Linear Effects of Dispersion on the Beam-Beam Interaction

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Linear Effects of Dispersion on the Beam-Beam Interaction

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Abstract. The linear effects due to the presence of a dispersion at the interaction point in the beam-beam collision are studied. A linear analysis shows a synchro-betatron coupling, which leads a synchrotron tune shift and a synchro-betatron resonance in addition to the well-known betatron tune shift. The beam sizes and the emittances are computed. The effects might be serious for the collision with dispersion.

Introduction

The effects of dispersion at the interaction point (IP) has been studied for long time [1]. The focus, however, was only on the effects of the synchrotron oscillation on the betatron oscillation [2]. By symplecticity, however, the reciprocal effect should exist as well. In this paper, we study the linear effects paying equal attention to synchrotron and betatron oscillations. The main new result is the existence of a negative synchrotron tune shift which can produce a serious instability. The synchrotron tune shift is larger when the beam-beam parameter ξ, the dispersion η or the energy spread σ_η are large and when the betatron function at the collision point β or the bunch length σ_z are small.

This effect might not be very important for conventional machines, in which the dispersion at the IP comes only from the machine errors. For future high performance colliders, however, it is widely believed that ν_z should be small to reserve large space in the tune space avoiding the synchro-betatron sidebands, σ_z should be small, the damping time should also be small which makes σ_η larger. All these choices make the effect more serious. Recently, in addition, the monochromatic collision [3] became an important issue for tau-charm factories [4,5]. In this scheme, at the IP, the beams have the same and rather large

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dispersion (but with opposite sign). We thus should consider the effect more seriously. This provides the motivation of this paper.

This paper is organized as follows. In Sect. 2 we introduce the dispersion in the presence of the synchrotron oscillation. Then in Sect. 3, the one-turn map is parametrized. The beam-beam interaction is introduced in Sect. 4. In Sect. 5, we report the stability analysis, showing how $\nu_s$ is shifted by the synchro-betatron coupling. We also find some additional effects on the bunch length and the energy spread in Sect. 6. Conclusions follow under Sect. 7.

**Dispersion with Synchrotron Oscillations**

Usually, the dispersion is defined for a particle with fixed energy (coasting beams) [6]. We extend this definition to the dynamics including synchrotron oscillations [7].

If the synchronous particle (center of the weak beam) travels in a ring with energy $E_0$ on the nominal closed orbit, then any other particle is described by the physical variables $(x, p_x, y, p_y, z, \delta)$, $x, y$ being the transverse deviations from the nominal closed orbit, $p_x, p_y$ their momenta, $z = s - ct$ the longitudinal displacement and $\delta = (E - E_0)/E_0$ the energy deviation.

In a ring without cavities, the transfer matrix $M_{cb}(s)$ (here the suffix $cb$ stands for "coasting beam") for the physical variables is the symplectic matrix

$$M_{cb}(s) = \begin{pmatrix} M_4 & 0 & a \\ b^t & 1 & -\ell \\ 0^t & 0 & 1 \end{pmatrix},$$

where $M_4$ is a $4 \times 4$ (symplectic, non diagonal) matrix, $a, b$ are 4-vectors and $\ell$ is a constant. $M_4$ and $a, b$ are determined by the lattice and related to each other as $a = M_4 b$ by the symplecticity of $M_{cb}$, where $J$ is the 4-dimensional symplectic metric.

$M_{cb}(s)$ can be block-diagonalized by introducing the dispersion matrix

$$G(s) = \begin{pmatrix} I & 0 & -\eta \\ (J\eta)^t & 1 & 0 \\ 0^t & 0 & 1 \end{pmatrix},$$

as

$$\hat{M}_{cb}(s) = G^{-1}(s) M_{cb}(s) G(s) = \begin{pmatrix} M_4 & 0 & 0 \\ 0^t & 1 & -\alpha \\ 0^t & 0 & 1 \end{pmatrix},$$

where $\eta = (\eta_x, \eta_\perp, \eta_y, \eta_\perp') = (I - M_4)^{-1} a$ is the dispersion vector and $\alpha = \ell - a^t J (I - M_4)^{-1} a$ is the momentum compaction factor. Note that the
above definition of $\eta$ is identical with the usual definition of dispersion for the coasting beam in Ref.[6]. As easily seen, the transfer matrix $M_{cb}(s_1, s_2)$ can be block-diagonalized as

$$M_{cb}(s_1, s_2) = G^{-1}(s_1)M_{cb}(s_1, s_2)G(s_2)$$

(4)

