INSTABILITIES AND BEAM INTENSITY LIMITATIONS IN CIRCULAR ACCELERATORS

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Abstract
The main aim of these lectures will be to give an insight into the physics of the mechanisms of instabilities and beam intensity limitations in circular accelerators. Three different techniques will be used to evaluate the various instabilities. The first will be using ‘few-particle models’, the second will use matrix techniques involving eigenvalues, and the third technique will use Sacherer’s modal analysis technique. For each instability the threshold or the growth rate will be evaluated with particular attention to the parameter dependence.

1. CALCULATION TECHNIQUES
1.1 Using matrix techniques (Eigenvalues)
The position of a particle (U) in two-dimensional phase space can be defined by its instantaneous position u and angle u’ (du/dt). Consequently n particles may be represented by a column matrix with N (2n) rows. A transition matrix may be derived which describes the transition of the n particles from one situation to another, i.e.

\[ U_2 = \begin{bmatrix} u_1 \\ u_1' \\ u_2 \\ u_2' \\ \vdots \\ u_n \\ u_n' \end{bmatrix} = \begin{bmatrix} t_{11} & t_{12} & t_{13} & t_{14} & \cdots & t_{1N} \\ t_{21} & t_{22} & \cdots & \cdots & \cdots & \cdots \\ t_{31} & \cdots & t_{33} & \cdots & \cdots & \cdots \\ t_{41} & \cdots & \cdots & t_{44} & \cdots & \cdots \\ t_{51} & \cdots & \cdots & \cdots & t_{55} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ t_{N1} & \cdots & \cdots & \cdots & \cdots & t_{NN} \end{bmatrix} \begin{bmatrix} u_1 \\ u_1' \\ u_2 \\ u_2' \\ \vdots \\ u_n \\ u_n' \end{bmatrix} \]

This transition matrix is of necessity of dimension N x N. Evaluation of the eigenvalues of the transition matrix allows determination of the growth rates, damping times, and frequency shifts experienced by the particles. For more than two particles it is usually necessary to evaluate the eigenvalues by computer.

1.2 Using frequency shifts
The well-known differential equation of a simple harmonic oscillator with a ‘normalized’ driving force is

\[ \ddot{u} + \omega_0^2 u = Gu = (G_R + jG_I)u \]

where \( \omega_0 \) is the natural frequency of the oscillation and \( G_R \) and \( G_I \) are the real and imaginary components of the normalized driving force. The equally well-known solution to this equation is

\[ u = u_0 \exp(j\omega_0 t) \]
where \[ \omega_n = \omega_0 + \Delta \omega, \text{ and } \Delta \omega = - \frac{G}{2\omega_0} = - \frac{G_R}{2\omega_0} - j \frac{G_I}{2\omega_0} \]
giving
\[ u = u_0 \exp(j\omega_0 t) \exp\left( - \frac{jG_R}{2\omega_0} t \right) \exp\left( + \frac{G_I}{2\omega_0} t \right). \quad (1) \]

Consequently the exponential coefficient of the motion (\( \alpha \)) is given by
\[ \alpha = \frac{1}{\tau} = -\text{lm} (\Delta \omega) = + \frac{G_I}{2\omega_0}. \quad (2) \]

\textit{Hence the motion is unstable} (\( \alpha > 0 \)) \textit{if the imaginary component of the frequency shift is of negative sign or if the imaginary component of the driving force (G_I) is of positive sign.}

1.3 \textbf{Evaluation of the normalized driving force}

In the case of beam instabilities the driving ‘force’ is normally an induced ‘force’ such as a voltage induced by the passage of the charged beam itself. Since the force is induced by the beam then it can only have components at frequencies corresponding to the modes of oscillation of the beam itself. Thus the force results from the spectrum of the beam oscillation ‘sampling’ the impedance \( Z \) seen by the beam. In certain cases the induced driving force is fairly obvious and can be derived almost by inspection, but in most cases the derivation is more complicated.

The induced voltage can be derived from the inverse Fourier transform of the product of the impedance and the bunch current (in frequency domain) i.e.
\[ V(t) = \mathcal{Z}^{-1}[V(\omega)] = \mathcal{Z}^{-1}[Z(\omega)I(\omega)]. \]

The voltage induced by the beam gives the field which must be integrated over the bunch spectrum (\( \rho \)) in order to get the total force i.e.
\[ G = jk \int V(t) \rho(t) dt = jk \int \mathcal{Z}^{-1}[Z(\omega)I(\omega)] \rho(t) dt. \]

Sacherer [1] has shown that, in general,
\[ G \propto jI_0 \frac{\sum Z(\omega)h(\omega)}{\sum h(\omega)} \quad (3) \]
where \( I_0 \) is the average bunch current and \( h(\omega) \) is the power spectrum of the bunch current distribution. Clearly, stability is determined by the sign of the real part of the impedance \( Z_R(\omega) \). The stability situation is therefore investigated by evaluating the frequency dependence of the impedance seen by the beam and the power spectrum of the beam oscillations which ‘sample’ the impedance to produce an induced normalized force.
2. ‘ROBINSON’ INSTABILITY

The Robinson instability [2] has been analysed in many and complicated ways since it was pointed out more than 30 years ago. This instability is driven by the fundamental accelerating modes of the RF cavities and is not a serious effect for modern accelerators since the cure is easy and well known. However, as will be seen later, the Robinson instability is a specific case of the more general coupled-bunch instability and therefore serves as a simplified introduction to instabilities and their evaluation in accelerators.

2.1 Robinson ‘physics’

The physics of the Robinson instability may be understood by simply examining the beam-induced voltage produced by the interaction of the beam with the fundamental accelerating mode of the RF cavities. It should be stated that, for the sake of simplicity, only the induced voltage from the passage of the bunch on the previous turn is taken into account. This is not an unreasonable approximation for room-temperature cavities but is certainly not applicable to the case of superconducting ones. The voltage induced by a bunch at time ‘zero’ is shown as a function of time in Fig. 1. After around four full oscillations of the induced voltage the bunch returns to sample the voltage which it induced in the cavity on the previous turn. In frequency domain this corresponds to operating at the peak of the resonance curve of the cavities. We now assume that the revolution frequency of the bunches \( f_b \) can be changed slightly and that we maintain the frequency of the induced voltage constant. If the bunch frequency is increased slightly then the bunch arrives sooner and experiences the induced sinusoidal voltage but with a negative gradient as shown in Fig. 1. From fundamental longitudinal phase space dynamics (phase stability) it is well known that this situation is stable, above transition energy. In the case where the bunch frequency is less than the mode frequency then the bunch arrives later and experiences a sinusoidal voltage with a positive slope. This situation is unstable and may be explained by considering a small energy oscillation with respect to the situation shown in Fig. 1. If the bunch at time zero has slightly more energy than the reference bunch then it will arrive somewhat later after one turn and (for \( f_b < f_{\text{mode}} \)) experience a voltage gain which is higher than the reference bunch. Clearly an unstable situation which is depicted in frequency domain by the lower trace on the left of Fig. 1.

