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**Hamiltonian diagonalization for the nonconformal scalar field
on an isotropic gravitational background**

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1 Introduction

The different aspects of quantum field theory in curved space-time and its applications in cosmology of the early Universe has been a subject of many studies in recent years (see, e.g., the monographs [1–3] and references therein). Specifically, for the cases of scalar, spinor and vector quantized fields on a gravitational background, the effects of particle creation from vacuum, vacuum polarization (including Casimir effect) and spontaneous symmetry breaking were investigated.

The majority of papers, in which spin zero case was considered, investigated the quantized minimally or conformally coupled scalar fields (in the last case field equation is conformally invariant in the limit of zero mass). The investigations concerning the scalar field on a gravitational background with arbitrary coupling has received only occasional attention in the literature. By way of example the paper [4] has to be mentioned, where the anomalous trace of the vacuum stress-energy tensor (SET) of a nonconformal field was found, and the paper [5] in which the total renormalized SET of such field was calculated in de Sitter space-time. Also, in recent paper [6] we have found the total vacuum SET of arbitrary coupled scalar field in a radiation dominated Friedmann model. But the general case of the homogeneous isotropic space remained untouched.

It is common knowledge now that on gravitational background the concept of particle loses the uniqueness, so that the different particle concepts may be introduced [1]. One of such concepts, suggested firstly in [7], was based on the diagonalization procedure of the metrical Hamiltonian of conformal scalar field in the isotropic gravitational background. In that paper the quasiparticles were defined whose time-dependent creation-annihilation operators diagonalize the instantaneous Hamiltonian

of quantized field at every moment (see also [8]). At a later time a number of papers appeared (see, e.g., [2, 9–12]) devoted to the calculation of a particle creation rate not only for scalar, but also for spinor and vector particles using the Hamiltonian diagonalization procedure. The other, so-called “adiabatic” particles were introduced in [13] using the WKB-asymptotics of solutions to classical field equations. This particle concept is based on the principle of minimization of the number of created pairs and its time derivatives.

As was shown by the further investigations, the adequate physical quantity which describes the vacuum quantum effects on gravitational background is not the particle number but the total renormalized expectation value of the SET operator [2]. This quantity provides the combined description of vacuum quantum effects in external gravitational field without separation into contributions of particle creation and vacuum polarization. By this means the concept of particles becomes of no physical importance. Nevertheless from the mathematical point of view the possibility of exact diagonalization of the quantized field Hamiltonian still remains interesting as it reduces the problem with some interaction to that of free noninteracting quasiparticles.

In this paper we perform the diagonalization procedure for the Hamiltonian of nonconformal scalar field with arbitrary coupling in homogeneous isotropic gravitational background. It is shown that in nonstationary case such a diagonalization is possible only for the values of coupling coefficient ξ in the interval between the minimal and conformal ones: $0 \leq \xi \leq 1/6$. The quasienergy of nonconformal field is found and is shown to differ from the effective oscillator frequency of the wave equation (note that these quantities coincide in the conformal case).

The paper is organized as follows. In Sec. 2 the quantization procedure of nonconformal field in isotropic background is briefly outlined. In Sec. 3 the diagonalization procedure for the Hamiltonian is performed. In Sec. 4 we discuss the different definitions of the vacuum state for nonconformal scalar field.

Throughout the paper units are used in which $\hbar = c = 1$.

2 Quantization of nonconformal scalar field in isotropic background

We will consider the complex scalar field $\varphi(x)$ of mass m obeying the equation

$$(\square + m^2 + \xi R)\varphi(x) = 0, \tag{1}$$

where R is a scalar curvature of space-time. For $\xi = 0$ Eq. (1) is the equation with minimal coupling. For $\xi = 1/6$ (in 4-dimensional space-time) Eq. (1) describes conformally coupled scalar field and is conformally invariant in the zero-mass limit.