Let us extend the dispersion to the case with the synchrotron oscillation. We insert an RF cavity which is located at a dispersion-free point, $s_0$. Then the one turn matrix is

$$M_{arc}(s) = M_{cb}(s + C, s_0)KM_{cb}(s_0, s),$$

(5)

$$K = \begin{pmatrix}
I & 0 & 0 \\
0^t & 1 & 0 \\
0^t & k & 1
\end{pmatrix}$$

(6)

where $C$ is the machine circumference and $k$ a positive number. From eq.(5), we have

$$G^{-1}(s)M_{arc}(s)G(s) =$$

$$G^{-1}(s)M_{cb}(s + C, s_0)G(s_0)G^{-1}(s_0)K G(s_0) G^{-1}(s_0) M_{cb}(s_0, s) G(s).$$

(7)

Note that $G(s_0) = I$, because $\eta = 0$ there by assumption, so that $G^{-1}(s_0) K G(s_0) = K$ and it is block-diagonal. As stated above, in addition, $G^{-1}(s)M_{cb}(s + C, s_0)G(s_0)$ and $G(s_0)^{-1}M_{cb}(s_0, s)G(s)$ are block-diagonal. We thus conclude that $G^{-1}(s)M_{arc}(s)G(s)$ in Eq.(7) is block-diagonal.

It was shown that block-diagonalization of $M_{arc}(s)$ can be obtained as (3) using the same $G(s)$ given by (2) having the same $\eta$. This is so even after inserting further cavities, as long as $\eta = 0$ at the insertion points.

**Parametrization of One - Turn Map**

In the following we restrict ourselves to a two dimensional problem $(y, p_y, z, \delta)$ with vertical dispersion at the IP. We will express the one-turn matrix $M_{arc}$ (from the IP to IP) in terms of Twiss parameters, tunes, and dispersions.

The block-diagonal matrix at IP is:

$$\tilde{M}_{arc} = G_0^{-1}M_{arc}G_0 = \begin{pmatrix}
\cos \mu_y & \beta_y \sin \mu_y & 0 & 0 \\
-\beta_y^{-1} \sin \mu_y & \cos \mu_y & 0 & 0 \\
0 & 0 & \cos \mu_s & -\beta_s \sin \mu_s \\
0 & 0 & \beta_s^{-1} \sin \mu_s & \cos \mu_s
\end{pmatrix},$$

(8)
where $\mu_{y,z} = 2\pi \nu_{y,z}$ (note that $\nu_s$ is positive for conventional electron machines, having positive momentum compaction factor $\alpha$),

$$
\beta_s = \frac{\sigma_s}{\sigma_\delta}, \quad \epsilon_s = \sigma_s \sigma_\delta,
$$

are the synchrotron beta-function and the longitudinal emittance respectively, $\sigma_s$ the bunch length and $\sigma_\delta$ the energy spread.

The general dispersion matrix (2) at IP gives

$$
G_0 = \begin{pmatrix}
1 & 0 & 0 & -\eta_y \\
0 & 1 & 0 & -\eta_y' \\
\eta_y' & -\eta_y & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix},
$$

(10)

where $\eta_y'$ is the vertical dispersion slope (we shall also assume here $\eta_y' = 0$ at IP).

In short, the revolution matrix in the arc (i.e. without beam-beam collision) for the physical variables can be expressed as

$$
M_{arc} = G_0 \hat{M}_{arc} G_0^{-1}.
$$

(11)

**Beam-Beam Collision**

In the weak-strong picture the dynamics of the single (test) particle in the weak beam is influenced by the strong beam, which is not affected at all. In the linear approximation the particle receives a kick at IP from the strong beam. This interaction is described by the matrix

$$
M_{bb} = \begin{pmatrix}
1 & 0 & 0 & 0 \\
-4\pi \xi_y / \beta_y & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix},
$$

(12)

which contains the vertical beam-beam parameter $\xi_y$, viz., for Gaussian bunches:

$$
\xi_y = \frac{r_e}{2\pi \gamma \sigma_y (\sigma_x + \sigma_y)} N \beta_y,
$$

(13)

$N$ being the number of particles in the strong beam, $r_e$ the classical electron radius, $\gamma$ the relativistic factor, $\sigma_y$ the vertical beam size.

$$
\sigma_y = \left[ \beta_y \epsilon_y + \eta_y^2 \sigma_\delta^2 \right]^{1/2},
$$

(14)

$\epsilon_y$ the vertical emittance, and all quantities evaluated at the IP.