![Fig. 1: Beam-induced voltage on right with the frequency spectrum of the resonance curves of the cavities on the left](image-url)
3. **CALCULATION OF ROBINSON: INDUCED VOLTAGE**

This simplified analysis of the Robinson instability requires the analysis of the motion of a single particle driven by the induced voltage in the fundamental mode of the cavities.

Figure 2 is a schematic of a single turn of an accelerator with a single cavity. A single turn for convenience starts at the exit of the cavity, which has an induced voltage due to the passage of the particles, and is followed by a longitudinal ‘drift’ to the entrance of the cavity. The cavity is assumed to have no length and simply gives an energy boost to the particles.

![Cavity](image)

**Fig. 2:** Schematic of an accelerator with a single cavity

The induced (mode) voltage generated at location 1 and remaining at location 2 is given by

\[
v_m = -V_m \exp \left\{ j\omega_m (t_2 - t_1) - \frac{(t_2 - t_1)}{T_f} \right\}
\]

(4)

where \( \omega_m \) is the radiancy of the excited mode, \( T_f \) is the filling time of the cavity, and the revolution time dependence on the relative energy offset (\( \Delta = \delta E/E \)) is

\[
(t_2 - t_1) = t_{rev} (1 + \Delta \eta),
\]

where \( t_{rev} \) is the average revolution time and

\[
\eta = \frac{1}{\gamma_f} - \frac{1}{\gamma^2},
\]

where \( \gamma_f \) is the relative transition energy.

Substituting

\[
\delta_0 = (\omega_m - h\omega_{rev}) t_{rev},
\]

where \( h \) is the RF harmonic number,

and

\[
\tau_0 = \frac{t_{rev}}{T_f},
\]

gives

\[
v_m = -V_m \exp \left\{ j(\delta_0 + 2\pi h)(1 + \eta \Delta) - \tau_0 (1 + \eta \Delta) \right\}
\]

and neglecting small terms

\[
v_m = -V_m \exp \left\{ j(\delta_0 + 2\pi h \eta \Delta) - \tau_0 \right\}
\]

(5)

which by linearizing gives
\[ v_m = -V_m e^{-\tau_0} \left[ \cos \delta_0 - 2\pi \eta_0 \Lambda \cdot \sin \delta_0 + j(\sin \delta_0 + 2\pi \eta_0 \Lambda \cdot \cos \delta_0) \right]. \quad (6) \]

Taking only the oscillating part of Eq. (6)
\[ v_m(\Lambda) = V_m e^{-\tau_0} 2\pi \eta_0 \Lambda \{ \sin \delta_0 - j \cos(\delta_0) \}. \quad (7) \]

The induced voltage may also be written in terms of \( \varphi \) by simply substituting
\[ \varphi = \frac{j\eta_0 \Lambda}{Q_s} \]

\((Q_s \) is the synchrotron frequency, defined later, Section 5) which gives
\[ v_m(\varphi) = V_m e^{-\tau_0} 2\pi Q_s \varphi \{ \cos \delta_0 - j \sin \delta_0 \}. \quad (8) \]

### 4. ROBINSON BY EIGENVALUES OF SINGLE-PARTICLE MOTION

Referring to Fig. 2, the energy at the exit of the cavity after one turn (location 3) is
\[ \Lambda_3 = \Lambda_2 + \frac{eV_{RF}}{E} \{ \sin(\Phi_3 + \varphi_2) - \sin(\Phi_3) \} + \Re \left\{ \frac{eV_m(\Lambda)}{E} \right\} \]
\[ \Lambda_3 = \Lambda_2 + \frac{eV_{RF}}{E} \{ \sin(\Phi_3 + \varphi_2) - \sin(\Phi_3) \} + \frac{eV_m}{E} e^{-\tau_0} 2\pi \eta_0 \Lambda \cdot \sin \delta_0 \]

where \( \varphi \) is measured with respect to the synchronous particle.

Consequently the linearized longitudinal phase space transition over a complete turn (single-turn matrix \( S \)) is given by
\[
\begin{bmatrix}
\Phi \\
\Lambda
\end{bmatrix}_3 = \begin{bmatrix}
\frac{1}{E} & 0 \\
\frac{eV_{RF} \cos \Phi_3}{E} & 1 + \frac{eV_m}{E} e^{-\tau_0} 2\pi \eta_0 \cdot \sin \delta_0
\end{bmatrix} \cdot \begin{bmatrix}
\frac{1}{E} & 0 \\
2\pi \eta_0 & 1
\end{bmatrix} \cdot \begin{bmatrix}
\varphi \\
\Lambda
\end{bmatrix}.
\]

For a real 2x2 matrix the absolute value (amplitude) of the two eigenvalues is equal to the square root of the determinant, hence if the determinant is greater than unity the motion is unstable with exponential growth
\[ \det(T) - 1 = \frac{2\tau_{\text{rev}}}{\tau}. \]

Hence the growth rate is
\[ \frac{1}{\tau} = \frac{eV_{RF} \eta V_m e^{-\tau_0}}{2E} \cdot \sin \delta_0. \quad (9) \]

Clearly the motion is **unstable when the product \( \eta \delta_0 \) is positive**, i.e. above transition energy \((\eta > 0)\) when \( \delta_0 \) is positive (i.e. when \( \omega_m > \hbar \omega_{\text{rev}} \)). This is of course the Robinson instability. However, the growth rate is not identical to that usually derived because of the approximation of only taking the induced voltage of the previous turn.
5. **ROBINSON BY SOLUTION OF THE FORCED EQUATION OF MOTION**

The general equation of longitudinal motion is

$$\dot{\phi} + \frac{\eta h \omega_{rev}^2}{2\pi E} [V_{RF}(\phi) - V_{RF}(0) + v_m] = 0$$

which, for a normal sinusoidal RF voltage becomes

$$\dot{\phi} + \Omega^2 \phi = -\left(\frac{eh\eta \omega_{rev}^2}{2\pi E}\right) v_m$$

(10)

where $\Omega$ is the synchrotron radiancy

$$\Omega = \omega_{rev} Q_s = \omega_{rev} \sqrt{\frac{eh V_{RF} \cos \Phi_s}{2\pi E}}.$$

Substituting for $v_m$ from Eq. (7) gives

$$\dot{\phi} + \Omega^2 \phi = -\left(\frac{eh\eta \omega_{rev}^2}{2\pi E}\right) V_m e^{-\tau_0} Q_s (\cos \delta_0 - j \sin \delta_0) \phi.$$

As shown previously the growth rate is given by the imaginary part of the driving force, i.e.

$$\frac{1}{\tau} = \frac{1}{2\Omega} \frac{eh\eta \omega_{rev}^2 Q_s V_m e^{-\tau_0}}{E} \cdot \sin \delta_0 = \frac{e\eta \omega_{RF} V_m e^{-\tau_0}}{2E} \cdot \sin \delta_0.$$

Fortunately this gives the same result as Eq. (9).

6. **SPECTRUM OF LONGITUDINAL OSCILLATIONS**

In Section 1 it was shown that the stability situation can be investigated by evaluating the frequency dependence of the impedance seen by the beam and the power spectrum of the beam oscillations which ‘sample’ the impedance to produce an induced normalized force. In this section the power spectrum of oscillations in the longitudinal plane are investigated.

