Mirror Maps in Chern-Simons Gauge Theory

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Abstract

We describe mirror symmetry in $N = 2$ superconformal field theories in terms of a dynamical topology changing process of the principal fiber bundle associated with a topological membrane. We show that the topological symmetries of Calabi-Yau sigma-models can be obtained from discrete geometric transformations of compact Chern-Simons gauge theory coupled to charged matter fields. We demonstrate that the appearance of magnetic monopole-instantons, which interpolate between topologically inequivalent vacua of the gauge theory, implies that the discrete symmetry group of the worldsheet theory is realized kinematically in three dimensions as the magnetic flux symmetry group. From this we construct the mirror map and show that it corresponds to the interchange of topologically non-trivial matter field and gauge degrees of freedom. We also apply the mirror transformation to the mean field theory of the quantum Hall effect. We show that it maps the Jain hierarchy into a new hierarchy of states in which the lowest composite fermions have the same filling fractions.

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1. Introduction

One of the most interesting consequences of $N = 2$ worldsheet supersymmetry in string theory is the occurrence of physically smooth spacetime topology changing processes. They result from a string duality called ‘mirror symmetry’ [1]–[6] in which two topologically distinct Calabi-Yau compactifications give rise to identical physical models. Heuristically, as the size of a given compactification is made small, huge curvature fluctuations modify the fabric of spacetime leading to a change in topology. The mirror transformation relating these two distinct geometrical formulations of the same physical situation connects strong to weak sigma-model coupling constants and, via a judicious choice of geometrical model, seemingly difficult physical questions can be analysed with perturbative ease. Mirror symmetry also leads to the notion of ‘quantum geometry’, which is the appropriate modification of standard, classical geometry to make it suitable for describing the spacetime physics implied by string theory, and it greatly simplifies the problem of computing the moduli space of $N = 2$ superconformal field theories which form the set of string vacua.

Several attempts have been made to provide a rigorous mathematical framework, for example using algebraic geometry and toric geometry, for the duality in Calabi-Yau moduli space implied by the existence of mirror manifolds. Conversely, mirror symmetry has been applied as a tremendously powerful calculational tool in these same branches of mathematics. In this paper we shall describe the mirror map in terms of a dynamical topology changing process acting on a principal fiber bundle over a 3-manifold $\mathcal{M}$. The main idea is to exploit the intimate relationship between two-dimensional conformal field theory and the three-dimensional topological field theory defined by the Chern-Simons action

$$kS_{\text{CS}}^{[G]}[A] = \frac{k}{4\pi} \int_{\mathcal{M}} \text{Tr} \left( A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right)$$

where $A$ is a connection on a principal fiber bundle over $\mathcal{M}$ with structure group $G$. The quantum field theory defined by (1.1) is equivalent to the Wess-Zumino-Novikov-Witten (WZNW) model at level $k \in \mathbb{Z}$ defined on a compact Riemann surface $\Sigma$. This can be seen at the level of the physical state space of (1.1) on the product manifold $\mathcal{M} = \Sigma \times \mathbb{R}$, which is naturally isomorphic to the finite-dimensional space of conformal blocks of the WZNW model [7], or in a path integral approach whereby the theory (1.1) defined on a 3-manifold $\mathcal{M}$ with boundary $\partial \mathcal{M} = \Sigma$ induces the chiral gauged WZNW model on $\partial \mathcal{M}$ [8].

The key feature of this correspondence is that various algebraic properties of two-dimensional conformal field theories on $\partial \mathcal{M}$ can be understood geometrically and dynamically in the three-dimensional approach. For instance, the $n$-point correlation functions of the conformal field theory can be decomposed

$$\left\langle \prod_{i=1}^{n} V_{L}^{(R_i)}(z_i, \bar{z}_i) \right\rangle = \left\langle \prod_{i=1}^{n} V_{L}^{(R_i)}(z_i) \right\rangle \left\langle \prod_{i=1}^{n} V_{R}^{(R_i)}(\bar{z}_i) \right\rangle$$

(1.2)
in terms of products of left and right conformal blocks, where $V^{(R_i)}_L(z_i)$ and $V^{(R_i)}_R(\bar{z}_i)$ are the holomorphic and anti-holomorphic chiral vertex operators corresponding to the left-right symmetric vertex operator $V^{(R_i)}(z_i, \bar{z}_i)$ in a representation $R_i$ of the group $G$. In the corresponding three-dimensional description we consider the 3-manifold $\mathcal{M} = \Sigma \times [0, 1]$ whose two boundaries $\Sigma_L$ and $\Sigma_R$ are connected by a finite time interval. A Chern-Simons gauge theory in $\mathcal{M}$ induces both left- and right-moving sectors of the two-dimensional conformal field theory, and an insertion of a vertex operator on the worldsheet $\Sigma$ is equivalent to insertions of the chiral vertex operators $V^{(R_i)}_L(z_i)$ and $V^{(R_i)}_R(\bar{z}_i)$ on the left- and right-moving worldsheets $\Sigma_L$ and $\Sigma_R$, respectively. The insertions corresponding to the correlation functions (1.2) are induced by path-ordered products of the open Wilson line operators $W^{(R_1, \ldots, R_n)}_{C_{z_1, \bar{z}_1}, \ldots, C_{z_n, \bar{z}_n}}[A^{(1)}, \ldots, A^{(n)}] = \prod_{i=1}^n \text{Tr} P \exp \left( i \int_{C_{z_i, \bar{z}_i}} A^{(i)a} R_{ia} \right) \quad (1.3)$

along the oriented paths $C_{z_i, \bar{z}_i} \subset \mathcal{M}$ with endpoints $z_i \in \Sigma_L$ and $\bar{z}_i \in \Sigma_R$. Correlators of insertions of the Wilson lines (1.3) in $\mathcal{M}$ induce phase factors from adiabatical rotation of charged particles coupled to the Chern-Simons gauge fields $A^{(i)}$ in the representations $R_i$. The quantum particles propagate along $C_{z_i, \bar{z}_i}$ from left- to right-moving worldsheets, so that the corresponding linkings of the Wilson lines from the adiabatical rotations in $\mathcal{M}$ are equivalent to braidings of the associated vertex operators on $\Sigma$.

The induced phases arising from the braiding operations are given by the Knizhnik-Zamolodchikov formula [11]

$$\Delta_R(G_k) = \frac{T_R(G)}{k + C_2(G)} \quad (1.4)$$

for the anomalous scaling dimensions of primary operators in the corresponding current algebra $G_k$ based on $G$ at level $k$, where $T_R(G)$ is the quadratic Casimir of the representation $R$ and $C_2(G)$ is the quadratic Casimir of the adjoint representation of $G$. In the three-dimensional approach the weights (1.4) can be obtained as the Aharonov-Bohm phases that arise from adiabatical rotation of charged particles, interacting with a Chern-Simons gauge field, about one another [12], or equivalently from the corresponding Aharonov-Bohm scattering amplitudes between dynamical charged particles [10]. The key property behind this equivalence is that a particle of charge $q$ minimally coupled to a Chern-Simons gauge theory is also a source of magnetic flux $\Phi_q$. For an abelian gauge field $A$, this follows from the Gauss law

$$\frac{k}{4\pi} B = \frac{k}{4\pi} \epsilon^{0ij} \partial_i A_j = J^0 \quad (1.5)$$

obtained by varying the action $kS^{U(1)}_{\text{CS}}[A]$ minimally coupled to the particle current $J^\mu$, with respect to the temporal component $A_0$ of $A$. From (1.5) we see that the charge $q$ also carries magnetic flux

$$\Phi_q \equiv \frac{1}{2\pi} \int_D d^2x \ B(x) = \frac{2q}{k} \quad (1.6)$$
where \( D \subset \mathcal{M} \) is a disc in a neighbourhood of the charged matter. The conformal weights (1.4) are therefore induced spins that are the Aharonov-Bohm phases from adiabatical rotation of particles of charge \( q = \sqrt{T_R(G)} \), i.e.

\[
\Psi(e^{2\pi i(x_1 - x_2)}) = e^{4\pi i \Delta R(G_k)} \Psi(x_1 - x_2) = e^{2\pi i q \Phi} \Psi(x_1 - x_2)
\]  

(1.7)

where \( \Psi \) is the 2-particle wavefunction.

In [13] these properties were exploited to obtain a dynamical description of models representing a particular region of the moduli space of \( N = 2 \) superconformal field theories. The \( N = 2 \) superconformal algebra is generated by the usual Virasoro stress-energy tensor \((T(z), \bar{T}(\bar{z}))\), an extra \( U(1) \) current \((J(z), \bar{J}(\bar{z}))\) of conformal dimension 1 (which is not present in the \( N = 0 \) and \( N = 1 \) algebras), and two fermionic supercurrents \((G^\pm(z), \bar{G}^\pm(\bar{z}))\) with \( U(1) \) charges \( \pm 1 \). Actually, there is a one-parameter family of supercurrents labelled by their various boundary conditions, which continuously interpolates among a family of isomorphic \( N = 2 \) superconformal algebras. This algebraic property of the conformal field theory was described dynamically in [13] in terms of the propagation of charged particles coupled to the relevant Chern-Simons gauge theory. The supercurrents also define non-singular and closed finite-dimensional chiral rings in the operator product algebra of primary fields [1]. A primary field lies in the chiral-chiral ring if it has non-singular operator product with \((G^+(z), \bar{G}^+(\bar{z}))\), and in the chiral-antichiral ring if its operator product with \((G^+(z), \bar{G}^-(\bar{z}))\) is regular.

In the following we will describe, within a geometric setting, some dynamical properties of the chiral rings of \( N = 2 \) superconformal field theories on generic compact Riemann surfaces. We will show how the topological symmetries of Calabi-Yau manifolds can be realized by geometrical transformations on the 3-manifold \( \mathcal{M} \) with respect to the coupling of charged matter to a Chern-Simons gauge theory. Such a matter-coupling represents a deformation of the corresponding conformal field theory. The “mild” symmetries, such as Hodge duality and Kähler symmetry, arise from orientation-reversing isometries of \( \mathcal{M} \), such as parity and time-reversal. We will show that mirror symmetry arises when one makes the topology of the gauge theory non-trivial. Namely, when the gauge group is compact, magnetic monopole-instanton induced transitions alter the topology of the given principal fiber bundle by changing its first Chern class. In this case the discrete symmetry group of the superconformal model in the given region of moduli space is realized kinematically as the magnetic flux symmetry group of the gauge theory. The Hilbert space of the topologically non-trivial Chern-Simons theory provides a representation of the chiral rings of the corresponding conformal field theories. Mirror symmetry is then a mapping between two non-perturbative dynamical processes in which topologically non-trivial matter field configurations are exchanged with topologically non-trivial configurations of the gauge fields that interpolate between inequivalent vacua of the gauge theory. We also show how these maps are related to ‘topological duality’ transformations that are reminiscent of \( T \)-duality transformations in string theory. These results show that, within
the topological membrane approach [9], in which the string theory is studied by filling in the worldsheet and viewing it as the boundary of a 3-manifold $\mathcal{M}$, mirror symmetry is induced by a non-perturbative dynamical process that relates two topologically distinct membranes to one another, both of which induce the same string theory on $\partial \mathcal{M}$. This process is illustrated schematically in fig. 1. In certain instances this maps a topologically non-trivial fiber bundle onto a trivial one, so that, from a mathematical perspective, the symmetry of the worldsheet theory implies that difficult questions concerning a principal fiber bundle can be enormously simplified by mapping it onto its mirror.

