THEORY OF RARE $B$ DECAYS

A. Ali
Deutsches Elektronen Synchrotron DESY, Hamburg
Notkestraße 85, D-22603 Hamburg, FRG

To be published in the Proceedings of the Seventh International Symposium on Heavy Flavor Physics, University of California Santa Barbara, California, July 7-11, 1997
THEORY OF RARE B DECAYS

A. Ali
Deutsches Elektronen-Synchrotron DESY, Notkestrasse 85, D-22603 Hamburg, FRG

We discuss some selected topics in rare $B$ decays in the context of the standard model and compare theoretical estimates with available data. Salient features of the perturbative-QCD and power corrections in the decay rate for $B \to X_s + \gamma$ are reviewed and this framework is used to determine the Cabibbo-Kobayashi-Maskawa (CKM) matrix element $|V_{ts}|$, yielding $|V_{ts}| = 0.033 \pm 0.007$ from the present measurements of the electromagnetic penguins. We review estimates of the ratio $R_{K^*} \equiv \Gamma(B \to K^* + \gamma)/\Gamma(B \to X_s + \gamma)$ in a number of theoretical models, which give a consistent account of this quantity. Issues bearing on the photon energy spectrum in $B \to X_s + \gamma$ are also discussed. The CKM-suppressed decays $B \to X_d + \gamma$, $B^\pm \to \rho^\pm + \gamma$, and $B^0 \to (\rho^0, \omega) + \gamma$ are reviewed with particular emphasis on the long-distance contributions in the exclusive decays. The impending interest in these decays in determining the parameters of the CKM matrix is emphasized. Finally, the semileptonic decays $B \to X_s \ell^+\ell^-$ are also discussed in the context of the SM.

1 $B(B \to X_s + \gamma)$ and $B \to K^* + \gamma$ in the Standard Model and experiment and determination of $|V_{ts}|$

1.1 Experimental status

Electromagnetic penguins were first sighted on the $B$ territory in 1993 by the CLEO collaboration through the exclusive decay $B \to K^* + \gamma$. This feat was followed by the daunting measurement of the inclusive decay $B \to X_s + \gamma$ in 1994 by the same collaboration. The present CLEO measurements can be summarized as:

- $B(B \to X_s + \gamma) = (2.32 \pm 0.57 \pm 0.35) \times 10^{-4}$,
- $B(B \to K^* + \gamma) = (4.2 \pm 0.8 \pm 0.6) \times 10^{-5}$,

which yield an exclusive-to-inclusive ratio:

$$R_{K^*} \equiv \frac{\Gamma(B \to K^* + \gamma)}{\Gamma(B \to X_s + \gamma)} = (18.1 \pm 6.8)\%.$$ (2)

Very recently, the inclusive radiative decay has also been reported by the ALEPH collaboration with a (preliminary) branching ratio:

$$B(H_b \to X_s + \gamma) = (3.29 \pm 0.71 \pm 0.68) \times 10^{-4}.$$ (3)

Since the ALEPH measurement is done at the $Z^0$ peak in the process $Z^0 \to b\bar{b} \to H_b + X \to (X_s + \gamma) + X$, the branching ratio in (3) involves a different
weighted average of the various $B$-mesons and $\Lambda_b$ baryons produced in $Z^0$ decays (hence the symbol $H_b$) than the corresponding one given in (1), which has been measured in the decay $Y (4S) \rightarrow B^+B^-$, $B^0 \overline{B}^0$. Theoretically, the inclusive radiative decay widths for the various beauty hadrons are expected to be nearly equal. Despite this, their branching ratios are not all equal reflecting the differences in the respective total decay rates (equivalently lifetimes).

In the context of SM, the principal interest in the decay rates in eqs. (1) and (3) lies in that they determine the ratio of the CKM matrix elements $|V_{ts}|/|V_{cb}|$. Since $|V_{cb}|$ and $|V_{tb}|$ have been directly measured, one can combine these measurements to determine $|V_{ts}|$. In addition, the quantity $R_K^*$ provides information on the decay form factor in $B \rightarrow K^* \gamma$. We review in this section first the branching ratio $\mathcal{B}(B \rightarrow X_s + \gamma)$ (and $\mathcal{B}(H_b \rightarrow X_s + \gamma)$) in the SM and then discuss estimates of $|V_{ts}|$ and $R_K^*$.

### 1.2 SM estimates of $\mathcal{B}(B \rightarrow X_s + \gamma)$ and $\mathcal{B}(H_b \rightarrow X_s + \gamma)$

The leading contribution to the decay $b \rightarrow s + \gamma$ arises at one-loop from the so-called penguin diagrams. With the help of the unitarity of the CKM matrix, the decay matrix element in the lowest order can be written as:

$$\mathcal{M}(b \rightarrow s + \gamma) = \frac{G_F}{\sqrt{2}} \frac{e}{2\pi^2} \lambda_t (F_2(x_t) - F_2(x_c)) q^\mu \epsilon_\nu \bar{s} \sigma_{\mu\nu} (m_b R + m_s L) b. \tag{4}$$

where $G_F$ is the Fermi coupling constant, $e = \sqrt{4\pi\alpha_{em}}$, $x_i = m_i^2/m_W^2$; $i = u, c, t$ are the scaled quark mass ratios, and $q_\mu$ and $\epsilon_\nu$ are, respectively, the photon four-momentum and polarization vector. The GIM mechanism is manifest in this amplitude and the CKM-matrix element dependence is factorized in $\lambda_t \equiv V_{tb}V_{ts}^*$. The (modified) Inami-Lim function $F_2(x_i)$ derived from the (1-loop) penguin diagrams is given by:

$$F_2(x) = \frac{x}{24(x-1)^4} \times \left[ 6x(3x^2-2) \log x - (x-1)(8x^2 + 5x - 7) \right]. \tag{5}$$

As the inclusive decay widths of the $B$ hadrons are proportional to $|V_{cb}|^2$, the measurement of $\mathcal{B}(B \rightarrow X_s + \gamma)$ can be readily interpreted in terms of the CKM-matrix element ratio $\lambda_t/|V_{cb}|$. For a quantitative determination, however, QCD radiative and power corrections have to be computed, which we discuss next.

