On convergence of the Schwinger — DeWitt expansion

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Abstract

The Schwinger — DeWitt expansion for the evolution operator kernel of the Schrödinger equation is studied for convergence. It is established that divergence of this expansion which is usually implied for all continuous potentials, excluding ones of the form \( V(q) = aq^2 + bq + c \), really takes place only if the coupling constant \( g \) is treated as independent variable. But the expansion may be convergent for some kinds of the potentials and for some discrete values of the charge, if the latter is considered as fixed parameter. Class of such potentials is interesting because inside of it the property of discreteness of the charge in the nature is reproduced in the theory in natural way.

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1 Introduction

The short-time Schwinger — DeWitt expansion is used in the quantum theory for various purposes [1, 2, 3, 4, 5]. As usually it is treated as asymptotic one, so as other expansions in different parameters: conventional perturbation theory [6, 7], the WKB-expansion, $1/n$-expansion [8] etc. We mean under the Schwinger — DeWitt expansion following representation of the solution of the Schrödinger equation for the evolution operator kernel

$$i \frac{\partial}{\partial t} \langle \vec{q}', t | \vec{q}, 0 \rangle = -\frac{1}{2} \sum_{\nu=1}^{3} \frac{\partial^2}{\partial q_{\nu}^2} \langle \vec{q}', t | \vec{q}, 0 \rangle + V(\vec{q}') \langle \vec{q}', t | \vec{q}, 0 \rangle \quad (1)$$

with initial condition

$$\langle \vec{q}', t = 0 | \vec{q}, 0 \rangle = \delta(\vec{q}' - \vec{q}). \quad (2)$$

The kernel $\langle \vec{q}', t | \vec{q}, 0 \rangle$ is written as

$$\langle \vec{q}', t | \vec{q}, 0 \rangle = \frac{1}{(2\pi i t)^{3/2}} \exp \left\{ \frac{i(\vec{q}' - \vec{q})^2}{2t} \right\} F(t; \vec{q}', \vec{q}), \quad (3)$$

and $F$ according to [1, 2, 3, 9, 10] is expanded in powers of $t$

$$F(t; \vec{q}', \vec{q}) = \sum_{n=0}^{\infty} (it)^n a_n(\vec{q}', \vec{q}), \quad (4)$$

Here and everywhere below dimensionless values defined in obvious manner are used. The potential $V(\vec{q})$ is continuous function.

Factor in front of $F$ in R.h.s. of (3) is the kernel for the free theory, i.e. for $V \equiv 0$. Behavior of relation $\langle \vec{q}', t | \vec{q}, 0 \rangle / \langle \vec{q}', t | \vec{q}, 0 \rangle |_{V=0}$ when $t \to 0$ was studied in [11] for wide class of potentials and for $t = -i\tau, \tau > 0$. It was established that this relation tends to 1 for $\tau \to 0$. This fact may serve as justification of representation (4) with $a_0 = 1$.

Because the expansion (4) is usually considered as asymptotic, it is naturally that only the problem of character of asymptotic growth was studied. E.g., estimates from above were obtained for the coefficients $a_n$ [5, 10]. These estimates show that $a_n < \Gamma(bn) (0 < b \leq 1)$ as $n \to \infty$. But in general case they do not prove that divergence certainly takes place for every potential. Specifically, in [12, 13] some potentials were established for which the Schwinger — DeWitt expansion is convergent for definite values of the charge. So, it is interesting to study the problem: when convergence is possible and when is not? Namely this problem is under consideration at present paper.

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2 Quantum mechanics in one-dimensional space

2.1 General prescription

Let us consider the Cauchy problem for the evolution kernel for the Schrödinger equation in one-dimensional space

\[ i \frac{\partial}{\partial t} \langle q', t | q, 0 \rangle = -\frac{1}{2} \frac{\partial^2}{\partial q'^2} \langle q', t | q, 0 \rangle + V(q') \langle q', t | q, 0 \rangle, \]  

(5)

\[ \langle q', t = 0 | q, 0 \rangle = \delta(q' - q). \]  

(6)

We represent the kernel in the form

\[ \langle q', t | q, 0 \rangle = \frac{1}{\sqrt{2\pi it}} \exp \left\{ i \frac{(q' - q)^2}{2t} \right\} F(t; q', q), \]  

(7)

where \( F \) is given by the short-time expansion

\[ F(t; q', q) = \sum_{n=0}^{\infty} (it)^n a_n(q', q). \]  

(8)

The coefficient functions \( a_n(q', q) \) may be determined from the sequence of recurrent relations

\[ a_0(q', q) = 1, \]  

(9)

\[ na_1(q', q) + (q' - q) \frac{\partial a_1(q', q)}{\partial q'} = a_1(q', q') = -V(q'), \]  

(10)

and

\[ na_n(q', q) + (q' - q) \frac{\partial a_n(q', q)}{\partial q'} = \frac{1}{2} \frac{\partial^2 a_{n-1}(q', q)}{\partial q'^2} - V(q') a_{n-1}(q', q) \]  

(11)

for \( n > 1 \). Eqs. (9)–(11) show that \( a_n \) can be calculated via \( a_{n-1} \) by means of integral relations

\[ a_n(q', q) = \frac{1}{\eta^{n-1}} \eta^n \left\{ \frac{1}{2} \frac{\partial^2}{\partial x^2} - V(x) \right\} a_{n-1}(x, q) \bigg|_{x=q+(q'-q)\eta}. \]  

(12)

Combinations of eqs. (12) for different numbers \( n \) allows us to represent \( a_n \) for given \( n \) through the potential \( V \) and its derivatives

\[ a_n(q', q) = -\frac{1}{\eta_{n-1}^{n-1}} \eta_n \left\{ \frac{1}{2} \frac{\partial^2}{\partial x_n^2} - V(x_n) \right\} \times \left\{ \frac{1}{2} \frac{\partial^2}{\partial x_{n-1}^2} - V(x_{n-1}) \right\} \cdots \left\{ \frac{1}{2} \frac{\partial^2}{\partial x_2^2} - V(x_2) \right\} V(x_1). \]  

(13)
Here \( x_i = q + (x_{i+1} - q) \eta_i \), \( x_{n+1} = q' \). Derivatives with respect to different \( x_i \) may be easily connected with each other

\[
\frac{\partial}{\partial x_i} = \eta_{i-1} \frac{\partial}{\partial x_{i-1}} = \eta_{i-1} \eta_{i-2} \frac{\partial}{\partial x_{i-2}}
\]

(14)

e等. We use (13) for evaluation the behavior of \( a_n \) for \( n \to \infty \).

