$N = 1$ Dual String Pairs and their Modular Superpotentials

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We review the duality between heterotic and F-theory string vacua with $N = 1$ space-time supersymmetry in eight, six and four dimensions. In particular, we discuss two chains of four-dimensional F-theory/heterotic dual string pairs, where F-theory is compactified on certain elliptic Calabi-Yau fourfolds, and the dual heterotic vacua are given by compactifications on elliptic Calabi-Yau threefolds plus the specification of the $E_8 \times E_8$ gauge bundles. We show that the massless spectra of the dual pairs agree by using, for one chain of models, an index formula to count the heterotic bundle moduli and determine the dual F-theory spectra from the Hodge numbers of the fourfolds and of the type IIB base spaces. Moreover as a further check, we demonstrate that for one particular heterotic/F-theory dual pair the $N = 1$ superpotentials are the same.

1. Introduction

Several types of strong-weak coupling duality symmetries in string theories were explored and established during recent years \cite{1-4}. As a result of these non-perturbative investigations it seems nowadays clear that all five known consistent string constructions, the two heterotic strings, the type IIA,B superstrings and the type I superstring, are related by some kind of duality transformation such that they are all equivalent and on equal footing. Moreover it was a very important and fascinating observation \cite{2} that a strongly coupled string theory, after including non-perturbative states, may look like a theory which lives in higher dimensions. Specifically, the Dirichlet (D) 0-branes of the IIA superstring behave in the same way as the Kaluza-Klein states of 11-dimensional supergravity, and in fact there is very strong evidence that the type IIA string in ten dimensions is dual to 11-dimensional supergravity compactified on a circle of radius $R$.

The full quantum version of 11-dimensional supergravity is now called $M$-theory; $M$-theory on circle of radius $R$ is supposed to describe the full strongly coupled type IIA superstring. However a fundamental definition of $M$-theory is up to now not clear. One promising attempt to define $M$-theory is given by a Hamiltonian theory of non-commuting matrices, known as M(atrix)-theory \cite{5}.

In \cite{6} Vafa made the proposal that non-perturbative string physics can lead to dimensions even beyond 11, namely he conjectured that the IIB superstring theory should be regarded as the toroidal compactification of twelve-dimensional F-theory. This would lead to supergravity theories with two time and ten spatial directions which are hardly understood (for progress along these lines see \cite{7}). However, if one is slightly less ambitious and does not want to enter the enormous difficulties in formulating a consistent 12-dimensional theory, F-theory can be regarded as to provide new non-perturbative type IIB vacua on D-manifolds in which the complexified coupling varies over the internal space. These compactifications then have a beautiful geometric interpretation as compactifications of F-theory on elliptically fibred manifolds, where the fibre encodes the behaviour of the coupling, the base is the D-manifold, and the points where the fibre degenerates specifies the positions of the D 7-branes in it. Compactifications of F-theory on elliptic Calabi-Yau twofolds (the K3), threefolds and fourfolds can be argued \cite{6} to be dual to certain heterotic string theories in 8,6 and 4 dimensions and have provided new insights into the re-
motion between geometric singularities and perturbative as well as non-perturbative gauge symmetry enhancement and into the structure of moduli spaces.

Let us consider the duality between the heterotic string and F-theory in slightly more detail. To compactify F-theory down to D dimensions we are dealing with a Calabi-Yau space of complex dimension \(6 - \frac{D}{2}\), called \(X^{6-\frac{D}{2}}\), which is elliptically fibred over a complex base \(B^{5-\frac{D}{2}}\) of complex dimension \(5 - \frac{D}{2}\). This non-Calabi-Yau space \(B^{5-\frac{D}{2}}\) is the type IIB compactification space from ten to \(D\) dimensions. Moreover one has to assume the \(X^{6-\frac{D}{2}}\) exists a fibration \(\mathcal{B}\) over the two dimensional surface \(B^{5-\frac{D}{2}}\) because one has to assume the \(X^{6-\frac{D}{2}}\) to be a K3 fibration over the two dimensional surface \(B^{4-\frac{D}{2}}\). So, we have the following fibration structure for \(X^{6-\frac{D}{2}}\):

\[
X^{6-\frac{D}{2}} \rightarrow T^2 \ B^{5-\frac{D}{2}} \rightarrow T^2 \ B^{4-\frac{D}{2}}. \tag{1}
\]

Then, using the eight-dimensional duality among F-theory on an elliptic K3 and the heterotic string on \(T^2\), one derives by an adiabatic argument that the dual heterotic vacua are given by Calabi-Yau compactifications on a threefold \(Z^{3-\frac{D}{2}}\), which is an elliptic fibration over the same surface \(B^{1-\frac{D}{2}}\):

\[
Z^{3-\frac{D}{2}} \rightarrow T^2 \ B^{2-\frac{D}{2}}. \tag{2}
\]

and where the choice of an \(E_8 \times E_8\) gauge vector bundle \(V_1 \times V_2\) has to be specified in order to match all F-theory fibration data.

In this paper we will consider \(D\)-dimensional dual pairs with \(N = 1\) space-time supersymmetry. Specifically, in the next section we briefly discuss \(D = 8\), i.e. F-theory on K3 being dual to the heterotic string on \(T^2\), then in section three we review \(D = 6\) F-theory on Calabi-Yau threefolds \(X^3\) [8] which is dual to the heterotic string on K3 and finally, the main part of our paper, we discuss \(D = 4\) F-theory on fourfolds \(X^4\) [9-13] and their heterotic duals on Calabi-Yau threefolds \(Z^3\). On the F-theory side as well as on the heterotic side the geometry of the considered manifolds relies on del Pezzo surfaces. To establish the four-dimensional F-theory/heterotic duality we match the massless spectra [12, 13] by using, for one chain of models, an index formula [11] to count the heterotic bundle moduli and determine the dual F-theory spectra from the Hodge numbers of the fourfolds \(X^4\) and of the type IIB base spaces. Moreover, for models which we construct by a \(\mathbb{Z}_2\) modding of dual 4D pairs with \(N = 2\) space-time supersymmetry we can show that the F-theory/heterotic \(N = 1\) superpotentials agree [12]. Here the superpotentials, which were computed on the F-theory side in [14,15], are given by certain modular functions, similar to the prepotentials which appear in \(N = 2\) dual typeII/heterotic string pairs [3,16].