The appropriate framework to incorporate QCD corrections is that of an effective theory obtained by integrating out the heavy degrees of freedom, which in the present context are the top quark and $W^\pm$ bosons. The effective
Hamiltonian depends on the underlying theory and for the SM one has (keeping operators up to dimension 6),

\[ H_{\text{eff}}(b \to s + \gamma) = -\frac{4G_F}{\sqrt{2}} \lambda_t \sum_{i=1}^{8} C_i(\mu) O_i(\mu), \]  

where the operator basis, the lowest order coefficients \( C_i(m_W) \) and the renormalized coefficients \( C_i(\mu) \) can be seen elsewhere. The perturbative QCD corrections to the decay rate \( \Gamma(B \to X_s + \gamma) \) consist of two distinct parts:

- Evaluation of the Wilson coefficients \( C_i(\mu) \) at the scale \( \mu = O(m_b) \).
- Evaluation of the matrix elements of the operators \( O_i \) at the scale \( \mu = O(m_b) \).

The Wilson coefficients are calculated with the help of the renormalization group equation whose solution requires the knowledge of the anomalous dimension matrix in a given order in \( \alpha_s \) and the matching conditions, i.e., the Wilson coefficients \( C_i(\mu = m_W) \), calculated in the complete theory to the commensurate order. The leading logarithmic (LL) anomalous dimension matrix has been calculated by several independent groups. First calculation of the next-to-leading order (NLO) anomalous-dimension matrix has been carried out by Chetyrkin, Misiak and Münz. The matching conditions to order \( \alpha_s \) have also been worked out in the meanwhile by several groups. Of these, the first six corresponding to the four-quark operators have been derived by Buras et al., and the remaining two \( C_7(\mu = m_W) \) and \( C_8(\mu = m_W) \) were worked out by Adel and Yao. These latter have been recalculated by Greub and Hurth, confirming the earlier result. Recently, these matching conditions have also been confirmed by Buras, Kwiatkowski and Pott.

The NLO corrections to the matrix elements are of two kinds:

- QCD Bremsstrahlung corrections \( b \to s\gamma + g \), which are needed both to cancel the infrared divergences in the decay rate for \( B \to X_s + \gamma \) and in obtaining a non-trivial QCD contribution to the photon energy spectrum in the inclusive decay \( B \to X_s + \gamma \).
- Next-to-leading order virtual corrections to the matrix elements in the decay \( b \to s + \gamma \).

The Bremsstrahlung corrections were calculated by Ali and Greub in the truncated basis (involving the operators \( O_1, O_2, O_7 \) and \( O_8 \)) and subsequently in the complete operator basis by the same authors and by Pott. The NLO
virtual corrections were completed by Greub, Hurth and Wyler. These latter calculations have played a key role in reducing the scale-dependence of the LL inclusive decay width. All of these pieces have been combined to get the NLO decay width \( \Gamma(B \to X_s + \gamma) \) and the details are given in the literature.

It is customary to express the branching ratio \( B(B \to X_s + \gamma) \) in terms of the semileptonic decay branching ratio \( B(B \to X_s + \gamma) \) of the semileptonic decay width \( \Gamma(B \to X_s + \gamma) \) and the details are given in the literature.

The theoretical part can be expressed as

\[
\frac{\Gamma(B \to \gamma + X_s)}{\Gamma_{SL}}^{th} = \left| \frac{\lambda_t}{V_{cb}^2} \right|^2 \frac{6\alpha}{\pi f(z)} F(|D|^2 + A). \tag{8}
\]

Here,

\[
f(z) = 1 - 8z + 8z^3 - z^4 - 12z^2 \ln z \quad \text{with} \quad z = \frac{m_{c,\text{pole}}^2}{m_b^2},
\]

and the power corrections can be expressed as

\[
B(B \to X_s + \gamma) = \frac{\Gamma(B \to \gamma + X_s)}{\Gamma_{SL}}^{th} B(B \to X_s), \tag{7}
\]

with

\[
\frac{\Gamma(B \to \gamma + X_s)}{\Gamma_{SL}}^{th} = \frac{\lambda_t^2}{V_{cb}^2} \frac{6\alpha}{\pi f(z)} F(|D|^2 + A). \tag{8}
\]

where \( m_{b,\text{pole}} \) is the b-quark pole mass and \( m_b \) is the b-quark mass in the \( \overline{\text{MS}} \) scheme. The function \( \kappa(z, \bar{\mu}_b) \) represents the QCD correction to the semileptonic decay width, which depends on the scale \( \bar{\mu}_b = O(m_b) \). The other functions \( D \) and \( A \) can be seen in the literature. It should be remarked that while the Bremsstrahlung function \( A \) has been calculated in the complete operator basis, the virtual corrections contributing to \( D \) are still known in the truncated approximation. However, since the numerical values of the Wilson coefficients \( C_3, ..., C_6 \) are very small, the contribution left out in the NLO expression in eq. (8) is expected to be small (not more than several percent).

In addition to the perturbative QCD improvements discussed above, also the leading power corrections, which start in \( 1/m_b^2 \), have been calculated to the decay widths appearing in the numerator and denominator of eq. (8). The power corrections in the numerator have been obtained assuming that the decay \( B \to X_s + \gamma \) is dominated by the magnetic moment operator \( O_7 \). Writing this correction in an obvious notation as

\[
\frac{\Gamma(B \to X_s + \gamma)}{\Gamma^0(B \to X_s + \gamma)} = 1 + \frac{\delta_b}{m_b^2}, \tag{10}
\]
one obtains $\delta_b = 1/2\lambda_1 - 9/2\lambda_2$, where $\lambda_1$ and $\lambda_2$ are, respectively, the kinetic energy and magnetic moment parameters of the theoretical framework based on heavy quark expansion (henceforth called HQET)\textsuperscript{25}. Using $\lambda_1 = -0.5 \, \text{GeV}^2$ and $\lambda_2 = 0.12 \, \text{GeV}^2$, one gets $\delta_b/m_b^2 \simeq -4\%$. However, it turns out that the leading order ($1/m_b^2$) power corrections in the heavy quark expansion are identical in the inclusive decay rates $\Gamma(B \rightarrow X_s + \gamma)$ and $\Gamma(B \rightarrow X_t\ell\nu)$, as far as $\lambda_1$ is concerned. The corrections proportional $\lambda_2$ differ only marginally. Thus, including or neglecting the $1/m_b^2$ corrections makes a difference of only 1\% in the ratio (8) and hence in $\mathcal{B}(B \rightarrow X_s + \gamma)$.

