A Scalar-Tensor Theory of Gravity Based on Non-Commutative Geometry

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Abstract

The unified description of gauge and Higgs fields based on non-commutative geometry by A. Connes has a similar structure to Kaluza-Klein theory, though it is a 4-dimensional theory. Its extension to one in curved spacetime is, then, expected to give a Brans-Dicke type gravity in which Higgs like scalar fields couples to gravitational field. In this paper, we study a B-D type gravity using the matrix coordinate method originated by R. Coquereaux. In particular, the possibility to introduce two kinds of scalar fields is discussed in details.

1. Introduction

Recently, many attempts have been made to extend the Connes unified description of gauge and Higgs fields based on non-commutative geometry\(^1\). In these works\(^2\) the spacetime is set as a product of the 4-dimensional spacetime by an extra discrete space. The scalar fields are, then, introduced as the connections associated with the gauge transformations defined in such an extended spacetime; the generalized curvature gives rise to the Lagrangian density for gauge fields including Higgs fields as their crew.

Contrary to expectations in earlier works, it have come to be known that the distinction between the standard gauge theory with Higgs fields and one based on non-commutative geometry is not large even for the number of free parameters\(^3\).
The approach from non-commutative geometry is, however, still interesting as a way to determine the form of Higgs potential theoretically.

Now, some of those unified description of gauge and Higgs fields have similar structure to the Kaluza-Klein theory\(^4\), although they are four-dimensional theories. This resemblance between NCG and KK theory allows us to extend such a theory to one in curved spacetime. Then, we may obtain Brans-Dicke type gravitational theories\(^5\), in which scalar fields couple to Einstein gravity. According to this idea, several attempt have been made\(^6\).

In the variation to formulate the gauge theory from non-commutative geometry, R. Coquereaux\(^7\) developed a useful approach to the Connes theory by regarding the spacetime as a product of the 4-dimensional spacetime by an extra space of finite matrices; then, the scalar fields are introduced as the connections associated with the gauge transformations defined in such an extended spacetime.

In a previous paper\(^8\), we have studied the introduction of such a matrix coordinate associated with a left-right \(Z_2\) symmetry of matter fields in the framework of quantum mechanic and applied the formulation to \(SU(2)_L \otimes SU(2)_R \otimes U(1)\) gauge theory. The purpose of this article is to formulate a gravitational theory defined on an extended manifold such that local coordinates have the matrix \(\eta\) as their fifth coordinate besides four spacetime coordinates \(\{x_\mu\}\).

In the next section, we shall give a short review of the introduction of matrix coordinate discussed in a previous paper and its embedding in curved four dimensional spacetime is discussed. Then, we shall discuss a Brans-Dicke type of gravitational theory, the scalar-tensor gravity, standing on such a matrix coordinate method. As an extension of the those formulation, a study also is made on the introduction of two kinds of scalar fields associated with the matrix coordinate; there, we shall study the structure of the effective potential of those fields, in the framework of classical theory, by taking inflationary models into account. Section 3 is devoted to summary and discussion.

2. Vielbein Formulation of Gravity with Scalar Fields in \((x^\mu, \eta)\) Space

As discussed in ref. 1, the \(2 \times 2\) matrix \(\hat{\eta} = M\eta = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\) can be introduced as a coordinate variable of matter field \(\Psi\), which cause the change between \(\psi_L \leftrightarrow \psi_R\). Here, \(M\) is a parameter with the dimension of mass. Indeed, in terms of the standard ket \(\rangle = |+\rangle + |-\rangle, (\eta|\pm\rangle = \pm M^{-1}|\pm\rangle\)), we get the expression
Figure 1: The effective potential $V(\rho, \sigma)$ with $\lambda = 0$.

Figure 2: $V(\rho, \sigma)$ for $\lambda > 0$.

Figure 3: The cross section of figures 1 and 2 by a line with $\rho = \text{const}$.

Figure 4: $V(\rho, \sigma)$ for $\lambda < 0$. 
Figure 5: The cross section of fig. 4 by a line with $\sigma = const. \neq 0$.

Figure 6: The effective potential $V(\rho, \sigma)$ with a finite temperature.
\[
\Psi = \frac{1}{\sqrt{2}} (\psi_L(x) + 2M\eta\psi_R(x)),
\]
which should be compared with the \( x \)-representation of state \( |\Psi\rangle = \Psi(\hat{x}), \langle \cdot | = \int dx\langle \cdot | x\rangle \).

On the analogy of \( \partial_x f(\hat{x}) = i[\hat{p}, f(\hat{x})] \) for ordinary variable, one can also define the derivative with respect to \( \eta \) by

\[
\partial_\eta A = i[\pi, A] \equiv i(\{\pi, A^*\} + \{\pi, A^o\}),
\]

where \( A^* \) and \( A^o \) are block-diagonal and off-block-diagonal parts of \( A \), respectively. The \( \pi \) is the momentum operator conjugate to \( \eta \) defined by \( \{\pi, \eta\} = -i1 \) in the sense of quantum mechanics; then, we get \( \partial_\eta \eta = 1 \). An explicit form of \( \pi \) is given by

\[
\pi = -\frac{M}{2}
\begin{pmatrix}
0 & 1 \\
1 & 0
\end{pmatrix}.
\]

From this definition for the 'derivative of matrix', one can verify easily the following:

\[
\partial_\eta^2 A = -[\pi^2, A]
\]

and

\[
\partial_\eta(AB) = (\partial_\eta A)B + A(\partial_\eta B), \quad (A^* = A^o)
\]

The first of the above equations allows us to read \( \partial_\eta^2 A = 0 \), since \( \pi^2 = -\frac{M^2}{4}1^o \).