Figure 1: In string theory, worldsheet mirror symmetry provides a map $M \leftrightarrow \tilde{M}$ between two Calabi-Yau target spaces with different topologies. In the topological membrane approach, worldsheet mirror symmetry provides a map $L(\Sigma) \leftrightarrow \tilde{L}(\Sigma)$ between two Chern-Simons gauge theories with different topological line bundles.

As a more concrete realization of these formalisms, we consider the most successful physical application of Chern-Simons gauge theory in condensed matter physics, namely the quantum Hall effect [14]. A quantum Hall system is a two-dimensional gas of electrons interacting with a uniform neutralizing background potential and with a strong perpendicular magnetic field. In certain instances, the transverse conductivity exhibits plateaus at certain values of the (fractional) number of filled Landau levels. An effective $(2 + 1)$-dimensional description of these phenomena is provided by coupling an additional (fictitious) gauge field to the electric charge current and adding to the Lagrangian a corresponding Chern-Simons term. The mirror map applied to these systems is especially transparent and we present transformation laws which map a given quantum Hall system onto its ‘mirror’. The Hall conductivity is changes by a simple rescaling, and, in certain cases, it is invariant under this transformation indicating a true ‘mirror symmetry’ of quantum Hall systems. We discuss some noteworthy physical aspects of such mappings, such as the role of the topologically non-trivial field configurations in these instances,
which could have bearings on experimental situations. In this way we show that not only is the phenomenon of mirror symmetry dynamical in origin, but it also exhibits universal applications in other branches of physics and mathematics.

The outline of this paper is as follows. In section 2 we briefly review the basic ideas of mirror symmetry and the construction of mirror manifolds via algebraic isomorphisms of $N = 2$ superconformal minimal models. In section 3 we describe the relevant superconformal field theories in terms of matter-coupled three-dimensional gauge theory and give the discrete geometric transformations of $\mathcal{M}$ which represent the symmetries of the Calabi-Yau spaces. In section 4 we describe the properties of compact Chern-Simons theory, emphasizing the appearance of monopole-instanton field configurations as objects inducing topology change on the principal fiber bundle of the gauge theory. Then in section 5 we present the mirror map in full detail in terms of charge deformations represented by Wilson line operators, and also by a detailed construction of the Hilbert space of the matter-coupled topological field theory. In section 6 we apply these constructions to the quantum Hall effect, and finally section 7 contains some concluding remarks.

2. Mirror maps in $N = 2$ superconformal field theories

In this section we shall briefly review some of the main ideas behind mirror symmetry in $N = 2$ superconformal field theories that will be used throughout this paper. A more detailed review can be found in [6].

2.1. Marginal operators and mirror symmetry

A deformation of a conformal field theory with action $S_{\text{CFT}}$ by some operators $\mathcal{O}_i$ (in the original theory) of conformal dimensions $(\Delta_i, \bar{\Delta}_i)$ is described by an action of the form

$$S[\mu] = S_{\text{CFT}} + \sum_i \lambda_i[\mu] \int_\Sigma d^2z \mathcal{O}_i(z, \bar{z})$$

(2.1)

where $\lambda_i[\mu]$ are coupling constants which depend on a scale $\mu$. The corresponding renormalization group flows allow one to smoothly interpolate between various two-dimensional renormalizable quantum field theories and the model with action $S_{\text{CFT}}$ considered as infrared or ultraviolet fixed points of the flows [15]. Of central importance to understanding the structure of the moduli space of $N = 2$ superconformal field theories are deformations by marginal operators, i.e. those with scaling dimensions $\Delta + \bar{\Delta} = 2$. These operators deform a given conformal field theory to a “nearby” one of the same central charge $c$ and thereby generate a family of isomorphic conformal field theories which are all continuously related to each other. The subset of marginal operators with conformal weights $(\Delta, \bar{\Delta}) = (1, 1)$ that continue to have dimensions $(1, 1)$ after perturbation by any other operator in the collection are said to be truly marginal. Such fields naturally span the
tangent space to any point in the moduli space and can be used to move around the moduli space without spoiling conformal invariance.

An abstract $N = 2$ superconformal field theory with $c = 3d$ has two types of truly marginal operators, $O_{(1,-1)}$ and $O_{(1,1)}$, which are constructed out of primary fields in the chiral-antichiral ring with $U(1)$ charges $(\mathcal{Q}, \overline{\mathcal{Q}}) = (1, -1)$ and in the chiral-chiral ring with $U(1)$ charges $(1, 1)$, respectively (see for example [3, 6]). They therefore differ only by the sign of a $U(1)$ charge in the anti-holomorphic sector. These operators can be given a geometrical interpretation in terms of a non-linear sigma-model on a Calabi-Yau target space $M$ of complex dimension $d = \frac{c}{3}$. The operator $O_{(1,-1)}$ corresponds to deformations of the Kähler structure of $M$ which, to lowest order, can be represented in terms of a harmonic $(1,1)$-form on $M$. The operator $O_{(1,1)}$, on the other hand, corresponds to deformations of the complex structure of $M$ which can be represented in terms of a harmonic $(1,d-1)$-form on $M$. Both deformations preserve the Calabi-Yau conditions on $M$ and so the $N = 2$ superconformal invariance of the original non-linear sigma model is maintained. The chiral-chiral ring is thus a deformation of the DeRham cohomology ring of the Calabi-Yau manifold $M$. For the chiral-chiral primary subspaces with $U(1)_L \times U(1)_R$ charges $[1]$

\[ \left( \mathcal{Q}^{(i)}, \overline{\mathcal{Q}}^{(j)} \right) = \left( i - \frac{c}{6}, \frac{c}{6} - j \right) \]  

the dimensions of these subspaces are equal to the Hodge numbers $h^{i,j}(M) = \dim \mathcal{H}^{i,j}(M)$, the dimensions of the spaces of harmonic $(i,j)$-forms on $M$. In fact, for the sigma-model with target space $M$, the chiral-chiral ring is the cohomology ring of $M$ deformed by instanton effects (i.e. complex curves in $M$) [16].

In terms of the abstract conformal field theory, the distinction between the two types of truly marginal operators $O_{(1,-1)}$ and $O_{(1,1)}$ is rather trivial, being just the sign of a conventional $U(1)$ charge. The $N = 2$ superconformal algebra is invariant under reflection of the corresponding current $\tilde{J}(\bar{z})$. On the other hand, their geometrical counterparts (the Dolbeault cohomology groups $H^{1,1}(M)$ and $H^{1,d-1}(M)$, respectively) differ far more significantly. This led to the conjecture [1] that to each Calabi-Yau manifold $M$ there corresponds a mirror Calabi-Yau manifold $\tilde{M}$ corresponding to the same conformal field theory but with $O_{(1,-1)} \in H^{1,d-1}(\tilde{M})$ and $O_{(1,1)} \in H^{1,1}(\tilde{M})$. Each type of marginal operator would then correspond to either a Kähler or a complex structure deformation of the mirror pair $(M, \tilde{M})$. Geometrically, this means that the Hodge numbers of $M$ are related to those of $\tilde{M}$ by

\[ h^{1,1}(M) = h^{1,d-1}(\tilde{M}) , \quad h^{1,d-1}(M) = h^{1,1}(\tilde{M}) \]  

This correspondence generalizes to the other Hodge numbers as well [1]. Now the chiral-antichiral ring is a deformed cohomology ring of the mirror Calabi-Yau manifold $\tilde{M}$ with the Hodge numbers

\[ h^{i,j}(\tilde{M}) = h^{i,d-j}(M) \]
and so the Hodge diamond for $\tilde{M}$ is mirror reflection through a diagonal axis of the Hodge diamond for $M$. Furthermore, the chiral-chiral ring coincides with the undeformed Dolbeault cohomology ring with values in the exterior algebra of the holomorphic tangent bundle of $\tilde{M}$. In this way, one can extract non-trivial information about instanton numbers for a Calabi-Yau space $M$ from a simple calculation of the Dolbeault cohomology on the mirror image $\tilde{M}$ of $M$ (see [3] for some examples).

### 2.2. Constructing the mirror manifold

The key to constructing the mirror manifold, which was first carried out in [2], of a conformal field theory $C$ associated with a Calabi-Yau manifold $M$ (or, more generally, the equivalence class of conformal field theories which are isomorphic to the non-linear sigma-model with target space $M$) is to find an operation which flips the sign of the right-moving $U(1)_R$ charge of each marginal operator in $C$ (compare (2.2) and (2.4)). This operation maps $C$ to an isomorphic conformal field theory $\tilde{C}$ (with corresponding Calabi-Yau space $\tilde{M}$) such that $O_{(1,-1)} \in H^{1,1}(M) \to \tilde{O}_{(1,1)} \in H^{1,d-1}(\tilde{M})$ and $O_{(1,1)} \in H^{1,d-1}(M) \to \tilde{O}_{(-1,1)} \in H^{1,1}(\tilde{M})$. Since $C \cong \tilde{C}$, the marginal operator $O_{(1,-1)}$ (and similarly $O_{(1,1)}$) can be thought of as either a Kähler (complex structure) deformation on $M$ or as a complex structure (Kähler) deformation on $\tilde{M}$. This observation typically simplifies the computation of the moduli space of complex structures of a Calabi-Yau manifold $M$ (and hence of the conformal field theory $C$) [3].

The $N = 2$ superconformal minimal models do admit such an operation, orbifolding, which yields an isomorphic conformal field theory related to the original one by a change in the sign of all $U(1)_R$ charges. Then, by transporting the orbifold operation on these minimal models to the region of moduli space corresponding to a Calabi-Yau sigma-model, this gives an explicit method for constructing the mirror manifold [16, 17]. So an important ingredient in the construction of mirror manifolds is an understanding of the $N = 2$ minimal model conformal field theories.

