_{\pm} (\pm) \bar{c}^\mp \zeta F^\pm [A, a] c^3 - \alpha (1 + \zeta) \eta \sum_{\pm} \bar{c}^\pm c^\mp c^\mp - \alpha (\zeta + \frac{\alpha}{\beta} \eta^2) \bar{c}^3 c^3 c^3. \tag{B.2} \]
Therefore, the U(1) invariant four-ghosts interaction term $\bar{c}^+ \bar{c}^- c^+ c^-$ coming from the expansion of $\ln \det Q$,

$$(\bar{c}^b c^b - \bar{c}^c c^c \delta^{ab})(\bar{c}^b c^a - \bar{c}^d c^d \delta^{ba}) = -2\bar{c}^1 \bar{c}^2 \bar{c}^2 c^2 = -2\bar{c}^+ \bar{c}^- c^+ c^- \quad (B.3)$$

is canceled by adding the BRST exact term, $-i\delta_B(\bar{c}^3 \bar{c}^- c^3 c^-) = -i\{Q_B, c^3 \bar{c}^+ c^-\}$. Such a term does not influence the physical state characterized by $Q_B|_{\text{phys}} = 0$. This is an implication of the renormalizability of YM theory in MAG.

C Magnetic monopole and Dirac string in SU(2) gauge theory

In this appendix, we discuss how the abelian objects, Dirac magnetic monopole and Dirac string, are produced due to singular gauge transformation in SU(2) non-Abelian gauge theory.  

The Non-Abelian field strength $F_{\mu\nu}$ is defined using the covariant derivative,

$$D_\mu := \partial_\mu - igA_\mu \quad (C.1)$$

as

$$F_{\mu\nu} = \frac{i}{g} [D_\mu, D_\nu] = \frac{i}{g} [\partial_\mu - igA_\mu, \partial_\nu - igA_\nu]. \quad (C.2)$$

This is rearranged as

$$F_{\mu\nu} = \frac{i}{g} [\partial_\mu, \partial_\nu] + [\partial_\mu, A_\nu] - [\partial_\nu, A_\mu] - ig[A_\mu, A_\nu]$$

$$= \frac{i}{g} [\partial_\mu, \partial_\nu] + \partial_\mu A_\nu - \partial_\nu A_\mu - ig[A_\mu, A_\nu]. \quad (C.3)$$

It should be remarked that the first term on the RHS in the final line can not be neglected when there is a singularity. We consider the local gauge transformation,

$$A_\mu \rightarrow A'_\mu := UA_\mu U^\dagger + \frac{i}{g} U\partial_\mu U^\dagger \quad (C.4)$$

Straightforward calculation using (C.4) leads to

$$F'_{\mu\nu} := \partial_\mu A'_\nu - \partial_\nu A'_\mu - ig[A'_\mu, A'_\nu] \quad (C.5)$$

$$= U(\partial_\mu A_\nu - \partial_\nu A_\mu - ig[A_\mu, A_\nu])U^\dagger + \frac{i}{g} U[\partial_\mu, \partial_\nu]U^\dagger. \quad (C.6)$$

This is consistent with (C.3), that is, the field strength transforms covariantly,

$$F_{\mu\nu} \rightarrow F'_{\mu\nu} := UF_{\mu\nu}U^\dagger. \quad (C.7)$$

This appendix is deeply indebted to H. Suganuma and H. Ichie [50].
In what follows we assume that $A_\mu$ is not singular and that the singularity in $A'_\mu$ comes from the gauge rotation $U$. In such a case, we call $U$ the singular gauge rotation. Therefore, the gauge-transformed field strength is composed of two parts, the regular and the singular part,

$$F_{\mu\nu}' = F_{\mu\nu}'^r + F_{\mu\nu}'^s,$$

$$F_{\mu\nu}'^r := U(\partial_\mu A_\nu - \partial_\nu A_\mu - ig[A_\mu, A_\nu])U^\dagger,$$

$$F_{\mu\nu}'^s := \frac{i}{g}U[\partial_\mu, \partial_\nu]U^\dagger. \quad (C.8)$$

First, we show that only the second part of the potential $A'_\mu(x)$,

$$A_\mu^s(x) := \frac{i}{g}U(x)\partial_\mu U^\dagger(x) \quad (C.9)$$

gives rise to the non-vanishing magnetic current. The diagonal part $a_\mu^s$ of the gauge potential $A_\mu^s$ is singular on the point where the Dirac string exists. The direction of the Dirac string can be changed arbitrarily by the gauge transformation. Hence the Dirac string is not a physical object. Actually, the magnetic charge is shown to obey the Dirac quantization condition. This can be seen as follows.