Figure 3 (from Ref. [3]) shows the time domain signal of a single bunch with Gaussian charge distribution circulating in an accelerator with a revolution time of $T_0$ and with vanishing synchrotron motion. This is the signal which would be detected by a current transformer with large frequency bandwidth. The current distribution is

$$I(t) = \frac{q}{\sqrt{2\pi \sigma t}} \exp\left\{-\frac{t^2}{2\sigma^2}\right\},$$

where $q$ is the bunch charge and $\sigma$ the bunch length in time.
The well-known Fourier transform of the sampled signal (also shown in Fig. 3) is given by

\[ I_p(\omega) = \frac{q}{T_0} \exp \left( -\frac{p^2 \omega_0^2}{2\sigma^2} \right) \]

where \( \sigma_\omega = \frac{1}{\sigma_t} \) and \( \omega_0 = 2\pi/T_0 \).

Since in this case there is no synchrotron motion included in the spectrum then longitudinal instabilities are excluded; however, by using the derived spectrum to sample the real part of the spectrum of the longitudinal impedance (as described in Section 1), the voltage loss per turn can be derived. Similarly by using the imaginary part of the impedance the shift in the incoherent synchrotron frequency can be evaluated [3].

The next level in complexity is to introduce the synchrotron motion which causes a modulation of the bunch signal in the time domain. Transformed into the frequency domain this produces the familiar sidebands above and below the lines at revolution frequency and separated by the synchrotron frequency. The lower plot in Fig. 4 shows the frequency spectrum as it would be seen on a real spectrum analyser where negative frequencies are ‘folded’ into positive frequencies. The real power spectrum of the bunch, with negative frequencies shown, is shown in Fig. 5 where the synchrotron sideband amplitudes (drawn as arrows) are all shown as equal for the sake of simplicity. These frequencies occur at

\[ \omega_p = \sum_{-\infty}^{\infty} (p + Q_s) \omega_{\text{rev}} \]

Also superimposed on this spectrum is the real part of the impedance of a narrow-band resonator impedance which could represent the fundamental mode of an accelerating structure, i.e.
\[ Z_R(\omega) = \frac{R_s / \omega}{1 + Q^2 \left( \frac{\omega_r}{\omega} - \frac{\omega}{\omega_r} \right)^2} \]

where \( R_s \) is the shunt impedance, \( Q \) the quality factor, and \( \omega_r \) the resonant frequency of the structure.

Note that for positive frequencies the impedance is positive in sign and the inverse for negative frequencies due to the \( Z/\omega \) dependence of the impedance in the longitudinal plane.

![Fig. 4: Signals from a bunch with synchrotron motion, in time and frequency domain](image)

![Fig. 5: Power spectrum with synchrotron sidebands drawn as arrows](image)

In Fig. 5 the resonant frequency of the cavity is drawn slightly above the harmonic number (drawn as 3 for the sake of ease of illustration). In this case it may be seen that the amplitude of the impedance sampled at the synchrotron frequency at positive frequencies is greater than those at negative frequencies.
This situation is depicted more simply in Fig. 6 where the negative frequencies are folded into the positive axis, and in order to represent the fact that the impedance is negative, the lines of negative frequencies are drawn with a downward arrow. It will be shown later that the downward arrows result in exponential growth of the synchrotron motion whereas positive arrows cause damping. Consequently in Fig. 6 (a) where the resonant frequency of the cavities is tuned slightly above the harmonic number, the summation of the lengths of the upward and downward arrows is upward, denoting overall damping. In Fig. 6 (b) the cavity is exactly tuned to the harmonic number and the summation is zero whereas Fig. 6 (c) with detuning below the harmonic number shows instability. This is again the Robinson instability derived by using impedances and the mode spectrum of the bunch motion.

7. GROWTH RATES

Sacherer [1] has shown that the complex frequency shift of the synchrotron sidebands produced by a general impedance is given by

$$\Delta \omega_m = -j \frac{m \omega_s}{(m+1)} \frac{I_b \omega_{rev}}{3B_0^3 h V_{RF} \cos \Phi_s} \frac{\sum h_{mk}(\omega_p)Z(\omega_p) \omega_p}{\sum h_{mm}(\omega_p)}$$

where $m$ is the mode of oscillation (see later) and equals 1 for dipolar oscillations, $I_b$ the bunch current, $B_0$ the bunching factor (bunch length $\tau_L$/revolution time) and the summation is over the mode spectrum of the bunch oscillations.

The bunching factor can be written, for Gaussian bunches (with r.m.s. length $\sigma_s$) as

$$B_0 = \frac{16}{3} \sqrt{\frac{2}{\pi}} \frac{\sigma_s}{2\pi R} = 0.678 \frac{\sigma_s}{R}$$

with $2\pi R$ the circumference of the machine.

For $m = 1$, and for a narrow-band resonator covering two synchrotron sidebands (as in Fig. 6) this may be written
\[ \Delta \omega_m = -j \frac{\omega_s}{2} \frac{I_{\text{total}}}{3B_0^2 hV_{\text{RF}} \cos \Phi_s} F_m \left( \frac{Z(p^+) - Z(p^-)}{p} \right) \]

where \( F_m \) is the form factor (amplitude of the envelope of the power spectrum of the bunch oscillations), \( p \) is \( \frac{f_p}{f_{\text{rev}}} = h \) for the fundamental RF, and \( Z(p^\pm) \) refers to the impedance at the upper and lower synchrotron sideband frequencies.

Consequently
\[
\text{Im}(\Delta \omega_m) = -j \frac{\omega_s}{6} \frac{I_{\text{total}}}{B_0^2 hV_{\text{RF}} \cos \Phi_s} F_m \left( \frac{Z_R(p^+) - Z_R(p^-)}{p} \right).
\]

Here \( Z_R \) is the real part of the longitudinal impedance, i.e.
\[
Z_R(\omega) = \frac{R_s}{1 + Q^2 \left( \frac{\omega - \omega_\tau}{\omega} \right)^2}.
\]

For such a resonator the impedance difference between the synchrotron sidebands can be evaluated as
\[
Z(p^+) - Z(p^-) \approx 16 R_s Q^2 \frac{Q_s}{h} \frac{\Delta \omega}{\omega_\tau^2}
\]

where \( \Delta \omega \) is the frequency detuning of the accelerating structure \((\omega_\tau - h\omega_0)\). This relationship only holds for
\[
Q \ll \frac{\omega_\tau}{2 \left( \Delta \omega + \omega_s \right)} \approx \frac{h}{2Q}
\]
and will therefore not be valid for superconducting cavities where the loaded \( Q \) value is very high. When this inequality is satisfied, the growth rate can be reduced to
\[
\frac{1}{\tau} = -\text{Im}(\Delta \omega_m) = \frac{4}{3\pi} \left( \frac{I_{\text{total}} F_m}{B_0^2} \right) \left( \frac{\pi \eta}{h^3 E} \right) \left( R_s Q^2 \Delta \omega \right)
\]

where the first bracket refers to the beam conditions, the second refers to the machine parameters, and the third to the parameters of the cavities.