As described in [6] (see also [13]), the $k$-th $N = 2$ superconformal minimal model is isomorphic to the coset

$$M_k \cong SU(2)_k \times SO(2)_2 / U(1)_{k+2} \quad (2.5)$$

of ordinary $N = 0$ wznw models. Heuristically, it represents the operation of removing a free boson at one radius by dividing out the $U(1)$ subgroup and putting back a free boson at a different radius. The $SO(2)_2$ part of the coset (2.5) can be used to represent the extra $U(1)$ current $(J(z), \bar{J}(\bar{z}))$ and, upon fermionization, also the fermion fields of the $N = 2$ superconformal algebra. The Virasoro central charge of (2.5) is

$$c_k = \frac{3k}{k+2} \quad (2.6)$$

The $N = 2$ superconformal primary fields are labelled (in part) by their conformal dimensions $(\Delta_{j,m}, \bar{\Delta}_{\bar{j},\bar{m}})$ and also by the extra $U(1)$ charges $(Q_m, \bar{Q}_{\bar{m}})$. In the holomorphic
Neveu-Schwarz sector of the theory they are given by

$$\Delta_{j,m} = \frac{j(j+1)}{k+2} - \frac{m^2}{k+2}, \quad Q_m = -\frac{2m}{k+2}$$  \hspace{1cm} (2.7)

where $m = -j, -j + 1, \ldots, j - 1, j$ are the magnetic quantum numbers of a spin-$j$ representation of $SU(2)$. Unitarity of this representation imposes an upper bound on the isospin quantum numbers as

$$j = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots, \frac{k}{2}$$  \hspace{1cm} (2.8)

The $N=2$ minimal model is invariant under a $\mathbb{Z}_2 \times \mathbb{Z}_{k+2} \times \overline{\mathbb{Z}}_2 \times \overline{\mathbb{Z}}_{k+2}$ discrete symmetry group. The $\mathbb{Z}_{k+2}$ part of this symmetry group arises from the constraint (2.8) which implies that the charges $Q_m$ are cyclic with period $k + 2$, and it acts on the superconformal primary fields by the conformal group element $e^{2\pi i j_0}$. It is essentially the discrete symmetry group of level $k+2$ parafermions [18]. The $\mathbb{Z}_2$ symmetry is a charge conjugation symmetry represented by the fermion number $s$ which is defined modulo 4. The orbifold $M_k/\mathbb{Z}_{k+2}$ is a new conformal field theory $\tilde{M}_k$ which is isomorphic to $M_k$ [18, 19]. Moreover, the map from $\tilde{M}_k$ to $M_k$ is simply $m \to -m$, which changes the sign of the $U(1)_R$ charge associated with each (anti-holomorphic) primary field. This is precisely the map required to construct the corresponding Calabi-Yau mirror manifold. In what follows we shall describe this mirror map from the point of view of Chern-Simons gauge theory.

3. Three-dimensional description of $N=2$ minimal models and topological symmetries

In [13] it was shown that the $N=2$ minimal models (2.5) can be described completely using three independent Chern-Simons gauge fields $A$, $B$ and $C$ with the gauge field action

$$\mathcal{I}_k[A, B, C] = kS^{[SU(2)]}_{CS}[A] + 2S^{[SO(2)]}_{CS}[B] - (k + 2)S^{[U(1)]}_{CS}[C]$$  \hspace{1cm} (3.1)

The basic observables (2.7) of $M_k$ can be described in three-dimesional terms by coupling the action (3.1) to charged matter. An important feature of this, in the context of the last section, is that the addition of charged matter to the bulk $\mathcal{M}$ will induce a deformation of the two-dimensional conformal field theory on the boundary $\Sigma$ [20, 21]. In critical string theory the statistical sum of a deformed conformal field theory with action (2.1) gives a generating function for correlators in an external field. In the case of a single charge deformation it is given by

$$Z = \int D\phi \exp \left( -S_{\text{CFT}}[\phi] + \lambda \int_\Sigma d^2z \ V(z, \bar{z}) \right)$$

$$= 1 + \sum_{n=1}^\infty \frac{\lambda^n}{n!} \int_\Sigma d^2z_1 \cdots \int_\Sigma d^2z_n \langle V(z_1, \bar{z}_1) \cdots V(z_n, \bar{z}_n) \rangle$$  \hspace{1cm} (3.2)
where the averages denote $n$-point correlation functions in the unperturbed conformal field theory. These correlators coincide with the expectation values of Wilson line operators (1.3) in the corresponding Chern-Simons gauge theory. A gas of open Wilson lines (1.3) therefore describes charged matter in $\mathcal{M}$ corresponding to a deformation of the two-dimensional conformal field theory.

The coupling of charged matter to the Chern-Simons gauge theory introduces propagating degrees of freedom in the bulk and thus ruins the topological property of the three-dimensional field theory. Furthermore, the deformation parameters $\lambda_{[\mu]}$ of the induced conformal field theory are functions of the parameters of the matter fields. Thus a (truly marginal) deformation can be described dynamically by varying the parameters of the charged matter in three dimensions (for example the chemical potentials). These ideas were explored in detail for the $N = 0$ and $N = 1$ minimal models in [22].

The coupling of matter to the action (3.1) is therefore an essential ingredient for the construction of the mirror map in three-dimensional terms. As described in [13], the first step in obtaining a three-dimensional description of the minimal models (2.5) is to minimally couple the $SU(2)$ gauge field $A$ in (3.1) to a matter current $J_{[\mu]}^{(j)}R^{(j)\alpha}$ in a spin-$j$ representation $R^{(j)}$ of $SU(2)$. Then the induced spin of this charged matter is $j(j+1)/(k+2)$ (see (1.4)). We also minimally couple the $U(1)$ Chern-Simons gauge field $C$ at level $-(k+2)$ to a current $J^{(m)\mu}$ carrying an abelian charge $q = m$ corresponding to the magnetic quantum numbers of this same spin-$j$ representation. Then the total induced spin $\Delta_{j,m}$ of the matter-coupled action

$$I_{k}^{(j,m)}[A, B, C] = I_{k}[A, B, C] + \int_{\mathcal{M}} \left( 2j(j+1)A_{\mu}^{a}J_{a}^{(j)\mu} + C_{\mu}J^{(m)\mu} \right)$$

(3.3)

coincides precisely with the conformal dimensions in (2.7) for the $N = 2$ minimal model in the holomorphic Neveu-Schwarz sector. Furthermore, the abelian magnetic flux (1.6) carried by the charge $q = m$ coupled to the gauge field $C$ (at level $-(k+2)$) coincides precisely with the $U(1)$ charges in (2.7), $\Phi_{m} = Q_{m}$.

The discrete symmetries of $M_{k}$ described in the previous section can be explained dynamically in the three-dimensional picture. First of all, the $U(1)_{R}$ sector of the minimal model depends on the fermion number $s$ and orbifolding by $\mathbb{Z}_{2}$ sends $s \rightarrow -s$ which is related to the gso projection [23]. This projection was described in detail in [13] in terms of the propagation of an extra charge coupled to the $SO(2)_{2}$ gauge field $B$ in (3.1). This field has the effect of mapping charged bosons into fermions and vice-versa through the Aharonov-Bohm effect (see (1.7)), and as such it represents the supercharges of the $N = 2$ supersymmetric theory. This therefore yields a dynamical picture of the discrete $\mathbb{Z}_{2}$ symmetry of $M_{k}$. As for the parafermionic symmetry of $M_{k}$, from (1.7) we likewise see that it is an anyonic symmetry of the matter-coupled Chern-Simons theory, i.e. that

*See [13] for a precise description of the relationship between the abelian flux $\Phi_{m}$ and the total $U(1)$ charge of the conformal group generator $J(z)$. 

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the charged particles acquire fractional exchange statistics parametrized by the phases $e^{2\pi im\Phi_m} \in \mathbb{Z}_{k+2}$. From a quantum mechanical perspective, all physical quantities are invariant under transformation of the wavefunctions by these statistical phases. In the next section we will see how this statistical exchange symmetry arises from a kinematical property of the pure gauge theory.

Moreover, we see that left- and right-moving $U(1)$ charges (which from (2.2) are associated with the Hodge numbers $h^{i,j}(M)$) not being equal requires $\Phi_L \neq \Phi_R$, i.e. charge non-conservation as the $U(1)$ particle propagates from left to right boundaries of $\mathcal{M} = \Sigma \times [0,1]$. Exactly this type of process arises in compact Chern-Simons gauge theory [24, 25] whereby a monopole-instanton transition can change the charge of a particle along a Wilson trajectory. In particular, the mirror map $m \rightarrow -m$ which changes the sign of the $U(1)_R$ charge in the $N = 2$ minimal model (2.5) corresponds in three-dimensional terms to flipping the sign of the charge $q = m$ coupled to the abelian Chern-Simons gauge field $C$. Thus the key to understanding the mirror map $m \rightarrow -m$ from a three-dimensional perspective lies in the nature of the gauge theory of the field $C$. This will be the topic of the next section.

An interesting aspect of the three-dimensional description is that it provides full geometric pictures for all of the topological symmetries of the associated Calabi-Yau space $M$ represented by its Hodge diamond. For instance, Hodge duality of the DeRham cohomology ring can be represented through the Kähler condition as

$$H^n(M) \cong \bigoplus_{i+j=n} H^{i,j}(M) \cong H^{2d-n}(M) \quad (3.4)$$

Comparing the Dolbeault cohomology groups on the left-hand side of the second isomorphism in (3.4) with those of the right-hand side, and using complex conjugation along with the Kähler condition, we find the equalities

$$h^{d-i,d-j}(M) = h^{i,j}(M) \quad , \quad h^{i,j}(M) = h^{j,i}(M) \quad (3.5)$$

among the various Hodge numbers of $M$. The first set of equalities in (3.5) represents a parity transformation $P : \mathcal{M} \rightarrow \mathcal{M}^*$ of the three-dimensional spacetime (which changes its orientation) under which the magnetic fluxes transform as pseudo-scalars, i.e.

$$\left( \Phi_m , \bar{\Phi}_m \right) \xrightarrow{P} \left( -\Phi_m , -\bar{\Phi}_m \right) \quad (3.6)$$

This property follows from (2.2) which implies that

$$\left( -\Phi_m^{(i)} , -\bar{\Phi}_m^{(j)} \right) = \left( \frac{d}{2} - i , j - \frac{d}{2} \right) = \left( (d - i) - \frac{d}{2} , \frac{d}{2} - (d - j) \right) = \left( \Phi_m^{(d-i)} , \bar{\Phi}_m^{(d-j)} \right) \quad (3.7)$$

The Chern-Simons action is parity-odd and this transformation can be absorbed in a reflection $(k+2) \rightarrow -(k+2)$ of the coefficient of the gauge field $C$ in (3.1). The second set of equalities relate a charge non-conservation process $q_i \rightarrow q_j$ as the charge propagates...
from left- to right-moving worldsheets on $\mathcal{M} = \Sigma \times [0,1]$ to the same non-conservation on propagation from right- to left-moving sectors of $\Sigma$, i.e.

$$(\Phi_m, \bar{\Phi}_m) \xrightarrow{T} (\bar{\Phi}_m, \Phi_m)$$

(3.8)

This is achieved by reversing the orientation of the worldlines of the particles in the Wilson lines (1.3), i.e. a time-reversal transformation $T : \mathcal{M} \to \mathcal{M}^*$, under which the Chern-Simons term is again odd. Hodge duality and the Kähler structure of $\mathcal{M}$ in the three-dimensional picture is thus represented by discrete orientation-reversing isometries of the 3-manifold $\mathcal{M}$. Thus, in addition to the mirror reflection symmetries across the diagonals that will be described in the following, the geometrical and dynamical properties of the three-dimensional quantum field theory (3.3) also account in a simple way for the horizontal and vertical symmetries (3.5) of the Hodge diamond of $\mathcal{M}$ (see fig. 2).