Therefore the complete one-turn matrix (including the beam-beam collision) for the physical variables is:

$$
M = M_{arc} M_{bb} = G_0 \hat{M}_{arc} G_0^{-1} M_{bb}.
$$

(15)
Stability Analysis

Let us denote the eigenvalues of $M$ as $\lambda_\pm = \exp \pm i \tilde{\mu}_\pm$. After some algebra we get

$$2 \cos \tilde{\mu}_\pm = \cos \mu_y + \cos \mu_z - 2 \pi \xi_y \sin \mu_y + 2 \pi \xi_y \chi \sin \mu_z \pm D^{1/2},$$  \hspace{1cm} (16)

$$D = (\cos \mu_y - \cos \mu_z - 2 \pi \xi_y \sin \mu_y - 2 \pi \xi_y \chi \sin \mu_z)^2 - 16 \pi^2 \xi_y^2 \chi \sin \mu_y \sin \mu_z,$$  \hspace{1cm} (17)

$$\chi = \frac{\eta_y^2}{\beta_y \beta_y} = \frac{\eta_y^2 \sigma_y}{\sigma_y \beta_y}$$  \hspace{1cm} (18)

being the synchrotron tune shift factor. The motion is stable if and only if $|\cos \tilde{\mu}_\pm| \leq 1$ and $D \geq 0$.

| $\eta_y$ | 1m |
| $\beta_y$ | .1m |
| $\xi_y$ | $1 \times 10^{-5}$m |
| $\sigma_y$ | .001 |
| $\nu_y$ | .03 |
| $\nu_s$ | .03 |

Table 1: Standard parameters used as the example. This gives $\chi = 1$.

To lowest order in $\xi_y$, we get

$$\nu_y \rightarrow \nu_y + \xi_y, \quad \nu_s \rightarrow \nu_s - \xi_y \chi.$$  \hspace{1cm} (19)

A negative synchrotron tune shift is thus predicted due to the combined effect of dispersion ($\chi$) and linearized beam-beam interaction ($\xi_y$).

The perturbative equation (19), implies that the linear instability occurs for

- $\nu_y \lesssim$ half integers (betatron instability)
- $\nu_s \gtrsim$ half integers (synchrotron instability).
- $\nu_s \gtrsim \nu_y +$ integers (synchro-betatron instability).

The instability regions in the ($\nu_y$, $\nu_s$) and ($\xi_y$, $\nu_s$) planes are shown in Figs. 1 and 2 in terms of the growthrate $\Gamma$ for a typical set of parameters reported in Table 1. The three unstable regions stated above are clearly seen. The same set of parameters will be used throughout the paper unless otherwise stated.

We use a rather big value for $\eta$ to emphasize the unstable regions. Some typical
values of $\eta$ are shown in Table 2 [8], where we wrote also the corresponding $\chi$ values.

FIGURE 1 (Left) The (growth rate $-1$) at $\nu_y = 0.1$ versus $\xi_y$ and $\nu_s$ and (right) its contour plot.

FIGURE 2 The (growth rate $-1$) at $\xi_y = 0.023$ versus $\nu_y$ and $\nu_s$ and
(right) its contour plot.

For more typical \( \eta \) value (see Table 2), the instability regions are less wide than for \( \eta = 1 \), but the resonances are still present.

The above result naturally suggests that a machines might be intrinsically more stable when \( \nu_s < 0 \).

<table>
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<tr>
<th></th>
<th>( B T C F )</th>
<th>( A l e x a n i n )</th>
<th>( Z h o l e n t s^{**} )</th>
<th>( T e n g )</th>
<th>( L e D u f f )</th>
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<td>.15m</td>
<td>.6m</td>
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<tr>
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<tr>
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<td>.0075m</td>
<td>.0075m</td>
<td>.01*m</td>
<td>.0058m</td>
<td>.01*m</td>
</tr>
<tr>
<td>( \xi )</td>
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<td>.01</td>
<td>.05</td>
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<td>.023</td>
</tr>
<tr>
<td>( \chi )</td>
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<td>.14</td>
<td>.03</td>
<td>.2</td>
<td>.07</td>
<td>.07</td>
</tr>
</tbody>
</table>

Table 2: Typical parameters for planned machines. (***) In this paper 1D—horizontal betatron motion is considered. (*) These values are reasonable guesses, these parameters being left unspecified in the original reference.

