Developments and new applications of numerical stochastic perturbation theory

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A review of new developments in numerical stochastic perturbation theory (NSPT) is presented. In particular, the status of the extension of the method to gauge fixed lattice QCD is reviewed and a first application to compact (scalar) QED is presented. Lacking still a general proof of the convergence of the underlying stochastic processes, a self-consistent method for testing the results is discussed.

1. NSPT for gauge fixed lattice QCD

Numerical stochastic perturbation theory (NSPT) was successful in performing high loops gauge invariant computations in lattice QCD [1] (for new, and potentially relevant, phenomenological implications see [2]). An extension to gauge fixed computations would of course be highly welcome. First results in this direction were reported in St. Louis, but the matter turned out to be not settled down. In the following, as a first step, the basics from [3] are recalled.

As it is well known, only gauge invariant quantities have a large time limit in standard stochastic quantization [4], while gauge–non–invariant quantities are affected by divergences. As usual, the situation is slightly different on the lattice due to the fact that the gauge degrees of freedom have been compactified, so that gauge–non–invariant quantities average to zero. Gauge fixed Langevin simulations rely on the old observation [5] that the standard evolution step

\begin{equation}
U'_\mu(n) = e^{-F_\mu([U],\eta)(n)} U_\mu(n)
\end{equation}

(\eta is the gaussian noise) can be interlaced with a gauge transformation

\begin{equation}
U^G_\mu(n) = G(n) U_\mu(n) G^\dagger(n + \hat{\mu})
\end{equation}

Since

\begin{equation}
F_\mu([U^G],\eta) = G F_\mu([U],G\eta G^\dagger) G^\dagger
\end{equation}

the convergence of the process for gauge invariant quantities is unaffected, while the gauge transformation \( G \) can be chosen in order to enforce the gauge condition one is interested in: the stochastic process is then attracted towards the submanifold defined by the gauge condition and gauge–non–invariant quantities no longer average to zero. This recipe is in a sense the lattice implementation of Zwanziger’s stochastic gauge fixing scheme [6].

NSPT rely on an expansion of the solution of Langevin equation in powers of the coupling, which actually decompactifies the fields (in a sense the fundamental fields of NSPT live in the Lie algebra), so that divergences in the gauge–non–invariant sector are back. Nevertheless one can adopt the stochastic gauge fixing scheme also in this context, with the caveat that also the gauge condition one wants to enforce has to be expanded as a series of conditions. To be definite, as the \( A_\mu \) field is expanded as

\begin{equation}
A_\mu(n) = A_\mu^{(0)} + g A_\mu^{(1)} + g^2 A_\mu^{(2)} + \ldots
\end{equation}

the form of Landau condition one needs to impose is

\begin{equation}
\partial_\mu A_\mu^{(k)}(n) = 0
\end{equation}

for every order \( k \) (partial derivatives, as usual, are to be understood as finite difference operators).

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The implementation presented in [3] goes as follows. One considers the functional (norm)

\[ N[G] = \sum_{n,\mu} \text{Tr}(A_G^\mu(n)A_G^\mu\dagger(n)) = \sum_k g^k N^{(k)} \]  

(5)

where an expansion for \( N \) is induced by the expansion of the field \( A_\mu \). If the gauge transformation is chosen as

\[ G = e^{-\alpha w} \quad w = \sum_k g^k \partial_\mu A_\mu^{(k)} \]  

(6)

extremum conditions for every \( N^{(k)} \) are recovered that exactly enforce (4). First simulations with this recipe were presented in [3] and they seemed good up to order \( g^4 \). As a matter of fact, if one lets the system evolve for a while, the signal for gauge–non–invariant quantities is not at all stable, but one recovers a collection of plateaux. While gauge fixing transformation is in charge to kill divergences due to “longitudinal” degrees of freedom, (6) lets some divergence still at work. This can be checked by inspecting the norms of the fields \( A_\mu^{(k)} \). While the norm of \( A_\mu^{(0)} \) is minimized and stable through the process (as a matter of fact, this norm is exactly \( N^{(2)} \), which is the only \( N^{(k)} \) one is ensured to minimize), higher orders suffer divergences in their norms.

A first attempt has been done in order to enforce (4) and keep every norm under control through the process. This can be obtained by adding new terms to \( w^{(k)} \) in (6) (the formulae are quite cumbersome and they will be written down elsewhere). The results are encouraging and can be summarized by inspecting fig. 1. In the upper half the norm of \( A_\mu^{(1)} \) is monitored through the evolution with and without these new terms ((4) is enforced in either case). The lower half shows the signal for order \( g^4 \) of the trace of the link, again with and without these new terms: the new gauge fixing procedure keeps the plateau stable (while the old procedure lets the system depart from it).

