Balls in Boxes and Quantum Gravity

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Four dimensional simplicial gravity has been studied by means of Monte Carlo simulations for some time\cite{1}, the main outcome of the studies being that the model undergoes a discontinuous phase transition\cite{2} between an elongated and a crumpled phase when one changes the curvature (Newton) coupling. In the crumpled phase there are singular vertices growing extensively with the volume of the system\cite{3} giving an infinite Hausdorff dimension, whereas the elongated phase has a Hausdorff dimension equal to two. This phase has all properties of a branched-polymer phase\cite{4}. We have postulated\cite{5} that this behaviour is a manifestation of the constrained-mean-field scenario as realised in the Branched Polymer\cite{6} (BP) or Balls-in-Boxes model\cite{7}. The models of\cite{5–7} share all the features of 4D simplicial gravity except that they exhibit a continuous phase transition. We note here that this defect can be remedied by a suitable choice of ensemble.

The partition function of the Balls-in-Boxes model\cite{7} is
\begin{equation}
Z(M,N) = \sum_{\{q\}} p(q_1) \cdots p(q_M) \delta_{q_1 + \cdots + q_M, N},
\end{equation}
which describes weighted partitions of \(N\) balls in \(M\) boxes. This model can be solved\cite{7} in the limit of an infinite number of boxes and fixed density of balls per box: \(M \to \infty\) and \(\rho = N/M = \text{const.}\) In this limit the partition function can be expressed in terms of the free energy density per box \(f(\rho)\)
\begin{equation}
Z(M,N) = e^{Mf(\rho) + \cdots}.
\end{equation}
By introducing the integral representation of the Kronecker delta function one finds by the steepest descent method that
\begin{equation}
f(\rho) = \mu_*(\rho) \rho + K(\mu_*(\rho))
\end{equation}
where \(K\) is a generating function given by
\begin{equation}
K(\mu) = \log \sum_{q=1}^{\infty} p(q) e^{-\mu q}
\end{equation}
and \(\mu_*(\rho)\) is a solution of the saddle point equation
\begin{equation}
\rho + K'(\mu_*) = 0
\end{equation}
For a suitable choice of the weights \(p(q)\) the system displays a two phase structure with a critical density \(\rho_{cr}\).

When \(\rho\) approaches \(\rho_{cr}\) from below \(\mu_*\) approaches \(\mu_{cr}\) from above. When \(\rho\) is larger than \(\rho_{cr}\), \(\mu\) sticks to the critical value \(\mu_{cr}\) and the free energy is linear in \(\rho\)
\begin{equation}
f(\rho) = \mu_{cr} \rho + \kappa_{cr}
\end{equation}
where \(\kappa_{cr} = K(\mu_{cr})\). The change of regimes \(\rho < \rho_{cr}\) (3) to \(\rho \geq \rho_{cr}\) (6) corresponds to the phase transition. To understand the physical nature of the transition it is convenient to consider the dressed one-box probability, defined as the probability that a particular box contains \(q\) balls. In the large \(M\) limit the saddle point equation gives
\begin{equation}
\pi(q) = \begin{cases} 
e^{-K(\mu_*) p(q) e^{-\mu_* q}} & \text{for } \rho < \rho_{cr} \\ e^{-\kappa_{cr} p(q) e^{-\mu_{cr} q}} & \text{for } \rho \geq \rho_{cr} \end{cases}
\end{equation}
The approach of the dressed probability to the limiting form (7) is not uniform. In particular
the average, \( \langle q \rangle = \sum_q q \pi(q) \), does not give \( \rho \) for \( \rho > \rho_{cr} \) as it should. One can easily correct for this by adding a “surplus anomaly” term to \( \pi \) for finite \( M \) in the phase above \( \rho_{cr} \) [8]
\[
\pi_M(q) = \pi(q) + \frac{1}{M} \delta(q - M(\rho - \rho_{cr})).
\] (8)
In effect, \( M - 1 \) boxes keep the critical form of the distribution and one box takes over the surplus of balls. One can easily check by performing directly finite size computations that this situation is indeed realized in the model.