For simplicity we will restrict ourselves by the quasi-Euclidean space-time with the metric

$$ds^2 = g_{ik} dx^i dx^k = a^2(\eta) \left[d\eta^2 - (dx^1)^2 - (dx^2)^2 - (dx^3)^2 \right], \quad (2)$$

where $a(t)$ is the scale factor, and conformal time η is connected with the proper synchronous time t by the relation $dt = a(\eta)d\eta$. The cases of space sections with positive or negative curvatures may be considered in a similar manner. Note that the dimension of a in (2) is length and $x^0 = \eta$, x^α are dimensionless.

The metrical SET of the field φ is obtained by the variation of Lagrangian, corresponding to Eq. (1), with respect to the metrical tensor g^{ik} and takes the form [1–3]

$$\begin{aligned} T_{ik} = & (1 - 2\xi) (\partial_i \varphi^* \partial_k \varphi + \partial_k \varphi^* \partial_i \varphi) \\ & + (4\xi - 1) g_{ik} \partial^l \varphi^* \partial_l \varphi - 2\xi [\varphi^* \nabla_i \nabla_k \varphi + (\nabla_i \nabla_k \varphi^*) \varphi] \\ & + \left[(1 - 4\xi) m^2 g_{ik} - 2\xi G_{ik} - 4\xi^2 R g_{ik} \right] \varphi^* \varphi, \end{aligned} \quad (3)$$

where G_{ik} is the Einstein tensor.

The metrical Hamiltonian of the field φ is expressed through (3) by the relation

$$\begin{aligned} H(\eta) = & \int_{\Sigma} \zeta^i T_{ik}(x) d\sigma^k = \int_{\eta=\text{const}} \zeta^0 T_{00}(x) g^{00} \sqrt{-g} d^3x \\ = & a^2(\eta) \int_{\eta=\text{const}} d^3x T_{00}(x), \end{aligned} \quad (4)$$

where Σ is a set of space-like hypersurfaces $\eta = \text{const}$ and the time-like conformal Killing vector field $\zeta^i = (1, 0, 0, 0)$ is orthogonal to them carrying out translations along the time coordinate η determined by it.

In the space-time with metric (2) it is possible to separate variables in Eq. (1) representing its solutions in the form

$$\varphi(x) = \varphi_J(x) = \frac{1}{\sqrt{2}} \frac{g_J(\eta)}{a(\eta)} \exp(i\lambda_\alpha x^\alpha), \quad (5)$$

where λ^α are the components of dimensionless momentum, $J \equiv (\lambda_1, \lambda_2, \lambda_3)$, and the function g_J satisfies the equation

$$g_J''(\eta) + \Omega_J^2(\eta) g_J(\eta) = 0, \quad (6)$$

$$\Omega_J^2(\eta) = \omega_J^2(\eta) - 6 \left(\frac{1}{6} - \xi \right) \frac{a''(\eta)}{a(\eta)}, \quad \omega_J^2(\eta) = m^2 a^2(\eta) + \lambda^2.$$

Here prime denotes the derivative with respect to η and $\lambda = (\lambda_1^2 + \lambda_2^2 + \lambda_3^2)^{1/2}$.

The Eq. (6) has the first integral of the form

$$g_J' g_J^* - g_J g_J^{*'} = 2i, \quad (7)$$

whose value normalizes the function g_J in the same way as in the absense of external field.

Initial conditions for g_J at some moment η_0 , which are in agreement with (7), are

$$g_J(\eta_0) = A_J^{-\frac{1}{2}}(\eta_0), \quad g_J'(\eta_0) = i A_J^{\frac{1}{2}}(\eta_0), \quad (8)$$

where the real function A_J may coincide with ω_J , Ω_J or some other effective frequency (see below). In consequence of space isotropy all these frequencies and therefore g_J depend only on λ .