For the later purpose, we also define the total derivative of a matrix \( A \) with respect to \( \eta \) by \( d_\eta A = \delta_\eta A = \delta_\eta \partial_\eta A \), where \( \delta_\eta \) is a quantity related to the total derivative of \( \eta \) by \( d_\eta \eta = \delta_\eta \partial_\eta \eta = \delta_\eta \), to which we require

\[
d_\eta(AB) = (d_\eta A)B + A(d_\eta B)
\]

Now, the LR symmetric gauge theories based on the above \( \eta \) calculus can be constructed by introducing covariant derivative operators \( D_A = \partial_A - igW_A \), \( (A = \mu, \eta) \), where \( W_\mu = W_\mu^a T_a^L + W_\mu^b T_b^R \) and \( W_\eta = -i(\phi^a Q_a + \phi^a Q_a^+) \). Here, \( \{T_a^L/R\} \) are generators of the gauge symmetry with block-diagonal matrices and \( Q_a, Q_a^+ \) are off-block-diagonal matrices that form a closed supersymmetry algebra with \( \{T_a^L/R\} \).

Then, \( \frac{1}{4} Tr F^{AB} F_{AB}^* \), \( (F_{AB} = \frac{1}{2}[D_A, D_B]) \) gives rise to the Lagrangian density for

\footnote{The most general form of the anti-hermitian matrix satisfying \( \{\pi, \eta\} = -i1 \) is
\[
\pi = iM \left(
\begin{pmatrix}
a & -\frac{1}{2} + ib \\
-\frac{1}{2} - ib & -a
\end{pmatrix}, \quad (a, b; \text{ real numbers}),
\right.
\]
to which \( \pi^2 = -M^2(1 + b^2 + a^2)1 \) is satisfied obviously.}
gauge fields $W_\mu$ and for scalar fields $\phi^a, \phi^{a*}$ in a form characterized by one coupling constant $g$.

In the case of gravitational theory, the gauge symmetry represented by block-diagonal matrices should be read as that of Lorentz transformation and so, we should define the covariant derivatives under local Lorentz transformations by

$$D_N \Psi = (\partial_N - \frac{i}{2} \omega_N^{(A)(B)} \Sigma_{AB}) \Psi,$$

(2.6)

where $N = (\mu, \eta)$ and

$$\Sigma_{\mu\nu} = \begin{bmatrix} \sigma_{\mu\nu} & 0 \\ 0 & \sigma_{\mu\nu} \end{bmatrix}, \quad \Sigma_{\mu\eta} = \begin{bmatrix} 0 & \gamma_{\mu} \\ \gamma_{\mu} & 0 \end{bmatrix}$$

(2.7)

This equation says that $\omega^{\mu\eta}$'s may be nothing but the counter part of Higgs fields in LR symmetric gauge theory, which cause the interchange between L and R components of matter fields.

The formalism that treats $(x^N) = (x^\mu, \eta)$ as coordinates in an extended spacetime suggests to formulate the present model in a similar way to the 5-dimensional Kaluza-Klein theory, though the present model is a 4-dimensional theory. In the K.-K. theory, the metric in the extra space $g_{ss}$ is understood as square of scalar fields. Under those consideration, we assume that the extended vielbein in $(\mu, \eta)$ spacetime are $2 \times 2$ even matrices having the following forms:

$$E^{(\nu)}_\mu(x) = e^{(\nu)}_\mu(x) 1 \neq 0, \quad E^{(\eta)}_\eta(x) \neq 0,$$

(2.8)

and

$$E^{(\mu)}_\mu = E^{(\eta)}_\eta = 0$$

(2.9)

where $e^{(\nu)}_\mu(x)$ is the vierbein in the 4-dimensional spacetime and only $E^{(\eta)}_\eta(x)$ is a function of scalar fields. In terms of vierbein, the metric in $(\mu, \eta)$ spacetime is defined by

$$g_{NM} = g^{(AB)}_{AB} E^{(A)}_N E^{(B)}_M,$$

(2.10)

where $-g^{(0)}_{00} = g^{(0)}_{\mu \mu} = 1$ and $g^{(0)}_{NM} = 0, (N \neq M)$.