2. Eight Dimensions

The heterotic string compactified on \(T^2\) to eight dimensions is well known. The heterotic moduli are given by 18 complex parameters which divide themselves into the K"ahler class plus complex structure of \(T^2\), called \(T\) and \(U\), and 16 Wilson line moduli, which determine the \(E_8 \times E_8\) gauge bundle, i.e. the unbroken gauge group in eight dimensions. The moduli space of the 18 complex moduli is given by the well known Narain case [17]

\[
\mathcal{M} = \frac{SO(2, 18)}{SO(2) \times SO(18)}, \tag{3}
\]

where the discrete \(T\)-duality group \(SO(2, 18, \mathbb{Z})\) has still to be modded out. At special points/loci the generic Abelian gauge group \(U(1)^{18} \times U(1)^2\) can be enhanced; e.g. at the line \(T = U\) there is an \(SU(2)\) gauge symmetry enhancement and for tuned Wilson lines one recovers the full \(E_8 \times E_8\) gauge symmetry.

The dual type IIB compactification to eight dimensions on the space \(B^1 = P^1\) with complex coordinate \(z\) is characterized by a complex dilaton field \(\tau\) which varies holomorphically over \(z\), i.e. \(\tau = \tau(z)\). To fulfill the corresponding equations of motion a non-perturbative background of 24 7-branes has to be turned on. Then Vafa has argued [6] that the heterotic string compactified on a two-torus in the presence of Wilson lines is dual to F-theory compactified on the family

\[
y^2 = x^3 + f_8(z)x + g_{12}(z) \tag{4}
\]
of elliptic K3 surfaces, where $f_5(z), g_2(z)$ are polynomials of order 8, 12 respectively. In particular F-theory on the two parameter subfamily

$$y^2 = x^3 + \alpha x^2 + (z^5 + \beta z^6 + z^7)$$

(5)

of K3’s with $E_8$ singularities at $z = 0, \infty$ is dual to the heterotic theory with Wilson lines switched off [8]. Therefore there must exist a map which relates the complex structure and Kähler moduli $U$ and $T$ of the torus on which the heterotic theory is compactified to the two complex structure moduli $\alpha$ and $\beta$ in (5). This map can be worked out explicitly, and one gets [18]

$$j(T)j(U) = -1728\frac{g_3^3}{27},$$

(6)

$$(j(T) - 1728)(j(U) - 1728) = 1728\frac{g_2^2}{4}(\alpha \beta) \cdot (\beta - \alpha)(\beta - \alpha).$$

Let us remark that our result can be regarded as the two-parameter generalization of the one-parameter torus in the well-known Weierstrass form

$$g_2 = \frac{1}{3} \pi^4 E_4, \quad g_3 = \frac{1}{27} \pi^6 E_6$$

$$g_2^2 = 4x^3 - g_2(\tau) x - g_3(\tau).$$

As a check of the result eq.(6) we compute the discriminant of the K3 eq. (5):

$$\Delta^{(K3)} = \left(\alpha^3 + \frac{27}{4} \beta^2 + 27\right)^2 - 27^2 \beta^2.$$  (8)

It is easy to see that $\Delta^{(K3)} = 0$ at the line $T = U$ of $SU(2)$ gauge symmetry enhancement. So the K3 becomes singular at this locus, and as long as both $\alpha$ and $\beta$ are not zero, the singularity is of type $A_1$.

3. Six Dimensions

Here we consider first the heterotic string on K3. The heterotic gauge bundle is specified by the numbers $(n_1, n_2)$ of $E_8 \times E_8$ instantons turned on. In addition one can consider also the non-perturbative background of $n_5$ heterotic 5-branes [19]. The choice of the three integers $n_1, n_2, n_5$ is restricted by the following Green-Schwarz anomaly matching condition

$$n_1 + n_2 + n_5 = 24.$$  (9)

The massless spectrum of the resulting six-dimensional $N = 1$ supergravity is then determined as follows. Besides the supergravity multiplet, we have first $N_H$,

$$N_H = 20 + \text{dim}_{\mathbb{Q}} M_{\text{inst}} + n_5,$$  (10)

hypermultiplet moduli fields which parametrize the Higgs branch of the theory. Here the 20 counts the number of $K3$ moduli, the second term in eq. (10) denotes the dimension of the quaternionic instanton moduli space of the embedded $E_8$ instantons, and the last term arises, since the position of each five brane on K3 is parametrized by a hypermultiplet. The dilaton $\phi$ plus the self-dual antisymmetric tensor field $B^+_{\mu\nu}$ are members of a tensor multiplet. In addition, considering the non-perturbative contribution of heterotic 5-branes, there are additional tensor multiplets in the massless spectrum [19], since on the world sheet theory of the 5-brane lives a massless tensor field. Hence the total number $N_T$ of six-dimensional tensor fields is

$$N_T = 1 + n_5.$$  (11)

Thus the Coulomb branch of the six-dimensional theory is characterized by a real $(1 + n_5)$-dimensional moduli space, parametrized by the scalar field vev’s of the tensor multiplets. Finally, one gets massless vector multiplets associated to the part of the gauge group $G_1 \times G_2$ of rank $r(V)$ which is left unbroken by the instantons. So $N_V = \text{dim}(G_1 \times G_2)$. If $n_a \geq 10$ ($a = 1, 2$) the gauge group $G_a$ is completely broken at a generic point in the hypermultiplet moduli space, whereas for smaller values of $n_a$, there will be always an unbroken non-Abelian gauge symmetry. Note that at special loci in the hypermultiplet moduli space the instantons may fit into a smaller subgroup of $E_8$ and the gauge group $G_a$ can be enhanced, which is just the (reverse) Higgs effect in field theory.