The solutions of (6) satisfying initial conditions (8) have the sense of positive-frequency solutions at the moment η_0 . The complex-conjugated to them will be negative-frequency ones. In such a manner the complete orthonormal set of solutions to Eq. (1) is:

$$\varphi_J^{(+)}(x) = \frac{1}{\sqrt{2}} \frac{g_\lambda}{a} e^{i\lambda_\alpha x^\alpha}, \quad \varphi_J^{(-)}(x) = \frac{1}{\sqrt{2}} \frac{g_\lambda^*}{a} e^{-i\lambda_\alpha x^\alpha}. \quad (9)$$

Now it is possible to present the field operator in Schrödinger representation as the decomposition with respect to the functions (9):

$$\varphi(x) = \frac{1}{(2\pi)^{3/2}} \int d^3\lambda \left[\varphi_J^{(+)}(x) a_J^{(+)} + \varphi_J^{(-)}(x) a_J^{(-)} \right], \quad (10)$$

where creation-annihilation operators obey the usual commutation relations

$$[a_J^{(-)}, a_{J'}^{(+)}] = [a_J^{(-)}, a_{J'}^{(+)}] = \delta_{JJ'}. \quad (11)$$

Here $a_J^{(-)}$, $a_J^{(+)}$ are annihilation operators for particle and antiparticle respectively; $a_J^{(-)}$, $a_J^{(+)}$ are the creation operators for antiparticle and particle, and the sign $\delta_{JJ'}$ means three-dimensional δ -function depending on the difference of momentum vectors.

The Schrödinger vacuum state may be defined as

$$a_J^{(-)}|0\rangle = a_J^{(+)}|0\rangle = 0. \quad (12)$$

In what follows we will see, however, that for the nonconformal field this state is not a vacuum of the Hamiltonian even at the initial moment $\eta = \eta_0$.

3 Hamiltonian diagonalization procedure

Let us now substitute Eqs. (9),(10) into (3) with $i = k = 0$ and represent the Hamiltonian (4) as a quadratic form in terms of creation-annihilation operators

$$H(\eta) = \int d^3\lambda \left\{ C_\lambda(\eta) \left[\tilde{a}_J^{(+)} a_J^{(-)} + \tilde{a}_{-J}^{(-)} a_{-J}^{(+)} \right] + D_\lambda(\eta) \tilde{a}_J^{(+)} a_{-J}^{(+)} + D_\lambda^*(\eta) \tilde{a}_{-J}^{(-)} a_J^{(-)} \right\}, \quad (13)$$

where the coefficients are

$$C_\lambda(\eta) = \frac{1}{2} \left(|g'_\lambda|^2 + \omega_\lambda^2 |g_\lambda|^2 \right) - 3 \left(\frac{1}{6} - \xi \right) \frac{a'}{a} \left[\frac{d}{d\eta} |g_\lambda|^2 - \frac{a'}{a} |g_\lambda|^2 \right],$$

$$D_\lambda(\eta) = \frac{1}{2} \left(g_\lambda'^2 + \omega_\lambda^2 g_\lambda^2 \right) - 3 \left(\frac{1}{6} - \xi \right) \frac{a'}{a} \left[2g_\lambda g'_\lambda - \frac{a'}{a} g_\lambda^2 \right]. \quad (14)$$

With the use of (6), (7) it is easily seen that

$$C_\lambda^2(\eta) - |D_\lambda(\eta)|^2 = \kappa_\lambda^2(\eta), \quad (15)$$

where

$$\kappa_\lambda^2(\eta) \equiv \lambda^2 + m^2 a^2 + 36\xi \left(\frac{1}{6} - \xi \right) \frac{a'^2}{a^2}. \quad (16)$$