The potential \( V(q) \) contains the coupling constant \( g \) as multiplier. If \( g \) is independent variable, as it is treated in conventional perturbation theory, then for convergence of the expansion (8) one is to demand that contribution into \( a_n \) proportional to \( g^l \) behaves itself for every \( l \) such that the series in powers of \( t \) is convergent. I.e., cancellations between contributions into \( a_n \), containing \( g \) in different powers are not possible. So, it is convenient to calculate not hole coefficients \( a_n \), but only the terms containing \( g^l \), which we denote as \( a_n^l \).

The representation (13) allows us to analyse the structure of \( a_n^l \) and to understand sources of arising the factorial growth of \( a_n^l \) for \( n \to \infty \). We start from previous consideration.

### 2.2 Previous consideration

If one opens brackets in (13) and takes only terms proportional to \( g^l \), i.e., containing \( l \) multipliers of type \( V(x_i) \), then one gets

\[
\frac{(n-1)!}{(n-l)!(l-1)!}
\]

addends of the form

\[
\int_0^1 D_n \eta \frac{1}{2n-l} W(x_n)W(x_{n-1}) \cdots W(x_2)V(x_1),
\]

(15)

where \( W(x_i) \) is either \( V(x_i) \) or \( \frac{\partial^2}{\partial x_i^2} \),

\[
D_n \eta = \eta_{n-1}^2 d\eta_n \eta_{n-1} d\eta_{n-2} \cdots \eta_2 d\eta_2 d\eta_1
\]

And besides number of multipliers \( W(x_i) = V(x_i) \) at every term is equal to \( l - 1 \), and number of multipliers \( W(x_i) = \frac{\partial^2}{\partial x_i^2} \) is equal to \( n - l \). Total order of derivatives acting on all \( l \) multipliers \( V(x_i) \) is equal to \( 2(n-l) \).

To produce differentiating in (15) it is convenient to represent the operators \( \frac{\partial}{\partial x_i} \) via other operators \( \frac{\partial}{\partial x_j} \) with a help of relations of type (14) so that only derivatives

\[
\frac{\partial^m V(x_j)}{dx_j^m} \equiv V^{(m)}(x_j)
\]
will be present at all expressions. During this transform additional factors of type \( \eta^i_k \) will appear (we call them \( \eta \)-factors). From every term of the form (15) one will get \( l^2(n-l) \) terms of type

\[
(-1)^l \int_0^1 D_n \eta \eta^{k_{n-1}} \ldots \eta^{k_2} \eta^{k_1} \frac{\partial^{2(n-l)}}{\partial x_{l+1}^2} V^{(m_1)}(x_{l+1}) \ldots V^{(m_l)}(x_1). \tag{16}
\]

Let us evaluate the contribution of \( \eta \)-factors \( \eta^{k_{n-1}} \ldots \eta^{k_2} \eta^{k_1} \) into (16). We put for a moment all \( V^{(m)}(x) \) independent on \( \eta_i (i = 1, \ldots, n) \). Then the integrals with respect to \( \eta_i \) may give maximal factor \( 1/n! \) (when in (16) every \( k_i = 0 \)) and minimal factor \( (n-1)!!/(2n-1)! \) (when every \( k_i \) is equal to maximal possible value \( k_i = 2(n-i) \)). Because

\[
\frac{(n-1)!}{(2n-1)!} \sim \frac{1}{4^n n!}
\]
as \( n \to \infty \), then it is not essentially what exact set of \( k_i \) is used in (16).

Among all terms of kind (15) the following term gives the largest contribution as \( n \to \infty \)

\[
(-1)^l \int_0^1 D_n \eta \sum_{\{m_i\}} \frac{(2(n-l))!}{m_1! \ldots m_l!} V^{(m_1)}(x_1) \ldots V^{(m_l)}(x_l), \tag{17}
\]

It is so, because after differentiating in (17) it will appear maximum number of the terms of type (16). If one ignores presence of \( \eta \)-factors evaluating (17), i.e., puts every \( \eta_i \) equal to 1 (but not in \( D_n \eta! \)), then one gets

\[
(-1)^l \int_0^1 D_n \eta \sum_{\{m_i\}} \frac{(2(n-l))!}{m_1! \ldots m_l!} V^{(m_1)}(x_1) \ldots V^{(m_l)}(x_l), \tag{18}
\]

where following notation is used

\[
\sum_{\{m_i\}} \equiv \sum_{m_1=0}^{2(n-l)} \sum_{m_2=0}^{2(n-l)-m_1} \ldots \sum_{m_{l-1}=0}^{2(n-l)-m_1-\ldots-m_{l-2}} \sum_{m_l=2(n-l)-\sum_{i=1}^{l-1} m_i}. \]

Suppose that in some domain the expansion

\[
V(x) = \sum_{m=0}^{\infty} \frac{V^{(m)}(q)}{m!} (x-q)^m = \sum_{m=0}^{\infty} C_m(q)(x-q)^m \tag{19}
\]
takes place. Then instead of (18) we have

\[
(-1)^l \frac{(2(n-l))!}{2n-l} \int_0^1 D_n \eta \sum_{\{m_i\}} C_{m_1}(x_1) \ldots C_{m_l}(x_l). \tag{20}
\]
Number of addends in $\sum_{\{m_i\}}$ is equal to
\[
\frac{(2n-l-1)!}{(2(n-l))!(l-1)!} < 2^{2n-l-1}.
\]
Hence, this cause cannot lead to asymptotic growth of type $n!$.