The local SU(2) matrix $U(x)$ can be written in terms of three Euler’s angles $\alpha$, $\beta$, $\gamma$,

$$U(x) = e^{i\gamma(x)\sigma_3/2}e^{i\beta(x)\sigma_2/2}e^{i\alpha(x)\sigma_3/2}$$

$$= \begin{pmatrix} e^{\frac{i}{2}(\alpha(x)+\gamma(x))} & e^{\frac{i}{2}(\alpha(x)-\gamma(x))} \sin \frac{\beta(x)}{2} & e^{-\frac{i}{2}(\alpha(x)+\gamma(x))} \cos \frac{\beta(x)}{2} \\ -e^{-\frac{i}{2}(\alpha(x)-\gamma(x))} \sin \frac{\beta(x)}{2} & e^{\frac{i}{2}(\alpha(x)-\gamma(x))} & e^{\frac{i}{2}(\alpha(x)+\gamma(x))} \cos \frac{\beta(x)}{2} \end{pmatrix}. \quad (C.10)$$

Using the residual U(1) invariance after MAG, we can choose $\gamma(x) = -\alpha(x)$. A convenient choice is to take $\alpha(x) = -\gamma(x) = \varphi(x)$, $\beta(x) = \theta(x)$ and identity the angle $\theta$ and $\varphi$ with the polar and the azimuthal angles in the three-dimensional polar coordinate of SU(2) so that

$$U(x)_{\theta,\varphi} = \exp(i\theta \vec{e}_\varphi \cdot \vec{\sigma}/2) = \begin{pmatrix} \cos \frac{\theta(x)}{2} e^{i\varphi(x)} \sin \frac{\theta(x)}{2} \\ -e^{-i\varphi(x)} \sin \frac{\theta(x)}{2} & \cos \frac{\theta(x)}{2} \end{pmatrix}$$

$$\vec{e}_\varphi := -\sin \varphi \vec{e}_X + \cos \varphi \vec{e}_Y, \quad (C.11)$$

where $(X, Y, Z)$ is identified with the space coordinates of $x^\mu = (0, r) = (0, X, Y, Z)$ and

$$0 < \theta := \arctan \frac{\sqrt{X^2 + Y^2}}{Z} < \pi, \quad 0 < \varphi := \arctan \frac{Y}{X} < 2\pi. \quad (C.13)$$

This choice does not lose generality, since we can always rotate the matrix using the residual U(1) degrees of freedom, see [39] for details.\footnote{If we take $\gamma(x) = \alpha(x)$ and write $\alpha(x) = \gamma(x) = \varphi(x)$, $\beta(x) = \theta(x)$, $U(x) = \begin{pmatrix} e^{i\varphi(x)} \cos \frac{\theta(x)}{2} & \sin \frac{\theta(x)}{2} \\ -\sin \frac{\theta(x)}{2} & e^{-i\varphi(x)} \cos \frac{\theta(x)}{2} \end{pmatrix}. \quad (C.14)
For the gauge rotation (C.12), the three-dimensional part of $A^i_\mu$ is

$$\tilde{A}^i(x) = \frac{1}{g r} (\cos \varphi(x) \bar{e}_\varphi + \sin \varphi(x) \bar{e}_0) T^1 + \frac{1}{g r} (\sin \varphi(x) \bar{e}_\varphi - \cos \varphi(x) \bar{e}_0) T^2$$

$$+ \frac{1}{g r} \tan \frac{\theta(x)}{2} \bar{e}_\varphi T^3,$$

(C.16)

where we have used

$$\nabla := \bar{e}_r \frac{\partial}{\partial r} + \bar{e}_\theta \frac{\partial}{r \partial \theta} + \bar{e}_\varphi \frac{\partial}{r \sin \theta \partial \varphi}.$$  

(C.17)

The diagonal abelian part is defined by

$$a'_\mu := 2 \text{tr}(T^3 A'_\mu).$$

(C.18)

In this case, $^{14}$

$$\tilde{a}^i(x) = \frac{1}{g r} \tan \frac{\theta(x)}{2} \bar{e}_\varphi = \frac{1}{g r} \frac{1 - \cos \theta}{\sin \theta} \bar{e}_\varphi,$$  

(C.21)

or

$$a^i_\mu(x) = (a_0^i(x), \tilde{a}^i(x)) = \frac{1}{g r(r + Z)} (0, -Y, X, 0).$$  

(C.22)