We will diagonalize the nondiagonal Hamiltonian (13) by the canonical Bogoliubov transformations of creation-annihilation operators. To do this we introduce the new time-dependent creation-annihilation operators $b_J^{(\pm)}(t)$, $\tilde{b}_J^{(\pm)}(t)$ connected with the Schrödinger ones by the equations

$$a_J^{(-)} = \alpha_\lambda^*(\eta) b_J^{(-)}(\eta) - \beta_\lambda(\eta) \tilde{b}_{-J}^{(+)}(\eta),$$

$$\tilde{a}_J^{(-)} = \alpha_\lambda^*(\eta) \tilde{b}_J^{(-)}(\eta) - \beta_\lambda(\eta) \tilde{b}_{-J}^{(+)}(\eta) \quad (17)$$

(and by Hermitian conjugated ones). Here the coefficients $\alpha_\lambda, \beta_\lambda$ satisfy the identity

$$|\alpha_\lambda|^2 - |\beta_\lambda|^2 = 1, \quad (18)$$

which provides the canonicity of the transformation (17), i.e. the fulfilment of the same commutators for b -operators as were presented in (11) for a -ones.

After substitution of (17) into (13), the Hamiltonian takes the form:

$$H(\eta) = \int d^3\lambda \left\{ \left[C_\lambda \left(|\alpha_\lambda|^2 + |\beta_\lambda|^2 \right) - 2\text{Re} \left(D_\lambda \alpha_\lambda \beta_\lambda^* \right) \right] \left(\tilde{b}_J^{(+)} b_J^{(-)} + \tilde{b}_{-J}^{(-)} b_{-J}^{(+)} \right) + \left(D_\lambda^* \beta_\lambda^2 - 2C_\lambda \alpha_\lambda \beta_\lambda + D_\lambda \alpha_\lambda^2 \right) \tilde{b}_J^{(+)} b_{-J}^{(+)} + \left(D_\lambda \beta_\lambda^{*2} - 2C_\lambda \alpha_\lambda^* \beta_\lambda^* + D_\lambda^* \alpha_\lambda^{*2} \right) \tilde{b}_{-J}^{(-)} b_J^{(-)} \right\}. \quad (19)$$

To obtain the diagonality of (19) it is necessary to put the coefficients near nondiagonal terms $\tilde{b}_J^{(+)}\tilde{b}_{-J}^{(+)}$ and $\tilde{b}_{-J}^{(-)}\tilde{b}_J^{(-)}$ equal to zero:

$$D_\lambda^*\beta_\lambda^2 - 2C_\lambda\alpha_\lambda\beta_\lambda + D_\lambda\alpha_\lambda^2 = 0 \quad (20)$$

or, in equivalent form

$$D_\lambda^*\left(\frac{\beta_\lambda}{\alpha_\lambda}\right)^2 - 2C_\lambda\frac{\beta_\lambda}{\alpha_\lambda} + D_\lambda = 0. \quad (21)$$

Solving the quadratic equation (21) we get

$$\left(\frac{\beta_\lambda}{\alpha_\lambda}\right)_{1,2} = \frac{C_\lambda \pm \sqrt{C_\lambda^2 - |D_\lambda|^2}}{D_\lambda^*}. \quad (22)$$

In conformal case $\xi = 1/6$ and the coefficient C_λ is positive due to (14). Then the lower sign in (22) should be chosen to satisfy the initial condition $\beta_\lambda(\eta_0) = 0$. In nonconformal case C_λ may be also negative and then the upper sign in (22) should be used.

If we suggest that the quantity (15) is positive (which is the case in conformal situation but, in general, is not valid for nonconformal one) then, using (18), (22) one obtains

$$\begin{aligned} |\beta_\lambda|^2 &= \mp \frac{C_\lambda}{2\sqrt{C_\lambda^2 - |D_\lambda|^2}} - \frac{1}{2}, \\ |\alpha_\lambda|^2 &= \mp \frac{C_\lambda}{2\sqrt{C_\lambda^2 - |D_\lambda|^2}} - \frac{1}{2}. \end{aligned} \quad (23)$$