The appearance of the surplus anomaly is a condensation phenomenon similar to Bose-Einstein condensation and the transition in the Kac-Berlin spherical model[9]. We call the phase \( \rho < \rho_{cr} \) fluid and \( \rho > \rho_{cr} \) condensed. The system may enter the condensed phase either by changing density or by modifying the weights \( p \).

For instance, the one parameter family of weights \( p(q) = q^{-\beta}, q \geq 1 \) gives the critical line
\[
\rho_{cr} = \frac{\zeta(\beta - 1)}{\zeta'(\beta)}
\] (9)
so one can change the phase either by varying \( \beta \) for fixed \( \rho \) (as in case of branched polymers where \( \rho = 2 \)), or by fixing \( \beta \) and varying the density \( \rho \).

The transitions in \( \rho \) and \( \beta \) in this variant of the model are continuous, unlike the simulations of simplicial gravity.

It is also possible to consider ensembles with varying density. The simplest candidate is an ensemble with a chemical potential coupled to the total number of balls, which can be treated as a box in contact with a reservoir of balls. However, this leads to a totally decoupled system which corresponds to \( M \) copies of the urn-model[10]. For \( \mu \geq \mu_{cr} \) the average number of balls in the urn diverges. More interesting in the context of simplicial gravity is the model where we keep the number of balls fixed and vary the number of boxes [8,11]. This gives direct analog of the ensemble used in the simplicial gravity simulations
\[
Z(\kappa, N) = \sum_M Z(M, N)e^{\kappa M}.
\] (10)
It is now more natural to consider curvature \( r = 1/\rho \) instead of density. The partition function

\[
\langle r \rangle = \int dr e^{N(f(r) + \kappa r)}
\] (11)
and the saddle point equation for this integral is
\[
\kappa + f'(r_*) = 0.
\] (12)
For \( \kappa > \kappa_{cr} \) this equation reduces to \( \kappa = K(\mu_{sp}(r_*)) \) which has a unique solution for \( r_* \). The value of \( r_* \) (the centre of a gaussian distribution) is the average curvature in the limit \( N \to \infty \). This situation continues as long as \( \kappa > \kappa_{cr} \).

For \( \kappa < \kappa_{cr} \) the saddle point equation (12) has no solution and therefore the integrand is no longer gaussian but a monotonic function of \( r \). In particular, for \( r < r_{cr} \) it is exponential, \( \exp N(\kappa - \kappa_{cr})r \), and for large \( N \) only this exponential part contributes in the integral (11) giving \( \langle r \rangle \sim 1/N \).

The average curvature \( \langle r \rangle \) is shown as a function of \( \kappa \) in Figure 1. The bold line is a limiting curve for \( N = \infty \). For \( \kappa > \kappa_{cr} \) it is the solution of the saddle point equation (12). It stops at \( r_{cr} \) and falls to zero. In the neighbourhood of the critical point the curves are steepest. This
part of the curves corresponds to the pseudocritical region where the two phases coexist. One expects a double peak histogram for $r$: one peak near the maximum of the gaussian phase and the other near the kinematic limit $1/N$. In Figure 2 we show the distributions of $r$ for two different sizes $N$, showing the coexistence of two phases that is characteristic of first order transitions.

To summarize, the Balls-in-Boxes model describes well basic features of simplicial gravity simulations such as the appearance of the singular vertices and the mother universe[5]. With the appropriate choice of ensemble one obtains a first order transition too. We note that arguments [14,15] have recently been given that the bare weights $p(q)$ in simplicial gravity can be approximated by $p(q) \sim e^{b q} e^{-a q^2}$, which gives a similar behaviour to the power law weights discussed here.

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REFERENCES


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