If we choose among all terms in (20) only one with $m_i = \lfloor 2(n-l)/l \rfloor$, $i = 1, \ldots, l$ ([$\ldots$] denotes integer part of number; for the sake of brevity we will consider that all $m_i$ are equal to each other; more strictly, it is necessary to put some $m_i$ equal to $\lfloor 2(n-l)/l \rfloor + 1$, because condition $\sum_{i=1}^{l} m_i = 2(n-l)$ is to be fulfilled, but for our asymptotic estimates this is not essential) and evaluate the functions $C_m(x)$ by the constants $C_m \equiv \max_{\{x\}} C_m(x)$, then we get for this term estimate
\[
\frac{(2(n-l))!}{2^{2n-l-1}n!} C_l^{[2(n-l)/l]}. \tag{21}
\]

Consider the contributions, containing $g$ in power $l = \lfloor n\alpha \rfloor$, where $\alpha$ is any number from the interval $0 < \alpha < 1/2$. For such $l$ the contribution (21) behaves itself for $n \to \infty$ as $\Gamma(n(1-2\alpha))$, which means that the series (8) diverges. So, we see that there are contributions into $a_n$, which has factorial growth as $n \to \infty$.

Now we accurately will study when such divergence really take place. For proving of divergence of the expansion (8) it is enough to prove that $a_n(q', q)$ rises as $\Gamma(bn)$ at least for any values of $q', q$ from analytic domain of the function $V(q)$. Practically it is convenient to consider $q' = q$. In this case in (20) all $x_i = q$ and $\sum_{\{m_i\}} C_{m_1}(q) \ldots C_{m_l}(q)$ does not depend on integration variables $\eta_i$ and integrals can be exactly calculated.

### 2.3 Possible cancellations

Generally speaking, different contributions into $a_n^{(l)}$, having asymptotic growth of type $\Gamma(bn)$, may cancel each other because of different signs of $V^{(m)}(q)$ for different $m$, so that hole coefficient $a_n^{(l)}$ will not have such behavior. At first, we show that such cancellations really do not take place.

The most simple case is one when all $V^{(m)}(q)$ either have the same sings or $V^{(m)}(q) = (-1)^m |V^{(m)}(q)|$. Then all structures $V^{(m_1)}(q) \ldots V^{(m_l)}(q)$ for every set of $m_i$ have the same sign and no any cancellations can occur.

Consider general case, when the signs of $V^{(m)}(q)$ depend on $m$ in arbitrary way. Evaluate $a_n^{(l)}(q, q)$
\[
|a_n^{(l)}(q, q)| = \left| \sum_{\{m_i\}} A_{\{m_i\}}^{nl} V^{(m_1)}(q) \ldots V^{(m_l)}(q) \right|, \tag{22}
\]
where the coefficients $A_{\{m_i\}}^{nl}$ take into account contributions of all terms of kind (16) (notation $\{m_i\}$ in index of $A$ is used instead of $m_1, \ldots, m_l$). As it is clear from
previous estimates, there are such $A_{\{m_i\}}^{nl}$ that behave themselves as $\Gamma(n(1-2\alpha))$ when $n \to \infty$ for $l = \lfloor n\alpha \rfloor$, $0 < \alpha < 1/2$. Assume that for some $q$, e.g., for $q = 0$, the signs of derivatives $V^{(m)}$ are such that $|a'_n(0,0)|$ increase not strongly then $n^c B^n$ ($c$ and $B$ are any positive constants) or even decrease for $n \to \infty$. Let us test conservation of this property for $q \neq 0$. We represent

$$V^{(m)}(q) = \sum_{k=0}^{\infty} \frac{V^{(m+k)}(0)}{k!} q^k$$  \hspace{1cm} (23)

and substitute (23) into (22)

$$|a'_n(q,q)| = \left| \sum_{k=0}^{\infty} q^k \sum_{\{m_i\}} A_{\{m_i\}}^{nl} \times \sum_{k_1=0}^{k} \sum_{k_2=0}^{k-k_1} \cdots \sum_{k_{i-1}=0}^{k-k_{i-1}-\cdots-k_{i-2}} \frac{V^{(m_1+k_1)}(0) \cdots V^{(m_i+k_i)}(0)}{k_1! \cdots k_i!} \right|_{k_i = k - \sum_{i=1}^{l-1} k_i}.  \hspace{1cm} (24)$$

Because $q$ is independent variable then the coefficients in front of every power of $q$ should have property mentioned above.

The coefficient in front of $q^0$ is equal to

$$\sum_{\{m_i\}} A_{\{m_i\}}^{nl} V^{(m_1)}(0) \cdots V^{(m_l)}(0)  \hspace{1cm} (25)$$

and according to our supposition it may increase not strongly then $n^c B^n$ because of cancellations (nevertheless, among $A_{\{m_i\}}^{nl}$ are such ones which increase as $\Gamma(n - 2l)$).