Beam Sizes and Emittances

In order to write the envelope equation for the physical variables in the presence of dispersion at the beam-beam collision point, it is convenient to introduce normalized variables first.

From the physical variables \( \mathbf{x} = (y, p_y, z, \delta) \) we introduce a set of normalized variables \( \mathbf{X} = (Y, P_y, Z, \Delta) \) by

\[
\mathbf{x} = G \mathbf{X},
\]

where \( B = \text{diag}(B_y, B_s) \)

\[
B_y = \begin{pmatrix}
\beta_y^{1/2} & 0 \\
0 & \beta_y^{-1/2}
\end{pmatrix}, \quad B_s = \begin{pmatrix}
\beta_s^{1/2} & 0 \\
0 & \beta_s^{-1/2}
\end{pmatrix},
\]

is the normalization matrix and \( G(s) \) the dispersion matrix (2). For these variables the mapping equations in the arc are

\[
\begin{pmatrix} Y \\ P_y \end{pmatrix}' = \lambda_y R_y \begin{pmatrix} Y \\ P_y \end{pmatrix} + \sqrt{\epsilon_y (1 - \lambda_y^2)} \begin{pmatrix} \hat{r}_1 \\ \hat{r}_2 \end{pmatrix},
\]

---

7
\[
\begin{pmatrix}
Z' \\
\Delta
\end{pmatrix} = \begin{pmatrix}
1 & 0 \\
0 & \lambda_s
\end{pmatrix} R_s \begin{pmatrix}
Z \\
\Delta
\end{pmatrix} + \begin{pmatrix}
0 \\
\sqrt{\epsilon_s(1 - \lambda_s^4)} \hat{r}_3
\end{pmatrix},
\]
(23)

\[
R_y = \begin{pmatrix}
\cos \mu_y & \sin \mu_y \\
-\sin \mu_y & \cos \mu_y
\end{pmatrix}, \quad R_s = \begin{pmatrix}
\cos \mu_s & \sin \mu_s \\
-\sin \mu_s & \cos \mu_s
\end{pmatrix}
\]
(24)

where \( < \hat{r}_i > = 0 \), \( < \hat{r}_i^2 > = 1 \) and \( \lambda_{y,s} = exp(-1/T_{y,s}) \) with \( T \) being the damping time measured by the revolution time. Note that the second mapping (23) is not symmetric in the synchrotron variables [9].

We define the envelope for \( \mathbf{X} \) as

\[
\Sigma_{ij} = < X_i X_j > .
\]
(25)

Without beam-beam force, the nominal equilibrium distribution in \( \mathbf{X} \) is the solution of the following equation

\[
\Sigma = (\Lambda R) \Sigma (\Lambda R)^t + (I - \Lambda^2) E,
\]
(26)

with

\[
R = diag(R_y, R_s), \quad E = diag(\epsilon_y, \epsilon_y, \epsilon_s, \epsilon_s), \quad \Lambda = diag(\lambda_y, \lambda_y, 1, \lambda_s^2).
\]
(27)

Under the ergodic assumption (time average = ensemble average) the solution of equation (26) is \( \Sigma = E \).

**FIGURE 3** Top: the beam sizes as functions of \( \xi_y \). Left: \( < y^2 > \) (solid line) and \( < z^2 > \) (dashed line). Right: \( < p_y^2 > \) (solid line) and \( < \delta^2 > \) (dashed line).
line). Bottom: the emittances as functions of $\xi_y$. Left: $\epsilon_3$ (solid line) and $\epsilon_8$ (dashed line). Right: increase of vertical emittance $\epsilon_y$ for small $\xi_y$ values. $\nu_3 = 0.03$ throughout. All quantities are normalized to their nominal ($\xi_y = 0$) values.