Recently, the power corrections proportional to $1/m_b^2$, resulting from the interference of the operators $O_2$ and $O_7$ in $B \rightarrow X_s + \gamma$, have also been worked out\textsuperscript{26,27,28}. Expressing this symbolically as

$$\frac{\Gamma(B \rightarrow X_s + \gamma)}{\Gamma^0(B \rightarrow X_s + \gamma)} = 1 + \frac{\delta_c}{m_c^2},$$

one finds\textsuperscript{28} $\delta_c/m_c^2 \simeq 0.03$.

There exist several (marginally) differing numerical values of the branching ratio $\mathcal{B}(B \rightarrow X_s + \gamma)$ in the SM. Using $|V_{cb}^*V_{tb}/V_{ub}| = 0.976 \pm 0.010$ obtained from the unitarity constraints\textsuperscript{29}, fixing the scale $\mu_b = m_b$, varying the scale $\mu_b$ in the range $2m_b \geq \mu_b \geq m_b/2$, and using current values of the various input parameters, but ignoring the $\delta_c/m_c^2$ term, the short-distance contribution has been estimated by Chetyrkin et al. as\textsuperscript{10}: $\mathcal{B}(B \rightarrow X_s + \gamma) = (3.28 \pm 0.33) \times 10^{-4}$, where all the errors have been combined in quadrature. Including the $\delta_c/m_c^2$ term, setting $\mu_b = \mu_b$ and varying $\mu_b$ in the stated range, but keeping the other parameters the same as in the work of Chetyrkin et al.\textsuperscript{10}, Greub and Hurth\textsuperscript{30} determine $\mathcal{B}(B \rightarrow X_s + \gamma) = (3.38 \pm 0.33) \times 10^{-4}$. Recently, a NLO value has been obtained by Buras et al.\textsuperscript{14}, yielding $\mathcal{B}(B \rightarrow X_s + \gamma) = (3.48 \pm 0.31) \times 10^{-4}$. The shift in the central value from the one given by Greub and Hurth\textsuperscript{30} is due to systematically discarding the next-next-leading order terms, which were kept in the earlier work\textsuperscript{10,30}, and the reduced error is due to treating the scale uncertainty in the numerator and denominator in eq. (8) independently.

The numerical values for $\mathcal{B}(B \rightarrow X_s\gamma)$ quoted above\textsuperscript{10,14,30} have been obtained by using the semileptonic branching ratio $\mathcal{B}(sl) = (10.4 \pm 0.4)\%$ taken from earlier data at $T(4S)$. This number has changed somewhat in the meanwhile and the current measurements are\textsuperscript{31}: $\mathcal{B}(sl) = (10.49 \pm 0.46)\%$ (at $T(4S)$) versus $\mathcal{B}(sl) = (11.16 \pm 0.20)\%$ (at $Z^0$). Updating $\mathcal{B}(sl)$ yields

$$\mathcal{B}(B \rightarrow X_s + \gamma) = (3.51 \pm 0.32) \times 10^{-4},$$

(12)

to be compared with the CLEO measurement $\mathcal{B}(B \rightarrow X_s + \gamma) = (2.32 \pm 0.67) \times 10^{-4}$. The corresponding inclusive branching ratio at the $Z^0$ is obtained by
using in eq. (7) the semileptonic branching ratio measured at the $Z^0$. This gives

$$ B(H_b \to X_s + \gamma) = (3.76 \pm 0.30) \times 10^{-4}, \quad (13) $$

to be compared with the ALEPH measurement $B(H_b \to X_s + \gamma) = (3.29 \pm 0.98) \times 10^{-4}$. The agreement between experiment and SM is good though the CLEO number is marginally $(2\sigma)$ lower than the SM branching ratio.

We would like to use the NLO SM-based theory and experiments to determine the CKM matrix element ratio $|V^*_{ts}V_{tb}/V_{cb}|$ and $|V_{ts}|$. The two inclusive measurements given in eqs. (1) and (3), and the corresponding NLO SM-estimates given in eqs. (12) and (13) yield,

$$ |V^*_{ts}V_{tb}/V_{cb}| = 0.79 \pm 0.11 \text{ (expt)} \pm 0.04 \text{ (th) (0Y(4s))}, $$

$$ |V^*_{ts}V_{tb}/V_{cb}| = 0.91 \pm 0.14 \text{ (expt)} \pm 0.04 \text{ (th) (0Z^0)}. \quad (14) $$

Averaging the two measurements following the PDG prescription\textsuperscript{29} gives the following weighted average for the CKM matrix element ratio:

$$ |V^*_{ts}V_{tb}/V_{cb}| = 0.84 \pm 0.09 \text{ (expt)} \pm 0.04 \text{ (th) \Rightarrow 0.84 \pm 0.10} \quad (15) $$

where the second row has been obtained by adding the theoretical and experimental errors in quadrature. With the CKM unitarity, one has $|V^*_{ts}V_{cb}/V_{tb}| \simeq |V_{cs}|$; this equality holds numerically (within present precision) if one compares the l.h.s. obtained from the decay $B \to X_s + \gamma$ given in eq. (15) with the present determination of the r.h.s. from charmed hadron decays\textsuperscript{29}, $|V_{ts}| = 1.01 \pm 0.18$. Using the value of $|V_{tb}|$ measured by the CDF collaboration\textsuperscript{32}, $|V_{tb}| = 0.99 \pm 0.15$ and noting\textsuperscript{33} that $|V_{cb}| = 0.0393 \pm 0.0028$, finally yields

$$ |V_{ts}| = 0.033 \pm 0.007, \quad (16) $$

where all the errors have been added in quadrature. This is probably as direct a determination of $|V_{ts}|$ as we will ever see, as the decay $t \to W + s$ is too daunting to measure due to the low tagging efficiency of the $s$-quark jet. With improved measurement of $B(B \to X_s + \gamma)$ and $V_{tb}$, one expects to reduce the present error on $|V_{ts}|$ by a factor of 2, possibly 3.