Since by our convention \( \eta \) is positive above transition, then stability is ensured by \( Z^+ \) being greater than \( Z^- \) which occurs when \( \Delta \omega \) is positive, i.e. when the cavity frequency is tuned above \( h\omega_{\text{rev}} \).

This relationship has a similar parameter dependence to that previously derived using the simpler model (Sections 4 and 5) but is not identical since in the simpler model only the influence of the cavity induced voltage on the previous turn is taken into account.
8. MODES AND SPECTRA OF MULTIPLE BUNCHES

The next level of complexity is to consider the spectrum of the dipolar motion of a beam with many bunches. Figure 7 shows the longitudinal phase space (energy deviation against RF phase) of the dipole modes of oscillation of four bunches. The different bunches are spaced horizontally left to right and the different modes of oscillation spaced vertically. The mode of oscillation \( n \) is defined by the phase advance between the motion of successive bunches, i.e.

\[
\Delta \mu = \frac{2\pi}{k_b} n.
\]

Consequently for the \( n = 0 \) mode of oscillation all four bunches move in an identical way, and there is zero phase difference between the motion of successive bunches. This is shown in Fig. 7 as are the other three possible modes of oscillation. It should be realized that in this phase space the motion is in the anticlockwise sense.

The bunches may also be subjected to ‘shape’ modes of oscillation. These are depicted for completeness in Fig. 8, where it can be seen that the \( m = 1 \) mode (dipolar mode which we have been considering) is a centre-of-gravity motion whereas the higher-mode numbers produce changes in the bunch shape.
In general the spectral lines of motion occur at frequencies

$$\omega_p = \omega_{\text{rev}}(k \cdot k_b + n + m Q_s) = \omega_{\text{rev}}(p + m Q_s)$$

where $p = n + k \cdot k_b$, $k = 0, \pm 1, \pm 2, \ldots$ and $n = 0, 1, 2, \ldots (k_b - 1)$.

An example of the spectrum is shown in Fig. 9. It is apparent that the spectrum simply repeats itself for different values of $k$. In addition, negative $k$ values correspond to negative frequencies. Superimposed on this diagram is a narrow-band resonator impedance (plotted as $Z_R/\omega_0$) which couples to mode $n = 1$ and $n = 3$. In this case since the real part of the impedance is negative at the negative frequency corresponding to mode $n = 1$, this mode will be unstable. The mode $n = 3$ is damped by the positive real part of the impedance. The previously described Robinson instability is a particular case of this coupled bunch instability but affecting only mode $n = 0$.

A more compact and useful way to draw spectra for coupled-bunch oscillations is shown in Fig. 10 where the negative frequencies are ‘folded’ into positive frequencies but drawn with negative amplitudes. In this way modes of oscillation with positive amplitudes are stable and those with negative amplitudes unstable. The growth rates can be evaluated in an identical way to the technique used previously for the Robinson instability.
In cases where the impedance is broadband then the summation of the impedance should be performed over frequency lines with the same mode number \((n)\).

**Fig. 10:** Longitudinal spectrum of four bunches with the real part of the impedance superimposed

### 9. TRANSVERSE MOTION

#### 9.1 Equation of motion and transverse impedance

The equation of motion of a single particle in a coasting beam in the transverse (horizontal or vertical) plane is

\[
\ddot{x} + (Q_P\Omega_{rev})^2 x = \frac{F_x}{m}
\]

where \(Q_P\) is the betatron tune and \(F_x/m\) is the normalized driving force given by

\[
\frac{F_x}{m} = \frac{e(E + B \times v)}{\gamma m_0}.
\]

The definition of the transverse impedance is

\[
Z^T = j \frac{2\pi \int (E + B \times v) ds}{\beta I \Delta}.
\]

Consequently the equation of motion for a single particle in a coasting beam is

\[
\ddot{x} + (Q_P\Omega_{rev})^2 x = -j \frac{eB}{\gamma m_0} \frac{Z^T I}{2\pi R} x.
\]
Here the normalized driving force produces a frequency shift

\[ \Delta \omega = \frac{j}{2Q\gamma \omega_{\text{rev}}} \frac{e\beta}{\gamma m_0} \frac{Z^T I}{2\pi R}. \]

This can be written

\[ \Delta \omega = \frac{j}{4\pi Q \beta} \frac{c}{(E/e)} Z^T I. \]

The transverse impedance is evaluated at the frequencies

\[ \omega = (p + Q \beta) \omega_{\text{rev}} + \Delta \omega. \]

Clearly the imaginary part of the transverse impedance will produce a real frequency shift whereas the real part will cause instability for negative resistances and damping for positive resistances.

### 9.2 Modes of transverse oscillations

An example from Ref. [4] of the line spectrum of a bunched beam with transverse motion is shown in Fig. 11 which is drawn for a machine with five relatively long bunches (the PS Booster) and a \( Q_\beta = 4.2 \). This spectrum is similar to that shown previously for the longitudinal plane except that the sidebands are separated from the revolution frequency lines by the non-integer part of the transverse tune (\( q_\beta \)).

![Spectrum for a machine with 5 long bunches (PS booster) and Q = 4.2. Envelope is drawn for head-tail mode = 0 and is shifted towards positive frequencies.](image)

**Fig. 11**: Transverse spectrum of five bunches
9.3 Chromaticity

In the case of transverse oscillations, the spectrum is significantly affected by the chromaticity (the tune dependence on the momentum offset). A particle which is performing synchrotron oscillations can be represented by the ellipse in Fig. 12: above transition a particle with higher energy than the synchronous particle drifts towards the ‘tail’ of the bunch. This means that in Fig. 12 a synchrotron oscillation moves on an approximate ellipse in the anticlockwise direction. Now consider what happens when there is a negative tune dependence on the particle momentum (or energy), i.e.

\[ Q_h = \frac{\Delta Q_h}{\Delta} \leq 0. \]

A particle at the head of the bunch has the same momentum deviation as the synchronous particle and therefore the same betatron frequency. As this particle moves along the synchrotron ellipse shown, the betatron frequency initially decreases and a betatron phase lag develops, shown as the downward arrows in the figure. When the particle reaches the tail of the bunch its phase lag is at a maximum. Continuing the motion towards the head of the bunch causes the particle to gradually regain betatron phase until it reaches the head of the bunch where the whole cycle restarts. It is also apparent that, if a large number of particles were distributed around the synchronous orbit with betatron phase advances as indicated in Fig. 12, then the pattern would remain stationary. The net result of all this is that the head and the tail of the bunch oscillate at the same frequency but with a phase difference.

\[ \text{Fig. 12: Betatron phase along a synchrotron orbit} \]

The total phase shift between the head and the tail is usually denoted by \( \chi_{\text{max}} \) and can be calculated as follows. Specify the longitudinal position of a particle within a bunch by its time delay (\( \tau \)) from the head of the bunch. This time delay changes by an amount \( \Delta \tau \) per turn

\[ \frac{d\tau}{dk} = \Delta \tau = \eta \Delta. \]
The betatron phase shift is
\[ \chi = 2\pi \int \Delta Q \frac{d\tau}{d\phi} \frac{dk}{d\phi} = \frac{2\pi Q_0}{\eta_{\text{rev}}} \frac{d\tau}{d\phi} = \frac{\omega_{\text{rev}} Q_0 \tau}{\eta}. \]

Thus the betatron phase varies linearly along the length of the bunch and reaches its maximum value at the tail where \( \tau = \tau_L \). The same linear dependence of the phase shift on the distance along the bunch is true for higher modes and the oscillation amplitude is given by a standing wave pattern \( |P_m(t)| \).

