Verification of $O(a)$ improvement

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The status of simulations using the non-perturbatively $O(a)$ improved Wilson action in the quenched approximation is reviewed. The impact of non-perturbative improvement on the hadronic mass spectrum and the size of residual lattice artefacts in spectral quantities and decay constants are assessed.

1. INTRODUCTION

One of the most important systematic effects in lattice simulations of QCD is the finiteness of the lattice spacing $a$. It is well known that physical observables computed using the Wilson action are subject to corrections of order $a$, which can be rather large. In order to obtain reliable results in the continuum limit it is desirable to reduce lattice artefacts, either through the Symanzik improvement programme [1,2], or by employing a renormalisation group approach [3,4]. Recently the ALPHA Collaboration has carried out the Symanzik improvement programme to leading order through a non-perturbative determination of the $O(a)$ improved fermion action and isospin currents [5-8]. This approach should lead to the complete removal of lattice artefacts of order $a$ in spectral quantities and matrix elements of local currents, so that the remaining cutoff effects are of order $a^2$. The non-perturbatively improved action has already been employed in a number of simulations in the quenched approximation [9-11]. Here, we assess the impact of non-perturbative improvement on the calculation of the mass spectrum and decay constants in the light hadron sector, analyse the scaling behaviour and estimate the size of residual lattice artefacts.

The general expression of the $O(a)$ improved fermion action reads [12]

$$S^I_F[U, \bar{\psi}, \psi] = S^W_F[U, \bar{\psi}, \psi] + c_{sw} \sum_{x, \mu, \nu} \bar{\psi}(x) \sigma_{\mu \nu} F_{\mu \nu} \psi(x) + c_{sw} \sum_{x, \mu, x', \nu} \bar{\psi}(x) \sigma_{\mu \nu} F_{\mu \nu} \psi(x) + \ldots,$$

where $S^W_F$ is the (unimproved) Wilson fermion action and $c_{sw}$ is an improvement coefficient. The improved and renormalised axial and vector currents are defined as [13]

$$(A^a_{\mu})^a_\mu = Z_A(1 + b_la m_q)\{A^a_{\mu} + c_A a \partial_\mu P^a\},$$

$$(V^a_{\mu})^a_\mu = Z_V(1 + b_v a m_q)\{V^a_{\mu} + c_V a \partial_\mu T^a_{\mu \nu}\},$$

where $Z_A, Z_V$ are the renormalisation factors of the respective currents, and $b_A, c_A, b_V, c_V$ are further improvement coefficients. The unimproved currents and densities are defined as

$$(A^a_{\mu})^a_\mu = \bar{\psi}\gamma_\mu \gamma_5 \tau^a \psi, \quad (V^a_{\mu})^a_\mu = \bar{\psi}\gamma_\mu \gamma_5 \psi, \quad P^a = \bar{\psi}\gamma_\mu \gamma_5 \tau^a \psi, \quad T^a_{\mu \nu} = i\bar{\psi}\sigma_{\mu \nu} \tau^a \psi.$$

where $\tau^a$ are the Pauli matrices acting in flavour space. The normalisations $Z_A, Z_V$ and improvement coefficients $c_{sw}, c_A, c_V, b_V$ have been determined non-perturbatively for bare couplings $g_0$ in the range $0 \leq g_0 \leq 1$ [6-8]. Results and proposals for non-perturbative determinations of $b_A$ at $g_0 \approx 1$ have been reported [14-16]. Furthermore, all of the above improvement coefficients have been calculated in perturbation theory to one-loop order [17,18].

The parameters of the simulations discussed here are listed in Table 1. In addition to the non-perturbative value of $c_{sw}$, the QCDSF collaboration have also used unimproved Wilson fermions for a direct comparison [9]. UKQCD have also used the tadpole improved value of $c_{sw}$ on the same set of configurations and at $\beta = 5.7$ [19,20], where a non-perturbatively determined value of $c_{sw}$ is not available. Table 1 shows that all collaborations have used the same two values of $\beta$. This implies that one cannot test as yet whether non-perturbative improvement indeed leads to a scaling behaviour of physical observables which is consistent with $O(a^2)$ corrections. However, if
Table 1
Simulations using non-perturbative improvement. The number of exceptional configurations discarded from the ensemble is shown in brackets.

