Global monopole in scalar tensor theory

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Abstract

The well known monopole solution of Barriola and Vilenkin (BV) resulting from the breaking of a global SO(3) symmetry is extended in general relativity along with a zero mass scalar field and also in Brans-Dicke (BD) theory of gravity. In the case of BD theory, the behaviour of spacetime and other variables such as BD scalar field and the monopole energy density have been studied numerically. For monopole along with a zero mass scalar field, exact solutions are obtained and depending upon the choice of arbitrary parameters, the solutions either reduce to the BV case or to a pure scalar field solution as special cases. It is interesting to note that unlike the BV case the global monopole in the BD theory does exert gravitational pull on a test particle moving in its spacetime.
1 Introduction

Monopoles are point like topological objects that may arise during phase transitions in the early universe[1, 2]. Depending on the nature of the scalar field it can be shown that spontaneous symmetry breaking can give rise to such objects which are nothing but the topological knots in the vacuum expectation value of the scalar field and most of their energy is concentrated in a small region near the monopole core. From the topological point of view they are formed in vacuum manifold when the latter contains surfaces which can not be continuously shrunk to a point. These monopoles have Goldstone fields with their energy density decreasing with distance as $r^{-2}$. These monopoles are found to have some interesting features in a sense that it exerts practically no gravitational force on its surrounding nonrelativistic matter but the spacetime around it has a deficit solid angle.

In a pioneering work Barriola and Vilenkin(BV) [3] showed the existence of such a monopole solution resulting from the breaking of global SO(3) symmetry of a triplet scalar field in a Schwarzschild background. Recently we [4] have also obtained a monopole solution in Kaluza-Klien spacetime which extends the earlier work of BV to its five dimensional analogue.

In the present work we make an attempt to study such a monopole in Brans-Dicke(BD) theory[5] of gravitation. The renewed interest of scalar tensor theory of gravitation is mainly due to two important theoretical developments in the study of early universe - one is the prediction of the dilaton field arising from the low energy limit of the string theory. The other is the recent theory of extended inflation which is believed to have solved the fine tuning problem by slowing down the expansion rate of universe from the exponential to polynomial.

In view of serious difficulties in solving the monopole problem in the original version of the BD theory, an attempt is made in the second part of the paper to get the solution in the Dicke’s revised unit[6] where the field equations are identical with the Einstein’s field equations with the BD scalar field appearing as a ”matter field” in the theory. But even in the revised version, we have not as yet obtained the solution in a closed form. However we have been able, so far, to reduce the mathematical formalism to a single differential equation involving only a single variable namely the BD scalar field and once the scalar field is known, the metric components and the monopole
energy density can be obtained from the other equations. We have, however, made a numerical study of the behaviour of the different variables and the results seem to be physically consistent.

In the third part of the paper we have studied the similar problem with the monopole interacting with a zero mass scalar field and have been able to obtain the exact solutions of the field equations. The BV monopole solutions are recovered from these solutions when the zero mass scalar field vanishes. On the other hand when the monopole field is switched off we get back the usual zero mass scalar field solution already existing in literature.

Our paper is organised as follows. In section 2 the field equations for a global monopole in Brans-Dicke theory in revised units are written, analysed, and their qualitative behaviour is studied numerically. In section 3 the same problem is addressed in Einstein’s theory in the presence of a zero mass scalar field for two distinctly different situations. The paper ends with a conclusion in section 4.

2 The Global Monopole In Branse Dicke Theory

The gravitational field equation for a global monopole in Brans-Dicke theory written in Dicke’s revised unit[6] is in general

\[ G_{\mu}^{\nu} = -T_{\nu}^{\mu} - \frac{2\omega + 3}{\phi^2} \left[ \phi^{\mu\nu} \phi_{\nu} - \frac{1}{2} \delta^{\mu\nu} \phi_{\alpha} \phi^{\alpha} \right] \]  

(1)

Where \( T_{\nu}^{\mu} \) is the energy momentum tensor due to a monopole field, \( \phi \) is the B-D scalar field and \( \omega \) is the B-D parameter.