The exclusive-to-inclusive ratio $R_K$ has been worked out in a number of models. This involves estimates of the matrix elements of the electromagnetic penguin operator, implicitly assuming the SD-dominance. Taken on their face
value, different models give a rather large theoretical dispersion on $R_{K^*}$. However, one should stress that QCD sum rules, models based on quark-hadron duality, and the improved lattice-QCD estimates of the 1997 vintage (being discussed by Lynn [36]) are theoretically more reliable. Concentrating only on them, some representative results are:

\begin{align*}
R_{K^*} &= 0.20 \pm 0.06 \quad \text{[Ball [34]]}, \\
R_{K^*} &= 0.17 \pm 0.05 \quad \text{[Colangelo et al. [34]]}, \\
R_{K^*} &= 0.16 \pm 0.05 \quad \text{[Ali, Braun & Simma [35]]}, \\
R_{K^*} &= 0.13 \pm 0.03 \quad \text{[Ali & Greub [15]]}, \\
R_{K^*} &= 0.16 \pm 0.04 \quad \text{[Flynn [36]]}.
\end{align*}

These estimates are consistent with each other and with the CLEO measurement $R_{K^*} = 0.181 \pm 0.06$. Summarizing this section, it is fair to conclude that SM gives a quantitative account of data in electromagnetic penguin decays in inclusive rates, yielding a first determination of $|V_{ts}|$ with an accuracy of $\pm 20\%$. Further, both $B(B \to X_s + \gamma)$ and the ratio $R_{K^*}$ are in agreement with the dominance of the short-distance physics in these decays.

## 2 Photon energy spectrum in $B \to X_s + \gamma$

Calculation of the photon energy spectrum is somewhat intractable as the main work-horse, namely HQET, does not quite make it to the very end of this spectrum. There is no alternative at present but to model the non-perturbative effects. We review the present state of the art.

The two-body partonic process $b \to s \gamma$ yields a photon energy spectrum which is just a discrete line, $1/(\Gamma) d\Gamma(b \to s \gamma) = \delta(1 - x)$, where the scaled photon energy $x$ is defined as $x \equiv 2 E_\gamma m_b/(m_b^2 - m_s^2)$. The physical photon energy spectrum is obtained by convoluting the non-perturbative effects and the perturbative QCD corrections, such as the ones arising from the decay $b \to s \gamma + g$. The latter gives a characteristic Bremstrahlung spectrum in $x$ in the interval $[0, 1]$ peaking near the end-points, $E_\gamma \to E_{\gamma\text{max}}$ (or $x \to 1$) and $E_\gamma \to 0$ (or $x \to 0$), arising from the soft-gluon and soft-photon configurations, respectively. Near the end-points, one has to improve the spectrum obtained in fixed order perturbation theory. This is done in the region $x \to 1$ by isolating and exponentiating the leading behaviour in $\alpha_{em} \alpha_s(\mu)^m \log^{2n}(1 - x)$ with $m \leq n$, where $\mu$ is a typical momentum in the decay $B \to X_s + \gamma$. In this region, which is dominated by the magnetic moment operator $O_7$, the spectrum can be symbolically expressed as

\begin{equation}
\frac{d\Gamma(\gamma)}{dx} = -C \frac{\alpha_s(\mu)}{3\pi} (C_{eff}^{\gamma f f})^2 \exp \left( \frac{\alpha_s(\mu)}{3\pi} \Omega_1 \right) \left[ \Omega'_2 + \Omega'_1 \left( 1 + \frac{\alpha_s(\mu)}{3\pi} \Omega_2 \right) \right],
\end{equation}

(17)
where $C$ is a normalization constant ($r \equiv m_s/m_b$).

$$C = (1 - r)^3(1 + r)\frac{\overline{m}_m = m_b^2 m_s (p o l e)^3}{32\pi^4} \alpha G_F^2 |\lambda|^2$$

The running of $\alpha_s$ is a non-leading effect, but as it is characteristic of QCD it modifies the Sudakov-improved end-point photon energy spectrum compared to its analogue in QED. The expressions for $\Omega_1(x, r)$, $\Omega_2(x, r)$ and their derivatives (denoted by primes) can be seen for non-zero $s$-quark mass in literature. The expressions (with $r \to 0$) are:

$$\Omega_1 = -2 \ln^2(1 - x) - 7 \ln(1 - x),$$

$$\Omega_2 = 10(1 - x) + (1 - x)^2 - \frac{2}{3}(1 - x)^3 - (1 - x)(3 + x) \ln(1 - x),$$

where the double logarithmic term in $\Omega_1$ is universal. The other terms are specific to the decay $B \to X_s + \gamma$. As long as the $s$-quark mass is non-zero, there is no collinear singularity in the spectrum. However, parts of the spectrum have large logarithms of the form $\alpha_s \log(m_b^2/m_s^2)$, which are important near the end-point $x \to 0$ and are, in principle, present for any photon energy and should be resummed. The order $\alpha_s$ corrected distribution for low photon energies is dominated by the operator $O_8$ and is given by:

$$\frac{d\Gamma_{88}}{dx}\bigg|_{x>0} = C \frac{\alpha_s(\mu)}{3\pi} \frac{(C_s^{eff})^2}{9} \tilde{\Gamma}(x, r),$$

where

$$\tilde{\Gamma}(x, r) = \left( \frac{4 + 4r}{x - rx} - 4 + 2r \right) \ln \frac{1 - x + rx}{r}$$

$$= \frac{(1 - x) \left[ 8 - (1 - r)x(16 - 9x + 7rx) + (1 - r)^2x^2(1 - 2x) \right]}{x(1 - x + rx)^2}.$$

The difference between the fixed order spectrum and its exponentiated version can be seen in the work of Kapustin et al. If low energy photons can be detected in $B \to X_s + \gamma$ (say, for $E_\gamma \leq 1$ GeV), then such an experiment could help measure $C_s^{eff}$. Unfortunately, with the SM values of the Wilson coefficients, the partial branching ratio for $B \to X_s + \gamma$ for low photon energies is too small to be measured even at $B$ factories. However, with an anomalously large $C_s^{eff}$, as has been entertained in the literature in other contexts, the $C_s^{eff}$-dependent part of the photon energy spectrum may get appreciably enhanced. It is worthwhile to measure the spectrum in the intermediate energies.
(1.0 GeV ≤ 2.0 GeV) to search for the effect of such anomalously enhanced $C_S^{eff}$ contribution in beyond-the-SM scenarios.