The vector potential $\tilde{a}^i$ is singular on the negative $Z$ axis and is not defined for $\theta = \pi$. Then the rotation is given by

$$\nabla \times \tilde{a}^i(x) = \tilde{B}_m + \tilde{B}_{DS} = \frac{\bar{r}}{g r^3} + \frac{4 \pi}{g} \delta(X) \delta(Y) \theta(-Z) \bar{e}_Z.$$

(C.23)

This implies that $\nabla \times \tilde{a}^i(x) = \frac{\bar{r}}{g r^3}$ except along the negative $Z$ axis. The singularity along the negative $Z$ axis is called the Dirac string. This can not be avoided as

For this choice of $\gamma$, the Dirac string appears on the positive $Z$ axis, since the $\beta = 0, \pi$ corresponds to

$$U(x)_{0, \varphi} = \begin{pmatrix} e^{i \varphi(x)} & 0 \\ 0 & e^{-i \varphi(x)} \end{pmatrix}, \quad U(x)_{\pi, \varphi} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. $$

(C.15)

$^{14}$The 4-dimensional expression is given by

$$a^i_\mu(x) = -\frac{1}{g} [\cos \beta(x) \partial_\mu \alpha(x) + \partial_\mu \gamma(x)].$$

(C.19)

The angle $\gamma(x)$ does not appear in the $U(1)$ invariant quantity. Actually, the magnetic current given by

$$k_\mu(x) = \frac{1}{g} \epsilon_{\mu \nu \rho \sigma} \partial_- [\partial_\mu \cos \beta(x) \partial_\nu \alpha(x)]$$

(C.20)

does not contain the angle $\gamma$. For more details, see [13].
long as one uses the single expression for the gauge potential in the whole space. A
method to avoid the singularity is using the Wu-Yang monopole [51]. It is impossible
to construct a single singularity-free potential which is defined everywhere. When
considering the total space, we need at least two expressions for the vector potential.

The magnetic monopole sits at \( \vec{r} = 0 \),

\[
\nabla \cdot \vec{B}_m = k_m^a(x), \quad k_m^a(x) = \frac{4\pi}{g} \delta^{(3)}(x).
\]

The four-dimensional expression of the magnetic current is

\[
\frac{1}{2} \epsilon_{\mu\nu\rho\sigma} \partial_\sigma f_{\mu\nu} = k_\mu(x), \quad k_\mu(x) = \frac{4\pi}{g} \delta_\mu^0 \delta^{(3)}(x).
\]

where the abelian-projected field strength is defined,

\[
f_{\mu\nu} := \partial_\mu a_\nu^s - \partial_\nu a_\mu^s := \text{tr}[T^s(\partial_\mu A_\nu^s - \partial_\nu A_\mu^s)]
\]

The magnetic flux \( \Phi \) obtained by integrating \( \vec{B}_m \) over any closed surface containing
the origin is

\[
\Phi_m := \int_S \vec{B}_m \cdot \vec{dS} = \frac{4\pi}{g}.
\]

On the other hand, the magnetic flux \( \Phi \) obtained by integrating \( \vec{B}_{Ds} \) over any closed
surface containing the origin is

\[
\Phi_{Ds} := \int_S \vec{B}_{Ds} \cdot \vec{dS} = -\frac{4\pi}{g}.
\]

We observe that the singular gauge potential \( A_\mu^s \) satisfies the following relation,

\[
\partial_\mu A_\nu^s - \partial_\nu A_\mu^s = i g \left\{ (\partial_\mu U)(\partial_\nu U^\dagger) - (\partial_\nu U)(\partial_\mu U^\dagger) \right\}
\]

\[
= i g \left\{ (\partial_\mu U)(\partial_\nu U^\dagger) - (\partial_\nu U)(\partial_\mu U^\dagger) \right\} + i g \left( U(\partial_\mu \partial_\nu U^\dagger) - U(\partial_\nu \partial_\mu U^\dagger) \right)
\]

\[
= i g \left\{ (\partial_\nu U)(\partial_\mu U^\dagger) - (\partial_\mu U)(\partial_\nu U^\dagger) \right\} + i g \left( U(\partial_\nu \partial_\mu U^\dagger) - U(\partial_\mu \partial_\nu U^\dagger) \right)
\]

\[
= i g \left\{ (\partial_\nu U)(\partial_\mu U^\dagger) - (\partial_\mu U)(\partial_\nu U^\dagger) \right\} + i g \left( U(\partial_\nu \partial_\mu U^\dagger) - U(\partial_\mu \partial_\nu U^\dagger) \right)
\]

\[
= i g [A_\mu^s, A_\nu^s] + i g (U[\partial_\mu, \partial_\nu]U^\dagger),
\]

where we have used

\[
UU^\dagger = 1, \quad \partial_\mu(UU^\dagger) = (\partial_\mu U)U^\dagger + U(\partial_\mu U^\dagger) = 0.
\]
Hence the abelian-projected field strength reads

\[ f_{\mu\nu} = \text{tr}(T^3 ig[A_\mu^s, A_\nu^s]) + \text{tr}(T^3 \frac{i}{g} U[\partial_\mu, \partial_\nu] U^\dagger). \]  