Let us now discuss F-theory compactifications on Calabi-Yau threefolds $X^3$ which are dual to the heterotic K3 compactifications just discussed before. $X^3$ has to be an elliptic fibration over a base $B^2$, i.e. $X^3 \to B^2$. As explained in [8] the spectrum of six-dimensional tensor, (Abelian) vector and hypermultiplets can be computed from
the Hodge numbers of $X^3$ together with the Hodge numbers of $B^2$ in the following way:

$$N_H = h^{(2,1)}(X^3) + 1,$$

$$N_T = h^{(1,1)}(B^2) - 1,$$

$$r(V) = h^{(1,1)}(X^3) - h^{(1,1)}(B^2) - 1.$$  \(12\) \(13\) \(14\)

These numbers have to match up the corresponding numbers on the heterotic side for a given dual string pair and imply in particular a mapping of the heterotic data, $(n_1, n_2, n_5)$ on the F-theory Hodge numbers.

Let us discuss in more detail a particular class of dual models. On the heterotic side, they are characterized by the absence of five-branes $n_5 = 0$; so there are no non-perturbative tensor fields, i.e. $N_T = 1$. The anomaly equation (9) then implies that $n_1 = 12 + k$ and $n_2 = 12 - k$ ($k \geq 0$). Assuming maximal Higgsing the spectrum of vector and hypermultiplets is then given for the following choices of $k$ as:

<table>
<thead>
<tr>
<th>$k$</th>
<th>$G_2$</th>
<th>$\text{dim} M_{\text{inst}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$E_8$</td>
<td>224</td>
</tr>
<tr>
<td>2</td>
<td>$SO(8)$</td>
<td>224</td>
</tr>
<tr>
<td>4</td>
<td>$E_6$</td>
<td>252</td>
</tr>
<tr>
<td>6</td>
<td>$E_6$</td>
<td>302</td>
</tr>
</tbody>
</table>
| 12  | $E_8$ | 472              | \(15\)

($G_1$ is always completely Higgsed.)

On the F-theory side, the compactifications which are dual to these perturbative heterotic string vacua are given by a class of elliptic threefolds $X_k^3$, where the bases $B_k^2$ are given by the $k$-th Hirzebruch surface $F_k$. These surfaces are all $P^1$ fibrations over $P^1$, and they are distinguished by how the $P^1$’s are twisted. For example, $F_0$ is just the direct product $P^1 \times P^1$. For all $F_k$, $h^{(1,1)}(F_k) = 2$. Therefore one immediately gets that $N_T = 1$, which corresponds to the universal heterotic dilaton tensor multiplet in six dimensions. In the following it will become very useful to describe the elliptically fibred Calabi-Yau spaces $X^3_k$ in the Weierstrass form [8]:

$$X^3_k: \quad y^2 = x^3 + \sum_{n=-4}^{4} f_{8-nk}(z_1) z_2^{4-n} x.$$

Here $f_{8-nk}(z_1)$, $g_{2-nk}(z_1)$ are polynomials of degree $8 - nk$, $12 - nk$ respectively, where the polynomials with negative degrees are identically set to zero. From this equation we see that the Calabi-Yau threefolds $X^3_k$ are K3 fibrations over $P^1_{z_1}$ with coordinate $z_1$; the K3 fibres themselves are elliptic fibrations over the $P^1_{z_2}$ with coordinate $z_2$. The Hodge numbers $h^{(2,1)}(X^3_k)$, which count the number of complex structure deformations of $X^3_k$ are given by the the number of parameters of the curve (16) minus the number of possible reparameterizations, which are given by 7 for $k = 0, 2$ and by $k + 6$ for $k > 2$. One can explicitly check the Hodge numbers derived in this way precisely match the heterotic spectra in table (15) using eqs. (12, 13, 14). The non-Abelian gauge symmetries are determined by the singularities of the curve (16) and were analyzed in detail in [20]. For example, for $k = 1, 2$ it is easy to see that the elliptic curve (16) is generically non-singular. Only tuning some parameters of the polynomials $f$ and $g$ to special values, the curve will become singular. These F-theory singularities correspond to the perturbative gauge symmetry enhancement in the dual heterotic models. On the other hand, for the cases $k > 2$ the curve (16) always contains generic singularities, since on the heterotic side the gauge group cannot be completely Higgsed.

Upon compactification on a two-dimensional torus $T^2$, the six-dimensional $N = 1$ duality between F-theory and the heterotic string leads to a $N = 2$ duality between F-theory on $X^3 \times T^2$ dual to heterotic on $K3 \times T^2$. This duality is then extended [6] by observing that the four-dimensional, $N = 2$ supersymmetric heterotic string on $K3 \times T^2$ is also dual [3] to the type IIA superstring compactified on the same Calabi-Yau threefold $X^3$. The number of $N = 2$ hypermultiplets is given by $N_H = h^{(2,1)}(X^3) + 1$, and the number of $U(1)$ vector multiplets is given by $N_V = h^{(1,1)}(X^3)$ in agreement with eqs. (12, 13, 14), taking into account that the $T^2$ compactifications leads to two additional $U(1)$ vector fields $T$ and $U$. We will later use this
\( N = 2 \) F-theory/heterotic duality to construct dual pairs with \( N = 1 \) space-time supersymmetry by a \( \mathbb{Z}_2 \) modding procedure.

4. Four Dimensions

As a continuation of the six-dimensional string vacua discussed in the previous chapter, \( N = 1 \) supersymmetric in four dimensions are obtained either by compactifying the heterotic string on a Calabi-Yau threefold \( Z \) or, in the dual description, by F-theory compactification on a fourfold \( X^4 \). Let us first discuss some properties of the heterotic vacua. Additional to the choice of the threefold \( Z \) the heterotic vacuum must be further specified by a particular choice of the gauge bundle \( V_1 \times V_2 \), which determines the breaking of the original gauge group \( E_8 \times E_8 \) to some subgroup \( G_1 \times G_2 \) and also, together with the data of \( Z \), the chiral \( N = 1 \) matter fields, which transform non-trivially under \( G_1 \times G_2 \). In general, the matter field representations will be in chiral representations with respect to the gauge group. Unlike six dimensions, where the gauge bundle can be characterized by two integers, the instanton numbers \( n_1 \) and \( n_2 \), the four-dimensional gauge bundle is characterized by the second Chern classes of \( V_a \), called \( c_2(V_a) \). In analogy to the Green-Schwarz condition eq. (9) in six dimensions, there is an anomaly constraint on the four-dimensional gauge bundle including non-perturbative 5-brane [11]:

\[
c_2(V_1) + c_2(V_2) + n_5 = c_2(Z). \tag{17}
\]

For perturbative heterotic vacua with \( n_5 = 0 \), this anomaly condition can be always satisfied by the so-called standard embedding of the spin connection into the gauge fields, i.e. identifying the \( SU(3) \) holonomy group of \( Z \) with a subgroup of one \( E_8 \). This choice leads to the class of so-called \((2,2)\) superconformal \( N = 1 \) heterotic vacua with gauge group \( E_8 \times E_8 \) and \( h^{(1,1)}(Z) \) chiral matter fields in the \( 27 \) representation of \( E_8 \). However this is by far not the only possible choice, and a general (perturbative) heterotic vacuum is given by a \((0,2)\) superconformal field theory. In fact, all models with a F-theory dual we know so far, are \((0,2)\) compactifications for which the gauge bundle is often very hard to be explicitly constructed.