With these values for the coefficients of the Bogoliubov transformations, the Hamiltonian coefficient near the diagonal terms in (19) takes the value

$$C_\lambda (|\alpha_\lambda|^2 + |\beta_\lambda|^2) - 2\text{Re}(D_\lambda\alpha_\lambda\beta_\lambda^*) = \mp\sqrt{C_\lambda^2 - |D_\lambda|^2} = \mp\kappa_\lambda. \quad (24)$$

In such a manner the Hamiltonian (19) will be put in a diagonal form

$$H(\eta) = \mp \int d^3\lambda \kappa_\lambda(\eta) \left[\tilde{b}_J^{(+)}(\eta)\tilde{b}_J^{(-)}(\eta) + \tilde{b}_{-J}^{(-)}(\eta)\tilde{b}_{-J}^{(+)}(\eta) \right] \quad (25)$$

with the quasienergies $-\kappa_\lambda(\eta)$ (when C_λ is negative) or $\kappa_\lambda(\eta)$ (when C_λ is positive) which is more usual.

Now let us suggest that the quantity (15) is negative. Then, according to (22)

$$\left(\frac{\beta_\lambda^*}{\alpha_\lambda^*}\right)_{1,2} = \frac{C_\lambda \mp i\sqrt{|D_\lambda|^2 - C_\lambda^2}}{D_\lambda}. \quad (26)$$

Multiplying (22) (for the negative value under the square root) by (26) we come to the result:

$$\left| \frac{\beta_\lambda}{\alpha_\lambda} \right|^2 = \frac{C_\lambda^2 + |D_\lambda|^2 - C_\lambda^2}{|D_\lambda|^2} = 1. \quad (27)$$

which is in contradiction with the canonicity identity (18).

Thus, according to Eq. (15), the positive-definiteness of the quantity $\kappa_\lambda^2(\eta)$, defined in (16), is a necessary and sufficient condition for the diagonalization of a nonconformal field Hamiltonian by the canonical Bogoliubov transformations. From (16) it is easily seen that as $\kappa_\lambda^2(\eta)$ should be nonnegative for all λ (including zero) and for all values of ma then this condition is equivalent to inequalities

$$0 \leq \xi \leq \frac{1}{6}. \quad (28)$$

This means that only for the coupling coefficient values between the minimal and conformal ones it is possible to diagonalize the quantized field Hamiltonian by the canonical transformations in isotropic gravitational background. For the other values of ξ , the nonconformal field contrary to conformal one, cannot be represented at every moment of time as some set of free quasiparticles which do not interact with external gravitational field. It is interesting also that when diagonalization is possible the effective oscillator frequency Ω_λ from the Eq. (6), as opposite to conformal case, does not coincide with the quasienergy κ_λ of the Hamiltonian (25) defined in (16).

4 Different definitions for the vacuum state

Let us consider now the field φ in a space-time with metric (2) and coupling coefficient ξ satisfying Eq. (28) so that the Hamiltonian diagonalization may be done. In addition to the Schrödinger vacuum state defined in (12) it is possible to introduce the instantaneous vacuum states of quasiparticles in accordance with

$$b_J^{(-)}(\eta)|0_\eta\rangle = \bar{b}_J^{*(-)}(\eta)|0_\eta\rangle = 0. \quad (29)$$

The vacuum state $|0\rangle$ contains pairs of quasiparticles with quantum numbers J and $-J$. To calculate their density we use the canonical transformation inverse to (17). The result is

$$\langle 0 | \bar{b}_J^{(+)}(\eta) b_J^{(-)}(\eta) | 0 \rangle = \langle 0 | b_{-J}^{(+)}(\eta) \bar{b}_{-J}^{(-)}(\eta) | 0 \rangle = |\beta_\lambda(\eta)|^2. \quad (30)$$

Due to the space isotropy, the spectrum of created pairs depends on λ only. Space homogeneity ensured that the spectral density of quasiparticles in a mode J is equal to the density of anti-quasiparticles in a mode $-J$.

