Lattice HQET Calculation of the Isgur-Wise Function

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We calculate the Isgur-Wise function on the lattice, simulating the light quark with the Wilson action and the heavy quark with a direct lattice implementation of the heavy-quark effective theory. Improved smearing functions produced by a variational technique, MOST, are used to reduce the statistical errors and to minimize excited-state contamination of the ground-state signal. Calculating the required matching factors, we obtain \( \xi'(1) = -0.64(13) \) for the slope of the Isgur-Wise function in continuum-HQET in the \( \overline{MS} \) scheme at a scale of 4.0 GeV.

1. The Tadpole-Improved Simulation

The Isgur-Wise function is the form factor of a heavy-light meson in which the heavy quark is taken to be much heavier than the energy scale, \( m_Q \gg \Lambda_{QCD} \). This calculation adds perturbative corrections to the simulation results of Draper & McNeile [1]. The Isgur-Wise function is calculated using the action first suggested by Mandula & Ogilvie [2]:

\[
iS = \sum_x \left\{ v_0 \left[ \psi_0^\dagger(x) \psi_0(x) - \psi_0^\dagger(x) \frac{U_t(x)}{u_0} \psi_0(x + \hat{t}) \right] + \sum_{j=1}^{3} \frac{-i v_j}{2} \left[ \psi_0^\dagger(x) U_j \psi_0 \psi_0(x + \hat{t}) - \psi_0^\dagger(x - \hat{j}) \frac{U^\dagger_j}{u_0} \psi_0 \psi_0(x - \hat{j}) \right] \right\}
\]

This leads to the evolution equation:

\[
G(x + \hat{t}) = \frac{U_t^\dagger(x)}{u_0} \left\{ G(x) - \sum_{j=1}^{3} \frac{i v_j}{2} \left[ \frac{U_j^\dagger(x)}{u_0} G(x + \hat{j}) - \frac{U^\dagger_j(x - \hat{j})}{u_0} G(x - \hat{j}) \right] \right\}
\]

where \( \bar{v}_j = \frac{v_j}{u_0} \) and \( G(\bar{x}, t = 0) = \frac{1}{u_0} f(\bar{x}) \). Better results can be obtained when a smeared source is used in place of a point source. The smearing function, \( f(\bar{x}) \), was calculated from a static simulation \( (v_j = 0) \) via the smearing technique MOST (Maximal Operator Smearing Technique [3]).

2. The Isgur-Wise Function

The Isgur-Wise function was extracted from the lattice simulation as a ratio of three-point functions which was suggested by Mandula & Ogilvie [2]. \( \left| \langle \text{lat} (v \cdot v') \rangle \right|^2 \) is the large \( \Delta t \) limit of

\[
\frac{\Delta t}{(v_0 + v'_0)^2} \frac{C_{3}^{wv'}(\Delta t) C_{3}^{wv'}(\Delta t)}{C_{3}^{wv}(\Delta t) C_{3}^{wv}(\Delta t)}
\]

where \( \Delta t \) is the time separation between the current operator and each \( B \)-meson interpolating field.

Draper & McNeile have presented [1] the non-tadpole-improved unrenormalized slope of the lattice Isgur-Wise function to demonstrate the efficacy of the computational techniques.

3. Tadpole Improvement for HQET

Tadpole improvement grew from the observation that lattice links, \( U \), have mean field value, \( u_0 \neq 1 \). Therefore, it is better to use an action written as a function of \( \langle \bar{d} d \rangle \). In the Wilson action, each link has a coefficient \( \kappa \); \( u_0 \) can be paired with \( \kappa \) to easily tadpole improve a posteriori any previous non-tadpole-improved calculation.
In the HQET, there is no common coefficient (analogous to $\kappa$) for both $U_i$ and $U_j$. Correspondence between tadpole-improved and non-tadpole-improved HQET actions with $v^2 = 1$ cannot be made via a simple rescaling of parameters, as is done for the Wilson action with $\kappa$.