The moduli of a general \((0,2)\) heterotic Calabi-Yau compactification count first the possible deformations of \( Z \) and, second, also the possible deformations of the gauge bundle \( V_1 \times V_2 \) (corresponding to the instanton moduli space in six dimensions). So the number of gauge singlet moduli, chiral plus antichiral fields is given by

\[
N_C = h^{(1,1)}(Z) + h^{(2,1)}(Z) + n_{\text{bundle}}, \tag{18}
\]

where \( n_{\text{bundle}} \) counts the bundle deformations. The number of \( N = 1 \) vector multiplets is simply given by the dimensions of the unbroken gauge group, \( N_V = \dim(G_1 \times G_2) \). In addition there will in general matter, charged under the gauge group. Like in six dimensions, for special values of the chiral moduli fields, the gauge group may be extended.

As already said in the introduction, we assume that \( Z \) is an elliptic threefold, i.e. there exist the fibration \( Z \rightarrow \tau \mathbb{P}^1 \). Because of this elliptic fibration structure there exist a \( \mathbb{Z}_2 \) involution \( \tau \) which acts trivially on the base \( \mathbb{P}^1 \) but acts as multiplication by \(-1\) on the elliptic fibres [11]. It follows that the gauge bundle moduli divide themselves into even and odd chiral superfields under \( \tau \): \( n_{\text{bundle}} = n_+ + n_- \). On the other hand, the difference \( n_- - n_+ \) can be computed from an index calculation [11] as follows. Namely consider the following index

\[
I = \sum_{i=0}^{3} (-1)^i \dim H^i_{\tau}, \tag{19}
\]

where \( \text{Tr}_{H^i_{\tau}(Z,\mathcal{O}_V)} \frac{1+i\pi}{2} = \dim H^i_{\tau} \). Here \( \dim H^i_{\tau} \) is \( n_- \), \( \dim H^i_{\tau} \) is \( n_+ \). Moreover, \( \dim H^1_{\tau} \) and \( \dim H^2_{\tau} \) are the numbers of unbroken gauge generators that are even or odd under \( \tau \). So in case the gauge group is completely broken we simply get

\[
I = n_- - n_+. \tag{20}
\]

Now consider the dual compactifications of F-theory on \( X^4 \). As discussed in the introduction, \( X^4 \) is assumed to have the following fibration structure: \( X^4 \rightarrow \mathcal{P} \mathbb{P}^1 \rightarrow \mathbb{P}^1 \). The cohomology of \( X^4 \) is characterized by four Hodge numbers, namely the number of Kähler deformations
$h^{(1,1)}(X^4)$, the number of complex structure deformations $h^{(3,1)}(X^4)$ and two Hodge numbers $h^{(2,1)}(X^4)$ and $h^{(2,2)}(X^4)$, where the last Hodge number will not play any role in the following. As discussed in [10], the requirement of tadpole cancellation demands that $n_3 = \frac{1}{2T}$ type IIB 3-branes have to be turned on, where $\chi$ is the Euler number of $X^4$. According to [10] there is the following relation between the Hodge numbers $h^{(1,1)}(X^4)$, $h^{(2,1)}(X^4)$ and $h^{(3,1)}(X^4)$: \[ \frac{1}{6} = 8 + h^{(2,1)}(X^4) + h^{(3,1)}(X^4). \] One can in fact show [11,21] that for a heterotic/F-theory dual string pair the numbers of 5-branes and 3-branes coincide: $n_3 = n_5$. Hence, if the Euler number of $X^4$ is non-vanishing we are dealing with a non-perturbative heterotic string vacuum.

Let us now discuss the map of the fourfold data to the heterotic data, provided by the Calabi-Yau threefold $Z$ plus the gauge bundle $V_1 \times V_2$. To start with, let us indicate the general structure of the F-theory/heterotic correspondence. First, the number of $X^4$ Kähler deformation $h^{(1,1)}(X^4)$ will correspond to the number of Kähler deformation $h^{(1,1)}(Z)$ of the heterotic threefold $Z$ and will also be related to the rank of the unbroken gauge group $r(V)$:

$$h^{(1,1)}(X^4) \leftrightarrow h^{(1,1)}(Z), r(V).$$

(21)

Second, the number of complex structure deformations $h^{(3,1)}(X^4)$ will be in correspondence to the number of complex structure deformations of $Z$ plus the number of even gauge bundle deformations:

$$h^{(3,1)}(X^4) \leftrightarrow h^{(2,1)}(Z) + n_e. \tag{22}$$

Finally, the Hodge number $h^{(2,1)}(X^4)$ is in correspondence with the odd bundle deformations:

$$h^{(2,1)}(X^4) \leftrightarrow n_o. \tag{23}$$

To be more precise, a refined analysis shows that like in the six-dimensional case also the topological data of the base $B^2$ enter the four-dimensional spectrum. Specifically as discussed in [22,12,13], looking at the dimensional reduction of the ten-dimensional type IIB spectrum, the following formulas are obtained

\[ r(V) = h^{(1,1)}(X^4) - h^{(1,1)}(B^3) - 1 + h^{(2,1)}(B^3). \tag{24} \]

Note that in this formula we did not count the chiral field which corresponds to the dual heterotic dilaton. Compactifying F-theory further to three dimensions on $X^4 \times S^1$ (or equivalently to two dimensions on $X^4 \times T^2$) these equations are consistent with M-theory compactification on $X^4$ to three dimensions, since there the sum $r(V) + N_C$ must be independent of the data of the base $B^2$ and is given by $r(V) + N_C = h^{(1,1)}(X^4) + h^{(2,1)}(X^4) + h^{(3,1)}(X^4) - 2$. In the following we will show that the F-theory spectra eqs.(24) and (25) will precisely match the heterotic spectra, in particular the number of heterotic moduli in eq.(18).