(C.31)

If \( U \) was not singular, the last term in (C.29) or (C.31) was absent, since \( A_\mu^s \) is a pure gauge which gives vanishing field strength for non-singular \( U(x) \),

\[ F_{\mu\nu} := \partial_\mu A_\nu^s - \partial_\nu A_\mu^s - ig[A_\mu^s, A_\nu^s] \equiv 0. \]  

(C.32)

The last term in (C.29) corresponds to the singularity due to Dirac string as shown shortly.

Now we clarify the physical meaning of the last term, \( \frac{i}{g}(U[\partial_\mu, \partial_\nu] U^\dagger)^{(3)} \). We show that

\[ U(x)[\partial_X, \partial_Y] U^\dagger(x) = -2\pi n i \delta(X) \delta(Y) \theta(-Z) \sigma_3. \]  

(C.33)

To prove this, we first show that

\[ [\partial_X, \partial_Y] \varphi(x) = 2\pi n \delta(X) \delta(Y). \]  

(C.34)

This is a result of the Stokes theorem; for the arbitrary 2-dimensional region \( S \) including \( (X,Y) = (0,0) \),

\[
\int_S dX dY [\partial_X, \partial_Y] \varphi = \int_S d^2 S \det \begin{pmatrix} \partial_X & \partial_Y \\ \partial_Y & \partial_X \end{pmatrix} = \int_S d^2 S \nabla \times (\nabla \varphi) \\
= \oint_{C=\partial S} \partial_\mu \varphi dx^\mu = \Delta \varphi = 2\pi n = 2\pi n \int_S dX dY \delta(X) \delta(Y),
\]  

(C.35)

where the integer \( n \) comes from the multi-valuedness of \( \varphi \).

When \( \theta = 0 \) (i.e., on the positive \( Z \) axis),

\[ U(x)_0,\varphi = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \]  

(C.36)

which does not give non-trivial contribution in (C.33). On the other hand, for \( \theta = \pi \) (i.e., on the negative \( Z \) axis),

\[ U(x)_{\pi,\varphi} = \begin{pmatrix} 0 & e^{-i\varphi(x)} \\ -e^{i\varphi(x)} & 0 \end{pmatrix}. \]  

(C.37)

Then, using (C.34),

\[ U(x)_{\pi,\varphi}[\partial_X, \partial_Y] U(x)_{\pi,\varphi}^\dagger = -i[\partial_X, \partial_Y] \varphi(x) \sigma_3 = -2\pi i n \delta(X) \delta(Y) \sigma_3. \]  

(C.38)

This proves the statement (C.33).

\[ ^{15} \text{This is derived also from Homotopy theory, } \Pi_2(SU(N)/U(1)^{N-1}) = \Pi_1(U(1)^{N-1}) = Z^{N-1}. \text{ In particular, } \Pi_2(SU(2)/U(1)) = \Pi_1(U(1)) = Z, \text{ see argument in Ref. [6].} \]
The relation (C.33) shows that the term $\frac{i}{g}(U[\partial_{\mu},\partial_{\nu}]U^\dagger)^{(3)}$ produces the magnetic field only along the negative $Z$ axis,

$$B_{Z}^{Ds} := \frac{i}{g}(U[\partial_{X},\partial_{Y}]U^\dagger)^{(3)} = \frac{4\pi n}{g}\delta(X)\delta(Y)\theta(-Z). \quad \text{(C.39)}$$

So this is identified with the Dirac string (not the magnetic monopole) extending from the origin to infinity along the negative $Z$ axis (due to the above choice of $U$) in three-dimensional space. Hence the divergence of $B_{Z}^{Ds}$ is non-zero at the origin,

$$k_0^{Ds} = \nabla \cdot B_{Z}^{Ds} := \partial_z \frac{i}{g}(U[\partial_{X},\partial_{Y}]U^\dagger)^{(3)} = -\frac{4\pi n}{g}\delta^3(x), \quad \text{(C.40)}$$

which should be compared with (C.28).