Consider, for example, the coefficient in front of first power of $q$

$$\sum_{\{m_i\}} A_{\{m_i\}}^{nl} \left\{ V^{(m_1+1)}(0)V^{(m_2)}(0) \cdots V^{(m_l)}(0) + V^{(m_1)}(0)V^{(m_2+1)}(0) \cdots V^{(m_l)}(0) + \cdots + V^{(m_1)}(0) \cdots V^{(m_l+1)}(0) \right\} =$$

$$l \sum_{\{m_i\}} A_{\{m_i\}}^{nl} V^{(m_1+1)}(0)V^{(m_2)}(0) \cdots V^{(m_l)}(0).  \hspace{1cm} (26)$$

In last equality the symmetry of $A_{\{m_i\}}^{nl}$ in indexes $m_i$ for $q' = q$ is taken into account. Expression (26) differs from (25) (besides nonessential factor $l$) only by replacement of $V^{(m_1)}(0)$ by $V^{(m_1+1)}(0)$. The derivatives of order $m_1 + 1$ have for some $m_1$ the same sign with $V^{(m_1)}(0)$, and for other $m_1$ — opposite sing. I.e., in (25), where because of exact choice of signs and values of derivatives $V^{(m_i)}(0)$ contributions rising as factorial exactly cancel each other, the signs of many terms have been arbitrary changed. So, initial exact tuning will disappear and there will not be full compensation of rising contributions. Even if in any special case sufficient compensation of rising contributions will be accidentally conserved for the terms considered, then we should
have in mind infinite number of terms with $k > 1$ in (24). There exist some terms among them for which disbalance between contributions with opposite signs occur. Hence, $|a_n^i(q, q)|$ will increase for $q \neq 0$ as $\Gamma(n(1 - 2\alpha))$ even in case, when $|a_n^0(0, 0)|$ increases not strongly then $n^\epsilon B^n$.

## 2.4 Proof of divergence

Because there are no essential cancellations between different contributions into $d_n^i(q, q)$ it is enough for proving of divergence of the series (8) to indicate any term in $a_n^i(q, q)$ rising as factorial. The way of doing this depends on analytical properties of the potential $V(q)$. At first we consider the case when $V(q)$ is entire function of the complex variable $q$. Then the series (19) is convergent for any $|q| < \infty$ and convergence range is infinite.

Let us take (17) and, calculating expression of type (18), use $\eta$-factor in the form $\eta_{n-1}^2 \eta_{n-2}^4 \ldots \eta_2^{2(n-2)} \eta_1^{2(n-1)}$, i.e., we change exact powers $k_i$ of the variables $\eta_i$ on maximal possible values $k_i = 2(n - i)$, which gives estimate from below for the contribution considered. We get

$$
\frac{1}{2n-l} \int_0^1 \mathcal{D}_n \eta \eta_{n-1}^2 \eta_{n-2}^4 \ldots \eta_2^{2(n-2)} \eta_1^{2(n-1)} \sum_{\{m_i\}} \frac{(2(n-l))!}{m_1! \ldots m_l!} V^{(m_1)}(q) \ldots V^{(m_l)}(q) =
$$

$$
\frac{1}{2n-l} \frac{(n - 1)!(2(n-l))!}{(2n-1)!} \sum_{\{m_i\}} C_{m_1}(q) \ldots C_{m_l}(q). \tag{27}
$$

At the sum $\sum_{\{m_i\}}$ we take the term with $m_i = [2(n-l)/l]$, $i = 1, \ldots, l$ and consider the contribution with $l = \lfloor n\alpha \rfloor$, $0 < \alpha < 1/2$. This gives for $a_n^{\lfloor n\alpha \rfloor}$ asymptotic estimate

$$
\frac{(n - 1)!(2n(1-\alpha))!}{2n-\lfloor n\alpha \rfloor(2n-1)!} C_{\lfloor 2(1-\alpha)/\alpha \rfloor}^{\lfloor n\alpha \rfloor}(q) \sim \Gamma(n(1 - 2\alpha)), \tag{28}
$$

which is fair for every entire function $V(q)$ so as for function $V(q)$ analytical at any bounded domain of variable $q$, because $C_{\lfloor 2(1-\alpha)/\alpha \rfloor}^{\lfloor n\alpha \rfloor}(q)$ does not depend on number $n$ and so behavior of Taylor’s coefficients $C_k(q)$ does not affect essentially on the estimate (28) (it can arise only the factor of type $n^\epsilon B^n$, which does not change the qualitative character of asymptotics (28)).

So, we established such contribution into $a_n$, that increases as $\Gamma(n(1 - 2\alpha))$ and leads to divergence of the series (8). In reality there exist many of such contributions. One can choose different $\alpha$, i.e., different $l$. The less $\alpha$, the more strong asymptotic increase of corresponding contributions into $a_n$ will take place. Nevertheless, one cannot put $\alpha = 0$ because $l \geq 1$. To understand what is maximal growth of $a_n$ for $n \to \infty$ one is to admit $\alpha$ to decrease slowly with increasing of $n$, e.g., as $\alpha = 1/\log n$ (see [10]). And what’s more, always $l = \lfloor n/\log n \rfloor > 1$ and $\Gamma(n(1 - 2\log n)) \sim n!$ for $n \to \infty$. 

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We see that the coefficients of the expansion (8) increase as \( n! \) for \( n \to \infty \) excluding cases when \( V(q) \) is polynomial of power \( L \). In this case \( V^{(m)}(q) \neq 0 \) only for \( m \leq L \) and so contribution into \( a_{n}^{l} \) is not equal to zero only if \( 2(n-1) \leq Ll \) or \( L \geq 2n/(L+2) \). This gives additional restriction on \( \alpha \): \( 2/(L+2) \leq \alpha \leq 1/2 \). Hence, maximal growth of \( a_{n}^{l} \) is reached when \( \alpha = 2/(L+2) \) or \( l = [2n/(L+2)] \). It provides the behavior [10]

\[
|a_{n}| \sim \Gamma \left( n \frac{L - 2}{L + 2} \right). \tag{29}
\]

For the potential \( V(q) \), which is determined by the function analytical at any bounded domain, corresponding considerations are more simple. In this case the expansion (19) converges at the circle of finite radius \( R(q) \) and \(|C_{m}(q)| \sim 1/R^{m}(q)\) when \( m \to \infty \).