### 4.1 Smooth Weierstraß models

In the following we will consider F-theory on a smooth elliptically fibred fourfold $X^4$. So $X^4$ can be represented by a smooth Weierstraß model without singularities. (We are following [13,23] in our discussion.) So there will be no generic Non-Abelian gauge group. We assume furthermore that generically there are no $U(1)$ factors, i.e. $r(V) = 0$. However for special moduli values there might be an enhanced gauge group with massless matter representations. However we will not discuss this gauge symmetry enhancement for this class of models. Since $r(V) = 0$, eq.(25) simplifies, and $N_C$ is given by

$$N_C = \frac{\chi}{6} - 10 + 2h^{(2,1)}(X^4). \tag{26}$$

The Euler number of $X^4$ can be computed in terms of the topological data of $B^3$ as [10]

$$\frac{\chi}{24} = 12 + \frac{15}{2} \int_{B^2} c_2^2(B^2). \tag{27}$$

Moreover, since $X^3$ is assumed to be a K3 fibration over $B^2$, i.e. $B^3$ is a $P^1$ fibration over $B^2$, one gets [11]

$$\frac{\chi}{24} = 12 + 90 \int_{B^2} c_2^1(B^2) + 30 \int_{B^2} t^2 \tag{28}.$$
where \( t = c_1(T) \) (\( T \) being the line bundle over \( B^2 \)) encodes the \( P^1 \) fibration structure of \( B^3 \). Note that \( t \) plays the role of the \( k \) of \( F_h \) in the string dualities in six dimensions between the heterotic string on \( K3 \) with \((12+k, 12-k)\) instantons embedded in each \( E \). So in total we get

\[
N_C = 38 + 360 \int_{\tilde{B}^2} c_1^2(\tilde{B}^2) + 120 \int_{\tilde{B}^2} t^2 + 2h^{(2,1)}(X^4).
\]  

Let us now compare this F-theory computation with the number of moduli on the heterotic side. Since \( Z \) is an elliptic fibration over the same (complex) two-dimensional base, the Hodge numbers of \( Z \) are computed to be

\[
h^{(1,1)}(Z) = 11 - \int_{\tilde{B}^2} c_1^2(\tilde{B}^2),
\]

\[
h^{(2,1)}(Z) = 11 + 29 \int_{\tilde{B}^2} c_1^2(\tilde{B}^2).
\]

The number of gauge bundle deformations is computed from the index as

\[
n_{\text{bundle}} = n_x - n_e + 2n_e = I + 2n_e
\]

\[
= 16 + 332 \int_{\tilde{B}^2} c_1^2(\tilde{B}^2) + 120 \int_{\tilde{B}^2} t^2 + 2n_e.
\]

We see that the number of heterotic moduli, i.e. the sum of eqs. (30), (31) and (32) precisely agrees with the number of \( \text{F-theory} \) moduli in eq. (29) after setting \( h^{(2,1)}(X^4) = n_e \).

Let us briefly discuss as an example a specific class of dual smooth Weierstrass models. For this class, the (complex) two-dimensional bases \( B^2 \) are given by the so-called del Pezzo surfaces \( dP_h \) the \( P^2 \)'s blown up in \( k \) points. Then, the three-dimensional bases, \( B^3 \) are characterized by \( k \) and \( t \), where \( t \) encodes the fibration structure \( B_{n,k} \rightarrow dP_h \). Moreover we restrict the discussion to the case \( k = 0, 1, 2, 3 \) and \( t = 0 \) where one has \( h^{2,1}(X^3) = 0 \), \( h^{1,1}(X^3) = 3 + k \), \( h^{3,1}(X^3) = \frac{1}{9} - 8 - (3 + k) = 28 + 361(9 - k) \), \( \int_{\tilde{B}^2} c_1^2(\tilde{B}^2) = 9 - k \) and this leads to \( \chi = 288 + 2160(9 - k) \) and

\[
N_C = 38 + 360(9 - k).
\]

On the heterotic side the Hodge numbers are

\[
h^{(1,1)}(Z) = 11 - (9 - k), \quad h^{(2,1)}(Z) = 11 + 29(9 - k),
\]

\( I = 16 + 332(9 - k) \) and \( n_e = 0 \). Obviously, the sum \( h^{(1,1)}(Z) + h^{(2,1)}(Z) + I \) agrees with eq. (33).

4.2. \( \mathbb{Z}_2 \) modding of \( N = 2 \) models

Now we will use the already well established duality between four-dimensional string compactifications with \( N = 2 \) space-time supersymmetry to construct new F-theory fourfolds and their dual heterotic vacua [12,13]. Specifically, the F-theory fourfolds will be \( \mathbb{Z}_2 \) modded versions of the product space \( X^3_k \times T^2 \), where the threefolds are elliptic fibrations over the Hirzebruch surfaces \( F_h \), as described in section 3. The \( \mathbb{Z}_2 \) modding breaks the space-time supersymmetry from \( N = 2 \) to \( N = 1 \). The resulting fourfolds are \( X^4_k = (X^3_k \times T^2)/\mathbb{Z}_2 \). On the heterotic side we perform the same \( \mathbb{Z}_2 \) modding, i.e. the heterotic Calabi-Yau threefold is \( Z = (K3 \times T^2)/\mathbb{Z}_2 \). In addition we have to specify how the \( \mathbb{Z}_2 \)-modding acts on the heterotic gauge bundle which specified by the instanton number \( k \). Note that in this way we will get \( N = 1 \) string vacua with non-trivial gauge groups. However the action of the \( \mathbb{Z}_2 \) is such that the charged matter field representations are non-chiral.