![Figure 2: Symmetries of the Hodge diamond correspond to discrete geometric transformations of the three-dimensional gauge theory. Each entry in the diamond represents a Hodge number $h_{i,j}$. Hodge duality and the Kähler condition can be viewed as a parity transformation $P$ and time-reversal $T$, respectively, of the 3-manifold $\mathcal{M}$. Mirror symmetry $\mu$ is a reflection about the main diagonal of the Hodge diamond and is described in the three-dimensional picture by a non-perturbative dynamical process.](image)

4. Topology of compact Chern-Simons gauge theory

The mirror map exchanges the marginal deformation operators $\mathcal{O}_{(1,1)}$ and $\mathcal{O}_{(1,-1)}$. The latter deformation represents the propagation of a charge $q$ coupled to the gauge field $C$
in (3.1) which flips its sign as it propagates through the bulk $\mathcal{M}$ from $\Sigma_L$ to $\Sigma_R$. As we will now explain, such a charge non-conservation process is non-perturbative in character and can be accounted for by a non-trivial topology for the Chern-Simons gauge theory. The exchange of the two marginal deformations will then correspond to the interchange of topologically trivial and non-trivial gauge propagations represented by the Wilson lines (1.3).

4.1. Topology change of the principal fiber bundle

Consider the $U(1)^k$ Chern-Simons gauge field $C$. It is a connection of a complex line bundle $L \to \mathcal{M}$ over the 3-manifold $\mathcal{M}$ with curvature $F(C) = dC$. If we consider the product manifold $\mathcal{M} = \Sigma \times [0,1]$, with $\Sigma$ a compact Riemann surface of genus $g$, then for each fixed time $t \in [0,1]$ the field equations in the absence of sources are $B \equiv F(C|_\Sigma) = 0$ (see (1.5)). The topological quantum field theory thus localizes onto the moduli space of flat gauge connections modulo gauge transformations on the Riemann surface $\Sigma$. From this fact it is natural to restrict the line bundle $L \to \mathcal{M}$ to a line bundle $L_\Sigma \to \Sigma$ with curvature the magnetic field $B$. It is classified topologically by its first Chern characteristic class

$$C_1(L_\Sigma) = [B/2\pi] \in H^2(\Sigma; \mathbb{Z}) \cong \mathbb{Z}$$

which is labelled by the first Chern number

$$c_1(L_\Sigma) = \frac{1}{2\pi} \int_\Sigma B \in \mathbb{Z}$$

(4.1)

When $c_1(L_\Sigma) \neq 0$, the gauge field $C|_\Sigma$ is not a function on $\Sigma$ but rather a section of the non-trivial line bundle $L_\Sigma \to \Sigma$. Its curvature can be written as $dC|_\Sigma = dC' + 2\pi c_1(L_\Sigma)$, where $C'$ is a single-valued one-form on $\Sigma$. The class of connections $C'$ is the $2g$-dimensional torus $H^1(\Sigma; \mathbb{R})/H^1(\Sigma; \mathbb{Z}) \cong \mathbb{R}^{2g}/\mathbb{Z}^{2g} \cong T^{2g}$, where the real cohomology accounts for all canonical locally gauge-equivalent connections while the integer cohomology accounts for equivalence under large gauge transformations.

We shall be interested in a particular process which shifts the Chern number (4.2) and hence maps onto a new line bundle $\tilde{L}_\Sigma \to \Sigma$. To see how this can arise, we consider the Hodge decomposition for the one-form $C|_\Sigma = C_i(x)dx^i$ on $\Sigma$ at fixed time $t$,

$$C|_\Sigma = \frac{4\pi}{k'} d\xi + *d\left(\frac{1}{\nabla_\perp^2} B\right) + 4\pi i \gamma(t)$$

(4.3)

where

$$k' = k + 2$$

(4.4)

Here $\xi$ is some function on $\Sigma$, $\nabla_\perp^2$ is the scalar Laplace-Beltrami operator with its zero modes removed, and $\gamma$ is a harmonic one-form on $\Sigma$, $d\gamma = d*\gamma = 0$. This harmonic form can be expanded as

$$\gamma(t) = \sum_{i=1}^{g} \left(\bar{\gamma}^i(t)\omega_i - \gamma^i(t)\bar{\omega}_i\right)$$

(4.5)
where $\omega_l$ are holomorphic harmonic 1-forms which generate $H^{1,0}(\Sigma)$. They obey the canonical period normalizations

$$\int_{a_l} \omega_l = \delta_{lm}, \quad \int_{b_m} \omega_l = \Omega_{lm}$$

(4.6)

where $a^l, b^l, l = 1, \ldots, g$, form a basis of canonical homology cycles, i.e. $a^l \cap b^m = \delta_{lm}$, $a^l \cap a^m = b^l \cap b^m = 0$, and $\Omega_{lm}$ is the $g \times g$ symmetric period matrix, with $\text{Im} \Omega > 0$, which is a function of the modular parameters of $\Sigma$. The metric on the space of holomorphic harmonic 1-forms is

$$G_{lm} \equiv i \int_{\Sigma} \omega_l \wedge \bar{\omega}_m = 2 \text{Im} \Omega_{lm}$$

(4.7)

Taking the exterior derivative of (4.3) we have

$$dC|_{\Sigma} = d^2 \theta + B(x) d^2 x$$

(4.8)

where $\theta = (4\pi/k')\xi$ is the pure gauge degree of freedom of $C$ on $\Sigma$. Since $d$ is nilpotent on $C^\infty(\Sigma)$, for smooth functions $\theta$ in (4.3) it follows that $\int_{\Sigma} dC|_{\Sigma}/2\pi = c_1(L_\Sigma)$ is just the Chern number of the original line bundle $L_\Sigma$. However, we could just as easily allow for a multi-valued function $\theta(x)$ on $\Sigma$. The simplest case is when $\theta(x) = n\theta(z, z_0)$ (with $z \equiv x^1 + ix^2$) is the angle function of the Riemann surface [26] with winding number $n \in \mathbb{Z}$ around a fixed point $z_0 \in \Sigma$. It is the multi-valued function which is related to the prime form of $\Sigma$,

$$\frac{1}{\sqrt{2}} \delta^{(2)}_\Sigma(z - z_0) = -\frac{1}{\pi} \log \mathcal{E}(z, z_0)$$

(4.9)

where

$$\mathcal{E}(z, z_0) = \Theta^{(g)} \left( \frac{1/2}{1/2} \right) \left( \frac{f^z_{z_0} \omega}{\sqrt{h(z)h(z_0)}} \right)$$

with $h(z) = \omega^l(z) \frac{\partial}{\partial u^l} \Theta^{(g)} \left( \frac{1/2}{1/2} \right) (u|\Omega) \bigg|_{u=0}$

(4.10)

and

$$\Theta^{(g)} \left( \frac{\alpha}{\beta} \right) (z|\Omega) = \sum_{\{n_l\} \in \mathbb{Z}^g} \exp \left[ i\pi(n_l + \alpha_l)\Omega^{lm}(n_m + \alpha_m) + 2\pi i(n_l + \alpha_l)(z^l + \beta^l) \right]$$

(4.11)

are the holomorphic, doubly semi-periodic Jacobi theta-functions of $\{z_l\} \in \mathbb{C}^g$, with $\alpha_l, \beta^l \in [0, 1]$. The prime form $\mathcal{E}(z, z_0)$ is antisymmetric in $(z, z_0)$ and for $z \sim z_0$ it behaves as $\mathcal{E}(z, z_0) \sim z - z_0$. Then

$$\theta(z, z_0) = \text{Im} \log \left( \frac{\mathcal{E}(z, z_0)}{\mathcal{E}(z, z')\mathcal{E}(z', z_0)} \right)$$

(4.12)

so that

$$d\theta = *d\log \mathcal{E}$$

(4.13)

where $z'$ is an arbitrary fixed reference point. The angle function satisfies the Laplace equation on $\Sigma$ and is not differentiable at the point $z_0$,

$$\nabla^2 \theta(z, z_0) = 0, \quad d^2 \theta(z, z_0) = 2\pi \delta^{(2)}_\Sigma(z - z_0) d^2 z$$

(4.14)
\( \theta(z, z_0) \) is a single-valued function on the universal covering space of \( \Sigma - \{ z_0 \} \) and along any closed contour \( C \subset \Sigma \) winding around the point \( z_0 \) it has the property \( \oint_C d\theta = 2\pi \mod 2\pi \), i.e. \( \theta(z, z_0) \in [0, 2\pi) \). The definition of \( \theta(z, z_0) \) in terms of the prime form \( E(z, z_0) \) is just the generalization to an arbitrary Riemann surface of the angle function on the plane, i.e. the angle of the line joining \( z \) to \( z_0 \) relative to a fixed reference axis (determined by \( z' \) here).

With this choice of function \( \theta \), (4.8) becomes

\[
d\tilde{C}|_\Sigma = \left[ 2\pi n \delta^{(2)}_\Sigma (z - z_0) + B(z) \right] d^2 z
\] (4.15)

and consequently

\[
\frac{1}{2\pi} \int_\Sigma d\tilde{C}|_\Sigma = n + c_1(L_\Sigma) \equiv c_1(\tilde{L}_\Sigma)
\] (4.16)

Thus the effect of making the pure gauge degree of freedom of \( C \) a compact variable is to shift the Chern number \( c_1 \) and hence the topological class (4.1) of the line bundle, i.e. it produces a different line bundle \( L_\Sigma \to \tilde{L}_\Sigma \). This topology changing process on the principal fiber bundle of the Chern-Simons gauge theory will be the essence behind the mirror transformation in the three-dimensional description.

### 4.2. Monopole-instantons in the Hamiltonian formalism

The topological consequences of the compactification of the Chern-Simons gauge theory above are due to the appearence of magnetic monopoles in the theory. The main idea is that one can achieve the multi-valued shift of the Chern numbers by considering a smooth scalar field \( \xi \) in the Hodge decomposition (4.3) and making the \( U(1) \) gauge group compact. The gauge group now contains the non-trivial topological information and its effect on the Hilbert space of the gauge theory is to effectively shift the functions \( \xi \) by the multi-valued angle function on \( \Sigma \). For this, we consider the abelian topologically massive gauge theory [27]

\[
S_{\text{TMGT}}[C] = \int_\mathcal{M} \left( -\frac{1}{4e^2} F(C) \wedge \ast F(C) + \frac{k'}{8\pi} C \wedge F(C) + C \wedge \ast J \right)
\] (4.17)

for the \( U(1) \) gauge field \( C \) coupled to a current \( J \) representing the propagation of a charged particle in the bulk. The kinetic term for \( C \) explicitly breaks the topological invariance of the pure gauge theory. It is included for full generality because radiative corrections by dynamical matter fields coupled to a Chern-Simons gauge field induce a Maxwell term for it. Furthermore, its presence allows for the construction of different string worldsheet actions, including the action for the heterotic string, using the topological membrane approach to string theory [28], and it also enables one to vary the choice of worldsheet complex structure in the induced conformal field theory on \( \Sigma \) via its coupling to the metric of \( \mathcal{M} \) [9]. The dimensionful parameter \( e^2 \) can be thought of as a regulator such that at the
end of calculations one takes the limit $\epsilon^2 \to \infty$ to recover the original pure Chern-Simons
gauge theory.