In the present formalism, any quantity has a structure of $2 \times 2$ matrix in L/R space. The forms of $\omega^{\mu\eta}, E^{(A)}_N$ and $g_{NM}$, imply that we may assign even-odd parity such as $(-1)^\mu = 1$ and $(-1)^\eta = -1$ to each vector index, since quantities with even (odd) number of $\eta$ indices are even (odd) matrices. Then, in the product of $dx^N$ and any tensor, say $T^{(A)}_{NM}$, the change of product order gives rise to the
sign factor \(^b\) such as \(T^{(A)\, MN} dx^K = (-1)^{K(A+N+M)} dx^K T^{(A)\, MN} \). The same sign factor arises when the differential operator in \(\partial_K \{ T^{(A)\, MN}(\cdots) \} \) acts on \(\cdots\) across \(T^{(A)\, MN}\) according to the Leibniz rule. Taking such a sign factor into consideration, further, we define the exterior product of one-forms and the exterior derivative of a \(p\)-form by

\[
dx^N dx^M = -(1)^{NM} dx^M dx^N ,
\]

\[
dx^{N_1} \cdots dx^{N_p} f_{N_1 \ldots N_p} = dx^{N_1} \cdots dx^{N_p} dx^M \partial_M f_{N_1 \ldots N_p} ,
\]

from which one can verify \(d^2 = 0\).

The parallelism of vielbein can be defined by the standard way; that is, we require the vanishing of covariant derivative

\[
\nabla_K E_M^{(B)} = \partial_K E_M^{(B)} - \Gamma_{KM}^N E_N^{(B)} + E_M^{(A)} \omega_K^{(A)} = 0
\]

(2.13)

Here, \(\omega_{(A)(B)}^{(C)}\) and \(\Gamma_{KM}^N\) are connection forms for tangent vector indices and world vector indices, respectively. As discussed in Appendix A, Eq. (??) can determine those connections so as to be

\[
\Gamma_{NK}^L = \frac{1}{2} (\partial_K g_{NM} + \partial_N g_{MK} - \partial_M g_{KN}) g^{ML}
\]

(2.14)

and

\[
\omega_{K\mu M} + \omega_{K M\mu} = 0, \quad \omega_{\rho\eta} = 0,
\]

\[
\omega_{\eta\eta} - \frac{1}{M} (\partial_{\eta} E_{\eta})^2 = 0
\]

(2.15)

(2.16)

where \(\omega_{KNM} = g_{AB}^{(0)} E_{N}^{(C)} \omega_{K(C)}^{(A)} E_{M}^{(B)}\).

In what follows, we shall derive the scalar curvature in this spacetime by calculating explicit forms of \(\Gamma_{KM}^N\) and \(\omega_{K(A)}^{(B)}\) in the following two kinds of \(E_{\eta}^{(n)}\):

\[
\text{case i}
\]

\[
E_{\eta}^{(n)} = \frac{1}{M} \phi(x) 1, \quad (g_{\eta\eta} = \frac{1}{M^2} \phi^2 (x) 1)
\]

(2.17)

\(\text{For any quantities } A \text{ other than } A = 1, dx^n, \text{ in general, the product } AB \text{ is not proportional to } BA \text{ even if } B \text{ is an even or odd matrix. Consequently, we can not introduce any specific sign factor to such a product.}\)
where $\phi(x)$ is a scalar field. In this case, the vielbein themselves are invariant under the operation of LR symmetry and the non-zero components of $\Gamma_{KM}^N$ are obtained easily in the following forms:

\begin{align}
\Gamma_{\mu\nu}^\rho &= \text{Christoffel in 4-dim. spacetime} \quad (2.18) \\
\Gamma_{\mu\eta}^\eta &= \Gamma_{\eta\mu}^\eta = \frac{\partial_{\mu}\phi}{\phi} \quad (2.19) \\
\Gamma_{\eta\mu}^\mu &= -\frac{1}{M^2} \phi \partial^\mu \phi \\n\end{align}

Substituting those expression for Eq.(2.13), the non-zero components of $\omega_{K(A)}^{(B)}$ are solved, mainly from $\nabla_\eta E^{(\mu)}_\eta = \nabla_\eta E^{(\eta)}_\mu = 0$, as

\begin{align}
\omega_{\mu(\nu)}^{(\rho)} &= \omega \text{ in 4-dim. spacetime} \quad (2.21) \\
\omega_{\eta(\mu)}^{(\eta)} &= M^{-1} \partial_{(\mu)} \phi \quad (2.22) \\
\omega_{\eta(\eta)}^{(\mu)} &= -M^{-1} (\partial^\nu \phi) E^{(\mu)}_\nu \\n\end{align}

Now, in terms of one-forms $\omega_{(A)}^{(B)} = dx^N \omega_{N(A)}^{(B)}$ and $E^{(A)} = dx^N E^{(A)}_N$, the scalar curvature in $(x^\mu, \eta)$ spacetime can be calculated by the manner given in Appendix A:

\begin{align}
R = E_{(B)} E^{(A)} : (d\omega_{(A)}^{(B)} + \omega_{(A)}^{(C)} \omega_{(C)}^{(B)}) \quad (2.24)
\end{align}

where the colon means the inner product between two forms defined by

\begin{align}
E_{(A)} E_{(B)} : E^{(C)} E^{(D)} = \delta_B^C \delta_A^D - (-1)^{CP} \delta_B^D \delta_A^C \\n\end{align}

Further, the volume element in $(x^\mu, \eta)$ space should be defined by

\begin{align}
E^{(0)} E^{(1)} E^{(2)} E^{(3)} E^{(\eta)} \sim \sqrt{|g_4| \delta \eta M^{-1}} \phi \\n\end{align}

Therefore, disregarding the factor const. $\times \delta \eta$, the Einstein-Hilbert action in $(x^\mu, \eta)$ space becomes
\[ S \sim \int d^4x \sqrt{|g_4|} M^{-1} \phi (R - \lambda) \]  \hspace{1cm} (2.27)

\[ = \int d^4x \sqrt{|g_4|} M^{-1} \phi (R_4 - \nabla_\mu \phi \partial^\mu \phi - \partial_\mu \phi \partial^\mu \phi \phi^2 - \lambda) \]

where \( \lambda \) is a cosmological constant in \((x^\mu, \eta)\) spacetime. The action (2.27) is the one for Brans-Dicke type of gravity.