First, consider the heterotic compactifications. Recall that the \( K3 \) can be represented by the elliptic curve eq. (4). The \( \mathbb{Z}_2 \) acts as quadratic re-definition on the coordinate \( z \), the coordinate of the base \( P^1 \) of \( K3 \), i.e. the operation is \( z \rightarrow -z \). This means that the modding is induced from the quadratic base map \( z \rightarrow z^2 \) with the two branch points 0 and \( \infty \). So the degrees of the corresponding polynomials \( f(z) \) and \( g(z) \) in eq. (4) are reduced by half. As a result the \( \mathbb{Z}_2 \)-modding reduces \( K3 \) to the del Pezzo surface \( dP_0 \). This corresponds to having on \( K3 \) a Nikulin involution of type \((10,8,0)\) with two fixed elliptic fibers in the \( K3 \) leading to the following representation of \( dP_0 \):

\[
dP_0: \quad y^2 = x^3 - f_4(z) x - g_6(z)
\]

Representing \( K3 \) as a complete intersection in the product of projective spaces as \( K3 = \left[ P^2_0 \right] \), the \( \mathbb{Z}_2 \) modding reduces the degree in the \( P^1 \) variable by half; hence the \( dP_0 \) can be represented as \( dP_0 = \left[ P^2_0 \right] \). Note that whereas the intersec-
ion lattice of $K3$ is given by $\Lambda = E_8 \oplus E_8 \oplus H \oplus H$, the intersection lattice of $dP_9$ is reduced to $\Lambda = E_8 \oplus H$.

Next we have to define how the $\mathbb{Z}_2$ modding acts on the torus $T^2$ which describes the compactification from six to four dimensions. Namely $T^2$ will appear as an elliptic fibre over the base $P^1$, with coordinate $z$ of $dP_9$. So the resulting space, the Calabi-Yau threefold $Z$, has two elliptic fibres over the same $P^1$ base, namely it has the following fibre product structure:

$$Z = dP_9 \times_P dP_9.$$  \hspace{1cm} (35)

The number of Kähler deformations of $Z$ is given by the sum of the deformations of the two $dP_9$'s minus one of the common $P^1$ base, i.e. $h^{1,1}(Z) = 19$. Similarly we obtain $h^{2,1}(Z) = 8 + 8 + 3 = 19$. This Calabi-Yau 3-fold is in fact well known being one of the Voisin-Borcea Calabi-Yau spaces. It can be obtained from $K3 \times T^2$ by the Voisin-Borcea involution, which consists in the ‘del Pezzo’ involution (type $(10,8,0)$ in Nikulin's classification) with two fixed elliptic fibers in the $K3$ combined with the usual $\mathbb{Z}_2$-involution with four fixed points in the $T^2$.

Writing $K3 \times T^2$ as $K3 \times T^2 = \begin{bmatrix} r^2 & 0 & 0 \\ r^2 & 0 & 0 \\ r^2 & 0 & 0 \end{bmatrix}$, the Voisin-Borcea involution changes this to $Z = dP \times_P dP$.

Finally, after having described the $\mathbb{Z}_2$ modding of $K3 \times T^2$ we will now discuss how this operation acts on the heterotic gauge bundle. We will consider a $N=1$ situation where after the $\mathbb{Z}_2$ modding the heterotic gauge group still lives on a four manifold, namely on the del Pezzo surface $dP_9$, which arises from the $\mathbb{Z}_2$ modding of the $K3$ surface. In effect, the instanton numbers $n_1$, $n_2$ will be simply reduced by half,

$$l_{1,2} = \frac{n_{1,2}}{2}. \hspace{1cm} (36)$$

So the total number of gauge instantons in $E_8 \times E_8$ will be reduced by two, i.e. $l_1 + l_2 = 12$ and we are considering $(l_1, l_2) = (6 + \frac{3}{2}, 6 - \frac{3}{2})$ instantons in $E_8 \times E_8$. It follows that the total dimension of the instanton moduli space is reduced by half by the $\mathbb{Z}_2$, which means that the quaternionic $N = 2$ moduli space of $N = 2$ hypermultiplet moduli is replaced by a complex moduli space of the $N = 1$ chiral moduli fields. So we see that we obtain as gauge bundle deformation parameters of the heterotic string on $Z$ the same number of massless, gauge neutral $N = 1$ chiral multiplets as the number of massless $N = 2$ hyper multiplets of the heterotic string on $K3$. This means that the $\mathbb{Z}_2$ modding keeps in the massless sector just one of the two chiral fields in each $N = 2$ hyper multiplet. These chiral multiplets describe the Higgs phase of the $N = 1$ heterotic string compactifications.

So we can now easily determine the total number of heterotic chiral moduli fields as like

$$N_C = 38 + \dim M_{\text{inst}}, \hspace{1cm} (37)$$

where $M_{\text{inst}}$, as a function of $k$, can be read off from table (15).

The gauge fields in $N = 1$ heterotic string compactifications on $Z$ are just given by those gauge fields which arise from the compactification of the heterotic string on $K3$ to six dimensions; they are invariant under the $\mathbb{Z}_2$ modding. However the complex scalar fields of the corresponding $N = 2$ vector multiplets in four dimensions do not survive the $\mathbb{Z}_2$ modding. The generic gauge groups for the considered values of $k$ can be found in table (15). Also observe that the two vector fields, commonly denoted by $T$ and $U$, which arise from the compactification from six to four dimensions on $T^2$ disappear from the massless spectrum after the modding. This is expected since the Calabi-Yau space has no isometries which can lead to massless gauge bosons. Finally, the $N = 2$ dilaton vector multiplet $S$ is reduced to a chiral multiplet in the $N = 1$ context. At special loci in the moduli space of the $N_C$ chiral fields, enlarged gauge groups plus charged, but non-chiral matter fields will appear. The specific groups and representations are listed in [13].
threefolds \( B_k^3 = X_k^3 / \mathbb{Z}_2 \) which can be written in Weierstrass form as follows:

\[
B_k^3: \quad y^2 = x^3 + \sum_{n=-1}^{4} f_{4-n}(z_1) x^{4-n}
+ \sum_{n=-6}^{6} g_{6-n}(z_1) x^{6-n}.
\]

(38)

The \( B_k^3 \) are now elliptic fibrations over \( F_{k/2} \) and still K3 fibrations over \( P_{1/2}^4 \).