Figure 13 from Ref. [4] shows the difference signal which would be detected at a pick-up as a function of the number of turns \((k)\) and the mode number \((m)\) plotted for three different values of the head–tail phase shift \((\chi)\).

This signal has the form
\[ P_m(t) \exp\left(j\omega z t + j2\pi k Q_0 \right) \]
where
\[ \omega_z = \frac{\chi_{\text{max}}}{\tau_L} = \frac{Q_0 \omega_{\text{rev}}}{\eta} \]
gives the frequency of the wiggles along the bunch. If the standing wave patterns \( P_m(t) \) are taken to be sinusoidal i.e.
\[ P_m(t) = \cos(m + 1)\pi \frac{t}{\tau_L} \quad \text{for} \ m = 0, 2, 4, \ldots \]
\[ P_m(t) = \sin(m + 1)\pi \frac{t}{\tau_L} \quad \text{for} \ m = 1, 3, 5, \ldots \]
the envelope of the line spectrum of the power spectrum is given by [4]

\[ h(\omega) = (m+1)^2 \frac{\tau_L^2}{2\pi^4} \left( \frac{\omega\tau_L}{\pi} \right)^2 \cos^2 \left( \frac{\omega\tau_L}{\pi} \right) \left[ \frac{\omega\tau_L}{\pi} - (m+1)^2 \right]^2 \]

which is drawn in Fig. 14 for the first four modes of oscillation.

Fig. 14: Power spectrum of modes 0 to 4 with \( \chi = 0 \)

When the chromaticity is finite, the travelling wave component, \( \exp (j\omega t) \), is present and shifts the power spectrum depending on the sign of \( \omega_{\xi} \) i.e.

\[ h(\omega) = h(\omega - \omega_{\xi}) \]

The actual spectrum is a line spectrum as already shown in Fig. 11 where it may be seen that the envelope has been moved towards higher frequencies. This means that \( \omega_{\xi} \) is positive, implying positive chromaticity above transition energy or negative chromaticity below.

9.4 Transverse growth rates

Sacherer [4] has derived the general result that the frequency shift is given by

\[ \Delta\omega_m = \frac{1}{(1+m)} \frac{j}{2 Q \beta_{0rev} \gamma m_0} \frac{I_b}{\tau_L} \frac{Z_T(\omega)h(\omega - \omega_{\xi})}{\sum h(\omega - \omega_{\xi})} \]

where once again the summations are over the mode spectrum of the bunches. Hence the growth rate is

\[ \frac{1}{\tau} = - \text{Im}(\Delta\omega_m) = \frac{1}{(1+m)} \frac{j}{2 Q \beta_{0rev} \gamma m_0} \frac{I_b}{\tau_L} \frac{Z_T(\omega)h(\omega - \omega_{\xi})}{\sum h(\omega - \omega_{\xi})} \]
where $Z_R^T$ is the real part of the transverse impedance. Consequently, in the transverse plane when the real part of the impedance is negative, an instability is provoked. In the case of a narrow-band resonator the real part of the transverse impedance is positive for positive frequencies and negative for negative frequencies. Hence transverse modes of oscillation at negative frequencies are unstable. It may therefore be appreciated that by moving the bunch spectrum towards higher frequencies (positive $\omega \xi$, $\chi$) the mode 0 is damped, whereas the higher order modes may become unstable. This means positive chromaticity above transition energy and negative below. This behaviour is shown graphically in Fig. 15.

\[ \omega \xi = 0 \]

\[ \omega \xi \text{ positive} \]

**Fig. 15:** Transverse spectra for two different values of $\omega \xi$

Figures 16 to 21 show the bunch line-spectra coupling to a resonator impedance and a resistive wall impedance: also shown are the resultant growth rates as a function of the chromatic shift. It is interesting to study these plots to understand the influence of $\omega \xi$ and of the tune values.
$\omega_\zeta = -2$  $\omega_\zeta = 0$  $\omega_\zeta = 2$

**Fig. 16:** Resonator impedance with varying $\omega_\zeta$ (for modes 0 and 1)

$\omega_\zeta = -2$  $\omega_\zeta = 0$  $\omega_\zeta = 2$

**Fig. 17:** Relative growth rates as a function of $\omega_\zeta$

$\omega_\zeta = -2$  $\omega_\zeta = 0$  $\omega_\zeta = 2$

**Fig. 18:** Resistive wall type impedance with varying $\omega_\zeta$ (for modes 0 and 1) ($Q_0 = 0.3$)
Fig. 19: Relative growth rates (resistive wall impedance) as a function of $\omega_\xi$ ($Q_\parallel = 0.3$)

Fig. 20: Resistive wall type impedance with varying $\omega_\xi$ (for modes 0 and 1) ($Q_\parallel = 0.7$)

Fig. 21: Relative growth rates (resistive wall impedance) as a function of $\omega_\xi$ ($Q_\parallel = 0.7$)
10. THE TRANSVERSE MODE COUPLING INSTABILITY (TMCI)

In the previous sections the classical head–tail instability was shown to be a resonant effect driven by broad band impedances and controlled by the chromatic parameter $\alpha G$. The ‘strong head–tail instability’ or the ‘transverse turbulent instability’ or more correctly the ‘transverse-mode coupling instability’ is a non-resonant instability which is unaffected by chromaticity and can be a severe and fundamental limitation to the intensity. This instability was first observed, but not understood in SPEAR, and later observed and explained in the PETRA machine [5] and in parallel by Talman [6]. The mechanism is identical to the classic head–tail in that the synchrotron motion is needed to interchange the head and tails of the bunch in order to drive the instability.

Probably the simplest way to gain a physical insight into the instability is to use a two-particle model, one at the head for the first half-synchrotron period and at the tail for the second half-synchrotron period. This situation is depicted in Fig. 22 along with the wakefield induced by the leading particle and experienced by the trailing one.

For the first half-synchrotron period particle 2 is trailing and while traversing the cavity, experiences the wakefield (only one assumed per turn) induced by particle 1 and therefore receives a deflecting kick. Similarly, during the second half-synchrotron period, particle 1 is trailing and receives a deflecting kick.