Finally, we give an alternative definition of the abelian-projected field strength,

$$f_{\mu\nu} = \text{tr}(T^3ig[A^s_{\mu},A^s_{\nu}]) + \text{tr}(T^3\frac{i}{g}U[\partial_{\mu},\partial_{\nu}]U^\dagger). \quad \text{(C.41)}$$

This is the abelian field strength obtained from the singular gauge potential and consists of the magnetic monopole part and the Dirac string part as shown above. In the RHS, the second term $\text{tr}(T^3\frac{i}{g}U[\partial_{\mu},\partial_{\nu}]U^\dagger)$ expresses the magnetic field on the Dirac string and vanishes elsewhere. Therefore, the remaining part $\text{tr}(T^3ig[A^s_{\mu},A^s_{\nu}])$ denotes the field strength of the magnetic monopole defined everywhere. Hence, the magnetic monopole part of the magnetic current defined by

$$k_\rho := \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}\partial^\sigma f_{\mu\nu} \quad \text{(C.42)}$$

is equivalent to

$$K_\rho = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}\partial^\sigma(g\epsilon^{ab3}A^a_{\mu}A^b_{\nu}). \quad \text{(C.43)}$$

In the three dimensional slices, this describes the magnetic monopole with magnetic charge

$$g_m := \int K_0(x)d^3x = \frac{4\pi}{g}n, \quad \frac{gg_m}{4\pi} = n, \quad \text{(C.44)}$$

where $n$ is an integer. This is nothing but the Dirac quantization condition.

In the original YM theory, as a result of the Jacobi identity,

$$\epsilon_{\mu\nu\rho\sigma}[D_{\nu},[D_{\rho},D_{\sigma}]] = 0, \quad \text{(C.45)}$$

the Bianchi identity always holds,

$$0 = \epsilon_{\mu\nu\rho\sigma}D_{\nu}F_{\rho\sigma}, \quad \epsilon_{\mu\nu\rho\sigma}\partial_{\nu}F_{\rho\sigma} = ig\epsilon_{\mu\nu\rho\sigma}A_{\nu}F_{\rho\sigma} = 2igA_{\nu}\tilde{F}_{\mu\nu}. \quad \text{(C.46)}$$

After gauge fixing, the Bianchi identity for the residual $U(1)$ is violated,

$$\epsilon_{\mu\nu\rho\sigma}\partial^\sigma f_{\mu\nu} \neq 0, \quad \text{(C.47)}$$
which leads to the magnetic monopole. In the original YM theory, the magnetic monopole does not exist. However, note that

\[ \epsilon_{\mu \nu \rho \sigma} \partial^\rho \mathcal{F}^{(3)}_{\mu \nu} \neq 0, \quad \mathcal{F}^{(3)}_{\mu \nu} := 2 \text{tr}(T^3 \mathcal{F}'_{\mu \nu}), \]  

(C.48)

since

\[ \epsilon_{\mu \nu \rho \sigma} \partial^\rho \mathcal{F}^{(3)}_{\mu \nu} = \epsilon_{\mu \nu \rho \sigma} \partial^\rho (\partial_\mu a'_\nu - \partial_\nu a'_\mu) - \epsilon_{\mu \nu \rho \sigma} \partial^\rho i g ([A'_\mu, A'_\nu])^{(3)} \]

\[ = \epsilon_{\mu \nu \rho \sigma} \partial^\rho \frac{i}{g} (U [\partial_\mu, \partial_\nu] U^\dagger)^{(3)}. \]  

(C.49)

The RHS is equal to the Dirac string contribution [45, 39].

Incidentally, the four-vector

\[ K^D_\rho = \frac{1}{2} \epsilon_{\mu \nu \rho \sigma} \partial^\rho \frac{i}{g} (U [\partial_\mu, \partial_\nu] U^\dagger)^{(3)}, \]  

(C.50)

denotes the trajectory

\[ K_\mu(x) = \int d\tau \frac{\partial y_\mu(\tau)}{\partial \tau} \delta^{(4)}(x - y(\tau, 0)), \]  

(C.51)

as the boundary \( x_\mu = y_\mu(\tau, 0) \) of the Dirac sheet described by \( y_\mu(\tau, \sigma) \) (world sheet of the Dirac string, i.e., 2-dimensional surface swept by the Dirac string in 4-dimensional space),

\[ \omega_{\mu \nu}(x) := \frac{i}{g} (U(x) [\partial_\mu, \partial_\nu] U^\dagger(x))^{(3)} = \int d\tau d\sigma \frac{\partial (y^\mu_\tau, y^\nu_\sigma)}{\partial (\tau, \sigma)} \delta^{(4)}(x - y(\tau, \sigma)). \]  

(C.52)
References


R.W. Haymaker, Dual Abrikosov vortices in U(1) and SU(2) lattice gauge theories, hep-lat/9510035.


39