Second, \( X_k^4 \) are of course no more products \( B_k^3 \times T^2 \) but the torus \( T^2 \) now is a second elliptic fibre which varies over \( P_{1/2}^4 \). Therefore the spaces \( X_k^4 \) have the form of being the following fibre products:

\[
X_k^4 = dP_3 \times P_{1/2}^4, B_k^3.
\]

(39)

All \( X_k^4 \) are K3 fibrations over the mentioned \( dP_3 \) surface. The Euler numbers of all \( X_k^4 \)’s are given by the value

\[
\chi = 12 \cdot 24 = 288.
\]

(40)

The corresponding (complex) three-dimensional IIB base manifolds \( B_k^3 \) have the following fibre product structure

\[
B_k^3 = dP_3 \times P_{1/2}^4, F_{k/2}. \quad (41)
\]

The fibration structure of \( X_k^4 \) provides all necessary informations to compute the Hodge numbers of \( X_k^4 \) from the number of complex deformations of \( B_k^3 \) in eq. (38). Without going into the details of the computations, the F-theory result is summarized in the following table [13]:

<table>
<thead>
<tr>
<th>( k )</th>
<th>( h^{1,1} )</th>
<th>( h^{2,1} )</th>
<th>( h^{3,1} )</th>
<th>( \tau(V) )</th>
<th>( N_C )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>12</td>
<td>112</td>
<td>140</td>
<td>0</td>
<td>262</td>
</tr>
<tr>
<td>2</td>
<td>12</td>
<td>112</td>
<td>140</td>
<td>0</td>
<td>262</td>
</tr>
<tr>
<td>4</td>
<td>16</td>
<td>128</td>
<td>152</td>
<td>4</td>
<td>200</td>
</tr>
<tr>
<td>6</td>
<td>18</td>
<td>154</td>
<td>176</td>
<td>6</td>
<td>340</td>
</tr>
<tr>
<td>12</td>
<td>20</td>
<td>240</td>
<td>260</td>
<td>8</td>
<td>510</td>
</tr>
</tbody>
</table>

(42)

Comparing with the corresponding heterotic numbers we get perfect agreement.

5. Comparison of F-theory/heterotic Superpotential

So far we have checked the \( N = 1 \) string-string duality for a class of models by showing that the massless spectra match up. Now we want to go one step further and demonstrate that for a particular dual pair also the \( N = 1 \) superpotentials agree. The superpotential \( W \) will depend on the chiral moduli fields, and as result of the minimization of \( W \) some of the moduli fields will be frozen to constant values. However \( N = 1 \) space-time will be left unbroken.

5.1. F-theory superpotential

As explained by Witten [14], a superpotential in F-theory is produced by wrapping 3-branes around certain divisors \( D^6 \subset X^4 \), or equivalently in the IIB language by wrapping 3-branes around 4-cycles in the base \( B^3 \). Then the superpotential is a sum over all contributing divisors like

\[
W(z) = \sum_{D} \exp <c(D^6), z>,
\]

(43)

where \( c(D^6) \) denotes the homology class of \( D^6 \) and \( z \) are moduli fields in the second cohomololy of \( X^4 \), \( z \in H^2(C^4) \). For concreteness consider the fourfold \( X_{k=0}^4 = (X_{k=0}^3 \times T^2) / \mathbb{Z}_2 = dP_3 \times P_1, F_0 \) which we discussed in section 4.2. The corresponding type IIB base \( B^3 \) is simply given by the direct product \( B^3_{k=0} = dP_3 \times P_1 \). This implies that the relevant divisors \( D^6 \) simply have the structure \( D^6 = C \times P^1 \times T^2_{11=12} \), where \( T^2_{11=12} \) is the elliptic fibre of \( X^4 \), and \( C \) is a rational curve on \( dP_3 \). So to obtain a non-vanishing superpotential \( W(z) \), we have to count all rational curves on \( dP_3 \) with self-intersection \( C^2 = -1 \). This was already done in [15], and the F-theory superpotential for this model has the form

\[
W(z, \tau, w_i) = e^{2\pi i z} \Theta_{E_8}(\tau, w_i)
= e^{2\pi i z} \sum_{i=1}^{8} \prod_{a=1}^{4} \theta_a(\tau, w_i).
\]

(44)

Here \( \Theta_{E_8} \) is the lattice partition function of \( E_8 \) and \( z, \tau \) and \( w_i \) are the moduli which correspond to the Kähler structure of \( dP_3 \). More precisely, viewing \( dP_3 \) as elliptic fibration, \( z \) is the Kähler modulus of the base \( P^1 \) and \( \tau \) corresponds to the modulus of the elliptic fibre. Note that the superpotential eq. (44) nicely reflects the intersection lattice of \( dP_3 \), \( \Lambda = E_8 \oplus \mathcal{H} \). Hence one can regard the \( w_i \) as kind of Wilson line moduli
fields. The superpotential \( W(z_0, \tau, w_i) \) is of modular weight 4 with respect to \( PSL(2, Z) \tau \) (the \( w_i \) transform as \( w_i \to \frac{w_i}{\tau + 2}\)).

5.2. Heterotic superpotential

As it is well known, a superpotential for (0,2) heterotic Calabi-Yau compactifications can be either generated by world-sheet instantons or by space-time instantons. Now according to [14] the superpotential generating divisors on the F-theory side correspond in our model to worldsheet-instantons on the heterotic side. So we want to consider the superpotential created by worldsheet-instantons/rational curves which is of the form

\[
W(z_i) = \sum n_d \exp(2\pi i d z_i),
\]

where \( z_i \in H^2(Z) \) and the \( n_d \) are the rational instanton numbers of degree \( d \). The rational instanton numbers of \( Z = (K3 \times T^2)/Z_2 = dP_0 \times P_1 \cdot dP_0 \) are essentially determined by the \( dP_0 \) geometry (for more details see [12]). Since the \( dP_0 \) base is common to the F-theory fourfolds and to the heterotic CY\(^{10,10}\) one can more or less immediately deduce the equality of the superpotentials. Specifically, the rational curve in \( Z \) projected on \( dP_0 \) have self-intersection \( C^2 = -1 \), which leads to same superpotential eq. (44) as on the F-theory side.\(^2\)

5.3. Modular correction factor to \( W \)

Now we will give some arguments that the superpotential eq. (44) has to be corrected by a modular function. In fact, the authors of [15] expect that this expression for the superpotential has to be corrected by an \( \eta(\tau)^8 \) denominator - leading to a completely modular invariant superpotential - when taking into account a correct counting of the sum of rational (-1)-curves including also reducible objects. We would like to argue that a different correction factor is required to get the correct modular weight for \( W \), namely a factor \( \eta(\tau)^{-12} \). In the following we will give two independent arguments in favor of this modular correction factor.