Consequently the equations of motion for this situation are

\[
\begin{align*}
\ddot{y}_1 + \omega_0^2 y_1 &= 0 & & \text{for} \ 0 < t < T_s / 2 \\
\ddot{y}_2 + \omega_0^2 y_2 &= \alpha y_1 \\
\ddot{y}_2 + \omega_0^2 y_2 &= 0 & & \text{for} \ T_s / 2 < t < T_s \\
\ddot{y}_1 + \omega_0^2 y_1 &= \alpha y_2 \\
\end{align*}
\]

where

\[
\alpha = \frac{Ne^2 W_0}{2m_0 \gamma}.
\]
Using Laplace transforms it is easy to show that the solution to an equation like
\[ \ddot{y}_2 + \omega_\beta^2 y_2 = \alpha y_1 \]
is
\[ y_2(t) = y_2(0) + y'_2(0) \frac{\sin \omega_\beta t}{\omega_\beta} + \alpha y'_1(0) \left\{ -\frac{1}{2\omega_\beta^2} \left( \sin \omega_\beta t + \omega_\beta \cos \omega_\beta t \right) \right\}. \]

Consequently the solution to the equations for the first half-synchrotron period is
\[ \begin{bmatrix} y_1(t) \\ y'_1(t) \\ y_2(t) \\ y'_2(t) \end{bmatrix} = \begin{bmatrix} A(T_s/2) & 0 \\ B(T_s/2) & A(T_s/2) \end{bmatrix} \begin{bmatrix} y_1(0) \\ y'_1(0) \\ y_2(0) \\ y'_2(0) \end{bmatrix} = T(0 \rightarrow T_s/2) \begin{bmatrix} y_1(t) \\ y'_1(t) \\ y_2(t) \\ y'_2(t) \end{bmatrix} \]

where
\[ A(t) = \begin{bmatrix} \cos \omega_\beta t & \sin \omega_\beta t \\ -\omega_\beta \sin \omega_\beta t & \cos \omega_\beta t \end{bmatrix} \]
and
\[ B(t) = \begin{bmatrix} \frac{\alpha t \sin \omega_\beta t}{2\omega_\beta} & \frac{\alpha \sin \omega_\beta t}{2\omega_\beta} \\ \frac{2\omega_\beta}{\alpha t \sin \omega_\beta t} + \frac{\alpha t \cos \omega_\beta t}{2\omega_\beta} & \frac{\alpha t \cos \omega_\beta t}{2\omega_\beta} \end{bmatrix}. \]

Similarly it can be shown that the transfer matrix for the second half-synchrotron period is
\[ T(T_s/2 \rightarrow T_s) = \begin{bmatrix} A(T_s/2) & B(T_s/2) \\ 0 & A(T_s/2) \end{bmatrix} \]
and hence the total transfer matrix for a complete synchrotron period is
\[ T(0 \rightarrow T_s) = \begin{bmatrix} A(T_s/2) & 0 \\ B(T_s/2) & A(T_s/2) \end{bmatrix} \begin{bmatrix} A(T_s/2) & B(T_s/2) \\ 0 & A(T_s/2) \end{bmatrix} = \begin{bmatrix} A^2 & AB \\ BA & A^2 + B^2 \end{bmatrix}. \]

All that remains to be done to determine stability is to evaluate the eigenvalues of this matrix \( T \). By eliminating resonant terms in the matrix \( B \) (it will be shown later that some interesting effects are dropped with this simplification), the eigenvalues \( \lambda \) may be evaluated \([7]\) from the characteristic equation:
\[ \lambda^4 + c_1 \lambda^3 + c_2 \lambda^2 + c_3 \lambda + 1 = 0. \]

Here
\[ c_1 = (2\eta^2 - 4) \cos \omega_\beta T_s \]
and

\[ c_2 = (\eta^2 - 2)^2 + 4\cos^2 \omega \beta T_s - 2 \]

where \( \eta = \frac{\alpha T_s}{4\omega \beta} \) (not to be confused with the frequency dispersion used previously).

Determination of the eigenvalues produces the rather simple result that there is instability if \( |\eta| > 2 \) and above this threshold the two modes of oscillation become degenerate.

This corresponds to a threshold current of

\[ I_{th} = \frac{16 f_{rev} E Q_s \omega \beta}{e c (2\pi R) W_0} \]

where the factor \( 1/\omega \beta \) is related to the \( \beta \) function. If the \( \beta \) function at the location of the wakefield is \( \beta_W \) then \( c/\omega \beta \) should be replaced by \( \beta_W \) giving

\[ I_{th} = \frac{16 f_{rev} E Q_s}{e (2\pi R) \beta_W W_0}. \]

Figure 23 shows the frequency spectrum of the two modes of oscillation as a function of the parameter \( \eta \).

Closer examination of the equations of motion shows that, above threshold, the instability which occurs causes a very severe disruption to the bunches. For \( \eta = 2 \) the amplitude of the motion of the trailing particle grows by a factor of 2 during the half-synchrotron period. Above this value the growth of the particles ‘bootstraps’ into a severe instability with a very fast growth rate.

In a real machine where accumulation is going on, increasing the intensity beyond the threshold results in an ‘explosion’ of the transverse beam dimensions accompanied by a large loss of beam current. Figure 24 shows such the behaviour of the bunch current (in a dedicated experiment on the PEP machine in 1985) as accumulation is continued beyond the threshold of the TMCI. In this plot accumulation is progressing normally until a single injection of around 1% of the total beam current causes the threshold to be surpassed and results in around 40% of the already accumulated beam being lost. It is also interesting to note that after the beam loss, accumulation begins again without problem.
The more precise equation for the threshold current (for more than two particles) is

\[ I_{th} = \frac{2\pi f_{rev}}{e} \frac{Q E_b}{\sum \beta_i k_i(\sigma_s)} \]

which is very similar to that derived for two particles with the slight change that the wakefield is replaced by the loss parameter \( k_i \) at each source of impedance and the summation is performed to include all sources. It is also important to note that the loss parameter is bunch-length dependent. This has been shown to be important in LEP where the threshold current has been increased by increasing the bunch length at injection energy by the use of wiggler magnets. In addition, LEP is operated with a very high \( Q_s \) at injection, as well as an increased injection energy, in order to increase the threshold.

11. TMCI WITH FEEDBACK

Using the two-particle model it was shown that instability occurs when mode 0 is reduced by about half of the synchrotron frequency (see Fig. 23). Since mode \( m = 0 \) is simply the centre-of-gravity motion of the bunch, then the frequency of this mode may be controlled by a feedback system [8]. If for example in Fig. 23 the frequency of mode 0 is maintained constant as the bunch current (\( \eta \)) is increased, then

---

**Fig. 23:** Variation of frequencies of modes \( m = 0 \) and \( m = -1 \) as a function of \( \eta \)

**Fig. 24:** Bunch current behaviour when the threshold of TMCI is exceeded
coupling to mode 1 should be delayed to higher currents. Such a ‘reactive’ feedback system must measure the centre of gravity of the bunch and produce a ‘kick’ which is proportional to the measured displacement. In practice this means there must be a multiple of \( \pi \) radians betatron phase advance between the location of the ‘pick-up’ measuring the displacement and the location of the fast kicker magnet which produces the deflection.