Fortunately, the evolution equation can be written (as noticed by Mandula & Ogilvie [4]) such that the $u_0$ is grouped with $v_0$. Thus, tadpole-improved Monte-Carlo data can be constructed from the non-tadpole-improved data by replacing $v_{nt} \to v^{\text{tad}}$ and by including two overall multiplicative factors:

$$G^{\text{tad}}(t; \tilde{v}^{\text{tad}}, v_0^{\text{tad}}) = u_0^{-t} v_0^{nt} G^{nt}(t; \tilde{v}^{nt}, v_0^{nt})$$

In addition to the multiplicative factors $u_0^{-t}$ and $v_0^{nt}/v_0^{\text{tad}}$, the tadpole-improvement of the simulation requires adjusting the velocity according to $v^{\text{tad}} = u_0 v^{nt}$, subject to $(v^{\text{tad}})^2 = 1$ and $(v^{nt})^2 = 1$. Thus,

$$v_{0}^{\text{tad}} = v_0^{nt} [1 + (1 - u_0^2)(v_0^{nt})^2]^{-\frac{1}{2}}$$

$$v_{j}^{\text{tad}} = v_j^{nt} [1 + (1 - u_0^2)(v_j^{nt})^2]^{-\frac{1}{2}}$$

4. Tadpole-Improved Renormalization

By comparing the unrenormalized propagator

$$\left[ v_{0}^{b} (v_{0}^{ik4} - 1) + \frac{v_{z}^{b}}{u_0} \sin(k_z) + M_{0}^{b} - \Sigma(k, v) \right]^{-1}$$

to the renormalized propagator

$$iH(k, v) = Z_Q \left[ v_{0}^{b} (ik_{z}) + v_{z}^{b} (k_{z}) + M_{0}^{b} \right]^{-1}$$

at $O(k^2)$ and using $(v^2)^2 = (v^b)^2 = 1$, the perturbative renormalizations can be obtained. Aglietti [5] has done this for a different non-tadpole-improved action, for the special case $\tilde{v} = v_2 \tilde{z}$.

With momentum shift, $p' = p + i \ln(u_0) \tilde{p}$ [6], with $\frac{1}{u_0} \exp(i k z) = \exp(i k z + i \ln(u_0))$ and with $X_p \equiv \frac{\partial}{\partial p} \Sigma(p)$, the tadpole-improved perturbative renormalizations are found to be

$$\delta M = -\Sigma(0) - u_0 \ln(u_0)$$

$$\delta Z_Q = Z_Q - 1 = -i u_0 X_4 - u_0 \sum_{j=1}^{3} v_j X_j$$

$$\delta v_{0} = -i v_{0} \frac{v_{j}}{v_0} X_4 - (1 + v_{j}^2) X_4 - u_0 \sum_{j \neq i} v_j X_j$$

$$\delta v_{j} = -i \sum_{j=1}^{3} v_j X_4 - u_0 v_0 \sum_{j=1}^{3} v_j X_j$$

$u_0$ is the perturbative expansion and

$$v_{j}^{\text{tad}} = v_{j}^{nt} \frac{Z_{v_j}^{\text{tad}}}{Z_{v_j}} , \quad v_{0}^{\text{tad}} = v_{0}^{nt} \frac{Z_{v_0}^{\text{tad}}}{Z_{v_0}}$$

$$Z_{v_j}^{\text{tad}} \equiv 1 - \frac{\delta v_{j}}{v_0}$$

If one fits to $\exp(-t)$ rather than $\exp(-(t+1))$, the tadpole-improved wave-function renormalization is reduced to $\delta Z_Q^{\text{tad}} = \delta Z_Q + \delta M^{\text{tad}} + u_0 \ln(u_0))/v_0$. Thus the $\ln(u_0)$ term cancels explicitly and, as in the static case [6], tadpole-improvement has no effect on $\delta Z_Q$, to order $\alpha$.