Canonical quantization of (4.17) in the Weyl gauge ($C_0 = 0$) gives the equal-time
commutation relations

$$ [\Pi_i(x), C_j(y)] = i \delta_{ij} \delta^{(2)}_\Sigma(x - y) \quad (4.18) $$

where

$$ \Pi_i = - \frac{1}{\epsilon^2} E_i - \frac{k'}{8\pi} \epsilon_{0ij} C^j \quad (4.19) $$
is the canonical momentum conjugate to the gauge field $C$, and $E_i = \dot{C}_i$ is the electric
field. The operator

$$ \mathcal{G} = \frac{1}{e^2} \partial_i E^i - \frac{k'}{4\pi} B - J^0 \quad (4.20) $$
generates time-independent local gauge transformations, and the elements of the local
gauge group are the operators

$$ U = \exp \left\{ -i \int_\Sigma d^2 x \; \theta(x) \left( \frac{1}{e^2} \partial_i E^i - \frac{k'}{4\pi} B - J^0 \right) \right\} \quad (4.21) $$

with $UCU^{-1} = C + d\theta$, i.e.

$$ U \xi U^{-1} = \xi + (k'/4\pi) \theta \quad (4.22) $$

The physical Hilbert space of the theory contains only those states $|\Psi\rangle$ which are gauge-
invariant, $U|\Psi\rangle = |\Psi\rangle$.

If the gauge group is compact, however, we must be more careful. As discussed in
[24, 25], we must also include in the gauge group the operators $V$ of the form (4.21) where
$\theta(x)$ is a multi-valued function on $\Sigma$. When $\theta$ is the angle function (4.12), we denote the
corresponding gauge group element by $V(x_0)$. Then integrating by parts in (4.21) implies
that the physical states must also be fixed points of the operators

$$ V(x_0) = \exp \left\{ -i \int_\Sigma d^2 x \; \theta(x) \left( \frac{1}{e^2} \partial_i E^i + \frac{k'}{4\pi} \epsilon_{0ij} A_j \right) \epsilon_{0ik} \partial^k \log \mathcal{E}(x, x_0) - \theta(x, x_0) J^0 \right\} \quad (4.23) $$

In this representation the operators (4.23) commute with non-compact gauge transforma-
tions and also amongst themselves for different points $x_0 \in \Sigma$, so that the invariance
condition $V(x_0)|\Psi\rangle = |\Psi\rangle$ can be imposed simultaneously for all $x_0$ (as required for ro-
tational and translational invariance of the physical Hilbert space). The identities (4.14)
along with the commutation relations (4.18) imply that

$$ [B(x), V^n(x_0)] = 2\pi n \delta^{(2)}_\Sigma(x - x_0) V^n(x_0) \quad (4.24) $$

Thus the operator $V^n(x_0)$ creates a pointlike magnetic vortex at the point $x_0$ on the
worldsheet $\Sigma$ with flux $\Phi = (1/2\pi) \int_\Sigma B = n$, where $n$ is the monopole number. Since
the electric field decays exponentially at large distances (the photon in (4.17) is massive),
from (1.6) we find that the operator $V^n(x_0)$ also creates electric charge

$$ \Delta Q = -nk'/2 \quad (4.25) $$
This change of charge/flux is, however, unobservable far from the vortex because local observables such as the electric field fall off exponentially and the Aharonov-Bohm phase is unity. Therefore $V(x_0)$ is the operator for a monopole-instanton [24] which is a dyon that interpolates between topologically inequivalent vacua of the topologically massive gauge theory. The magnetic monopoles appear here as topologically non-trivial configurations of the gauge group when the vacuum is projected onto the gauge-invariant subspace of the Hilbert space.

The magnetic monopole-instantons give an explicit kinematical realization of the discrete symmetry of the $N = 2$ superconformal minimal model as the magnetic flux symmetry group of the compact topologically massive gauge theory. The Bianchi identity

$$dF(C) = 0 \quad (4.26)$$

ensures the existence of a conserved topological current $*dC$, whose associated global charge is precisely the flux $\Phi$. It generates the magnetic flux group of the gauge theory (a subgroup of the gauge group in the source free $e^2 \to \infty$ limit). However, because the monopole-instantons change the magnetic flux by an integer, only a discrete subgroup of the flux group, consisting of the operators

$$U_\ell = e^{2\pi i \ell \Phi}, \quad \ell \in \mathbb{Z} \quad (4.27)$$

remains a symmetry group of the theory. Since $\Phi$ is quantized in units of $1/k'$, it is only the operators (4.27) for $\ell = 0, 1, \ldots, k' - 1$ that are represented non-trivially on this subspace of the physical Hilbert space. Therefore the magnetic flux symmetry group of compact topologically massive gauge theory is $\mathbb{Z}_{k'}$, which is precisely the holomorphic (or anti-holomorphic) statistical component of the discrete symmetry group $\mathbb{Z}_{k'} \times \mathbb{Z}_{k'} \times \mathbb{Z}_2 \times \mathbb{Z}_2$ of $M_k$. In the next section we shall describe the orbifold conformal field theory $M_k/\mathbb{Z}_{k'}$ explicitly in the three-dimensional language. This orbifolding is required since the effects of the magnetic monopoles are unobservable.

The presence of monopole-instantons also shifts the spectrum of allowed charges in the compact theory. To see this, we first write the gauge group generators in (4.21) in terms of the canonical momenta $\Pi_i$, and in the functional Schrödinger representation where $\Pi_i = i\delta / \delta C_i$ it is easy to see that physical states acquire a non-trivial projective phase under gauge transformations

$$V(x_0) \Psi[C] = \exp \left\{ i \int_\Sigma d^2x \theta(x, x_0) \left( \frac{k'}{8\pi} B + J^0 \right) \right\} \Psi[C + d\theta] \quad (4.28)$$

where $\Psi[C] = \langle C | \Psi \rangle$ with respect to the field basis. The cocycle of the gauge group in (4.28) should be single-valued under the shift $\theta \to \theta + 2\pi$. This implies that in the compact theory the charge $Q = \int_\Sigma d^2x J^0$ must be quantized according to

$$Q = q + \frac{k'}{8\pi} \int_\Sigma B \quad (4.29)$$
where $q$ is an integer representing the particle winding number around the monopole-instanton at $x_0 \in \Sigma$, i.e. invariance of the quantum field theory under large gauge transformations. When $k' = 0$ we recover the usual Dirac charge quantization condition of compact quantum electrodynamics. The extra term in (4.29) is due to the monopole-instanton background which, according to (4.24), carries $n$ units of magnetic flux. Thus the spectrum of allowed charge in the compact gauge theory is

$$Q = q + nk'/4$$

(4.30)

We can therefore consider the charge non-conservation process illustrated in fig. 3. A particle of charge $Q_L = q + nk'/4$ is inserted on the left boundary and then propagates through the bulk. It then interacts with the monopole-instanton background which itself carries charge (4.25). The effect of the monopole-instanton is to change the charge of the particle, creating a new state of charge $Q_R = q - nk'/4$ which then propagates to the right boundary.

Figure 3: A monopole-instanton can change the charge by $nk'/2$ along a Wilson trajectory.

4.3. Functional Schrödinger representation

It is instructive to examine these properties using the functional Schrödinger representation of the Hilbert space. Here we shall use the approach of [26] for the most part. For this, we consider the limit $1/e^2 \to 0$ whereby the full topologically massive gauge theory (4.17) reduces to the pure Chern-Simons theory for $C$ which is an exactly solvable three-dimensional topological field theory. This Hilbert space corresponds to the vacuum sector of the topologically massive gauge theory and it coincides with the moduli space of flat gauge connections on the Riemann surface $\Sigma$ [7]. We recall from subsection 4.1 that it was precisely this space which characterized the topology of the Chern-Simons gauge theory. For a monopole number $n \neq 0$, the contributions from the various vacuum
sectors are summed over in the partition function to yield a dynamical representation of this moduli space.

We decompose the current $J$ similarly to (4.3) as

$$\ast J = -d\chi + d\psi + i \sum_{l=1}^{\sigma} \left( j^l \omega_l - j^l \tilde{\omega}_l \right)$$  \hspace{1cm} (4.31)$$

The continuity equation $d\ast J = (\partial J^0 / \partial t) d^3 x + d\ast J \wedge dt = 0$ then yields $\psi = -\nabla_-^{-2} (\partial J^0 / \partial t)$. Using (4.3) we find that the equal-time canonical commutation relations (4.18) in the limit $1/e^2 \to 0$ can be written as

$$[\xi(z), B(w)] = -i\mathcal{P} \frac{\delta^{(2)}(z - w)}{(2\pi)^2} \quad \text{or} \quad B(z) = i\mathcal{P} \frac{\delta}{\delta \xi(z)}$$  \hspace{1cm} (4.32)$$

and

$$[\gamma_l, \bar{\gamma}_m] = \frac{1}{4\pi k} G_{lm} \quad \text{or} \quad \bar{\gamma}_l = -\frac{1}{4\pi k} \frac{\partial}{\partial \gamma_l} \delta - \frac{1}{4\pi k} \frac{\partial}{\partial \gamma_l}$$  \hspace{1cm} (4.33)$$

where $\mathcal{P}$ is the projection operator onto the space orthogonal to the zero modes of the scalar Laplace-Beltrami operator. The elements of the local gauge group are the operators

$$U = \exp \left\{ i \int_{\Sigma} d^2 z \theta \left( \frac{ik'}{4\pi} \mathcal{P} \frac{\delta}{\delta \xi} + J^0 \right) \right\}$$  \hspace{1cm} (4.34)$$

The Hamiltonian operator separates into a local part and a commuting topological part

$$H = -\int_{\Sigma} C \wedge \ast J = H_{\text{loc}} + H_{\text{top}}$$  \hspace{1cm} (4.35)$$

where

$$H_{\text{loc}} = \int_{\Sigma} d^2 z \left( i\chi \mathcal{P} \frac{\delta}{\delta \xi} - \frac{4\pi \xi}{k} \frac{\partial J^0}{\partial t} \right), \quad H_{\text{top}} = i \left( 4\pi j_l \gamma_l + \frac{1}{k} j_l \frac{\partial}{\partial \gamma_l} \right)$$  \hspace{1cm} (4.36)$$

We can therefore use separation of variables and write the wavefunctions as

$$\Psi(\xi, \gamma; t) = \Psi_{\text{loc}}(\xi, t) \Psi_{\text{top}}(\gamma, t)$$  \hspace{1cm} (4.37)$$

The Gauss’ law constraint $\mathcal{G} \approx 0$ at $1/e^2 \to 0$ acts only on the local part of the wavefunction and is solved by

$$\Psi_{\text{loc}}(\xi, t) = \exp \left( \frac{4\pi i}{k} \int_{\Sigma} d^2 z \xi(z) J^0(z, t) \right) \Psi_{\text{loc}}(t)$$  \hspace{1cm} (4.38)$$

The solution to the first Schrödinger equation $i\partial \Psi_{\text{loc}}(\xi, t) / \partial t = H_{\text{loc}} \Psi_{\text{loc}}(\xi, t)$ is then given by

$$\Psi_{\text{loc}}(t) = \exp \left( \frac{4\pi i}{k} \int_{t_0}^{t} dt' \int_{\Sigma} d^2 z \chi(z, t') J^0(z, t') \right)$$  \hspace{1cm} (4.39)$$

The wavefunctions (4.38) carry the usual one-dimensional unitary representation of the local $U(1)$ gauge group which acts on (4.38) by

$$\Psi_{\text{loc}}(\xi + (k'/4\pi) \theta, t) = e^{i \int_{\Sigma} d^2 z \theta(z) J^0(z, t)} \Psi_{\text{loc}}(\xi, t)$$  \hspace{1cm} (4.40)$$
The topological part $\Psi_{\text{top}}(\gamma, t)$ of the full wavefunction will be examined in the next section.