The equations for the two-particle model with feedback are

\[
\begin{align*}
\ddot{y}_1 + \omega_B^2 y_1 &= \sigma (y_1 + y_2) & \text{for } 0 < t < T_s / 2 \\
\ddot{y}_2 + \omega_B^2 y_2 &= \alpha y_1 + \sigma (y_1 + y_2) \\
\ddot{y}_2 + \omega_B^2 y_2 &= \sigma (y_1 + y_2) & \text{for } T_s / 2 < t < T_s \\
\ddot{y}_1 + \omega_B^2 y_1 &= \alpha y_2 + \sigma (y_1 + y_2)
\end{align*}
\]

where \( \sigma \) is the feedback parameter and is proportional to the tune shift (\( \Delta Q_{FB} \)) produced by the feedback on the motion of \((y_1 + y_2)\) at low intensities. Hence

\[
\sigma = -\Delta Q_{FB} \omega_{rev} \omega_{\beta}.
\]

It can be shown from the equations of motion that, in the absence of wakefields,

\[
\Delta \omega_{FB} = -\frac{\sigma}{\omega_{\beta}}.
\]

The solution to the coupled equations of motion with feedback can be performed in an identical manner to that used without feedback. In order to understand the stability limits with feedback [7] it is convenient to identify a feedback dimensionless quantity (\( \xi \)) similar to the \( \eta \) parameter defined previously. Remember that \( \eta \) is simply proportional to the bunch current while \( \xi \) is proportional to the gain of the feedback system. The first idea for feedback called for compensation of the drop in frequency of the \( m = 0 \) mode as a function of current. This corresponds to making the feedback parameter \( \xi \) equal to \(-\eta/2\). Figure 25 shows the calculated regions of instability [7] (hatched areas) in the \((\xi, \eta)\) plane. From this plot, it can be seen that, without feedback \((\xi = 0)\) and by increasing the bunch current \((\eta)\), the unstable region is reached when \( \eta = 2 \). This is depicted by the horizontal line starting at the origin. The line marked \((\xi = -\eta/2)\) shows the situation where the gain is controlled to compensate the linear part of the shift of mode 0. In this case the unstable region is reached at about four times the ‘current’ for zero feedback. Other stable regions are attainable with the feedback system e.g. the region indicated by \( \sqrt{\xi (\xi + \eta)} = \pi \). This analysis clearly shows that the threshold for the transverse (microwave) instability can be increased by the use of a reactive feedback system.
Fig. 25: Stability plots with feedback parameter $\xi$ varied

Figure 26 shows the behaviour of the frequencies of modes 0 and 1 with increasing ‘current’ and in the presence of feedback. With the feedback parameter set to zero we obtain the same plot as previously shown in Fig. 23. However, if the feedback parameter is adjusted so as to push the modes closer at vanishing currents then the instability occurs at a lower threshold current ($\xi = +\pi/4$). In the cases where the feedback enhances the separation of the modes then the threshold is increased ($\xi = -\pi/4$ and $-\pi/2$).

Fig. 26: Behaviour of modes as a function of $\eta$ and $\xi$
12. IMPROVEMENTS IN THE MODEL

The analytical two-particle model discussed so far has two principal simplifying assumptions.

- The use of the continuous differential equations of motion implies that both the wakefields and the feedback are not localized elements but spread out continuously over the circumference of the machine. This of course implies that there is no information concerning the betatron phase advance between the wakefields, and between the feedback and the wakefields.

- In neglecting some of the ‘resonant’ terms in the analysis of Sections 10 and 11 some important resonances disappear.

Clearly, in order to improve the model it is necessary to introduce localized elements [9], [10].

12.1 Localized elements

Figure 27 shows one possible schematic representation of localized elements. For this situation it has been shown [10] that the single-turn transfer matrix is given by

\[
S = L_{KP} \left[ L_{WK} \left( WL_{BW} \right)^{N_{W}^{-1}} WL_{PW} + K \right]
\]

where \( L_{KP} \) is the betatron phase rotation between the Kicker and the Pickup, and the other phase rotations are as indicated in Fig. 27. The feedback (kicker) matrix is given by

\[
K = \begin{bmatrix}
0 & 0 & 0 & 0 \\
\sigma & 0 & \sigma & 0 \\
0 & 0 & 0 & 0 \\
\sigma & 0 & \sigma & 0
\end{bmatrix}.
\]

For two particles the wakefield matrix is given by

\[
W_{1(2)} = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & \alpha(0) & 0 \\
0 & 0 & 1 & 0 \\
0(\alpha) & 0 & 0 & 1
\end{bmatrix}
\]

where the values in brackets refer to the second half-synchrotron period and \( \alpha \) has already been defined as the wakefield strength parameter. For more than two particles there are \( 2n \) \( (n = \text{number of particles}) \) wakefield possibilities if the particles are equally spaced longitudinally.
For two particles it is clear that the matrix which must be inspected for stability is

\[ S_T = S_1^{1/(2Q_s)} S_2^{1/(2Q_s)} \]

where the subscripts refer to the first and second half-synchrotron periods. In this procedure the synchrotron tune must be chosen such that \( 1/(2Q_s) \) is an integer. For more than two particles the matrix for a complete synchrotron period is

\[ S_T = S_1^M S_2^M \ldots S_p^M \]

where \( Q_s = 1/pM \), \( S_p \) is the single-turn matrix which corresponds to the wakefield situation \( p \), and \( p \) is the number of different wakefield situations (= \( 2n \), where there are \( n \) particles equally spaced longitudinally).

The evaluation of the eigenvalues of \( S_T \) is too cumbersome to be done analytically but can be performed with ease on a computer.

A set of results for two particles is shown in Fig. 28 where the \( Q_s \) is 0.0833 (i.e., \( 1/12 \)) and there is only a single wake per turn. With the feedback parameter set to zero one can clearly see the resonances at betatron tunes equal to multiples of \( Q_s \).
12.2 Summary of the results obtained with the ‘few-particle model’

The model described in the last section was used to study [11] the influence of the various parameters on the transverse mode coupling instability. The following summarizes the main results obtained.

- **One wakefield per turn.** The threshold for instability is strongly dependent on the betatron tune value. The feedback enhances the threshold for a range of tune values. However, as the feedback gain increases, the tune range diminishes rapidly. The enhancement of the threshold is strongly influenced by the betatron phase shift between the wakefield and the kicker.

- **Many wakefields equally spaced over one betatron wavelength.** In this case for zero feedback gain there is absolutely no tune dependence (resonances). This indicates that the ‘coherent synchrotron–betatron resonances’ are more or less restricted to the case of a single wake per turn. In most circular accelerators the transverse impedance is indeed localized, but localized to many hundreds of positions around the circumference. Hence for most accelerators these resonances will not appear. The maximum enhancement of the threshold current is greatly reduced when many wakefields (equally spaced over one betatron wavelength) are assumed. With many wakefields the kicker cannot be positioned in betatron phase for optimum enhancement for each wakefield.
13. COMPUTER SIMULATION OF TMCI

The few-particle models have been shown to predict the physical properties of the TMCI. However, as with all simplified models, they are limited in many ways. The previous section showed how some of the limitations can be removed to allow the introduction of localized elements in the model. The procedure uses a brute force technique for evaluation of the threshold of the instability. Although the model contains much of the physics there are, however, some simplifying assumptions which could be worrisome. Most of the remaining simplifications can be removed by a full-blown computer simulation which evaluates the thresholds in realistic conditions without the restriction of simplified wakefields and ‘few’ particles. In these simulations [12], [13] several hundreds or even thousands of superparticles are ‘tracked’ through a simplified machine lattice containing wakefields and feedback systems. The wakefields are evaluated offline using electromagnetic computer codes. Figure 29 shows the simulation of the PEP machine which was done in preparation for a dedicated experiment [14] designed to investigate the TMCI threshold in the presence of feedback. On the left is plotted the FFT of the centre-of-gravity motion for various values of the bunch current. It can be seen that as the current is increased the mode 0 decreases in frequency and approaches mode –1. It is also apparent that as the current increases the amplitude of the mode –1 grows. Instability is reached when the amplitude of the two modes is approximately equal. (This behaviour is often seen in machines with short bunches. In machines with longer bunches where there is bunch lengthening with current, the situation becomes more complicated and the modes m = 0 and m = –1 can become parallel without ever crossing.) An almost identical behaviour was subsequently measured during the dedicated experiment. The motion of the modes is plotted on the right side of Fig. 29. Note the obvious similarity with the predicted motion for two particles in Fig. 23. These computer simulations predicted very well the PEP behaviour without feedback and less well the behaviour with feedback. Feedback in PEP allowed the threshold current to be increased by more than a factor of two.