<table>
<thead>
<tr>
<th>Collab.</th>
<th>$\beta$</th>
<th>$L^3 \times T$</th>
<th>Statistics</th>
</tr>
</thead>
<tbody>
<tr>
<td>QCDSF [9]</td>
<td>6.0</td>
<td>$16^3 \times 48$</td>
<td>$\sim 1000$</td>
</tr>
<tr>
<td></td>
<td>6.2</td>
<td>$24^3 \times 48$</td>
<td>$\sim 200$</td>
</tr>
<tr>
<td>UKQCD [10]</td>
<td>6.0</td>
<td>$16^3 \times 48$</td>
<td>497(3)</td>
</tr>
<tr>
<td></td>
<td>6.2</td>
<td>$24^3 \times 48$</td>
<td>70(1)</td>
</tr>
<tr>
<td>APETOV [11]</td>
<td>6.0</td>
<td>$16^3 \times 48$</td>
<td>50(1)</td>
</tr>
<tr>
<td></td>
<td>6.2</td>
<td>$24^3 \times 48$</td>
<td>50</td>
</tr>
</tbody>
</table>

one assumes this to be the leading scaling behaviour after improvement, one can still estimate the size of residual lattice artefacts at a given value of $\alpha$.

2. SPECTRAL QUANTITIES

In Fig. 1 we show the Edinburgh plot. In addition to the non-perturbatively improved data, we also display the tadpole improved results by UKQCD at $\beta = 5.7$ and the data by GF11 using the unimproved action at $\beta = 5.7, 5.93, 6.17$ [21]. One observes that improvement yields consistently lower values for $m_N/m_P$ compared to the unimproved action. The most dramatic effect is observed at $\beta = 5.7$ when one compares the unimproved results (full triangles) to the tadpole improved ones (full circles). In fact, it seems that the mass behaviour obtained using tadpole improvement at $\beta = 5.7$ is indistinguishable from the non-perturbatively improved action. However, as we shall see later, the residual lattice artefacts in the tadpole improved data at $\beta = 5.7$ are still large. Thus, the Edinburgh plot disguises rather than exposes lattice artefacts and should therefore not be used to draw conclusions about the scaling behaviour.

It is a well-known fact that lattice simulations using unimproved actions fail to reproduce the experimentally observed behaviour of the vector-pseudoscalar mass splitting, i.e. $m_V^2 - m_P^2 \sim \text{const}$, which holds up to the mass of the charm quark. This is usually ascribed to the influence of lattice artefacts in the computation of these splittings. Data for $m_V^2 - m_P^2$ by UKQCD [10] and QCDSF [9] using the non-perturbatively improved action show that the splittings are close to the experimental values for the $(\rho, \pi)$ and $(K^*, K)$ systems (see Fig. 2 for a plot of the UKQCD data). Furthermore, by comparing the results at $\beta = 6.0$ and 6.2 one observes that the dependence on the lattice spacing is small. However, for quark masses above $m_{\text{strange}}$ the two collaborations see a slight downward trend in the data, so that one can expect that the $D^* - D$ splitting is still not reproduced correctly.

We now discuss the chiral limit and the critical value of the hopping parameter, $\kappa_c$. Usually $\kappa_c$ is
defined at the point where the pseudoscalar mass vanishes, \( m_{PS} = 0 \). In accordance with the quark mass behaviour of \( m_{PS}^2 \) implied by the PCAC relation, one can determine \( \kappa_c \) from a linear fit to

\[
(m_{PS})^2 = a^2 B \frac{1}{2} \left( \frac{1}{\kappa} - \frac{1}{\kappa_c} \right) = a^2 B m_q. \tag{5}
\]

Both QCDSF and UKQCD have reported that this linear ansatz in 1/\( \kappa \) results in poor fits with large correlated \( \chi^2/\text{dof} \) and have therefore resorted to using model functions which also contain quadratic terms in 1/\( \kappa \). However, in order to be consistent with \( O(a) \) improvement, the quark mass \( m_q \) in eq. (5) should be replaced by

\[
\tilde{m}_q = m_q (1 + b_n m_q), \tag{6}
\]

where \( b_n = -\frac{1}{2} - 0.0062 g_0^2 \) in one-loop perturbation theory [18]. This modification of the fitting ansatz in the determination of \( \kappa_c \) has so far not been used.