When the gauge group is compact, we must also include in it the operators representing monopole-instantons which can change the charge of a particle along a Wilson trajectory. To see this explicitly, we write the elements of the compact gauge group as the exponential of the Gauss’ law

$$V(z_0) = \exp \left\{ -i \int_{\Sigma} \xi d^2 \theta(z, z_0) \right\}$$

(4.41)

Substituting in the Hodge decomposition (4.3) and integrating by parts gives

$$V(z_0) = U \exp \left\{ -i \int_{\Sigma} \xi d^2 \theta(z, z_0) \right\}$$

(4.42)

where $U$ is given by (4.34). Using (4.14), it follows that on the subspace of the Hilbert space of states invariant under the non-compact part of the gauge group, we must also include the actions of the operators

$$V^n(z_0) = \exp \left\{ -2\pi in \int_{\Sigma} d^2 z \xi(z) \delta_{\Sigma}(z - z_0) \right\}$$

(4.43)

The effect of these operators acting on the wavefunctions (4.38) is to shift the charge density by $nk'/2$ at the position $z_0 \in \Sigma$ of the monopole-instanton,

$$V^n(z_0)\Psi_{\text{loc}}(\xi, t) = \exp \left\{ \frac{4\pi i}{k'} \int_{\Sigma} d^2 z \xi(z) \left( J^0(z, t) - \frac{nk'}{2} \delta_{\Sigma}(z - z_0) \right) \right\} \Psi_{\text{loc}}(t)$$

(4.44)

This gives an analytic representation of the monopole-instanton charge inducing process depicted in fig. 3. We start in the holomorphic sector $\Sigma_L$ with an initial wavefunction $\Psi_L(\xi) \equiv \Psi_{\text{loc}}(\xi, 0)$ of charge $Q$ quantized according to (4.30). At a certain time we apply the compact gauge transformation corresponding to the interaction with the monopole-instanton background localized around the point $z_0 \in \Sigma$, arriving finally in the anti-holomorphic sector $\Sigma_R$ with final wavefunction $\Psi_R(\xi) \equiv V^n(z_0)\Psi_{\text{loc}}(\xi, 1)$ of charge $Q - nk'/2$. For a point particle moving on $\Sigma$ with trajectory $z(t)$, we have

$$V^n(z_0)\Psi_{\text{loc}}(\xi, t) = \exp \left\{ \frac{4\pi i}{k'} \left( Q\xi(z(t)) - \frac{nk'}{2} \xi(z_0) \right) \right\} \Psi_{\text{loc}}(t)$$

(4.45)

so that this process is achieved if we take the interaction of the particle with the monopole-instanton at $z(1) \equiv z_0$.

5. Mirror maps and duality in Chern-Simons theory

The map between marginal operators $\mathcal{O}_{(1,1)} \leftrightarrow \mathcal{O}_{(1,-1)}$, which is the key to constructing the mirror manifold associated with the $N = 2$ superconformal minimal model, is trivial
from the perspective of the two-dimensional conformal field theory as it is simply the change of sign of the $U(1)_R$ charge associated with each marginal operator. However, it provides a non-trivial and unexpected map between the corresponding Calabi-Yau manifolds which have different topologies. We shall now see that the mirror map also implies a non-trivial map between Chern-Simons gauge theories of different topologies. In this way the phenomenon of mirror symmetry, which asserts the existence of two topologically inequivalent target spaces associated with the same conformal field theory, also implies the existence of two inequivalent topological membranes that induce precisely the same conformal field theory. We shall also describe the relationship between the mirror map and a certain topological duality symmetry of the three-dimensional gauge theory.

5.1. Deformed cohomology rings and the mirror transformation

We begin by describing the chiral rings of the minimal model $M_k$ as a deformation of the conformal field theory represented by the coupling of charged particles to the compact $U(1)_{k+2}$ Chern-Simons gauge theory. As explained in [13], the left- and right-moving magnetic fluxes must be quantized as odd integers. The odd-integer flux quantization condition yields the fermion-boson transmutation property of the theory, because it produces an additional factor of $-1$ in the Aharonov-Bohm phases (1.7) which maps bosons into fermions and vice versa.

As described in section 2, the pair of integer-valued fluxes $(\Phi_L, \Phi_R) = (i, -j) \in \mathbb{Z}^{-1} \times \mathbb{Z}^{-1}$ (5.1)
labels the Hodge numbers $h^{i,j}$ of the corresponding geometrical representation. Here $d$ is some integer which, for illustrative purposes, we identify as a “dimension” for the time being. With this convention $j < 0$ label $h^{i,d-|j|}$. At the end of this subsection we shall see how to map the minimal model onto a genuine Calabi-Yau manifold where $d$ will be a true complex dimension.

For generic $\Phi_L \neq \Phi_R$ we have corresponding charges $Q^{(i,-j)}_L \neq Q^{(i,-j)}_R$ and so we need to incorporate non-trivial monopole-instanton effects to produce a charge non-conservation process. The charges $Q^{(i,-j)}_L$ are quantized according to (4.30) and $Q^{(i,-j)}_R = Q^{(i,-j)}_L - n^{(i,-j)} k'/2$ for some monopole numbers $n^{(i,-j)} \in \mathbb{Z}$. Given the charge-flux relationship (1.6), we have explicitly

\[ q^{(i,-j)} = -\frac{k'}{4} (i - j), \quad n^{(i,-j)} = -(i + j) \quad (5.2) \]

The three-dimensional description of the chiral-chiral ring is now evident. First note that the horizontal and vertical symmetries (3.5) of the Dolbeault cohomology ring imply that $(i, j) = (j, i)$ and $(-i, -j) = (i, j)$. It therefore suffices to consider only positive values of $i$ and $j \in [-i, i]$, since, as discussed in section 3, the other pairs of fluxes can then be

\*This property will be especially important in section 6 where we study the quantum Hall effect.
obtained by parity and time-reversal transformations of the 3-manifold \( M = \Sigma \times [0, 1] \).

We now decompose the complex line bundle \( L_\Sigma \to \Sigma \) described in the previous section into a Whitney sum of \((d/2 + 1)^2\) line bundles,

\[
L_\Sigma = \bigoplus_{i=0}^{d/2} \bigoplus_{j=-i}^{i} L_{\Sigma}^{(i,j)}
\]  

The line bundle \( L_{\Sigma}^{(i,-j)} \to \Sigma \) has Chern number \( c_1(L_{\Sigma}^{(i,-j)}) = n^{(i,-j)} = -(i+j) \), so that

\[
c_1(L_\Sigma) = -\frac{d}{2}(\frac{d}{2} + 1)(2d + 5) \tag{5.4}
\]

Note that even though many of the component line bundles in (5.3) have the same topological class, it is necessary to incorporate them all to describe the full cohomology ring.

For a given monopole number \( n \), with \(-d \leq n \leq 0\), it follows from (3.4) that the line bundles in the topological class labelled by \( n \) generate the \( n \)-th Betti number \( b_n = \text{dim } H^n \) via

\[
b_n = \sum_{n^{(i,-j)} = n} h_{i,j} \tag{5.5}
\]

Corresponding to (5.3) we have an orthogonal decomposition of the Chern-Simons gauge field into a sum \( C = \sum_{i,j} C^{(i,j)} \), where \( C^{(i,j)} \) is a gauge connection on \( L_{\Sigma}^{(i,j)} \). We then introduce \( \frac{d}{2} + 1 \) charges \( Q^{(i)} = -k'/2i \), which for each \( i \) minimally couples to the \( 2i + 1 \) Chern-Simons gauge fields \( C^{(i,j)} \).

† Thus the relevant part of \( M_k \) which describes the deformed cohomology ring is given in three-dimensional terms as

\[
k' S^{(d)}[C] = \sum_{i=0}^{d/2} \sum_{j=-i}^{i} k' S_{\text{CS}}^{(U(1))[C^{(i,j)}]} + \int_M \sum_{i=0}^{d/2} J^{(-k'/2)\mu} \sum_{j=-i}^{i} C^{(i,j)}_\mu \tag{5.6}
\]

There are then two types of deformations corresponding to bundles with vanishing or non-vanishing Chern classes. The deformations representing the spaces of harmonic \((i, d - i)\)-forms and \((i, j)\)-forms with \( j \neq d - i \) are the Wilson line operators

\[
W^{(i,i)}[C] = \exp \left( iQ^{(i)} \int_L^{R} C_{\mu}^{(i,i)}(x) \, dx^\mu \right), \quad W^{(i,-j)}[C] = \exp \left( iQ^{(i)} \int_L^{R} C_{\mu}^{(i,-j)}(x) \, dx^\mu \right)
\]  

where the cross on the integral indicates the monopole-instanton transition which changes the charge by \( n^{(i,-j)} k'/2 \) along the particle trajectory as it propagates from \( \Sigma_L \) to \( \Sigma_R \). In the first process in (5.7) the magnetic flux of the particle is conserved along its motion corresponding to the triviality of the line bundle \( L_{\Sigma}^{(i,i)} \cong \Sigma \times S^1 \). In the second process the motion is such that the fibers of \( L_{\Sigma}^{(i,-j)} \) are twisted non-trivially. The finite-dimensional

††We stress that \( C \) is still regarded here as a single \( U(1) \) gauge field which is expanded into its components associated with the topological decomposition (5.3). At the end of this subsection we will combine several \( U(1)_{k+2} \) gauge fields \( C \) to make contact with Calabi-Yau sigma-models.
Hilbert space of the three-dimensional quantum field theory (5.6) then describes the chiral-chiral ring of $M_k$ which leads to a deformed cohomology ring. This Hilbert space will be discussed in more detail in the next subsection.