Fig. 29: Computer simulation of TMCI
14. SYNCHROBETATRON RESONANCES

The particle oscillation energy in the longitudinal plane can be coupled into the transverse plane by so-called ‘synchrobetatron’ coupling. The purpose in this section is to describe the physical processes which cause this coupling and finish by presenting some recent experimental results.

There are many well-known mechanisms for coupling the longitudinal and transverse motion of the particles. Of these, usually the most severe is momentum dispersion at the location of the RF cavities [15]. Consequently, most colliders are designed with zero horizontal momentum dispersion in all the RF straight sections and, of course, zero vertical dispersion everywhere. However, measurements have shown that the residual dispersion, produced by machine imperfections, is significant and particularly dangerous in the vertical plane.

The performance of LEP is fundamentally limited by the transverse-mode-coupling instability, (described in previous sections) which limits the current per bunch. In order to raise the threshold of this instability it has been proposed to increase significantly the synchrotron tune \( Q_s \) at injection energy. Since these high values of \( Q_s \) cannot be maintained during the energy ramp, it is inevitable that synchrobetatron resonances must be crossed. This experimental study [16] was initiated in order to understand the resonance behaviour and to allow a proposal of a scheme for crossing of synchrobetatron resonances during the energy ramp.

The mechanism for synchro-betatron coupling generated by dispersion at the RF cavities is depicted graphically in Fig. 30. Assume that a particle initially arrives at an RF cavity with the same energy as the synchronous particle and zero amplitude in betatron phase space as shown in Fig. 30. During the traversal of the cavity the particle gains energy and a new energy orbit is established about which the particle must oscillate. The particle which was initially at \( x = 0 \) in Fig. 30 has a new equilibrium orbit due to the energy increase. This is equivalent to the origin of the betatron axis being displaced by an amount equal to

\[
\Delta x = D_x \frac{\Delta E}{E}
\]

where \( D_x \) is the dispersion at the cavity and \( \Delta E \) is the energy gained (relative to the synchronous particle) by the particle traversing the cavity. It may be seen that if the frequency of the oscillations in the transverse plane is an integer multiple of those in the longitudinal plane, then the amplitude of the betatron oscillations build up with each traversal of the cavity. More generally the resonant condition is

\[
kQ_x + mQ_y + nQ_s = p
\]

where \( k, m, n, \) and \( p \) are integers.
14.1 Experimental observation of synchrobetatron resonances

The measurement technique used was rather simple and automated by use of a ‘tune scan’. The horizontal and vertical tunes are incremented in a prescribed way by variation of the main quadrupole chains. At each incremental step the tunes are measured along with the beam sizes (as measured from the synchrotron light monitor), the bunch currents, and the current lifetime.

Vertical tune scans were made at three intensity levels. The results are shown in Fig. 31 where the vertical beam size is plotted as a function of the measured vertical tune value. For ease of viewing a baseline offset was added to each scan, creating a ‘mountain range’ where the increase in the varied parameter is depicted by an increase in the vertical offset.

![Diagram](image)

**Fig. 30:** Mechanism for synchrobetatron coupling driven by momentum dispersion at the RF cavities

**Fig. 31:** Strength of SBRs as a function of tune and current
The vertical and horizontal tune shifts due to the LEP transverse impedance were subsequently measured as a function of intensity. These results were used to evaluate the incoherent or zero-current tune values ($Q_{\text{vinc}}$). In addition it is known that although the coherent (measured) synchrotron tune remains constant with intensity, the incoherent synchrotron tune increases with bunch current. If the hypothesis is made that, for example, the resonance $Q_y = 3Q_s$ is a single-particle resonance and therefore occurs at incoherent tune values, then by varying the value of $Q_{\text{sinc}}$, a perfect resonant condition can be found for various values of bunch current. This was done and the results are shown in Fig. 32.

**Fig. 32:** Vertical beam size plotted against incoherent vertical tune shift normalized to a set value of the incoherent synchrotron tune which reproduces the resonant condition

The residual vertical dispersion in the RF straight sections is the main mechanism for coupling longitudinal motion into the vertical phase plane. In normal operation of LEP this dispersion is minimized by careful correction of the closed orbit. The dispersion can be measured by subtracting closed orbits measured with energy deviations. For the results presented thus far the measured r.m.s. dispersion was 8 cm. It is also well known that the application of ‘asymmetric’ orbit bumps through the interaction region creates a dispersion bump all around LEP. For this reason fairly large ‘asymmetric’ bumps were applied in all even interaction regions to increase the r.m.s. dispersion. By this technique the measured residual vertical r.m.s. dispersion was increased from 8 cm to 44 cm. The tune scan with this increased dispersion caused loss of nearly all the beam current as the tune was swept across the ‘second sideband’ ($Q_y = 2Q_s$). The tune scan in going from high tune values downward towards and eventually across the second sideband is shown in Fig. 33 along with the tune scan at lower dispersion.
It has been known for some time [17] that orbit displacement in RF cavities is a driving mechanism for synchrobetatron resonances. In another experiment the closed orbit through the RF cavities around interaction points 2 and 6 was well corrected so as to produce the minimum displacement through the room-temperature cavities. A vertical closed-orbit ‘bump’ (of amplitude 10 mm peak) was then applied to these straight sections. The tune scans for these two different situations are shown in Fig. 34. Examination of these plots shows that the second sideband \( (Q_y = 2Q_s) \) is very slightly less excited, whereas the excitations of the third, fourth, and fifth sidebands are significantly enhanced.
In order to increase the threshold for the TMCI it is foreseen to operate LEP at very high values of $Q_s$ at injection energy. For this reason an experiment was performed in which the value of $Q_s$ was increased to nearly double its value used in the previous experiments. The results of these tests, shown in Fig. 35, indicate that the excitation of the synchrotron sidebands increases but only slightly as a function of the synchrotron tune.

**Fig. 35**: Variation of the synchrotron tunes

**ACKNOWLEDGEMENTS**

The author would like to thank D. Brandt, A. Hofmann, K. Hübner, and K.-H. Kissler for carefully reading the manuscript and pointing out some inconsistencies. The author has also benefited greatly from the references cited.

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