The envelope matrix $\sigma$ associated with the physical variables $\mathbf{x}$ is defined as

$$\sigma_{ij} = \langle x_i x_j \rangle,$$  \hspace{1cm} (28)

and from (20),

$$\sigma = GB\Sigma(GB)^t.$$  \hspace{1cm} (29)

In terms of $\mathbf{x}$ eq.(26) reads

$$\sigma = \tilde{\Lambda}M_{arc}\sigma(\tilde{\Lambda}M_{arc})^t + (GB)(I - \Lambda^2)E(GB)^t$$  \hspace{1cm} (30)

where

$$M_{arc} = GBR(GB)^{-1} = G\tilde{M}_{arc}G^{-1}, \hspace{0.5cm} \tilde{\Lambda} = GBA(GB)^{-1}.$$  \hspace{1cm} (31)

\begin{figure}[h]
\centering
\includegraphics[width=0.4\textwidth]{figure1.png}
\includegraphics[width=0.4\textwidth]{figure2.png}
\caption{$\nu_3 = -0.03$. The same functions of Fig. 3 are plotted.}
\end{figure}
In the presence of the linear beam-beam interaction the one turn matrix for \( x \) is \( M = M_{\text{arc}} M_{\text{bb}} \), Eq. (15). The equilibrium distribution for the envelope in \( x \) in the presence of the beam-beam interaction is then the solution of the equation

\[
\sigma = \tilde{A}M\sigma(\tilde{A}M)^t + (GB)(I - \Lambda^2)E(GB)^t,
\]

from which we get all relevant physical quantities. In particular, the emittances are

\[
\epsilon_{y,s} = \text{Abs}[[\text{Eigenvalues}[J\sigma]]].
\]

In Fig. 3 we plot \( \langle y^2 \rangle = \sigma_{11}, \langle z^2 \rangle = \sigma_{33}, \langle p_y^2 \rangle = \sigma_{22}, \langle \delta^2 \rangle = \sigma_{44} \) and the emittances \( \epsilon_{y,s} \) as functions of \( \xi_y \). Note the different behaviour of \( \sigma_{11} \) and \( \sigma_{33} \): the first one increases with \( \xi_y \), while the second one decreases.

The rapid increase of \( \langle p_y^2 \rangle \) is also shown, as well as that of the vertical emittance. Note that \( \sigma_y \) is dominated by \( \sigma_\delta \). With the parameters listed in Table 1 the instability threshold is at \( \xi_y = 0.021 \).

Since for \( \nu_s < 0 \) the threshold of the instability is shifted far away, it seems interesting to consider this case (see Fig. 4). The main difference with respect to a positive \( \nu_s \) is the reversed behavior of \( \sigma_{33} \) and \( \epsilon_s \): they increase with \( \xi_y \).

This shows that, as a relevant result of the linear analysis, the longitudinal dynamics is influenced by the transverse one. This effect has been usually overlooked in the literature.

**Discussion and Conclusion**

In high luminosity monochromatic colliders, common sense leads to require

i) small nonzero (usually positive) synchrotron tunes \( \nu_s \), to keep most of the parameter space clear from (nonlinear) synchrotron-betatron resonances; ii) bunch length smaller than the betatron function at the interaction point to avoid geometrical loss of luminosity. A suitable lower bound for the beam energy-spread is obviously implied. For efficient monochromatic collision we should require

\[
\beta_y \epsilon_y \ll \eta_y^2 \sigma_\delta.
\]

On the other hand, in view of eq. (19), for machines with positive nominal \( \nu_s \), the synchrotron tune shift factor \( \chi \) should be kept as small as possible, viz.

\[
\chi = \frac{\eta_y^2}{\beta_y \delta} = \frac{\eta_y^2 \sigma_\delta}{\sigma_\delta \beta_y} \ll 1
\]

which by eq. (9) entails

\[
\epsilon_y \ll \frac{\eta_y^2 \sigma_\delta}{\beta_y} \ll \epsilon_s.
\]
This might be a useful formula for the monochromatization scheme. (This is too simple for some cases. Actually, $\chi \xi > \nu_s$ is enough.)

With the negative nominal $\nu_s$, one can enjoy large safe area in the parameter space. This should be seriously considered as an alternative, obtained by adopting a negative momentum compaction factor. It might be worthwhile to note, in addition, that this choice would lead to less severe bunch lengthening [10].

A preliminary fully nonlinear simulation analysis, which is presently in progress, shows good agreement with the linear results described above.

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