We will calculate \( a_{n}^{1}(q,q) \). It is clear from (12), that

\[
a_{n}^{1}(q,q) = - \int_{0}^{1} D_{n} \eta \frac{1}{2^{n-1}} \frac{\partial^{2(n-1)}}{\partial x_{n}^{2} \ldots \partial x_{2}^{2}} V(x_{1}) \bigg|_{x_{i} = q} = \\
- \frac{1}{2^{n-1}} \int_{0}^{1} D_{n} \eta \eta_{n-1}^{2} \eta_{n-2}^{4} \ldots \eta_{1}^{2(n-1)} V^{2(n-1)}(q) = - \frac{(n-1)!}{2^{n-1}(2n-1)} C_{2(n-1)}(q). \tag{30}
\]

Taking into account behavior of the coefficients \( C_{m}(q) \), we get estimate

\[
|a_{n}^{1}(q,q)| \sim \frac{(n-1)!}{2^{n-1}(2n-1) R^{2(n-1)}(q)} \sim n! \tag{31}
\]

for \( n \to \infty \). Because \( a_{n}^{1} \) contains only one term, the problem of possible cancellations does not arise at all.

Thus, if one suppose that the charge \( g \) is independent variable and so cancellations between \( a_{n}^{l} \) with different numbers \( l \) do not occur, then for every potential \( V(q) \) the coefficients \( a_{n} \) of the Schwinger — DeWitt expansion (7), (8) behave themselves as \( \Gamma(bn) \) when \( n \to \infty \) and, hence, the expansion is always divergent (naturally, excluding trivial case \( V(q) = aq^{2} + bq + c \)). The only way to get convergent expansion is following. One is to consider the charge \( g \) not as independent variable, but as a fixed parameter. Then cancellations between \( a_{n}^{l} \) with different \( l \) are possible. For some potentials and for special values of the charge \( g \) these cancellations are sufficient to provide convergence of the Schwinger — DeWitt expansion. The examples of such potentials was presented in [12, 13]

\[
V(q) = - \frac{g}{\cosh^{2} q}, \tag{32}
\]

\[
V(q) = \frac{g}{q^{2}}, \tag{33}
\]

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\[ V(q) = \frac{g}{\sinh^2 q}, \]  
\[ V(q) = \frac{g}{\sin^2 q}, \]  
\[ V(q) = aq^2 + \frac{g}{q^2}. \]

The potentials (33) – (36) have singularity at \( q = 0 \) which does not allow us to use for them directly the formalism described above. But this formalism can be easily modified for application to singular potentials [14]. One should take instead of initial condition (6) the following one

\[ \langle q', t = 0 | q, 0 \rangle = \delta(q' - q) + A\delta(q' + q) \]

which may provide fulfillment of boundary condition for the wave function \( \psi(q) \) at \( q = 0 \) (\( \psi(q) \) should vanish at \( q = 0 \)) by appropriate choice of constant \( A \). Constant \( A \) is determined by requirement that the kernel does not have singularity at \( q = 0 \) or \( q' = 0 \) (\( t \neq 0 \)). In correspondence with (37) the kernel is represented through two functions \( F^{(\pm)} \) as

\[ \langle q', t | q, 0 \rangle = \frac{1}{\sqrt{2\pi it}} \exp \left\{ \frac{i(q' - q)^2}{2t} \right\} F^{(-)}(t; q', q) + \]
\[ A\frac{1}{\sqrt{2\pi it}} \exp \left\{ \frac{i(q' + q)^2}{2t} \right\} F^{(+)}(t; q', q), \]

where \( F^{(\pm)} \) can be expanded analogously to (8).

The function \( F^{(-)} \) may be calculated in the same way as ordinary function \( F \) discussed before. Calculation of \( F^{(+)\mbox{}} \) slightly differs from one of \( F^{(-)} \), but in our consideration function \( F^{(+)\mbox{}} \) is not essential. If we wish to prove divergence of representation (38), (8) for the kernel it is enough to prove divergence of the expansion (8) only for function \( F^{(-)} \). Independently on the behavior of \( F^{(+)\mbox{}} \) representation (38), (8) will be divergent in this case. So, statement about divergence of the Schwinger — DeWitt expansion for arbitrary charge \( g \) remains fair and for singular potentials too.

## 3 Quantum mechanics in three-dimensional space

### 3.1 General consideration

Formalism used in the one-dimensional case can be easily transferred on the three-dimensional space. Let us write corresponding formulas. Equations for \( a_n(q', \bar{q}) \) defined by (1)–(4) are

\[ na_1(\bar{q}', \bar{q}) + \sum_{\nu=1}^3 (q'_\nu - q_\nu) \frac{\partial a_1(q', \bar{q})}{\partial q'_\nu} = a_1(q'', \bar{q}'') = -V(\bar{q}''), \]  

where \( V(q) \) is the potential function as before.
and

\[ na_n(q', \bar{q}) + \sum_{\nu=1}^{3} (q'_\nu - q_\nu) \frac{\partial a_n(q', \bar{q})}{\partial q'_\nu} = \frac{1}{2} \Delta q' \alpha_{n-1}(q', \bar{q}) - V(q') a_{n-1}(q', \bar{q}) \]  

(40)

for \( n > 1 \). The solution of (39)–(40) has a form

\[ a_n(q', \bar{q}) = -\int_0^1 \eta_n^{-1} d\eta_n \int_0^1 \eta_{n-1}^{n-2} d\eta_{n-1} \ldots \int_0^1 \eta_2 d\eta_2 \int_0^1 \eta_1 \left\{ \frac{1}{2} \Delta_n - V(\bar{x}_n) \right\} \times \]

\[ \left\{ \frac{1}{2} \Delta_{n-1} - V(\bar{x}_{n-1}) \right\} \ldots \left\{ \frac{1}{2} \Delta_2 - V(\bar{x}_2) \right\} V(\bar{x}_1), \]

(41)

where \( \bar{x}_i = q + (\bar{x}_{i+1} - \bar{q}) \eta_i, \bar{x}_{n+1} = q', \Delta_i = \sum_{\nu=1}^{3} \frac{\partial^2}{\partial x'_\nu} \) is Laplasian acting on functions of the variable \( \bar{x}_i \), index \( \nu = 1, 2, 3 \) is Cartesian index. Correspondence between various differential operators is analogous to (14).