Carrying out, here, the conformal transformation such as \( g_{\mu \nu} = M \phi^{-1} \phi^*_{\mu \nu} \), we have c)

\[ S \sim \int d^4x \sqrt{|g_4|} \left( R^*_4 - \frac{7}{2} \frac{g^{\mu \nu} \partial_\mu \phi \partial_\nu \phi}{\phi^2} - M \frac{\lambda}{\phi} \right) \]  \hspace{1cm} (2.28)

Here, \( R^* \) is the scalar curvature constructed out of \( g^*_\mu \nu \): in the last expression, the total-divergence terms have also been disregarded. The cosmological term in (2.28) is a dynamical one; we may expect the smallness of the cosmological constant at the present time in relation to the time development of \( \phi \) as a model by Y. Fujii and T. Nishioke\(^9\).

However, the \( \lambda \) term is too simple to expect a role associated with the phase transition in the early universe and so, some modification will be necessary to get a more realistic model of dynamical cosmological constant.

case ii

\[ E_\eta(x) = \frac{1}{M} \begin{pmatrix} \phi_1(x) & 0 \\ 0 & \phi_2(x) \end{pmatrix}, \quad (g_{\eta \eta} = \frac{1}{M^2} \begin{pmatrix} \phi_1^2(x) & 0 \\ 0 & \phi_2^2(x) \end{pmatrix}) \]  \hspace{1cm} (2.29)

The symmetry in the second case is not a simple \( Z_2 \) symmetry but the system is invariant under the operation of \( \eta \) followed by the interchange \( \phi_1 \leftrightarrow \phi_2 \). In particular, Eq. (2.16) gives rise to a non vanishing \( \omega_\eta \).

From this expression of metric, we can get the non-zero components of connections \( \Gamma_{KM}^{N} \)'s in the following forms:

\[ ^{c)} \text{Under the conformal transformation } g_{\mu \nu} = F^{-1} \phi^*_{\mu \nu}, \quad (g^{\mu \nu} = F g^{* \mu \nu}), \text{ we can get } \sqrt{|g|} = F^{-2} \sqrt{|g^*|} \text{ and } R_4 = F \left[ R^*_4 - \frac{3}{2} g^{* \mu \nu} \partial_\mu f \partial_\nu f + \text{total derivative} \right] \text{ with } f = \ln F. \]
\[
\Gamma_{\mu\eta}^{\eta} = \Gamma_{\eta\mu}^{\eta} = \begin{pmatrix}
\frac{\partial_\mu \phi_1}{\phi_1} & 0 \\
0 & \frac{\partial_\mu \phi_2}{\phi_2}
\end{pmatrix}
\]  
(2.30)

\[
\Gamma_{\eta\eta}^{\mu} = -\frac{1}{M^2} \begin{pmatrix}
\phi_1 \partial_\mu \phi_1 & 0 \\
0 & \phi_2 \partial_\mu \phi_2
\end{pmatrix}
\]  
(2.31)

\[
\Gamma_{\eta\eta}^{\eta} = -\frac{M}{4} \begin{pmatrix}
0 & \frac{\phi_1^2 - \phi_2^2}{\phi_1^2}
-\frac{\phi_1^2 - \phi_2^2}{\phi_2^2} & 0
\end{pmatrix}
\]  
(2.32)

Then, with the aid of Eq.(2.13), the non-zero components of \(\omega_{K(B)}(C)\) become

\[
\omega_{\eta(\mu)}^{(\eta)} = M^{-1} \begin{pmatrix}
\partial_{(\mu)} \phi_1 & 0 \\
0 & \partial_{(\mu)} \phi_2
\end{pmatrix}
\]  
(2.33)

\[
\omega_{\eta(\eta)}^{(\mu)} = -M^{-1} \begin{pmatrix}
\partial_{(\mu)} \phi_1 & 0 \\
0 & \partial_{(\mu)} \phi_2
\end{pmatrix}
\]  
(2.34)

\[
\omega_{\eta(\eta)}^{(\eta)} = -\frac{M^2}{4} \begin{pmatrix}
0 & \frac{(\phi_1^2 - \phi_2^2)}{\phi_1^2 \phi_2^2}
-\frac{(\phi_1^2 - \phi_2^2)}{\phi_1^2 \phi_2^2} & 0
\end{pmatrix}
\]  
(2.35)