2. How can you trust NSPT?

Of course a stable signal does not mean at all that a result is correct. Actually there is an embarrassing lack of a general proof of the convergence of the (quite cumbersome) stochastic processes underlying NSPT, so that in a sense one could worry about whether to trust a NSPT result or not. A general strategy to check a result can be outlined as follows: check up by explicit computation the trivial order and then set up relations among different perturbative orders. To be definite, one can think of Schwinger–Dyson equations as obtained by

\[ \frac{1}{Z} \int D\phi \frac{\delta}{\delta\phi} (O[\phi] e^{-S[\phi]}) = 0 \]  

(7)

By expanding this equation in the coupling one recovers relations among different perturbative orders. As it is well known, this is a standard approach to perturbation theory. In this context one wants to look at these relations to test whether they holds once every expectation value involved has been obtained by means of the Langevin technique.

Unfortunately for (lattice) QCD (7) reads

\[ \frac{1}{Z} \int DU \nabla(O[U]e^{-S_W[U]}e^{-S_{GF}[U]}\Delta_{FP}) = 0 \]  

(8)

Figure 1. Above: the norm of the field \( A_\mu^{(1)} \) with and without corrections to (6). Below: the signal for order \( g^4 \) of the trace of the link in the same simulations.
where \( \nabla \) is the group derivative and \( \Delta_{FP} \) the Fadeev–Popov determinant, which is tremendous to take into account without introducing ghosts.

Nevertheless, from the standard formula for a QCD expectation value

\[
\frac{1}{Z(g)} \int DADcD\pi O[A] e^{-S[A]} e^{-S_c[A,c,\pi]} \quad (9)
\]

(where the action has been split in two terms \( S \) and \( S_c \), the latter being dependent on the ghosts), one can get a recursive relation for the coefficients of the expansion \( \sum_k g^k O^{(k)} \). Namely, by deriving with respect to the coupling \( g \), one gets (\( O' \) means \( d/dg O \))

\[
(k+1)O^{(k+1)} = O^{(k)} - (OS')^{(k)} + \sum_{l+m=k} S^{(l)}O^{(m)} + \sum_{l+m=k} S_c^{(l)}O^{(m)} - (OS_c')^{(k)} \quad (10)
\]

One can obtain by the Langevin technique everything except the last two contributions, which depend on the ghosts. For example, testing the order \( g^4 \) result for the trace of the link only requires to compute the ghost corrections to the gluon propagator.

3. NSPT for compact QED

A first implementation of the Schwinger–Dyson test program has been done within the context of a new application of NSPT, namely to compact QED. The final goal of this project is a collaboration with the Tor Vergata group, aiming at merging NSPT with the bermions approach [7] to push one loop further the computation of electron’s \( g-2 \). An important point to notice is that the extrapolation in the number of flavors is perturbatively well defined. Being QED not asymptotically free, a major problem to deal with is an explicit inclusion of counterterms.

In order to gain experience, we started simulating scalar QED defined by the action

\[
S = \frac{1}{\alpha^2} \sum_p (1-\cos \phi_{\mu\nu}) - \sum_{n\mu} \phi^*(n)U_\mu(n)\phi(n+\mu) + U_\mu(n-\mu)\phi(n-\mu) + \sum_n (M^2 + 8)\phi^*(n)\phi(n)
\]

\( \phi_{\mu\nu} \) being defined by the plaquette \( U_{\mu\nu} = \exp(i\phi_{\mu\nu}) \).

We have till now performed computations up to order \( \alpha^3 \) in the gauge invariant sector, every result having been tested by Schwinger–Dyson equations. In many instances the overall picture is the same as for QCD; in particular, some form of stochastic gauge fixing is needed also in the gauge invariant sector in order to keep fluctuations under control.

4. More to come . . .

Another application which is expected to come soon (some preliminary work has just started after the conference) is a high order perturbative expansion of the QCD running coupling \( \alpha_{SF} \) as defined in the SF scheme [8]. One is interested in pushing further the connection between different schemes and in evaluating the high order behavior itself of this definition of the coupling (in [1] the coupling \( \alpha_{pl} \) defined via the plaquette has been found to be affected by a renormalon).

REFERENCES