Implementation of non-perturbative effects is at present a model dependent enterprise and data are not precise enough to distinguish various models proposed in the literature\textsuperscript{17,37,42}. We shall confine ourselves to the discussion of the photon energy spectrum calculated in a simple model, in which the $b$ quark in the $B$ hadron is assumed to have a Gaussian distributed Fermi motion\textsuperscript{44} determined by a non-perturbative parameter, $p_F$. This model describes well the lepton energy spectrum in semileptonic decays $B \to X\ell\nu\ell$ and it has also received some theoretical support in the HQET approach subsequently.

The photon energy spectrum based on this model, including the QCD perturbative improvements, has been used both by the CLEO\textsuperscript{2} and ALEPH collaboration\textsuperscript{4} in the analysis of their data on $B \to X_s + \gamma$. An analysis of the CLEO photon energy spectrum has also been undertaken\textsuperscript{17} to determine the non-perturbative parameters of this model, namely $m_b(\text{pole})$ and $p_F$. The latter is related to the kinetic energy parameter $\lambda_1$ defined earlier in the HQET approach. The minimum $\chi^2$ of the CLEO data is obtained for $p_F = 450$ MeV and $m_b(\text{pole}) = 4.77$ GeV. However, the ±$1\sigma$ errors on these quantities are large (a similar conclusion has been drawn in terms of $\lambda_1$ and $m_b(\text{pole})$ by Li and Yu\textsuperscript{42}). The interesting question here is to determine if the non-perturbative aspects in the decays $B \to X\ell\nu\ell$ and $B \to X_s + \gamma$ can be described in terms of a universal shape function. To pursue this further requires lot more data which we hope will soon be forthcoming.

3 Inclusive radiative decay $B \to X_d + \gamma$ and constraints on the CKM parameters

The quantity of interest in the decay $B \to X_d + \gamma$ is the high energy part of the photon energy spectrum, which has to be measured requiring that the hadronic system $X_d$ recoiling against the photon does not contain strange hadrons to suppress the large-$E_\gamma$ photons from the decay $B \to X_s + \gamma$, which is now the largest background. Assuming that such an experiment is feasible, one can determine from the ratio of the branching ratios $B(B \to X_d + \gamma)/B(B \to X_s + \gamma)$ the parameters of the CKM matrix (in particular $\rho$ and $\eta$ in the Wolfenstein parameterization\textsuperscript{43}).

In close analogy with the $B \to X_s + \gamma$ case discussed earlier, the complete set of dimension-6 operators relevant for the processes $b \to d\gamma$ and $b \to d\gamma g$
can be written as:

\[ \mathcal{H}_{\text{eff}}(b \to d) = -\frac{4G_F}{\sqrt{2}} \xi_t \sum_{j=1}^{8} C_j(\mu) \hat{O}_j(\mu), \]  

(22)

where \( \xi_j = V_{jb} V_{jd}^* \) with \( j = u, c, t \). The operators \( \hat{O}_j, \ j = 1, 2 \), have implicit in them CKM factors. We shall use the Wolfenstein parametrization\(^{43}\), in which case the matrix is determined in terms of the four parameters \( A, \lambda = \sin \theta_C, \rho \) and \( \eta \), and one can express the above factors as:

\[ \xi_u = A \lambda^3 (\rho - i\eta), \quad \xi_c = -A \lambda^3, \quad \xi_t = -\xi_u - \xi_c. \]  

(23)

We note that all three CKM-angle-dependent quantities \( \xi_j \) are of the same order of magnitude, \( O(\lambda^3) \). It is convenient to define the operators \( \hat{O}_1 \) and \( \hat{O}_2 \) entering in \( \mathcal{H}_{\text{eff}}(b \to d) \) as follows\(^{16}\):

\[ \hat{O}_1 = -\frac{\xi_c}{\xi_t} (\bar{c}_L \gamma^\mu b_L)(\bar{d}_L \gamma_\mu c_L) - \frac{\xi_u}{\xi_t} (\bar{u}_L \gamma^\mu b_L)(\bar{d}_L \gamma_\mu u_L), \]

\[ \hat{O}_2 = -\frac{\xi_c}{\xi_t} (\bar{c}_L \gamma^\mu b_L)(\bar{d}_L \gamma_\mu c_L) + \frac{\xi_u}{\xi_t} (\bar{u}_L \gamma^\mu b_L)(\bar{d}_L \gamma_\mu u_L), \]  

(24)

with the rest of the operators \( (\hat{O}_j; \ j = 3...8) \) defined like their counterparts \( O_j \) in \( \mathcal{H}_{\text{eff}}(b \to s) \), with the obvious replacement \( s \to d \). With this choice, the matching conditions \( C_j(m_W) \) and the solutions of the RG equations yielding \( C_j(\mu) \) become identical for the two operator bases \( O_j \) and \( \hat{O}_j \). The branching ratio \( B(B \to X_d + \gamma) \) in the SM can be generally written as:

\[ B(B \to X_d + \gamma) = D_1 \lambda^2 \]

\[ \{(1 - \rho)^2 + \eta^2 - (1 - \rho)D_2 - \eta D_3 + D_4\}, \]  

(25)

where the functions \( D_i \) depend on various parameters such as \( m_t, m_b, m_c, \mu \), and \( \alpha_s \). These functions were calculated in the LL approximation some time ago\(^{16}\) and since then their estimates have been improved\(^{45}\), making use of the NLO calculations discussed earlier. We shall assume, based on model calculations\(^{46,47,48}\) and the \( 1/m_c^2 \) power corrections discussed earlier in the context of \( B \to X_s + \gamma \), that the LD contributions are small also in \( B(B \to X_d + \gamma) \).