The amount of quasiparticle pairs per unit space volume is

$$n(\eta) = \frac{1}{2\pi^2 a^3(\eta)} \int d\lambda \lambda^2 |\beta_\lambda(\eta)|^2. \quad (31)$$

Using Eqs. (14), (15) and (23) the spectral density of quasiparticles may be written in the form

$$|\beta_\lambda(\eta)|^2 = \mp \frac{1}{4\kappa_\lambda} \left\{ |g'_\lambda|^2 + \left[\lambda^2 + m^2 a^2 + 6 \left(\frac{1}{6} - \xi \right) \left(\frac{a'}{a} \right)^2 \right] |g_\lambda|^2 - 6 \left(\frac{1}{6} - \xi \right) \frac{a'}{a} \frac{d}{d\eta} |g_\lambda|^2 \pm 2\kappa_\lambda \right\}. \quad (32)$$

It would be reasonable to expect that for the proper choice of the effective frequency $A_\lambda(\eta_0)$ from (8), the quantity $\beta_\lambda(\eta_0)$ turns into zero, i.e. at initial moment the Hamiltonian becomes diagonal (as it takes place for the comformal case [2, 9–12]). Let us make an attempt to find the explicit form of the function $A_\lambda(\eta_0)$ from this requirement.

Putting $\eta = \eta_0$ in (32) and using condition $\beta_\lambda(\eta_0) = 0$ one comes to the quadratic equation for $A_\lambda(\eta_0)$:

$$A_\lambda^2(\eta_0) \pm 2\kappa_\lambda(\eta_0) A_\lambda(\eta_0) + \left[\lambda^2 + m^2 a^2(\eta_0) + 6 \left(\frac{1}{6} - \xi \right) \left(\frac{a'(\eta_0)}{a(\eta_0)} \right)^2 \right] = 0. \quad (33)$$

However, this equation in nonconformal case has only the complex solutions

$$[A_\lambda(\eta_0)]_{1,2} = \mp \kappa_\lambda(\eta_0) \pm 6i \left| \left(\frac{1}{6} - \xi \right) \frac{a'(\eta_0)}{a(\eta_0)} \right|. \quad (34)$$

This means that no a real effective frequency $A_\lambda(\eta_0)$ exists such that $\beta_\lambda(\eta_0) = 0$. As a consequence, a similar statement is true for the vacuum states of quasiparticles $|0_\eta\rangle$. In nonconformal case they differ from the Schrödinger vacuum $|0\rangle$ for all η including the initial moment η_0 . The single exception is the case when $a'(\eta_0)/a(\eta_0) = 0$ (e.g., for flat asymptotics). In this case

$$A_\lambda(\eta_0) = \mp \kappa_\lambda(\eta_0) = \mp \omega_\lambda(\eta_0), \quad \beta_\lambda(\eta_0) = 0, \quad |0_{\eta_0}\rangle = |0\rangle. \quad (35)$$

(Naturally, for the flat asymptotic the lower signs should be chosen here.)

In fact, the Schrödinger vacuum state $|0\rangle$ is defined by the Eqs. (8)–(12) in a non-unique way as it depends on the particular choice of $A_J(\eta_0)$. As it was mentioned in Sec. 2, both quantities ω_J and Ω_J from (6) may be used as A_J . If desired to have the vacuum $|0\rangle$ most similar to the instantaneous quasiparticle vacuum $|0_{\eta_0}\rangle$, it should be used $A_J(\eta_0) = \kappa_J(\eta_0)$. But at best the state $|0\rangle$ does not coincide exactly with any quasiparticle vacua. This raises the question what initial vacuum state, one of the possible Schrödinger states $|0\rangle$ or probably $|0_{\eta_0}\rangle$, is preferable from the physical point of view? To suggest some answer to this question let us calculate the spectral density of quasiparticles, defined at the moment η (and, therefore, diagonalizing the Hamiltonian at this moment) in initial quasiparticle vacuum state $|0_{\eta_0}\rangle$.