Because structure of the representation (41) is the same as one for (13), then proof of divergence of the expansion (3), (4) may be done analogously to one-dimensional case. But now some complications will arise because of many-component character of the variables \( \bar{x}_i \).

So as earlier we will consider the contributions into \( a_n \) proportional to \( q' \), denoting them as \( a^l_n \). We treat at the beginning \( g \) as independent variable, so there are no cancellations between \( a^l_n \) with different \( l \). To show divergence of the series (4) it is enough to show divergence only for any special values of \( q', \bar{q} \), e.g., for \( q' = \bar{q} \).

Let us evaluate contribution into \( |a^l_n(q', \bar{q})| \) of the form

\[ \left| \frac{1}{2^{n-l}} \int_0^1 D_n \eta \Delta_n \Delta_{n-1} \ldots \Delta_{l+1} V(\bar{x}_l) \ldots V(\bar{x}_1) \right| = \left| \frac{1}{2^{n-l}} \int_0^1 D_n \eta \sum_{\nu=1}^{3} \sum_{\nu_{l+1}=1}^{3} \frac{\partial^2 (n-l)}{\partial x'_{n,\nu} \partial x'_{l+1,\nu+1}} V(\bar{x}_l) \ldots V(\bar{x}_1) \right|_{q' = \bar{q}}. \]

(42)

Due to transition to the derivatives of type

\[ V^{(m_1, m_2, m_3)}(\bar{x}_j) \equiv \frac{\partial^{m_1+m_2+m_3} V(\bar{x}_j)}{\partial x_{j_1}^{m_1} \partial x_{j_2}^{m_2} \partial x_{j_3}^{m_3}} \]

\( \eta \)-factors of the form \( \eta_{n-1}^{k_1} \ldots \eta_{l}^{k_1} \) arise, which after integration in case \( q' = \bar{q} \) give additional factor varying from \( n!(n-1)!/(2n-1)! \) to 1. This factor is not essential for asymptotic estimates, because it does not affect factorial growth. We take minimal possible value of this factor and get estimate from below

\[ \left| \frac{1}{2^{n-l}} \frac{(n-1)!}{(2n-1)!} \sum_{k_1=0}^{n-l} \sum_{k_2=0}^{n-l-k_1} \frac{(n-l)!}{k_1!k_2!k_3!} \frac{\partial^{2(n-l)} \rho^l(q)}{\partial q_{k_1}^{2k_1} \partial q_{k_2}^{2k_2} \partial q_{k_3}^{2k_3}} \right|_{k_3 = n-l-k_1-k_2}. \]

(43)
For the sake of simplicity we take among all terms of (43) only that ones, in which differentiating with respect to only one component of the vector $\vec{q}$, e.g., $q_1$, presents. Then in (43) one should put $k_1 = n - l$, $k_2 = k_3 = 0$ and produce differentiation. One gets after this

$$\frac{1}{2^{n-l}} \frac{(n-1)!}{(2n-1)!} \left| \sum_{\{m_i\}} \frac{(2(n-l))!}{m_1! \ldots m_l!} V^{(m_1,0,0)}(q) \ldots V^{(m_l,0,0)}(q) \right|. \tag{44}$$

Notation $\sum_{\{m_i\}}$ coincides with one in (18).

Taking into account the Taylor expansion (its validity at any domain is assumed)

$$V(\vec{x}) = \sum_{m_1=0}^{\infty} \sum_{m_2=0}^{\infty} \sum_{m_3=0}^{\infty} \frac{V^{(m_1,m_2,m_3)}(\vec{q})}{m_1! m_2! m_3!} (x_1 - q_1)^{m_1}(x_2 - q_2)^{m_2}(x_3 - q_3)^{m_3} = \sum_{m_1=0}^{\infty} \sum_{m_2=0}^{\infty} \sum_{m_3=0}^{\infty} C_{m_1,m_2,m_3}(\vec{q})(x_1 - q_1)^{m_1}(x_2 - q_2)^{m_2}(x_3 - q_3)^{m_3} \tag{45}$$

one can rewrite (44) in the form

$$\frac{1}{2^{n-l}} \frac{(n-1)!}{(2n-1)!} \left| \sum_{\{m_i\}} C_{m_1,0,0}(\vec{q}) \ldots C_{m_l,0,0}(\vec{q}) \right|. \tag{46}$$

From terms of the sum $\sum_{\{m_i\}}$ we choose only one, in which all $m_i$ are equal to $\lfloor 2(n-l)/l \rfloor$ and consider $l = \lfloor n\alpha \rfloor$ with $0 < \alpha < 1/2$. Then we get

$$|a_n^{[\alpha]}(\vec{q}, \vec{q})| \sim \frac{(n-1)!}{2^n - \lfloor n\alpha \rfloor} \frac{[2n(1-\alpha)]!}{(2n-1)!} \left( C_{[2(1-\alpha)/\alpha],0,0}(\vec{q}) \right)^{[\alpha]} \sim \Gamma(n(1-2\alpha)). \tag{47}$$