The scalar curvature in this case is obtained in a form of following 2 \times 2 matrix; after a little long calculation (Appendix B), we can get

\[
R = R_4 - 2 \begin{pmatrix}
\frac{\nabla_{(\mu)} \partial_{(\mu)} \phi_1}{\phi_1} & 0 \\
0 & \frac{\nabla_{(\mu)} \partial_{(\mu)} \phi_2}{\phi_2}
\end{pmatrix} - 2 \begin{pmatrix}
\frac{\partial_{(\mu)} \phi_i \partial_{(\mu)} \phi_1}{\phi_1^2} & 0 \\
0 & \frac{\partial_{(\mu)} \phi_2 \partial_{(\mu)} \phi_2}{\phi_2^2}
\end{pmatrix}
\]  
(2.36)

\[
-\frac{M^4}{8} \begin{pmatrix}
\frac{\phi_1^2 - \phi_2^2}{\phi_1^2 \phi_2^2} & 0 \\
0 & \frac{\phi_1^2 - \phi_2^2}{\phi_1^2 \phi_2^2}
\end{pmatrix}
\]

where \(\nabla_{(N)} = E_{(N)}^K \nabla_K\). The standard form of scalar curvature should be defined by taking the trace of the above curvature as \(R = \frac{1}{2} Tr R\) and then, we get

\[
R = R_4 - \sum_{i=1}^{2} \left\{ \frac{\nabla_{(\mu)} \partial_{(\mu)} \phi_i}{\phi_i} + \frac{\partial_{(\mu)} \phi_i \partial_{(\mu)} \phi_i}{\phi_i^2} \right\}
\]

\[
- \frac{M^4}{16} \frac{(\phi_1^2 - \phi_2^2)(\phi_1^2 + \phi_2^2)}{\phi_1^2 \phi_2^2}
\]  
(2.37)
In order to make clear the difference between $\phi_2$ and $\phi_1$, it is convenient to introduce the variables $\Phi^2 = \phi_1 \phi_2$ and $\phi^2 = \frac{\Phi}{\phi_2}$. Then, the invariant volume element corresponding to Eq.(2.26) should be proportional to $\sqrt{\text{det}(E_{\mu}^{(\nu)})} \sqrt{\text{det}(E_{\eta}^{(\eta)})}$ and so, it can be represented as $d^4x \sqrt{|g_4|} M^{-1}\Phi$. Therefore, disregarding total-derivative terms, we can put the Einstein-Hilbert’s action in this case in the following form:

$$S = \frac{c^3}{16\pi G} \int d^4x \sqrt{|g_4|} M^{-1}\Phi \left\{ R_4 - \frac{2}{\Phi^2} \partial_{(\mu)} \Phi \partial^{(\mu)} \Phi - \frac{4}{\phi^2} \partial_{(\mu)} \phi \partial^{(\mu)} \phi \right\}$$

$$- \frac{M^4 c^2 (\phi^4 - 1)^2}{16\hbar^2} \left( \frac{\phi^4 + 1}{\Phi^2 \phi^6} - \frac{\lambda}{\phi^2} \right)$$

$$= \frac{c^3}{16\pi G} \int d^4x \sqrt{|g_4|} \left\{ R_4^* - \frac{7}{2} \frac{\partial_{(\mu)} \Phi \partial^{(\mu)} \Phi}{\Phi^2} - \frac{4}{\phi^2} \partial_{(\mu)} \phi \partial^{(\mu)} \phi \right\}$$

$$- \frac{M^5 c^2 (\phi^4 - 1)^2}{16\hbar^2} \left( \frac{\phi^4 + 1}{\Phi^3 \phi^6} - \frac{M\lambda}{\Phi} \right)$$

(2.38)

(2.39)

In the last expression, we have made the conformal transformation $g_{\mu\nu} = (\Phi/M)^{-1} g_{\mu\nu}^*$. The above action is invariant under the discrete transformation $\phi_1 \leftrightarrow \phi_2$ in addition to local coordinate transformations and will be reduced to the action (2.28) according as $\phi_1 \rightarrow \phi_2$.

To get a standard form of the action, it is convenient to introduce the variables defined by $\Phi = Me^{\sqrt{8\pi} \rho / m_p}$ and $\phi = e^{\sqrt{8\pi} \sigma / m_p}$, $(m_p = \sqrt{\hbar c/G})$ and then, the right-hand side of Eq.(2.40) can be rewritten as

$$S = \frac{c^3}{16\pi G} \int d^4x \sqrt{|g_4^*|} R_4^*$$

$$+ \frac{c^2}{\hbar} \int d^4x \sqrt{|g_4^*|} \left\{ -\frac{1}{2} \partial_{(\mu)} \rho \partial^{(\mu)} \rho - \frac{1}{2} \partial_{(\mu)} \sigma \partial^{(\mu)} \sigma - V(\rho, \sigma) \right\},$$

(2.40)

where $V(\rho, \sigma)$ is the potential for $\rho$ and $\sigma$ having the form
\[ V(\rho, \sigma) = \frac{m_p^2 M^2 c^2}{2 \pi \hbar^2} - e^{-3 \sqrt{\lambda} \rho / m_p} \sinh^2 \left( \frac{2 \sqrt{2 \pi} \sigma}{m_p} \right) \cosh \left( \frac{2 \sqrt{2 \pi} \sigma}{m_p} \right) \]

In Eqs.(2.38)~(2.41), we have restored the dimensional constants \( \hbar \) and \( c \) again to its expression. In particular, the effective potential (2.41) is reduced to one in the case (i) by putting \( \sigma = 0 \).