To get an estimate of \( B(B \to X_d + \gamma) \) at present, the CKM parameters \( \rho \) and \( \eta \) have to be constrained from the unitarity fits, which yield the following ranges (at 95% C.L.)\(^{49}\):

\[ 0.20 \leq \eta \leq 0.52, \]

\[ -0.35 \leq \rho \leq 0.35. \]  

(26)
The (nominally) preferred CKM-fit values at present are \((\rho, \eta) = (0.05, 0.36)\), for which one gets
\[
B(B \to X_d + \gamma) = (1.63 \pm 0.16) \times 10^{-5},
\]
where the error estimate follows from the one for \(B(B \to X_s + \gamma)\). Allowing the CKM parameters to vary over the entire allowed domain, one gets (at 95% C.L.)
\[
6.0 \times 10^{-6} \leq B(B \to X_d + \gamma) \leq 3.0 \times 10^{-5}.
\]
The present theoretical uncertainty in this rate is a factor 5, which shows that even a modest measurement of \(B(B \to X_d + \gamma)\) will have a very significant impact on the CKM phenomenology. To the best of our knowledge, there is no experimental bound available on \(B(B \to X_s + \gamma)\), but we hope that this decay will be measured in future at the B factories and CLEO.

4 CKM-suppressed exclusive decays \(B(B \to V + \gamma)\)

Exclusive radiative B decays \(B \to V + \gamma\), with \(V = K^*, \rho, \omega\), are also potentially very interesting for the CKM phenomenology. Extraction of CKM parameters would, however, involve a trustworthy estimate of the SD- and LD-contributions in the decay amplitudes. We have argued that the decays \(B \to X_s + \gamma\) and \((B^\pm, B^0) \to (K^{*\pm}, K^{*0}) + \gamma\) are consistent with the dominance of the SD-contribution. There exist good reasons to believe that also the CKM-suppressed exclusive radiative decays are dominated by SD-physics, though one has to work out the LD-contribution on a case-by-case basis. More importantly, data on the various charged and neutral B meson radiative decays can be used directly to put meaningful bounds on the LD-contributions. Hence, despite skepticism in some quarters and in this conference, we believe that exclusive radiative B decays are worth measuring.

The SD-contribution in the exclusive decays \((B^\pm, B^0) \to (K^{*\pm}, K^{*0}) + \gamma, (B^\pm, B^0) \to (\rho^\pm, \rho^0) + \gamma, B^0 \to \omega + \gamma\) and the corresponding \(B_s\) decays, \(B_s \to \phi + \gamma\), and \(B_s \to K^{*0} + \gamma\), involve the magnetic moment operator \(\mathcal{O}_7\) and the related one obtained by the obvious change \(s \to d, \hat{O}_7\). The transition form factors governing the radiative B decays \(B \to V + \gamma\) can be generically defined as:
\[
\langle V, \lambda | \frac{1}{2} \bar{\psi} \gamma_\mu q \gamma^\nu b | B \rangle = i \epsilon_{\mu \nu \rho \sigma} e^{(\lambda)}_{\nu} p^\rho f^\sigma_{B \to V} (0).
\]
Here \(V\) is a vector meson with the polarization vector \(e^{(\lambda)}\) and \(\psi\) stands for the field of a light \(u, d\) or \(s\) quark. In (29) the QCD renormalization of the
The operator is implied. Keeping only the SD-contribution leads to obvious relations among the exclusive decay rates, exemplified here by the decay rates for \((B^\pm, B^0) \to \rho + \gamma\) and \((B^\pm, B^0) \to K^* + \gamma\):

\[
\Gamma((B^\pm, B^0) \to (\rho^\pm, \rho^0) + \gamma) \approx \kappa_{u,d} \left| \frac{|V_{ud}|}{|V_{ts}|} \right|^2, \quad \text{(30)}
\]

where \(\kappa_i \equiv [F_S(B_i \to \rho\gamma)/F_S(B_i \to K^*\gamma)]^2\), which is unity in the \(SU(3)\) limit. (This is not being recommended as the \(SU(3)\)-breaking effects have been calculated in a number of papers\(^{34,35}\).) Likewise, assuming dominance of SD physics gives relations among various decay rates

\[
\Gamma(B^\pm \to \rho^\pm \gamma) = 2 \Gamma(B^0 \to \rho^0 \gamma) = 2 \Gamma(B^0 \to \omega \gamma), \quad \text{(31)}
\]

where the first equality holds due to the isospin invariance, and in the second \(SU(3)\) symmetry has been assumed.

The LD-amplitudes in radiative \(B\) decays from the light quark intermediate states necessarily involve other CKM matrix elements. In the CKM-suppressed decays \(B \to V + \gamma\) they are dominantly induced by the matrix elements of the four-Fermion operators \(\hat{O}_1\) and \(\hat{O}_2\). Estimates of these contributions have been obtained in the light-cone QCD sum rule approach\(^{51,52}\). Using factorization, the LD-amplitude in the decay \(B^\pm \to \rho^\pm + \gamma\) can be written in terms of the form factors \(F^L_1\) and \(F^L_2\),

\[
A_{long} = -\frac{eG_F V_{ub} V^*_{us}}{\sqrt{2}} C_2 \left[ C_1 + \frac{1}{N_c} C_5 \right] m_{\rho} \varepsilon^{\rho}_{\nu}(\rho) \varepsilon^{\nu}_{\mu}(\mu) \times \left\{ -i \left[ g^\mu(\rho \cdot p) - \ddot{p}^\mu q^\nu \right] \cdot 2 F^L_1(q^2) + \epsilon^\rho_{\mu\alpha\beta} p_{\alpha} q_{\beta} \cdot 2 F^L_2(q^2) \right\}. \quad \text{(32)}
\]

The two form factors are found to be numerically close to each other in the QCD sum rule approach, \(F^L_1 \approx F^L_2 \equiv F^L\), hence the ratio of the LD- and the SD-contributions reduces to a number\(^{52}\)

\[
A_{long}/A_{short} = R^{B^\pm \to \rho^\pm \gamma}_{L/S} \equiv \frac{V_{ub} V^*_{us}}{V_{tb} V^*_{ts}}. \quad \text{(33)}
\]

where

\[
R^{B^\pm \to \rho^\pm \gamma}_{L/S} \equiv \frac{4\pi^2 m_{\rho}(C_2 + C_1/N_c)}{m_b C_7 e^\gamma} \frac{F^{B^\pm \to \rho^\pm \gamma}}{F^{B^\pm \to \rho^0 \gamma}} = -0.30 \pm 0.07. \quad \text{(34)}
\]