### 3.2 Proof of divergence

One is to show now that there are no essential cancellations between different contributions into $a_n^l$ which diminish factorial contributions of type (47). Such cancellations cannot be caused by different signs of derivatives $V^{(m_1,m_2,m_3)}$ of different orders $m_\nu$. Reasonings proving this fact almost exactly repeat reasonings of previous Section. Only notations become slightly complicated. Instead of (22) one should write

$$|a_n^l(\vec{q}, \vec{q})| = \sum_{\{m_{i,1}\}} \sum_{\{m_{i,2}\}} \sum_{\{m_{i,3}\}} A_{\{m_{i,1}\}}^l V^{(m_{i,1},m_{i,2},m_{i,3})}(\vec{q}) \ldots V^{(m_{i,1},m_{i,2},m_{i,3})}(\vec{q}) \left|, \tag{48}$$

and in expression analogous to (24) instead of expansion in $q^k$ one should take expansion in the structures $q_1^{k_1} q_2^{k_2} q_3^{k_3}$ and consider the coefficients for every set $\{k_\nu\}$.

In the three-dimensional case one more cause of cancellations of contributions arises. It is following. If the potential is harmonic function, i.e.,

$$\Delta V = 0, \tag{49}$$

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then, as it is clear from (41), many of addends in \(a_n\) are equal to zero. One is to show that it is enough of the contributions of the terms remaining different from zero to keep the asymptotic behavior (47).

In (44) the coefficient

\[
\frac{(2(n-l))!}{m_1! \ldots m_l!}
\]

has meaning of number of terms of the form \(V^{(m_1,0,0)}(\vec{q}) \ldots V^{(m_l,0,0)}(\vec{q})\) in \(\Sigma_{\{m_i\}}\). We chose \(m_i = \lfloor 2(n-l)/l \rfloor\) for every \(i\). Let us evaluate how many terms among total number

\[
\frac{(2(n-l))!}{\lfloor 2(n-l)/l \rfloor !}
\]

are different from zero, if \(V(\vec{q})\) satisfies Eq. (49). Here \(l\) multipliers \(V(\vec{q})\) are under action of \(n-l\) operators \(\Delta_i\) (we trace only differentiating with respect to \(x_{i_1}\)). The contribution will be different from zero if every multiplier \(V(\vec{x})\) in it is under action of differential operator which includes not more then one derivative from every operator \(\Delta_{l+1}, \ldots, \Delta_n\) (see (42)), i.e.,

\[
\frac{\partial^{2(n-l)/l}}{\partial x_{i_1} \ldots \partial x_{i_{2(n-l)/l}/l}}
\]

where \(i_j = l+1, \ldots, n\) and there are no equal numbers among \(i_j\). Let such differential operator acts on \(V(\vec{x}_1)\). The set \(\{i_j\}\) can be chosen by

\[
\frac{(n-l)!}{(n-l-m_i)!m_i!}
\]

ways. Considering in such manner \(l/2\) multipliers \(V(\vec{x}_i)\) (we mean \(l\) as even for simplicity and \(m_i = \lfloor 2(n-l)/l \rfloor\) here) we get the factor

\[
\frac{(n-l)!}{(n-l-m_i)!m_i!} \frac{(n-l-m_i)!}{(n-l-2m_i)!m_i!} \ldots \frac{(n-l-(\frac{1}{2}-1)m_i)!}{(n-l-\frac{1}{2}m_i)!m_i!} = \frac{(n-l)!}{(m_i!)^{l/2}}.
\]

Next \(l/2\) multipliers \(V(\vec{x}_i)\) are under action of derivatives from all operators \(\Delta_{l+1}, \ldots, \Delta_n\), which was not used before. This gives one more factor (51). As a result we obtain the estimate of number of terms different from zero

\[
\frac{(n-l)!}{(m_i!)^l} = \frac{(n-l)!}{\lfloor 2(n-l)/l \rfloor !^l}
\]

(not all contributions are taken into account here, but for our purposes this underestimate is quite sufficient). For \(l = \lceil n\alpha \rceil\) we get from (52)

\[
\frac{(n(1-\alpha)!)^2}{\lfloor 2(1-\alpha)/\alpha \rfloor !^{n\alpha}} \sim [2n(1-\alpha)]!.
\]
I.e., taken into consideration different from zero contributions in (52) really provide asymptotic behavior (47) causing divergence of the expansion (4).

Eq. (53) takes into account only part of increasing contributions. Maximal growth corresponds to the case $\alpha \to 0$. Assuming slight dependence of $\alpha$ on $n$ and putting $\alpha = 1/\log n$ one gets, so as in previous Section, that for $n \to \infty$

$$|a_n| \sim n!.$$

(54)

Such growth takes place for every potential excluding polynomial ones. If $V(\vec{q})$ is polynomial of order $L$, then $C_{m_1,m_2,m_3} = 0$ for $\sum_{\nu=1}^{3} m_{\nu} > L$. So, there is boundary from below for possible values of $l$: $l \geq 2n/(L + 2)$. Maximal growth of $a_{n}^{l}$ takes place when $l_{m} = [2n/(L + 2)]$. This leads to estimate

$$|a_{n}| \sim \Gamma \left( n \frac{L - 2}{L + 2} \right),$$

(55)

which coincides with corresponding result for one-dimensional theory. For $L \leq 2$ (harmonic oscillator, linear potential, free case) the expansion converges. For $L \geq 3$ (anharmonic oscillator) it is divergent.