Another method defines \( \kappa_c \) at the point where the quark mass defined through the PCAC relation vanishes, i.e. \( m_{PCAC} = 0 \). Here it is important to realise that even after non-perturbative improvement chiral symmetry is only approximately restored, so that

\[
\partial_\mu \left\{ A_\mu(x) + c_\Lambda a_\mu P(x) \right\} = 2m_{PCAC} P(x) + O(a^2). \tag{7}
\]

Therefore, the values of \( \kappa_c \) determined by requiring either \( m_{PS}^2 = 0 \) or \( m_{PCAC} = 0 \) will differ by terms of order \( a^2 \). The compilation of results in Table 2 shows that in the range of \( \beta \) under study the difference in \( \kappa_c \) using either method is of the order of \( 10^{-4} \) and thus statistically significant.

We now analyse the scaling behaviour of the mass of the vector meson by comparing the approach to the continuum limit for unimproved and improved actions. To this end, we note that the continuum limit should be studied for constant physical volume, so that finite-volume effects do not distort the scaling behaviour. Since chiral extrapolations are poorly understood, we will use the available lattice data for the vector and pseudoscalar masses \( m_{V} \), \( m_{PS} \) as well as the string tension \( \alpha \sqrt{\sigma} \) in order to interpolate the dimensionless ratio \( m_{V}/\sqrt{\sigma} \) to \( m_{PS}/\sqrt{\sigma} \).

![Table 2](image)

<table>
<thead>
<tr>
<th>( \beta )</th>
<th>UKQCD</th>
<th>QCDSF</th>
<th>APETOV</th>
</tr>
</thead>
<tbody>
<tr>
<td>6.0</td>
<td>0.135335+5_{-2}</td>
<td>0.13531(1)</td>
<td>-</td>
</tr>
<tr>
<td>6.2</td>
<td>0.135895+14_{-55}</td>
<td>0.13589(2)</td>
<td>0.135861(19)</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>( \beta )</th>
<th>ALPHA</th>
<th>( m_{PCAC} = 0 )</th>
<th>APETOV</th>
</tr>
</thead>
<tbody>
<tr>
<td>6.0</td>
<td>0.135196(14)</td>
<td>-</td>
<td></td>
</tr>
<tr>
<td>6.2</td>
<td>0.135795(13)</td>
<td>0.135802(6)</td>
<td></td>
</tr>
</tbody>
</table>

1.125. For \( \sqrt{\sigma} = 440 \text{MeV} \) this implies \( m_{PS} = m_{K} = 495 \text{MeV} \). If \( m_{PS}^2 \) and \( m_{V} \) are linear functions of the quark masses then one expects that \( m_{V} \approx m_{K} = 892 \text{MeV} \) or \( m_{V}/\sqrt{\sigma} \approx 2.027 \). In Fig. 3 we have plotted the interpolated data for \( m_{V}/\sqrt{\sigma} \) versus \( \alpha \sqrt{\sigma} \) for unimproved [21] and improved actions [10, 19, 20]. For the unimproved action one observes that \( m_{V} \approx m_{K} \) is satisfied, but only after the extrapolation to the continuum limit. In contrast, the data obtained using the non-perturbatively or tadpole improved actions at \( \beta = 6.0 \), 6.2 show very little dependence on \( a \) and are rather close to \( m_{V} \approx m_{K} \). This, however, can no longer be claimed for the tadpole improved data at \( \beta = 5.7 \), where instead one observes large residual lattice artefacts. From the slope of the linear fit to \( m_{V}/\sqrt{\sigma} \) for the unimproved data, we infer the size of residual lattice artefacts at \( a \approx 0.1 \text{fm} \) to be \((12 \pm 1)\% \). For the non-perturbatively improved data we estimate the leading corrections of \( O(a^2) \) to be only \( \approx 2\% \).