The mirror theory is obtained by letting $j \rightarrow -j$, so that the corresponding particle winding numbers and monopole numbers are related to (5.2) by

$$\tilde{q}^{(i,j)} = k' \quad n^{(i,-j)} \quad , \quad \tilde{n}^{(i,j)} = \frac{4}{k'} q^{(i,-j)}$$  \hspace{1cm} (5.8)

From (5.8) we see that the mirror map in the topological membrane description is a process which interchanges particle winding numbers (associated with large gauge transformations which wind around the monopole-instanton) and monopole numbers (associated with the Chern classes, i.e. elements of $H^2(\Sigma; \mathbb{Z}) \cong \mathbb{Z}$). It is an order-2 transformation that is not an exact symmetry of the Chern-Simons gauge theory, because of the factors of $k'/4$ that appear in (5.8). We can view this map as a topology changing transformation of the topological membrane. It interchanges topological components of the Whitney bundle (5.3), because it essentially maps $W^{(i,-j)}[C] \leftrightarrow \tilde{W}^{(i,j)}[\tilde{C}]$ which corresponds to a non-trivial change of topology among the components

$$L^{(i,-j)}_{\Sigma} \leftrightarrow \tilde{L}^{(i,j)}_{\Sigma}$$  \hspace{1cm} (5.9)

This topology change on the line bundle $L_{\Sigma} \rightarrow \Sigma$ is in effect the one described in the previous section corresponding to a change of the Chern cohomology classes $c_1(L^{(i,-j)}_{\Sigma})$. The resulting quantum field theory associated with the mirror line bundle $\tilde{L}_{\Sigma} \rightarrow \Sigma$ then describes the chiral-antichiral ring of $M_k$. From this point of view one sees why the mirror map is “almost” a symmetry of the three-dimensional theory, in that the topology of $L_{\Sigma}$ is insensitive to the particular Whitney decomposition (5.3) (since (5.4) depends on the dimension $d$) and doesn’t see this internal topology change of its components. Thus spacetime topology change in the three-dimensional picture is a non-trivial topology change of the principal fiber bundle of the matter-coupled Chern-Simons gauge theory (5.6).

As an explicit example, let us examine the deformations by the marginal operators $\mathcal{O}_{(1,1)}$ and $\mathcal{O}_{(1,-1)}$ in this description. The operator $\mathcal{O}_{(1,1)}$ has $Q = \tilde{Q} = +1$ and can be represented by the Wilson line

$$\mathcal{O}_{(1,1)} \sim W^{(1,1)}[C]$$  \hspace{1cm} (5.10)

describing the propagation of a particle of charge $Q^{(1)} = q^{(1,1)} = -k'/2$ (and $n^{(1,1)} = 0$) between left and right boundaries. On the other hand, the operator $\mathcal{O}_{(1,-1)}$ has $Q = +1$ and $\tilde{Q} = -1$ so that the charge of the particle changes as it propagates between $\Sigma_L$ and $\Sigma_R$. This is possible if a monopole-instanton with $n^{(1,-1)} = -2$ (and $q^{(1,-1)} = 0$) induces magnetic flux $\Phi_L = -n^{(1,-1)}/2 = +1$ on the left boundary and $\Phi_R = n^{(1,-1)}/2 = -1$ on the right boundary. This field is thus represented by the Wilson line

$$\mathcal{O}_{(1,-1)} \sim W^{(1,-1)}[C]$$  \hspace{1cm} (5.11)

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In terms of the boundary induced conformal field theory, the mirror map is

$$O_{(1,1)} \leftrightarrow \tilde{O}_{(1,-1)}, \quad O_{(1,-1)} \leftrightarrow \tilde{O}_{(1,1)}$$  \hspace{1cm} (5.12)

Since both the original conformal field theory and its mirror are induced from a Chern-Simons theory, the mirror transformation (5.12) gives a map between gauge theories in (5.6),

$$W^{(1,1)}[C] \leftrightarrow \tilde{W}^{(1,-1)}[\tilde{C}] \quad \text{or} \quad q^{(1,1)} = -\frac{k'}{2}, \quad n^{(1,1)} = 0 \quad \leftrightarrow \quad q^{(1,-1)} = 0, \quad n^{(1,-1)} = -2$$

$$W^{(1,-1)}[C] \leftrightarrow \tilde{W}^{(1,1)}[\tilde{C}] \quad \text{or} \quad q^{(1,-1)} = 0, \quad n^{(1,-1)} = -2 \quad \leftrightarrow \quad q^{(1,1)} = -\frac{k'}{2}, \quad n^{(1,1)} = 0$$  \hspace{1cm} (5.13)

The mirror map here is thus a transformation between two dynamical non-perturbative processes. One is defined on a topologically trivial line bundle but incorporates non-trivial particle winding modes, while the other contains no particle windings but has a dynamical mechanism interpolating between topologically inequivalent vacua of the Chern-Simons gauge theory (5.6). In this way, the mirror map relates gauge theories defined on topologically trivial principal fiber bundles to those on topologically non-trivial ones. Given that the observables and correlation functions of Chern-Simons theory are intimately related to the topology of 3-manifolds and the geometry of Riemann surfaces [7], this mirror map on the topological membrane could serve as a powerful computational tool in these branches of mathematics.

This is not quite the whole story yet. We have to transport the $N = 2$ superconformal field theory defined by the minimal model $M_k$ to the region of the moduli space where the central charge $c = 3d$ coincides with that of the non-linear sigma-model on a Calabi-Yau manifold of complex dimension $d$. This is achieved using the Gepner construction [29] (see also [17, 16]). Consider the conformal field theory that is the tensor product of $N = 2$
minimal models,
\[
M(k_1, \ldots, k_s) = (\bigotimes_{\ell=1}^s M_{k_{\ell}})^{U(1)_{\text{odd}}}
\]
where the superscript denotes the orbifold projection onto a spectrum of odd integral total \(U(1)\) charges. The central charge of (5.14) is \(c = \sum_{\ell=1}^s c_{k_{\ell}}\) so that, according to (2.6), the Chern-Simons coefficients \(k_{\ell}\) in (5.14) obey the non-linear constraint
\[
\sum_{\ell=1}^s \frac{3k_{\ell}}{k_{\ell} + 2} = 3d
\]
The conformal field theory (5.14) is equivalent, by the appropriate transport via a marginal deformation in the moduli space, to the non-linear sigma-model on the Calabi-Yau space \(M\) defined by the zero locus
\[
z_{k_1+2}^+ + \ldots + z_{k_s+2}^+ = 0
\]
which is considered as an equation in \(\mathbb{WCP}^{s-1}(D_{k_1+2}, \ldots, D_{k_s+2})\), the weighted complex projective space, where \(D\) is the least common divisor of the \(k_{\ell}'\). The mirror of (5.14), obtained by changing the sign of all \(U(1)_{R}\) charge eigenvalues, is the orbifold \(M/G\), where \(G\) is the subgroup of \(\prod_{\ell=1}^s \mathbb{Z}_{k_{\ell}'}\) which acts on \(M\) by
\[
(z_1, \ldots, z_s) \mapsto (e^{2\pi i n_1/Q_1} z_1, \ldots, e^{2\pi i n_s/Q_s} z_s)
\]
where \(n_1, \ldots, n_s\) are arbitrary integers such that \(\sum_{\ell=1}^s n_{\ell}/Q_{\ell}\) is an integer. This latter condition enforces the \(U(1)\) projection in (5.14) and geometrically it is the condition of preserving the holomorphic \(d\)-form of \(M\).

For the three-dimensional description, we consider \(s\) independent copies of the matter-coupled minimal model action \(\sum_{\ell=1}^s \mathcal{I}_{d_{\ell}(j_{\ell}, m_{\ell})}[A_{\ell}, B_{\ell}, C_{\ell}]\) described in section 3, where the action for the \(C\)-field part of the coset is \(\sum_{\ell=1}^s k_{\ell}' S^{(d_{\ell})}[C_{\ell}]\) as described above. The action of the discrete group \(G\) on \(M\) is encoded through the fractional statistics acquired by the external charged particles, or equivalently by the magnetic flux symmetry group of \(\prod_{\ell=1}^s U(1)_{k_{\ell}'}\). This then relates, via the mirror map, the topology changing process described above on the Whitney product bundles \(\bigotimes_{\ell=1}^s L_{\Sigma}^{[\ell]}\) to the spacetime topology change \(M \leftrightarrow \tilde{M} = M/G\) in terms of Calabi-Yau spaces. In this way we obtain an intriguing relationship between topologically inequivalent principal fiber bundles over a 3-manifold and inequivalent Calabi-Yau manifolds, both leading to identical physical models. In both of these “spacetime” descriptions of the same conformal field theory, the mirror map is only a symmetry of the model at the level of the worldsheet theory.

### 5.2. Orbifold constructions and topological duality

We now turn once again to the Schrödinger representation of the gauge theory. This will enable an explicit construction of the orbifold \(\tilde{M}_k = M_k/\mathbb{Z}_{k'}\), and will show that mirror symmetry is related to a ‘topological duality’ property of the matter-coupled Chern-Simons gauge theory. We consider the propagation of a single particle of charge \(Q\) and
trajectory \( z(t) \) on the Riemann surface \( \Sigma \) coupled to the Chern-Simons gauge field \( C \), and we focus on the topological part of the wavefunctional of subsection 4.3. It satisfies the Schrödinger wave equation \( i \partial \Psi_{\text{top}}(\gamma,t)/\partial t = H_{\text{top}} \Psi_{\text{top}}(\gamma,t) \) which is solved by

\[
\Psi_{\text{top}}(\gamma,t) = \exp \left( 4\pi \gamma \int_0^t dt' \tilde{j}_i(t') + \frac{4\pi}{k'} \int_0^t dt' j_i(t') \int_0^t dt'' \tilde{j}_i(t'') \right) \times \tilde{\Psi}_{\text{top}} \left( \gamma + \frac{1}{k'} \int_0^t dt' j_i(t') \right) \tag{5.18}
\]

In the absence of sources the holomorphic wavefunctions \( \tilde{\Psi}_{\text{top}}(\gamma) \) must incorporate the invariance of the gauge theory under the large gauge transformations

\[
\gamma^m \rightarrow \gamma^m + s^m + \Omega^{nl} t_l \tag{5.19}
\]

where \( s^m \) and \( t_m \) are integers representing the winding numbers of the gauge field around the canonical homology cycles of \( \Sigma \). The invariant wavefunctions are combinations of the Jacobi theta-functions (4.11) [26]

\[
\tilde{\Psi}_{\text{top}}^{(r)} \left( \begin{array}{c} \alpha \\ \beta \end{array} \right) (\gamma|\Omega) = e^{-2\pi k' \gamma^m \gamma_m} \Theta^{(g)} \left( \frac{(\alpha + k' + r)/k'}{\beta} \right) (k' \gamma|k' \Omega) \tag{5.20}
\]

where \( r_l = 1, 2, \ldots k' \), and \( \alpha_l, \beta_l \in [0, 1] \).

For a point charge \( Q \), the current is \( J^0(z, t) = Q \delta_S^2(z - z(t)) \), \( J^\ell(z, t) = \frac{1}{2} \bar{Q} \zeta(t) \delta_S^2(z - z(t)) \). The decomposition (4.31) can then be solved for the local components \( \chi \) and \( \psi \) in terms of the angle function of \( \Sigma \), while the harmonic components are given by

\[
j^\ell(t) = Q \zeta(t) \omega^\ell(z(t)) \tag{5.21}
\]

The Hilbert space has finite dimension \( (k')^q \) and is spanned by the full set of wavefunctions (4.37) which are given by [26]

\[
\Psi_t^{(Q)} \left( \begin{array}{c} \alpha \\ \beta \end{array} \right) (\xi, \gamma; t|\Omega) = \exp \left\{ -2\pi k' \gamma^m \gamma_m + 4\pi \gamma^m \int_0^t dt' \tilde{j}_m(t') - j_m(t') \right\} \times \Theta^{(g)} \left( \frac{(\alpha + k' + r)/k'}{\beta} \right) \left( k' \gamma^m + \int_0^t dt' j^m(t') \biggm| k' \Omega \right) \tag{5.22}
\]

where

\[
\theta(t) = \text{Im} \left\{ \log \left( \frac{f(t)}{E(z(t), z'(t))^2} \right) + 2 \int_{z'}^{z(t)} \omega^l \int_{z(0)}^{z(t)} (\omega_l + \bar{\omega}_l) \right\} + \pi \tag{5.23}
\]

is the “angle function” of the curve \( z(t) \) on \( \Sigma \), with \( f(t) \sim \lim_{\epsilon \to 0} E(z(t), z(t) + \epsilon) \) a framing of the trajectory \( z(t) \). Note that in (5.22) we have also incorporated the invariance of
the physical states under compact gauge transformations as prescribed in the previous section.