The analogous LD-contributions to the neutral \(B\) decays \(B^0 \to \rho \gamma\) and \(B^0 \to \omega \gamma\) are expected to be much smaller:

\[
\frac{R^{B^0 \to \rho \gamma}_{L/S}}{R^{B^\pm \to \rho^\pm \gamma}_{L/S}} = \frac{e_d a_2}{e_u a_1} \simeq -0.13 \pm 0.05, \quad \text{(35)}
\]
where the numbers are based on using \( a_2/a_1 = 0.27 \pm 0.10 \) and \( e_d/e_u = -1/2 \)

is the ratio of the electric charges for the \( d \)- and \( u \)-quarks. This would then yield \( R_{L/S}^{B^0 \to \rho^0} \simeq R_{L/S}^{B^0 \to \omega^0} = 0.05 \).

To get a ball-park estimate of the ratio \( A_{\text{long}}/A_{\text{short}} \), we take the central

value from the CKM fits, yielding \( |V_{ub}|/|V_{td}| \simeq 0.33 \), which in turn gives,

\[
\left| A_{\text{long}}/A_{\text{short}} \right|_{B^0 \to \rho^0 \gamma} \simeq 0.1 ,
\]

\[
\frac{A_{\text{long}}^{B^0 \to \rho^0 \gamma}}{A_{\text{short}}^{B^0 \to \rho^0 \gamma}} \leq 0.02 .
\]

That the LD-effects remain small in \( B^0 \to \rho \gamma \) decay has also been supported

in an analysis based on the soft-scattering of on-shell hadronic decay products

\( B^0 \to \rho^0 \rho^0 \to \rho \gamma \) \cite{54}, though this paper estimates them somewhat higher

(between 4\% and 8\%).

The relations (31), which obtain ignoring LD-contributions, get modified

by including the LD-contributions to

\[
\frac{\Gamma(B^0 \to K^0 \gamma)}{2\Gamma(B^0 \to \rho^0 \gamma)} = \frac{\Gamma(B^0 \to K^0 \gamma)}{2\Gamma(B^0 \to \omega^0 \gamma)} = 1 + \Delta(R_{L/S}) ,
\]

where \( R_{L/S} \equiv R_{L/S}^{B^0 \to \rho^0 \gamma} \)

\[
\Delta(R_{L/S}) = 2 R_{L/S} V_{ud} \frac{\rho(1 - \rho) - \eta^2}{(1 - \rho)^2 + \eta^2} + (R_{L/S})^2 V_{ud} \frac{\rho^2 + \eta^2}{(1 - \rho)^2 + \eta^2} .
\]

The ratio of the CKM-suppressed and CKM-allowed decay rates for charged

\( B \) mesons gets likewise modified due to the LD contributions. Following earlier
discussion, we ignore the LD-contributions in \( \Gamma(B \to K^* \gamma) \). The ratio of the
decay rates in question can therefore be written as:

\[
\frac{\Gamma(B^\pm \to \rho^\pm \gamma)}{\Gamma(B^\pm \to K^{\pm \star} \gamma)} = \kappa_u \lambda^2 [(1 - \rho)^2 + \eta^2] (1 + \Delta(R_{L/S}))
\]

The effect of the LD-contributions is modest but not negligible, introducing an
uncertainty comparable to the \( \sim 15\% \) uncertainty in the overall normalization
due to the \( SU(3) \)-breaking effects in the quantity \( \kappa_u \).

Neutral \( B \)-meson radiative decays are less-prone to the LD-effects, as ar-
gued above, and hence one expects that to a good approximation (say, better
than 10%) the ratio of the decay rates for neutral $B$ meson obtained in the approximation of SD-dominance remains valid:\(^{35}\)

$$\frac{\Gamma(B^0 \to \rho \gamma, \omega \gamma)}{\Gamma(B \to K^* \gamma)} = \kappa_d \lambda^2 [(1 - \rho)^2 + \eta^2], \quad (40)$$

where this relation holds for each of the two decay modes separately. This ratio is at par with the mass-difference ratio $\Delta M_d/\Delta M_s$ in the neutral $B$-meson sector, as both involve a reliable estimate of the $SU(3)$-breaking effects but otherwise reflect the dominance of the SD-physics.

Finally, combining the estimates for the LD- and SD-form factors\(^{52,35}\), and restricting the Wolfenstein parameters in the allowed range given earlier, yields

$$B(B^{\pm} \to \rho^{\pm} \gamma) = (1.5 \pm 1.1) \times 10^{-6},$$

$$B(B^0 \to \rho \gamma) \approx B(B^0 \to \omega \gamma) = (0.65 \pm 0.35) \times 10^{-6}, \quad (41)$$

where we have used the experimental value for the branching ratio $B(B \to K^* + \gamma)$\(^{4}\). The large range reflects to a large extent the poor knowledge of the CKM matrix elements and hence experimental measurements of these branching ratios will contribute greatly to determine the Wolfenstein parameter $\rho$ and $\eta$. Present experimental limits (at 90% C.L.) are\(^3\): $B(B^{\pm} \to \rho^{\pm} \gamma) < 1.1 \times 10^{-5}$, $B(B^0 \to \rho \gamma) < 3.9 \times 10^{-5}$ and $B(B^0 \to \omega \gamma) < 1.3 \times 10^{-5}$. The constraints on the parameters ($\rho, \eta$) following from them are, however, not yet competitive to the one following from unitarity and lower bound on the mass difference in the $B^0_s - \overline{B}^0_s$ sector\(^{49}\).