Since the spacetime here is static and spherically symmetric, the metric is given in curvature coordinates in the form

\[ ds^2 = e^\nu dt^2 - e^\beta dr^2 - r^2 d\theta^2 - r^2 \sin^2 \theta d\phi^2 \]  

(2)

where \( \nu \) and \( \beta \) are functions of \( r \) alone.

In the original work of Barriola and Vilenkin[3] there is no contribution to the energy momentum tensor from the B-D scalar field but the lagrangian is due to global SO(3) symmetry for a triplet scalar field, whose symmetry breaking gives rise to a global monopole. Under reasonable conditions the energy momentum tensor due
to the monopole field outside the core now becomes [3]

\[ T^\mu_1 = T^\nu_r = \frac{\eta^2}{r^2} \quad (3) \]

\[ T^\theta_\theta = T^\phi_\phi = 0 \quad (4) \]

where \( \eta \) is the symmetry breaking scale of the theory.

The above forms of \( T^\mu_\nu \) are consistent with the Bianchi identity. One should note here that in the original version of the B-D theory[5] the relation \( T^\mu_\nu = 0 \) is separately satisfied which again leads to the same expressions (3) and (4) for \( T^\mu_\nu \). However, in the revised units, the B-D scalar field has additional contributions to the energy momentum tensor in the field equations and consequently the Bianchi identity in this case will certainly not lead to the identical expressions for energy momentum tensors as for monopole field only. We therefore prefer to write

\[ T^\mu_1 = T^\nu_r = \rho(r) \quad (5) \]

\[ T^\theta_\theta = T^\phi_\phi = 0 \quad (6) \]

where \( \rho(r) \) is a function of \( r \) to be determined from the field equations.

With (2),(5) and (6), equation (1) yields explicitly the following relations:

\[ e^{-\beta} \left( \frac{1}{r^2} - \frac{\beta'}{r} \right) - \frac{1}{r^2} = -\rho - \frac{k}{2} \psi'' e^{-\beta} \quad (7a) \]

\[ e^{-\beta} \left( \frac{1}{r^2} + \frac{\nu'}{r} \right) - \frac{1}{r^2} = -\rho + \frac{k}{2} \psi'' e^{-\beta} \quad (7b) \]

\[ e^{-\beta} \left( \frac{\nu''}{2} + \frac{2}{4} - \frac{\nu' \beta'}{4} + \frac{\nu' - \beta'}{2r} \right) = -\frac{k}{2} \psi'' e^{-\beta} \quad (7c) \]

where \( \psi = \ln(\phi) \) and \( k = \frac{2e^{+3}}{2} \) and prime denotes differentiation with respect to \( r \).

The wave equation for the BD scalar field is given by

\[ \frac{\psi''}{r} + 2 \frac{\psi'}{r} + \left( \frac{\nu' - \beta'}{2} \right) \psi' = -\rho \frac{e^{-\beta}}{k} \quad (8) \]

By (7a)-(7b)-2x(7c) one gets

\[ \psi'' + \frac{2 \psi'}{r} + \frac{\nu'^2}{2} = \frac{\nu' \beta'}{2} \quad (9) \]
One may attempt to solve the eqn(9) for two different situations: (i) \( \nu' = 0 \), which means \( \nu = \text{constant} \) and (ii) \( \nu' \neq 0 \), which in turn leads to differential equation:

\[
e^{\nu'} \nu'^2 = \frac{b^2}{r^2} e^\beta,
\]

where \( b \) is an arbitrary constant of integration.

In the first case one may choose \( \nu = 1 \) without the loss of any generality, which in view of Bianchi identity relation immediately leads to the equation

\[
\rho' + 2 \frac{\rho}{r} + \rho \psi' = 0
\]

The above equation has the first integral

\[
\rho r^2 = a^2 e^{-\psi}
\]

where \( a \) is another arbitrary integration constant.

In the second case that is for \( \nu' \neq 0 \) it is extremely hard task to obtain any differential equation which involves only a single variable. Hence we abandon this case in our present work and proceed with the other case, that is, \( \nu' = 0 \) for further calculations.