The potential \( V(\rho, \sigma) \), the dynamical cosmological term, is stable in its vanishing direction and so, the \( \sigma \) and \( \rho \) will tend to 0 and \( \infty \) respectively according as the cosmological development (Fig.1).

3. Summary and Discussion

In this paper, we have tried to formulate a scalar-tensor gravity from a point of view of non-commutative geometry. Then we have regard the scalar fields as bridges between local Lorentz transformations for different chiral components of matter fields. For this purpose, we have followed the matrix coordinate method developed by R. Coquereaux. The matrix coordinate is, then, treated as if it is a fifth coordinate in the Kaluza-Klein theory, although the resultant theory is a four-dimensional gravitational theory. In this case, we have set the fifth component of vielbein \( E^{(n)} \) as a diagonal matrix and so, the connection \( \omega_{(n)}(n) \) becomes an off diagonal matrix.

The Lagrangian for fermion fields, which we have not discussed in the context of this paper, then, should be

\[ L_f = i E \bar{\Psi} \gamma^\nu E_{(\nu)} \mu D_{\mu} \Psi - \kappa E \bar{\Psi} E_{(n)} \eta D_{\eta} \Psi, \quad (E = \text{det}(E^{(n)})) \]

(3.1)

to which one can again verify the invariance under the transformations

\[ \Psi \rightarrow \hat{\eta} \Psi, \quad E_{(n)} \eta \rightarrow \hat{\eta} E_{(n)} \eta \]

(3.2)

together with local Lorentz transformations. Therefore, there exists a discrete symmetry in the present model before symmetry breaking.

In defining the \( E^{(n)} \), we have studied two possibilities to introduce the scalar fields. First, we tried to regard \( E^{(n)} \) as a function of one scalar field \( \phi \) and succeeded in deriving naturally the Brans-Dicke type of scalar-tensor gravitational theory. Secondly, we extended the \( E^{(n)} \) so as to include two kinds of scalar fields \( \phi_1, \phi_2 \);
then, there arise an effective potential for scalar fields being interesting from the viewpoint of cosmology. The effective potential can be well described in terms of scalar fields $\rho, \sigma$ defined by $\sqrt{\phi_1 \phi_2} = M e^{\beta \rho / m_p}$ and $\sqrt{\phi_1 / \phi_2} = e^{\sqrt{15 \sigma / \sigma}}$, $(m_p = \sqrt{\hbar c / G})$. The form of the effective potential $V(\rho, \sigma)$ depends on the presence of cosmological term $\lambda$ in $(x^\mu, \eta)$ spacetime.

Figure 1 shows the form of $V(\rho, \sigma)$ without presence of such a cosmological term. One can see that the potential is stable for $\sigma \to 0$ and $\rho \to \infty$. The situation is not changed when we add a positive cosmological term ($\lambda > 0$) as in Fig. 2. In these cases, the cross section by a line with $\rho = \text{const.}$ has a single stable point at $\sigma = 0$ (Fig.3). On the other hand, if we add negative cosmological term ($\lambda < 0$), the potential will be modified as in Fig.4; the cross section by a line with $\rho = \text{const.}$ again has the profile in Fig.3. In addition, in this case, the cross section by a line with $\sigma = \text{const.} \neq 0$ have the profile in Fig.5. Those results suggest that if we start with a one-scalar-field model reduced from the present model by a constraint $a\rho + b\sigma = c$, we can get an effective potential being suitable for the chaotic inflation or the new inflation models.

Another interesting observation is the following: if we introduce a finite temperature effect ($\beta \neq 0$) only for the degrees of freedom of $\sigma$, then the potential will be modified in such a way that

$$V_{\text{eff}}(\rho, \sigma) \sim V(\rho, \sigma) + \frac{1}{\beta \Omega} \ln \left[ 2 \sinh \left( \frac{\beta h}{2} \sqrt{V''(\rho, \sigma)} \right) \right]$$

by regarding the $\rho, \sigma$ as global variables in the Lagrangian $L \sim \Omega \mathcal{L}$. Here $V''$ is the second derivative of $V$ with respect to $\sigma$ and $\Omega$ is a parameter corresponding to a (scale)$^3$ of the universe. Figure 6. represents the modified potential for a low temperature; the structure is similar to Fig.5 even for the line $\sigma = 0$.

Therefore, we can say that the present model for scalar-tensor theory of gravity out of non-commutative geometry has interesting structure to study the early stage of universe; although the practical application needs more detailed analyses.

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Appendix A

The parallelism of vielbein $E_{(A)}^N$ is defined, as usual, by the requirement of vanishing covariant derivative, which should be written as

$$\nabla_K E_{(A)}^N = \partial_K E_{(A)}^N + (-1)^{\kappa(A+M)} E_{(A)}^M \Gamma_{KMN}^N - \omega_{(A)}^{\mu N} E_{(B)}^N = 0, \quad (A.1)$$

where the $\omega$ terms are introduced to represent the change of $E_{(A)}^N$ under local Lorentz and LR transformations associated with the parallel displacement. The factor $(-1)^{\kappa(A+M)}$ in Eq. (A.1) does not play any effective role in practical calculations, since $A + M$ becomes even for non-vanishing components of $E_M^{(A)}$ and $E^{(M)}_{(A)}$; henceforth, we disregard such a factor.