5 Inclusive rare decays $B \to X_s \ell^+ \ell^-$ in the SM

The decays $B \to X_s \ell^+ \ell^-$, with $\ell = e, \mu, \tau$, provide new avenues to search for physics beyond the standard model\(^{55,56,57,58,59}\). The branching ratio $B(B \to X_s + \gamma)$ constrains the magnitude of $C^{\text{eff}}_7$ but the sign of $C^{\text{eff}}_7$ is not determined by the measurement of $B(B \to X_s + \gamma)$. This sign is in general model dependent. It is known that in SUSY models, both the negative and positive signs are allowed as one scans over the allowed SUSY parameter space. The $B \to X_s \ell^+ \ell^-$ amplitude in the standard model (as well as in several extensions of it such as SUSY) depends on the coefficient $C^{\text{eff}}_7$ and additionally on the coefficients of two four-Fermi operators, $C_9$ and $C_{10}$. It has been argued\(^{56}\) that the signs and magnitudes of all three coefficients $C^{\text{eff}}_7$, $C_9$ and $C_{10}$ can, in principle, be determined from the decays $B \to X_s + \gamma$ and $B \to X_s \ell^+ \ell^-$ by
measuring the dilepton invariant mass and Forward-Backward charged lepton asymmetry.

The SM-based rates for the decay $b \to s \ell^+ \ell^-$, calculated in the free quark decay approximation, have been known in the LO approximation for some time. The NLO contribution reduces the scheme-dependence of the LO result in these decays. In addition, long-distance (LD) effects, which are expected to be very important in the decay $B \to X_s \ell^+ \ell^-$, have also been estimated from data on the assumption that they arise dominantly due to the charmonium resonances $J/\psi, \psi', ...$ through the decay chains $B \to X_s J/\psi(\psi', ...) \to X_s \ell^+ \ell^-$. The resulting dilepton distribution is then a coherent sum of the resonating (LD) and mildly varying (SD) contributions. Data near the resonances can be used to better parametrize the LD contribution in future than is the case now. Recently, these LD-contributions have also been predicated upon assuming that far from the resonant-region they can be determined in terms of the $1/m_b^2$ contributions in the HQET approach. This remains an interesting conjecture but impossible to test experimentally, as, first of all, they represent just a class of power corrections and, more importantly, this theoretical framework breaks down near the resonances.

The leading ($1/m_b^2$) power corrections to the partonic decay rate and the dilepton invariant mass distribution have been calculated in the HQET approach; these results have, however, not been confirmed in a recent independent calculation, which finds that the power corrections in the branching ratio $B(B \to X_s \ell^+ \ell^-)$ are small (typically $-1.5\%$). The corrections in the dilepton mass spectrum and the FB asymmetry are also small over a good part of this spectrum. However, the end-point dilepton invariant mass spectrum is not calculable in the heavy quark expansion and will have to be modeled. As an alternative, non-perturbative effects in $B \to X_s \ell^+ \ell^-$ have also been estimated using the Fermi motion model discussed earlier. These effects, which model power corrections in $1/m_b$, are also found to be small over most of the phase space except for the end-point dilepton mass spectrum where they change the underlying parton model distributions significantly and have to be taken into account in the analysis of data.

Taking into account the spread in the values of the input parameters, $\mu$, $\Lambda$, $m_t$, and $B_{SL}$ discussed in the previous section in the context of $B(B \to X_s + \gamma)$, the following branching ratios for the SD-piece have been estimated:

\begin{align*}
B(B \to X_s e^+ e^-) &= (8.4 \pm 2.3) \times 10^{-6}, \\
B(B \to X_s \mu^+ \mu^-) &= (5.7 \pm 1.2) \times 10^{-6}, \\
B(B \to X_s \tau^+ \tau^-) &= (2.6 \pm 0.5) \times 10^{-7},
\end{align*}

(42)
where theoretical errors and the error on $B_{SL}$ have been added in quadrature. The experimental upper limit for the inclusive branching ratio for the decay $B \to X_s \mu^+ \mu^-$ was quoted by the UA1 collaboration some time ago\textsuperscript{65}, $B(B \to X_s \mu^+ \mu^-) < 5.0 \times 10^{-5}$. This limit has been put to question in a recent CLEO paper\textsuperscript{66}. From the CLEO data, a limit $B(B \to X_s \mu^+ \mu^-) < 5.8 \times 10^{-5}$ has been set on this decay, with $B(B \to X_s e^+ e^-) < 5.7 \times 10^{-5}$. Combining the di-electron and di-muon upper limits, CLEO quotes $B(B \to X_s \ell^+ \ell^-) < 4.2 \times 10^{-5}$ (all limits are at 90% C.L.). This is a factor 6 away from the SM estimates. As far as we know, there is no experimental limit on the mode $X_s \tau^+ \tau^-$. For a more detailed discussion of the modes $B \to X_s \ell^+ \ell^-$, some related exclusive decays such as $B \to (K, K^*) \ell^+ \ell^-$, and other rare $B$-decay modes, we refer to recent reviews\textsuperscript{8,67}. The CKM-suppressed decays $B \to X_d \ell^+ \ell^-$, which are expected to be typically a factor 20 below their corresponding CKM-allowed decay rates, and their role in determining the CKM parameters have been discussed elsewhere\textsuperscript{68,69}.

Acknowledgments

I would like to thank Christoph Greub, Jim Smith, Tomasz Skwarnicki and Mark Williams for helpful discussions. I also thank Mark for sending me a copy of the ALEPH paper on $b \to s + \gamma$ submitted to the EPS conference. The warm hospitality of the organizing committee is greatly appreciated.

References

3. T. Skwarnicki, these proceedings.
4. “A measurement of the inclusive $b \to s \gamma$ branching ratio”, P.G. Colrain and M.I. Williams (ALEPH Collaboration), contributed paper \# 587, EPS Conference, Jerusalem, August 1997.
25. T. Mannel, these proceedings.
32. M. Narain, these proceedings.
33. L. Gibbons, these proceedings.
34. P. Ball, TU-München Report TUM-T31-43/93 (1993);
36. J. Flynn, these proceedings.
45. A. Ali, H.M. Asatrian, and C. Greub, to be published.
50. A.J. Buras, these proceedings.
59. J. Hewett, these proceedings.
66. S. Glenn et al. (CLEO Collaboration), CLNS 97/1514; CLEO 97-21.
69. A.I. Sanda, these proceedings.