Using the Bogoliubov transformations (17), it is not difficult to derive the transformations which relate the quasiparticle operators, defined at a moment η , to such operators, defined at η_0 :

$$\begin{aligned} b_J^{(-)}(\eta) &= [\alpha_\lambda^*(\eta_0)\alpha_\lambda(\eta) - \beta_\lambda^*(\eta_0)\beta_\lambda(\eta)] b_J^{(-)}(\eta_0) \\ &\quad + [\alpha_\lambda(\eta_0)\beta_\lambda(\eta) - \beta_\lambda(\eta_0)\alpha_\lambda(\eta)] b_{-J}^{(+)}(\eta_0), \\ b_J^{*(-)}(\eta) &= [\alpha_\lambda^*(\eta_0)\alpha_\lambda(\eta) - \beta_\lambda^*(\eta_0)\beta_\lambda(\eta)] b_J^{*(-)}(\eta_0) \\ &\quad + [\alpha_\lambda(\eta_0)\beta_\lambda(\eta) - \beta_\lambda(\eta_0)\alpha_\lambda(\eta)] b_{-J}^{*(+)}(\eta_0). \end{aligned} \quad (36)$$

Calculating the quasiparticle pairs spectral density in initial vacuum state $|0_{\eta_0}\rangle$ with use of (36) we will get the result

$$\begin{aligned} \langle 0_{\eta_0} | b_J^{*(+)}(\eta) b_J^{(-)}(\eta) | 0_{\eta_0} \rangle &= \langle 0_{\eta_0} | b_{-J}^{(+)}(\eta) b_{-J}^{*(-)}(\eta) | 0_{\eta_0} \rangle \\ &= |\beta_\lambda(\eta)|^2 + |\beta_\lambda(\eta_0)|^2 + 2|\beta_\lambda(\eta_0)|^2 |\beta_\lambda(\eta)|^2 - 2\text{Re}[\alpha_\lambda(\eta_0)\beta_\lambda^*(\eta_0)\alpha_\lambda^*(\eta)\beta_\lambda(\eta)]. \end{aligned} \quad (37)$$

Note that the quantity $|\beta_\lambda(\eta)|^2$ is expressed through the solutions to the Eq. (6) by the Eqs. (15), (23). From (22) it follows also that

$$\alpha_\lambda^* \beta_\lambda = \mp \frac{D_\lambda}{2\sqrt{C_\lambda^2 - |D_\lambda|^2}} = \mp \frac{D_\lambda}{2\kappa_\lambda}. \quad (38)$$

Substituting all these to the right-hand side of (37) one will express the quasiparticle density in quasiparticle initial vacuum as

$$\langle 0_{\eta_0} | b_J^{*(+)}(\eta) b_J^{(-)}(\eta) | 0_{\eta_0} \rangle = \frac{1}{2\kappa_\lambda(\eta_0)\kappa_\lambda(\eta)} \{C_\lambda(\eta_0)C_\lambda(\eta) - \text{Re}[D_\lambda^*(\eta_0)D_\lambda(\eta)]\}. \quad (39)$$

This expression, contrary to (23), (32), has one and the same view for two different solutions (22) to Eq. (21).

Considering the limiting case $\eta \rightarrow \eta_0$ in (37), (39) we will get the zero result as one would expect in the situation when the initial vacuum is one of the quasiparticle vacuum states. On the other hand, the state $|0\rangle$ is not a vacuum of the quasiparticles even at the initial moment η_0 . This gives us possibility to emphasize that the instantaneous vacuum of quasiparticles defined at some moment η_0 is a better candidate for the role of initial vacuum state of nonconformal scalar field with $0 \leq \xi \leq 1/6$ on the isotropic gravitational background.

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