On the other side, the covariant derivative of $E_M^{(B)}$ can be determined through

$$\nabla_K (E_{(A)}^N E_{(B)}^N) = (\nabla_K E_{(A)}^N) E_{(B)}^N + E_{(A)}^N (\nabla E_{(B)}^N) = 0; \text{ then, one can easily verify that}$$

$$\nabla_K E_{(B)}^M = -E_{M}^{(A)} (\nabla_K E_{(A)}^N) E_{(B)}^N = (2.13) \quad (A.2)$$

To determine the connection $\Gamma_{KMN}^N$, we take notice of the relation

$$E_{N}^{(A)} (\partial_K E_{M}^{(B)}) = (\partial_K E_{M}^{(B)}) [E_{N}^{(A)}]_K, \quad (A.3)$$

where

$$[E_{N}^{(A)}]_K = E_{N}^{(A)} + \delta_{K}^{\eta} (\eta E_{N}^{(A)}) \hat{\eta} - E_{N}^{(A)})$$

$$= E_{N}^{(A)} + \delta_{K}^{\eta} \frac{2}{M} (\partial_K E_{N}^{(A)}) \hat{\eta} \quad (A.4)$$

Then, with the help of Eqs.(2.10) and (2.13), one can obtain

$$\delta_K g_{NM} = g_{AB}^{(0)} \{ (\partial_K E_{N}^{(A)}) E_{M}^{(B)} + (\partial_K E_{N}^{(B)}) [E_{M}^{(A)}]_K \}$$

$$= \Gamma_{KN}^{L} g_{LM} + \Gamma_{KM}^{L} g_{LN} - g_{AB}^{(0)} E_{N}^{(C)} \omega_{K(C)}^{(A)} E_{M}^{(B)}$$

$$+ \delta_{K}^{\eta} g_{AB}^{(0)} \{ \frac{2}{M} (\partial_K E_{M}^{(B)}) (\partial_{\eta} E_{N}^{(A)}) \} \hat{\eta} \quad (A.5)$$

The last term in the right-hand side of this equation spoils manifest general invariance in $(x^\mu, \eta)$ spacetime; we need not worry about this point, since it is enough to get the general invariance in $(x^\mu)$ spacetime only.
We, here, require that
\[ g^{(0)}_{AB} \{ E^{(C)}_{(N)}(\omega^{(A)}_{K(C)} E^{(B)}_{M}) - \delta^{n}_{M} (\partial_{K} E^{(B)}_{M}))(\partial_{n} E^{(A)}_{N})\} = 0 \]  \hspace{1cm} (A.6)

; that is, Eqs.(2.15) and (2.16). Then, the equation for the metric can be written in the standard closed form:
\[ \nabla K g_{NM} = \partial_{K} g_{NM} - \Gamma_{KN}^{L} g_{LM} - \Gamma_{KM}^{L} g_{NL} = 0, \]  \hspace{1cm} (A.7)

which yields obviously the expression (2.14) under the requirement \( \Gamma_{NM}^{L} = \Gamma_{MN}^{L} \).

Now, let us derive the expression for scalar curvature, which is not trivial in the matrix calculus: Using Eq.(2.12), the exterior derivative of the one form \( E^{(B)} = dx^{M} E^{(B)}_{M} \) can be written as
\[ dE^{(B)} = dx^{M} dx^{K} \Gamma_{KM}^{N} E^{(B)}_{N} - E^{(A)}_{M} \omega^{(B)}_{(A)} \]  \hspace{1cm} (A.8)

where \( \omega^{(B)}_{(A)} = dx^{K} \omega^{(B)}_{K(A)} \). Then, the nilpotency of exterior derivative gives rise to \(^{4}\)
\[ d^{2} E^{(B)} = dx^{M} dx^{K} dx^{L} \{ \partial_{L}(\Gamma_{KM}^{N} E^{(B)}_{N} - \partial_{L}(E^{(A)}_{M} \omega^{(B)}_{K(A)})) \}
\]
\[ = dx^{M} dx^{K} dx^{L} \{ \partial_{L} \Gamma_{KM}^{N} + (-1)^{L(K+M+J)} \Gamma_{KM}^{J} \Gamma_{LJ}^{N} \} E^{(B)}_{N} \]
\[ - E^{(A)} \{ d \omega^{(B)}_{(A)} + \omega^{(C)}_{(A)} \omega^{(B)}_{(C)} \} = 0, \]  \hspace{1cm} (A.9)

from which we get
\[ d \omega^{(B)}_{(A)} + \omega^{(C)}_{(A)} \omega^{(B)}_{(C)} = E^{(A)}_{M} dx^{K} dx^{L} \{ \partial_{L} \Gamma_{KM}^{N} + (-1)^{L(K+M+J)} \Gamma_{KM}^{J} \Gamma_{LJ}^{N} \} E^{(B)}_{N}. \]  \hspace{1cm} (A.10)