With \( \nu' = 0 \) one gets after some simple and straightforward calculation the following two equations:

\[
e^\beta = \frac{(k_2 \psi'^2 r^2 - 1)}{(a^2 e^{-\psi} - 1)}
\]

\[
(a^2 - e^\psi) \psi'' - \frac{k}{2} r (a^2 - e^\psi)^3 + \frac{a^2}{2} \psi'^2 + \frac{2}{r} (a^2 - e^\psi) \psi' - \frac{a^2}{k r^2} = 0
\]

One should note that the actual BD scalar field is \( \phi = e^\psi \).

The equation(12) is the key equation to our subsequent analysis. If this equation can be integrated, the complete set of equations will be solvable, as in view of equation(10) and (11), \( \rho \) and \( e^\beta \) may be known in terms of \( \psi \). However equation(12) is a highly nonlinear equation which may not be amenable to an analytic solution in a closed form.

Hence we try to solve the equation (12) numerically in order to estimate the behaviour of scalar field \( \phi, e^\beta \) as well as the energy density of the monopole \( \rho \) outside the core. We rewrite the equation(12) as,

\[
\frac{d^2 \phi}{dx^2} = 2.304 \frac{d \phi}{dx} + \frac{(a^2 - 2 \phi)}{(2a^2 - 2\phi) \phi} \frac{(d \phi)^2}{dx^2} - \frac{0.434k}{2\phi^2} \frac{d \phi}{dx} + \frac{a^2 \phi}{0.434k(a^2 - \phi)}
\]

where \( x = \log_\phi(\frac{r}{\delta}) \) where \( \delta \) = the core radius of the monopole, \( \phi = e^\psi \), the BD scalar field. The variables \( e^\beta \) and \( \rho \) take the form

\[
e^\beta = \frac{0.095k (\frac{d\phi}{dx})^2 - \phi^2}{a^2 \phi - \phi}
\]
We have tried to solve the equation (13) numerically by fourth order Runge-Kutta method and plot $\phi, \phi', \rho, e^\beta$ against $r$.

We have taken the initial condition that for a large distance from the monopole core i.e when $r$ takes a very large value, $\phi' \rightarrow 0$ and $\phi$ approaches a very small value. We have taken the distance upto $10^5$ times the core radius $\delta$ of the monopole.

In figure (1) we have plotted both $\phi$ and $\phi'$ and further in figure (2) and (3) we plot $e^\beta, \rho$ respectively against $\log_{10}(\delta/r)$.

The behaviour of $e^\beta$ in fig (2) shows that the spacetime outside the monopole core becomes flat for a large distance from the core. The fig (3) shows that the monopole energy density $\rho$ rapidly decreases with respect to the distance from the core. This is qualitatively consistent with the corresponding BV case where also the energy density of the monopole outside the core falls off with $r$. The behaviour of scalar field in fig (1) also shows that $\phi$ approaches a constant value and $\phi'$ approaches zero for a large distance from the core.

**Gravitational force in the field of a monopole**

Before ending up this section we would like to point out an important property of the BD scalar field-the gravitational force acting on surrounding matter of the monopole.

In the original version of BD theory our metric (2) becomes

$$ds^2 = e^{-\psi}(e^\beta dt^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2)$$

as $\nu = constant$ for our case and $\psi, \beta$ are given by (13) and (14) respectively. The radial component of the acceleration acting on a test particle in the gravitational field of the monopole is given by:

$$\dot{v^1} = v^0_\beta v^0$$

(17)

Since for a comoving particle $v^\mu = \frac{1}{\sqrt{g_0}} \delta_0^\mu$ we have $v^0 = e^{\psi/2}$ and $\dot{v}^1 = -\frac{1}{2} e^{(\psi/2-\beta)} \psi'$. Assuming $\psi' < 0$ which is obvious from fig (1), the particle accelerates away in the radial direction in order to keep it at rest which implies that a monopole has an attractive influence on the test particle. This attractive force is due to the presence of the BD scalar field $\psi$. When $\psi' = 0$ the BD scalar field is absent and we have
only the monopole remaining which however exerts no gravitational influence on the surrounding matter as shown by BV[3].

3 Global monopole interacting with a zero mass scalar field

In this section we have considered a monopole in the presence of a massless scalar field which also contribute to the energy momentum tensor in addition to that of the monopole field. Exact analytical solutions may be obtained in this case. However, for economy of space the details of mathematical steps are omitted and the main results are sketched below:

Case (i)

In this case we take the metric as

\[ ds^2 = e^{\nu} d\ell^2 - e^{\beta} dr^2 - r^2 d\Omega^2 \]  

(18)

where \( \nu, \beta \) are functions of \( r \) only.