So as in one-dimensional case, for the potentials determined by analytic, but not entire functions, i.e., by functions for which Taylor series (45) have finite convergence range, one can prove divergence by rather simple way. Consider

$$a_{n}^{1}(\vec{q}, \vec{q}) = -\frac{1}{2^{n-1}} \int_{0}^{1} D_{n} \eta \, \Delta_{n} \Delta_{n-1} \cdots \Delta_{2} V(\vec{x}_{1}) =$$

$$= -\frac{1}{2^{n-1}} \int_{0}^{1} D_{n} \eta \, \eta_{n-1}^{2} \eta_{n-2}^{4} \cdots \eta_{1}^{2(n-1)} \times$$

$$\sum_{k_{1}=0}^{n-1} \sum_{k_{2}=0}^{n-1-k_{1}} \frac{(n-1)!}{k_{1}! k_{2}! k_{3}!} V(2k_{1}, 2k_{2}, 2k_{3})(\vec{q}) \bigg|_{k_{3}=n-1-k_{1}-k_{2}}.$$

(56)

If $\sum_{\nu=1}^{3} 2k_{\nu} = 2(n - 1) \to \infty$ then the coefficients of the Taylor series for $V(\vec{q})$ behave themselves as

$$V(2k_{1}, 2k_{2}, 2k_{3})(\vec{q}) \sim \frac{(2k_{1})!(2k_{2})!(2k_{3})!}{R_{1}^{2k_{1}}(\vec{q}) R_{2}^{2k_{2}}(\vec{q}) R_{3}^{2k_{3}}(\vec{q})},$$

(57)

where $R_{\nu}(\vec{q})$ are conjugated convergence ranges of the expansion (45) (here we mean that for all $\nu$: $0 < R_{\nu}(\vec{q}) < \infty$). This gives estimate for $n \to \infty$

$$|a_{n}^{1}(\vec{q}, \vec{q})| \sim \frac{(n-1)!}{2^{n-1}(2n - 1)} \frac{1}{R(\vec{q})^{2(n-1)}} \sim n!.$$

(58)

Here $R(\vec{q}) = \max\{R_{1}(\vec{q}), R_{2}(\vec{q}), R_{3}(\vec{q})\}$. 

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If $V(\vec{q})$ is harmonic function then $a_1^n(\vec{q}, \vec{q}) = 0$. Nevertheless, in this case one can easily calculate $a_2^n(\vec{q}, \vec{q})$

$$a_2^n(\vec{q}, \vec{q}) = \frac{1}{2n-2} \int_0^1 D_n \eta \sum_{\nu_3=1}^3 \sum_{\nu_2=1}^3 \frac{\partial^{n-2}V(\vec{x}_2)}{\partial x_{n,\nu_2} \cdots \partial x_{3,\nu_3}} \frac{\partial^{n-2}V(\vec{x}_1)}{\partial x_{n,\nu_3} \cdots \partial x_{3,\nu_3}} \bigg|_{\vec{x}_i=\vec{q}}$$

$$= \frac{1}{2n-2} \int_0^1 D_n \eta \eta_{n-1}^2 \eta_{n-2}^2 \cdots \eta_3^{2(n-3)} \eta_2^{2(n-2)} \eta_1^{n-2} \times \sum_{k_1=0}^{n-2} \sum_{k_2=0}^{n-2-k_1} \frac{(n-2)!}{k_1!k_2!k_3!} \left( V^{(k_1,k_2,k_3)}(\vec{q}) \right)^2 \bigg|_{k_3=n-2-k_1-k_2}$$

Putting $k_1 = n - 2$, $k_2 = k_3 = 0$ we obtain estimate for $n \to \infty$

$$|a_2^n(\vec{q}, \vec{q})| > \frac{(n-2)!}{2^{n-2}(2(n-1))!} \left( V^{(n-2,0,0)}(\vec{q}) \right)^2 \sim \frac{(n-2)!}{2^{n-2}(2(n-1))!} \left( \frac{(n-2)!}{(n-2)^2 (\vec{q})} \right)^2 \sim n!.$$  

Hence, for harmonic potentials factorial growth of the coefficients $a_n$ takes place too.

So, analogously to the one-dimensional case, in the three-dimensional space the Schwinger — DeWitt expansion is divergent for all potentials (excluding trivial polynomials of order not higher than two) if the charge $g$ is considered as independent variable. If the charge is considered as fixed parameter, then for some kinds of the potentials and for some discrete values of the charge the expansion (3)–(4) may be convergent.

### 4 Conclusion

The results of our research may be summarized as following.

If we consider for the beginning the coupling constant $g$ of continuous potential $V(q)$ as independent variable, then the coefficients $a_n$ of representation (3)–(4) for the evolution operator kernel increase for $n \to \infty$ as

$$a_n \sim \Gamma \left( n \frac{L-2}{L+2} \right)$$

for the potentials being expressed via the polynomial of order $L$ and as

$$a_n \sim n!$$

for other ones. Hence, from this viewpoint the Schwinger — DeWitt expansion is divergent for all potentials excluding polynomials of order not higher than two.
This is not surprising fact. The expansion (3), (4) is usually considered as asymptotic. What about our considerations we can see that conclusion about divergence of the series (4) is valid only if we treat the charge $g$ as independent variable. But if the charge is treated as fixed parameter, then proof of divergence becomes not valid because of possibility of cancellations for terms with different powers of $g$ in $a_n(q', q)$. So, there is opportunity to avoid divergence. And really some potentials for which the expansion (4) is convergent for some discrete values of the charge $g$ are exist. Examples of such potentials are known from previous papers [12, 13]. In this case the function $F$ is analytic function of variable $t$ at $t = 0$ contrary to the case of independent charge, when the point $t = 0$ is essential singular point of $F$.

Discreteness of the charge for the class of the potentials for which the expansion is convergent, probably, may be connected with discreteness of the charge in the nature. In this correspondence, the potentials of this class are of special interest. Operating with them we get rid of some kind of divergences in the theory and, at the same time, have a theory with discrete charge. So, it seems to be necessary to look for other potentials of this class and study them carefully.

References