We can now turn the tables and ask how well various prescriptions to fix \( c_{sw} \) satisfy the “phenomenological” improvement condition

\[
m_{V}/\sqrt{\sigma} = \text{const.} \tag{8}
\]

Besides the data obtained using tadpole and non-perturbative estimates for \( c_{sw} \), one can use further spectrum data obtained using \( c_{sw} = 0 \) [21] and \( c_{sw} = 1 \) [22]. In Fig.4 we plot \( m_{V}/\sqrt{\sigma} \) at \( m_{PS} = m_{K} \) versus \( c_{sw} \). From the plot one infers that the condition (8) is satisfied within statisti-
Figure 3. Scaling behaviour of the vector mass at $m_{\text{PS}} = m_K$. The dashed line indicates the extrapolated result using the GF11 data.

Figure 4. $m_{V}/\sqrt{s}$ as a function of $c_{sw}$. The sets of points from right to left have been obtained using non-perturbative, tadpole, tree-level and no improvement. Data at $\beta = 5.7$ are shown only for unimproved and tadpole improved actions. The dashed line shows the extrapolated GF11 data.

A major advantage of non-perturbative improvement is that the renormalisation factor $Z_A$ has been determined with an accuracy of around 1%. At $\beta = 6.0$ and 6.2 one finds [7]

$$Z_A = \begin{cases} 0.791(9), & \beta = 6.0 \\ 0.807(8), & \beta = 6.2 \end{cases}$$

(13)

Note that the error on $Z_A$ must be combined with the statistical error on $f_{\text{PS}}^{(0)} + c_A f_{\text{PS}}^{(0)}$.

The improvement coefficient $c_A$ has also been determined non-perturbatively [6]:

$$c_A = \begin{cases} -0.083, & \beta = 6.0 \\ -0.037, & \beta = 6.2 \end{cases}$$

(14)

The effect of $c_A$ on the decay constant can be studied by comparing $f_{\text{PS}}^{(0)}$ with $f_{\text{PS}}^{(0)} + c_A f_{\text{PS}}^{(0)}$. Here the contribution from $c_A f_{\text{PS}}^{(0)}$ leads to a decrease in the pseudoscalar decay constant of $\sim 4\%$ at $\beta = 6.2$ and even $\sim 15\%$ at $\beta = 6.0$. This effect is particularly pronounced for $\beta < 6.2$.

The improvement coefficient $b_A$ is relevant for the determination of the kaon decay constant $f_K$. However, unlike $b_V$, the coefficient $b_A$ has so far not been determined non-perturbatively. In one-loop perturbation theory one finds [18]

$$b_A = 1 + 0.1522 \delta_0^2 + O(\delta_0^4).$$

(15)
In order to study the influence of \( b_A \), one can evaluate \((f_{\text{PS}})_R\) around the strange quark mass for different choices of \( b_A \). Here we compare

- \( b_A = 0 \)
- \( b_A = 1 + 0.1522 g_0^2 \)
- \( b_A = b_N \)

The seemingly ad hoc choice of \( b_A = b_N \) (here we use the non-perturbative determination of \( b_N \)) is motivated by the observation that the one-loop coefficients for \( b_A \) and \( b_V \) are approximately equal[18]. By applying different choices of \( b_A \) to the analysis of the UKQCD data around \( m_K \), one finds that \( b_A \) leads to an increase in \((f_{\text{PS}})_R\) of at most \( \sim 2\% \) at \( \beta = 6.2 \) and \( \sim 3\% \) at \( \beta = 6.0 \), which is fairly small. Since the choice \( b_A = b_N \) gives essentially the same mass behaviour compared to choosing \( b_A = 1 + 0.1522 g_0^2 \), one concludes that perturbative estimates of \( b_A \) are quite acceptable for quark masses up to and around \( m_{\text{strange}} \).