Solving the field equations we get

\[ e^\nu = (1 - \eta^2 - 2\frac{M}{r})^n \]  

(19)

\[ e^\beta = (1 - \eta^2 - 2\frac{M}{r})^{n-2} \]  

(20)

Where \( M \) and \( n \) are arbitrary constant of integration and \( \eta \) is the symmetry breaking scale for the monopole.

The energy density of the monopole is given by

\[ \rho_{\text{mon}} = \frac{1}{r^2} [1 - (1 - \eta^2)(1 - \eta^2 - 2\frac{M}{r})^{n-1}] \]  

(21)

The zero mass scalar field is given by

\[ \psi^2 = \frac{4M(n - 1)}{(1 - \eta^2)r^3}(1 - \eta^2 - 2\frac{M}{r})^{-1} \]  

(22)

For \( n = 1 \) the variation of the scalar field \( \psi \) vanishes, and we get back the BV type monopole with \( \rho_{\text{mon}} = \frac{\eta^2}{r^2} \).
On the other hand pure scalar field solution without the presence of a monopole field cannot be recovered in this form of metric, as is evident from the expressions (21) and (22).

Case(ii)

Here we take the metric in the form

\[ ds^2 = e^\nu dt^2 - e^\beta dr^2 - r^2 R^2 d\Omega^2 \]  

(23)

where \( \nu, \beta \) and \( R \) are functions of \( r \) only.

The solutions are

\[ e^\nu = (1 - \eta^2 - 2\frac{M}{r})^n \]  

(24a)

\[ e^\beta = (1 - \eta^2 - 2\frac{M}{r})^{-n} \]  

(24b)

\[ R^2 = (1 - \eta^2 - 2\frac{M}{r})^{1-n} \]  

(24c)

\[ \rho_{mon} = \frac{\eta^2}{r^2}(1 - \eta^2 - 2\frac{M}{r})^{n-1} \]  

(24d)

\[ \psi' = \sqrt{\frac{M}{r^2}} \sqrt{n^2 - 1}(1 - \eta^2 - 2\frac{M}{r})^{-1} \]  

(24e)

It is clear from the expressions (24d) and (24e) that when \( n = 1 \), we have \( \rho_{mon} = \frac{\eta^2}{r^2} \) and \( \psi' = 0 \), so that the BV solution is recovered. But on the other hand when we put \( \eta^2 = 0 \), we obtain \( \rho_{mon} = 0 \) and \( \psi' \neq 0 \) meaning that we have a pure massless scalar field distribution.

4 Conclusion

In this work we have extended the monopole solution of BV to the scalar tensor theory. In BD theory we have not been able to get the analytical solution. So we have taken recourse to numerical methods to study the behaviour of BD scalar field. From this field, the behaviour of metric coefficient and the energy density of the monopole are also found out. It is encouraging to point out that the nature of variation of BD scalar field, metric coefficient and also the energy density of the monopole are quite physically realistic in the sense that the spacetime is asymptotically flat. Also the monopole energy density outside the core decreases sharply with the distances.
For the sake of completeness, we have also studied the similar problem with monopole interacting with a massless scalar field, and have obtained exact solutions in this case. Under suitable choice of the parameters in some cases our solutions reduce to either the BV solution or to that of a pure zero mass scalar field.

To end up, a final remark may be in order. It is well known that in Einstein’s theory, the BV type of monopole does not practically exert any gravitational pull on any surrounding matter. However, our analysis shows that the monopole in BD theory does exert gravitational force on a test particle. So this observation is in striking contrast with the analogue in Einstein’s theory.

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Figure Captions

1. In Fig 1, we have plotted the BD scalar field $\phi$ and its first derivative $\phi'$ with respect to $\log(\delta/r)$.

2. In Fig 2, we have plotted the metric coefficient $e^\rho$ with respect to $\log(\delta/r)$.

3. In Fig 3, we have plotted the monopole energy density $\rho$ with respect to $\log(\delta/r)$.
References