Using, here, the 'inner product' between two forms defined in Eq.(2.25) and \( E^{(A)} = g^{(0)}_{AB} E^{(B)} \), we have
\[ R = E^{(B)} E^{(A)} : (d \omega^{(B)}_{(A)} + \omega^{(C)}_{(A)} \omega^{(B)}_{(C)}) \]  \hspace{1cm} (A.11)

\[ = g^{(0)}_{CA} E^{(B)}_{L} E^{(C)}_{K} E^{(A)}_{M} - (-1)^{AB}(A \leftrightarrow B)) \times \{ \partial_{L} \Gamma_{KM}^{N} + (-1)^{L(K+M+J)} \Gamma_{KM}^{J} \Gamma_{LJ}^{N} \} E^{(B)}_{N}. \]  \hspace{1cm} (A.12)

\(^{4}\)From our definition of the exterior derivative, one can verify \( d(AB) = -(dA)B + A(dB) \) provided \( B \) is a one form.
The resultant $R$ is a $2 \times 2$ matrix in LR space and so, the usual scalar curvature should be defined by $R = \frac{1}{2} Tr R$. Therefore, we get

$$R = \frac{1}{2} Tr [E(B)E(A) : (d\omega(A)^{(B)} + \omega(A)^{(C)}\omega(C)^{(B)})]$$

$$= \frac{1}{2} Tr (\delta_{N}^{L} g^{KM} - (-1)^{KL} \delta_{N}^{K} g^{LM}) \{ \partial L \Gamma_{KM}^{N} + (-1)^{L(K+M+J)} \Gamma_{KM}^{J} \Gamma_{LJ}^{N} \}$$

The right-hand side of the second equality includes the definition of the scalar curvature in 4-dimensional spacetime and so, Eq.(A.13) is nothing but an extended scalar curvature in $(x^\mu, \eta)$ spacetime.

**Appendix B**

We, here, give the explicit forms of non-trivial terms coming into the calculating $E(B)E(A) : (d\omega(A)^{(B)} + \omega(A)^{(C)}\omega(C)^{(B)})$ in the case of two scalar fields: First, $d\omega$ terms are

$$E(\mu)E(\eta) : d\omega(\eta)^{(\mu)} = \begin{pmatrix} -\frac{\partial(\omega)^{(\mu)}\phi_1}{\phi_1} & 0 \\ 0 & -\frac{\partial(\omega)^{(\mu)}\phi_2}{\phi_2} \end{pmatrix}$$

$$E(\eta)E(\mu) : d\omega(\mu)^{(\eta)} = \begin{pmatrix} -\frac{\partial(\omega)^{(\mu)}\phi_1}{\phi_1} & 0 \\ 0 & -\frac{\partial(\omega)^{(\mu)}\phi_2}{\phi_2} \end{pmatrix}$$

$$E(\eta)E(\eta) : d\omega(\eta)^{(\eta)} = -\frac{M^4}{2} \begin{pmatrix} \frac{(\phi_1-\phi_2)^2}{\phi_1\phi_2} \\ \frac{(\phi_1-\phi_2)^2}{\phi_1\phi_2} \end{pmatrix}$$

Secondly, as for $\omega\omega$ terms, we have

$$E(\mu)E(\nu) : \omega(\nu)^{(\eta)}\omega(\eta)^{(\mu)} = 0$$

$$E(\mu)E(\eta) : \omega(\mu)^{(\nu)}\omega(\nu)^{(\mu)} = \omega(\mu)^{(\nu)} \begin{pmatrix} -\frac{\partial(\omega)^{(\nu)}\phi_1}{\phi_1} & 0 \\ 0 & -\frac{\partial(\omega)^{(\nu)}\phi_2}{\phi_2} \end{pmatrix}$$

$$E(\mu)E(\eta) : \omega(\eta)^{(\eta)}\omega(\eta)^{(\mu)} = 0$$
\begin{align*}
E(\eta)E^{(\mu)} : \omega(\mu)^{(\nu)}\omega(\mu)^{(\eta)} &= \omega(\mu)^{(\nu)} \begin{pmatrix}
\frac{\partial(\omega)}{\phi_1} & 0 \\
0 & \frac{\partial(\omega)}{\phi_2}
\end{pmatrix} \\
E(\eta)E^{(\mu)} : \omega(\mu)^{(\eta)}\omega(\eta)^{(\eta)} &= 0 \\
E(\eta)E^{(\eta)} : \omega(\eta)^{(\mu)}\omega(\mu)^{(\eta)} &= -2 \begin{pmatrix}
\frac{\partial(\omega)}{\phi_1}\phi(\mu)\phi_1 & 0 \\
0 & \frac{\partial(\omega)}{\phi_2}\phi(\mu)\phi_2
\end{pmatrix} \\
E(\eta)E^{(\eta)} : \omega(\eta)^{(\eta)}\omega(\eta)^{(\eta)} &= -\frac{M^4}{8} \begin{pmatrix}
(\phi_1 - \phi_2)^4 & 0 \\
0 & (\phi_1 - \phi_2)^4 \phi_1^2 \phi_2^2
\end{pmatrix}
\end{align*}
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