We now analyse the scaling behaviour of \((f_{\text{PS}})_R\). In Fig. 5 we plot \((f_{\text{PS}})_Rr_0\) versus \((m_{\text{PS}}r_0)^2\), where \( r_0 \) is the hadronic radius defined in [23]. Data for \( r_0/a \) were taken from [24]. If lattice effects are small the data in Fig. 5 should lie on a universal curve. The results for \( f_Kr_0 \) computed at \( \beta = 6.0 \) and 6.2 show a slight dependence on the lattice spacing. In order to study residual lattice artefacts, we employ a similar procedure as in the case of the vector mass and extrapolate \((f_{\text{PS}})_Rr_0\) to \((m_Kr_0)^2\). Fig. 6 shows the resulting values of \( f_Kr_0 \) as a function of \((a/r_0)^2\). There is good agreement between the data from all three collaborations, and in principle their results could be combined. Furthermore, it appears that a linear extrapolation in \( a^2 \) yields a continuum result which is compatible with the experimentally observed value (although there is \textit{a priori} no reason why the quenched approximation should reproduce the measured result). In contrast, the authors of [25] have found that, for the unimproved action, the continuum value of \( f_K \) is significantly lower than the experimental result.

From the slope in \((a/r_0)^2\) one estimates that residual lattice artefacts in \( f_Kr_0 \) amount to ca. 10\% at \( a \simeq 0.1 \text{fm} \). Compared to the previously discussed case of the vector meson, this is a fairly large correction. Given the substantial contribution of \( c_Aq_{\text{PS}}^{(0)} \) to \((f_{\text{PS}})_R\) at \( \beta = 6.0 \), one can ask whether \( c_A \) has a large effect on the scaling behaviour. If one formulates a similar improvement condition for \( c_A \) as in eq. (8), for instance

\[ f_Kr_0 = \text{const.} \tag{16} \]

one can study how well it is satisfied for different choices of \( c_A \). Using the UKQCD data it turns out that the above condition “favours” smaller
values of $c_A$ at the lower end of the $\beta$-range. However, at this stage one should not jump to conclusions before a more thorough investigation of improvement conditions has been performed.

4. CONCLUSIONS

First results from simulations using the non-perturbatively improved Wilson action and currents in the quenched approximation show that lattice artefacts are drastically reduced. Unlike the case of unimproved actions, the results for the vector meson mass are practically independent of $a$ at $\beta = 6.0$ and 6.2, so that residual $a^2$ effects are around 2% at $a \approx 0.1$ fm. A real test of the scaling behaviour is still lacking and will only become possible when more and different values of the lattice spacing are studied.

The analysis of data for the pseudoscalar decay constant has shown that a non-perturbative determination of $b_A$ is required for quark masses above $m_{\text{strange}}$. The improvement coefficient $c_A$ has a large influence on the scaling behaviour, which motivates further investigation. On the whole, it appears that non-perturbative improvement leads to better agreement between the continuum result for $f_K$ and the experimental value.

The systematic nature of non-perturbative improvement makes it easily applicable to other situations: results for $c_{sw}$ computed for two flavours of dynamical quarks have been reported [26]. Furthermore, the formalism has been applied to quenched QCD with an improved gauge action [27].

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REFERENCES

4. P. Hasenfratz, these proceedings, hep-lat/9709110
8. M. Guagnelli and R. Sommer, these proceedings, hep-lat/9709088
9. M. Göckeler et al., hep-lat/9707021; P. Stephenson, these proceedings; D. Pleiter, these proceedings, hep-lat/9709016
10. P.A. Rowland (UKQCD Collab.), these proceedings
11. T. Mendes, these proceedings; M. Guagnelli, private communication
14. G. Martinelli et al., hep-lat/9705018
15. G. de Divitiis, these proceedings
16. S. Sharpe, these proceedings
26. K. Jansen and R. Sommer, these proceedings, hep-lat/9709022
27. R. Edwards and T. Klassen, these proceedings