(i) So far we have done a “naive” counting over the rational curves of \( dP_0 \). However one must also include degenerate curves. For this purpose we will use the precise rational curve counting on the del Pezzo \( dP_0 \) provided by mirror symmetry [24]. If we compare the partition function of [24] with the superpotential having all \( w_i \) locked to zero, which is given by \( W(\tau, w_i = 0) = \varphi_0 E_4(\tau) \), we find the asserted correction factor. In summary, we conclude that the true superpotential has the form

\[
W(z_0, \tau, w_i) = \frac{e^{2\pi i z_0} \Theta_E(\tau, w_i)}{\eta^{12}(\tau)}.
\]

This superpotential has modular weight -2 with respect to \( \tau \to \frac{\sigma^+ + i}{\sigma^- + i} \).

(ii) The second argument in favor of this correction factor is based on analyzing the modular transformation properties of \( W(\tau) \) in the orbifold limit of the heterotic threefold \( Z = (K3 \times T^2)/Z_2 \). Namely, representing \( K3 = T^4/Z_2 \), \( Z \) can be constructed as a \( Z_2 \times Z_2 \) orbifold, i.e. the orbifold limit of \( Z \) is given by \( Z = T^6/(Z_2 \times Z_2) \). The \( T^6 \) torus compactification possesses three moduli \( T_j \) \( (j = 1, 2, 3) \), which are the Kähler moduli of the three subtori \( T^3_j \) \( (T^3 = \prod_{j=1}^3 T^3_j) \). As explained in [12] we can relate the orbifold moduli \( T_j \) in a well defined way to the Calabi-Yau moduli with the result that the modulus \( \tau \) of the elliptic fibre of \( dP_0 \) corresponds in the orbifold limit to the deformation along the diagonal: \( \tau = T_1 = T_3 \). Now let us consider the modular transformation properties of the effective \( N = 1 \) supergravity action. In \( N = 1 \) supergravity the Kähler potential \( K \) and the superpotential \( W \) are connected, and the matter part of the \( N = 1 \) supergravity Langrangian [25] is described by a single function \( G(\phi, \bar{\phi}) = K(\phi, \bar{\phi}) + \log |W(\phi)|^2 \), where the \( \phi \)'s are chiral superfields. The target space duality transformations act as discrete reparametrization on the scalars \( \phi \) and induce simultaneously a Kähler transformation on \( K \). Invariance of the effective action constrains \( W \) to transform as a modular form of particular weight [26]; specifically under \( PSL(2, Z)T_1 \times PSL(2, Z)T_3 \), the superpotential must have modular weights -1. Now, with the identification \( \tau = T_1 = T_3 \), the Kähler
potential at lowest order in perturbation theory is given as \( K = -2 \log(\tau - \tau') \). Then, concerning the transformation properties of the superpotential under the diagonal modular transformations \( PSL(2, \mathbb{Z})_\tau \), invariance of the \( G \)-function requires that \( W \) has modular weight -2, i.e. that under \( \tau \to \frac{a\tau + b}{c\tau + d} \) one has \( W \to \frac{W}{(c\tau + d)^2} \). So we confirm in this way the superpotential eq. (46). Let us also remark on the factor \( g_0 \) in eq. (46). In the orbifold limit the possible \( z_0 \)-dependence of the superpotential is again restricted by \( T \)-duality. However the duality group with respect to the modulus \( z_0 \) is no longer the full modular group \( PSL(2, \mathbb{Z}) \) but only a subgroup of it, since the \( R \to 1/R \) duality is broken by the freely acting \( \mathbb{Z}_2 \) in this sector. So the superpotential is not required to transform as a modular function, but it should be just a periodic function in \( g_0 \).

Finally, let us determine [15,12] the minimum of the superpotential \( W(z_0, \tau, w_i) \) eq. (46). In fact, minimizing this superpotential leads to a supersymmetry preserving locus \( (W = 0, dW = 0) \) consisting in locking pairs of the \( w_i \) on the four half-periods of the elliptic curve \( E_\tau \). Expanding in \( \phi_i = \xi_i - \mu_i \) around the minima \( \mu_i \) gives \( W|_{SU(1)} \sim e^{2\pi i \phi_i/\phi_i^2} (\tau, \phi)|\phi_\tau \) -dependence of the superpotential. In the orbifold limit the superpotential \( W(z_0, \tau, w_i) \) lifts the vacuum degeneracy of this model; the vacuum expectation values of the Wilson line fields \( \phi_i \), \( \phi_i \), are set to zero after the minimization, i.e. the vacuum expectation values are not free, continuous parameters in the presence of this superpotential. Only \( \xi_i \) and \( \tau \) survive as moduli fields. However \( N = 1 \) space-time supersymmetry remains unbroken in the presence of the superpotential.

6. Conclusions

We have checked the \( N = 1 \) string-string duality for two classes of \( F \)-theory fourfolds compared to heterotic threefolds plus \( E_8 \times E_8 \) gauge bundles. The check relies on matching the massless spectra of dual \( N = 1 \) string pairs. In addition, for one model we have supported some strong evidence that the perturbative heterotic superpotential matches with its \( F \)-theory counterpart. It would be very interesting to analyze models with (modular) non-perturbative, \( S \)-dependent heterotic superpotentials [1,27] and their \( F \)-theory duals. In this context non-perturbative symmetries in underlying \( N = 2 \) models, like the \( S-T \) exchange symmetry, may play an important role. Also the question of breaking space-time supersymmetry in \( F \)-theory is very important. Moreover we would like to get a better understanding of \( F \)-theory/heterotic dual pairs with chiral matter representations with respect to the unbroken gauge group. In addition, it is a very important question, how transitions among \( N = 1 \) string vacua, possibly with a different number of chiral multiplets, take place [28].

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