5. Perturbative Renormalizations

We will present our computations of the renormalization factors elsewhere, but include this comment: Although the tadpole-improved functions include factors of $u_0^{nt}$ [7], these effects are higher order in $\alpha$ and are dropped. Only the velocity renormalization is explicitly affected by the perturbative expansion of $u_0$:

$$Z_{v_j}^{\text{tad}} = \left( 1 + \frac{\delta v_{j}}{v_0} \right) \frac{q_0^2 C_F}{16\pi^2}$$

The perturbative renormalizations favor a scale of $q^* a = 1.9(1)$ for $\alpha$, which yields $\alpha \approx 0.19(1)$.

6. Velocity Renormalization

Mandula & Ogilvie [4] consider the perturbative velocity renormalization expanded in orders of $\tilde{v}$. Our numbers for the velocity renormalization agree with theirs.

Another option is to consider, as did both Mandula & Ogilvie and Hashimoto & Matsufuru [8], the non-perturbative renormalization of the velocity. From Hashimoto’s & Matsufuru’s graph, we estimate their $Z_{v,\text{np}}^{\text{tad}} \approx 1.05(5)$. From Mandula & Ogilvie’s result, we notice that $Z_{v,\text{np}}^{\text{tad}} = \left( \frac{v_0 \delta v_0}{u_0^3} \right)$.

4Note: $(v^a = 1) \Rightarrow \left( v_0 \delta v_0 = u_0^3 \sum_j \frac{v_j}{u_0} v_{j}^{\text{tad}} \right)$. 

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\(u_0 \times 1.01(1)\). This is very close to the effect of tadpole-improving, and implies \(Z_\xi^{\text{adj}} = 1.01(1)\).

We therefore use \(Z_\xi^{\text{adj}} \approx 1\) as the non-perturbative velocity renormalization in our calculation to renormalize the slope of the Isgur-Wise function.

7. Renormalization of \(\xi(v \cdot v')\)

We claim that we can convert our Monte-Carlo data into tadpole-improved results and can calculate a renormalized tadpole-improved slope for the Isgur-Wise function.

We use the notation \(Z_\xi' = 1 + \delta Z_\xi\) for the renormalization of the Isgur-Wise function, with \(\delta Z_\xi\):

\[
g_2^2 \frac{1}{2} (1 - v \cdot v') \ln(\mu a)^2 - f'(v, v') \]

with \(r(v \cdot v')\) defined in [9] and primes on \(Z\) and \(f\) to indicate the “reduced value.”

For simplicity, we use the local current, which is not conserved on the lattice; \(Z_\xi'(1) \neq 1\). However, the construction in §2 guarantees that the extracted renormalized Isgur-Wise function is properly normalized, \(\xi_{\text{ren}}(1) = 1\).

8. Conclusions

After renormalization of our tadpole-improved results, we obtain \(\xi_{\text{ren}}(1) = -0.64(13)\) for the slope of the Isgur-Wise function in continuum HQET in the \(\overline{\text{MS}}\) scheme at a scale of 4.0 GeV. Without renormalization, the slope is \(\xi_{\text{unren}}(1) = -0.56(13)\). Without tadpole-improvement, the slope is \(\xi_{\text{ren}}(1) = -0.43(10)\).

We found that the tadpole-improved action (and therefore the tadpole-improved data) cannot be obtained from the non-tadpole-improved action (or data) by a simple rescaling of any parameter. However, the form of the evolution equation allows the construction of the tadpole-improved Monte-Carlo data from the non-tadpole-improved data as described in §3. After tadpole improvement, non-perturbative corrections to the velocity are negligible. Furthermore, tadpole improvement greatly reduces the perturbative corrections to the slope of the Isgur-Wise function.

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| Tadpole Improved              |
| \(\Delta t\)                 |
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| Tadpole Improved              |
| \(\Delta t\)                 |
| 2                             |
| 3                             |
| 4                             |

Table 1

The negative of the slope at the normalization point, \(\xi'(1)\), from both the unrenormalized and the renormalized ratio of three-point functions. This ratio gives the (un)-renormalized Isgur-Wise function \(\xi(v^2)\) at asymptotically-large times \(\Delta t\).

REFERENCES

4. J.E. Mandula and M.C. Ogilvie, hep-lat/9602004, hep-lat/9703020.