Considering the assembly of point charges $Q^{(i)}$ described in the previous subsection coupled to the independent gauge fields $C^{(i,j)}$, and incorporating the rest of the Hodge diamond according to the symmetries (3.5), the direct sum of all resulting Hilbert spaces forms a vector space $\mathcal{H}(M_k)$ which is the three-dimensional version of the deformed cohomology ring with multiplication of the corresponding wavefunctionals (5.22). The dimension of this ring is

$$\dim \mathcal{H}(M_k) = (d + 2)^2(k + 2)^g \equiv \sum_{n=-d}^{d} b^n$$

(5.24)

For a given geometrical model, the coefficients on the left-hand side of (5.24) can be fixed to coincide with the topological dimension on the right-hand side.

To describe the orbifold projection by $\mathbb{Z}_k'$, we use the fact that quantities which are multi-valued due to their windings around the location $z_0 \in \Sigma$ of the monopole-instanton can be considered as single-valued functions on the universal cover of the punctured Riemann surface $\Sigma - \{z_0\}$. This introduces an extra homology cycle which winds around the puncture at $z_0$. Correspondingly, this induces extra (real) harmonic components $\gamma^0$ and $j^0$ into the Hodge decompositions of the gauge field and the particle current. The effects of winding around the location of the monopole-instanton are then given by large gauge transformations analogous to (5.19) with $t_l = 0$. When the gauge field configurations wind $s^0$ times around the puncture $z_0$ we find

$$\Psi_r^{(Q)} \left( \begin{array}{c} \alpha \\ \beta \end{array} \right) (\xi, \gamma^0 + s^0, \gamma^m; t\Omega) = \exp \left( -4\pi ik's^0 - 2\pi i\alpha s^0 \right) \Psi_r^{(Q)} \left( \begin{array}{c} \alpha \\ \beta \end{array} \right) (\xi, \gamma^0, \gamma^m; t\Omega)$$

(5.25)

Combining this with the local and global $U(1)$ gauge invariance, we see that the wavefunctions (5.22) augmented by $\gamma^0$ carry a representation of the magnetic flux symmetry group $\mathbb{Z}_k'$ in such a way that on the whole they represent gauge-invariance with respect to the semi-direct product gauge group\footnote{Note that the $\alpha$'s and $\beta$'s are all free parameters and can be set to zero for a qualitative discussion of the wavefunction transformation properties. They can be fixed by requiring worldsheet modular invariance of the physical states [26].}

$$U_{\text{orb}} = U(1) \otimes_S \mathbb{Z}_{k+2}$$

(5.26)

with the magnetic flux symmetry regarded as a discrete automorphism group of the $U(1)$ gauge group. The Chern-Simons theory with gauge group (5.26) describes the orbifold current algebra $U(1)_{k+2}/\mathbb{Z}_{k+2}$ [8, 9, 30], which in the present case is enough to yield the three-dimensional description of the orbifold conformal field theory $\tilde{M}_k = M_k/\mathbb{Z}_k'$. This shows how the compact Chern-Simons gauge theory naturally constructs the required mirror of the $\mathbb{N} = 2$ minimal model.
We can also consider topologically non-trivial motions of the particle around \( z_0 \) in a time span \( t \). In this case the current changes according to

\[
f'_0 \, dt' \, j^0(t') \to f'_0 \, dt' \, j^0(t') + s^0
\]

with \( s^0 \) the winding number of the particle. Then the wavefunctions transform as

\[
\Psi_r^{(Q)} \left( \begin{array}{c} \alpha \\ \beta \end{array} \right) (\xi, \gamma; t|\Omega) = \exp \left( \frac{16\pi i}{k'} r_0 s^0 + \frac{4\pi i Q}{k'} (\alpha_0 - \alpha^{(0)}) s^0 + \frac{4iQ^2}{k'} [f(t) - f(0)] \right) \\
\times \Psi_r^{(Q)} \left( \begin{array}{c} \alpha \\ \beta \end{array} \right) (\xi, \gamma; 0|\Omega)
\]

with \( \alpha^{(0)} = Q \int_{r_0}^{z_0} \omega^0 \). Comparing the transformations (5.25) and (5.28) we see that there is a duality between large gauge transformations and the motion of particles around the puncture \( z_0 \), whereby we exchange \( k' \leftrightarrow 4/k' \).

The above topological duality map between the large gauge transformations and the particle windings around the monopole-instanton is similar to the Chern-Simons mirror map (5.8). A particle of charge \( Q = q + nk'/4 \) winds \( q \) times around the puncture \( z_0 \in \Sigma \) and invariance under large gauge transformations around \( z_0 \) gives the quantization of the monopole number \( n \). Topological duality then exchanges \( q \leftrightarrow n \) and \( k' \leftrightarrow 4/k' \) which, modulo the transformation of \( k' \), is the mirror map (5.8). The transformation properties of the Chern-Simons wavefunctions above make explicit the realization of the mirror map from an anyonic symmetry. However, the mirror map is not an exact duality symmetry of the three-dimensional theory except at the very special point \( k' = 4 \) of the moduli space where it corresponds to an exact interchange of particle winding and monopole numbers. In the topological membrane approach to string theory, where the Chern-Simons coefficient is related to the radius \( R \) of the target space compactification by \( k' = 4R^2/\alpha' \) [20, 25] with \( 1/2\pi \alpha' \) the string tension, the point \( k' = 4 \) corresponds to the self-dual point of the \( T \)-duality transformation \( R \leftrightarrow \alpha'/R \). This is precisely the three-dimensional realization of the well-known fact (see for example [3]) that, for some points in the moduli space, mirror symmetry is equivalent to \( T \)-duality. However, this is a very special case and in general there is no relationship between mirror symmetry and \( T \)-duality in the three-dimensional picture.

Note the appearance of an extra self-linking term in (5.28) (the last term in the exponential) as compared to (5.25). As we shall see in the next subsection, this term essentially incorporates the effects of the monopole-instanton transition and is crucial to the duality properties in the following sense. When \( \Sigma = S^2 \) is the Riemann sphere (where there are no homology cycles and the prime form is \( \mathcal{E}(z, z_0) = z - z_0 \) with \( z' = \infty \)), the topological duality \( q \leftrightarrow n \) can be thought of as interchanging the cohomology groups \( H^1(S^2 - \{z_0\}; \mathbb{Z}) \cong \mathbb{Z} \) and \( H^2(S^2 - \{z_0\}; \mathbb{Z}) \cong \mathbb{Z} \). This isomorphism is accomplished by application of the three-dimensional Hodge duality operator \( * \) on \( \mathcal{M} = \Sigma \times [0, 1] \) (signalling again the relation to a \( T \)-duality). For genus \( g > 0 \) this is not the case, because
The dual link of the graph $H^1(\Sigma - \{z_0\}; \mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z}^{2g}$ and $H^2(\Sigma - \{z_0\}; \mathbb{Z}) \cong \mathbb{Z}$ are no longer isomorphic. However, we will see below that the self-linking term in (5.28) depends on the genus $g$ of $\Sigma$ in such a way that it effectively “absorbs” this residual cohomology of the topological duality transformation.

The duality properties discussed in this section can be seen more generally and explicitly on a generic 3-manifold $\mathcal{M}$ without boundary from a simple path integral approach. For this, we consider the generating functional

$$Z[J] = \int DC \ D\tilde{C} \ D\lambda \ \exp \left\{ i \int_{\mathcal{M}} \left( \frac{k'}{8\pi} C \wedge dC + \frac{1}{8\pi} \lambda \wedge (k'dC - d\tilde{C}) + \frac{1}{k'} \tilde{C} \wedge \ast J \right) \right\}$$

(5.29)

defined as a path integral over three gauge fields $C$, $\tilde{C}$ and $\lambda$, where $J$ is an external source current. Integrating first over $\tilde{C}$ in (5.29) gives the constraint $d\lambda = (8\pi/k') \ast J$. Integrating over $\lambda$ enforces this constraint, giving the partition function for the matter-coupled Chern-Simons gauge field $C$ with statistics parameter $k'$,

$$Z[J] = Z[J] = \int DC \ \exp \left\{ i \int_{\mathcal{M}} \left( \frac{k'}{8\pi} C \wedge dC + C \wedge \ast J \right) \right\}$$

(5.30)

However, if we instead integrate first over $\lambda$ we obtain the constraint equation $k'dC = d\tilde{C}$, which is solved by

$$C = \frac{1}{k'} \left( \tilde{C} + \gamma \right)$$

(5.31)

where $\gamma$ is a harmonic one-form on $\mathcal{M}$. Integrating over $C$ we thus obtain another matter-coupled Chern-Simons theory

$$Z[J] = \tilde{Z}[\tilde{J}] = \int D\tilde{C} \ \exp \left\{ i \int_{\mathcal{M}} \left( \frac{1}{8\pi k'} \tilde{C} \wedge d\tilde{C} + \tilde{C} \wedge \ast \tilde{J} \right) \right\}$$

(5.32)

where $\tilde{J} = \frac{1}{k'} J$ is the dual source current. The equivalence between the two representations (5.30) and (5.32) shows more precisely the effect of the mirror map on the matter-coupled gauge theory. Moreover, the relationship between the Chern-Simons field $C$ and its dual $\tilde{C}$ in (5.31) shows exactly how the topological duality corresponds to a change in topology of the gauge theory (by adding a harmonic component $\gamma$).

### 5.3. Duality in topologically massive gauge theory

It is possible to see the effects of the monopole-instanton transition from a purely three-dimensional perspective. For this, we consider the duality properties of the full topologically massive gauge theory (4.17). Although the ground state of this theory has rather simple duality transformation laws, things are more complicated in the regulated theory for finite $\epsilon^2$. The duality properties of this theory were originally discussed in [31] and elucidated on in [20]. Using a path integral approach in the spirit of the previous subsection, it is possible to prove the equivalence of the two generating functionals

$$Z_M[J] = \tilde{Z}_M[J] \ e^{i \mathcal{L}[C]}$$

(5.33)
where

\[
Z_M[J] = \int DC \exp \left\{ \frac{i}{\sqrt{4\pi}} \int_M \left( -\frac{1}{4} F(C) \wedge \star F(C) + \frac{k'}{8\pi} C \wedge F(C) + C \wedge \star J \right) \right\} \tag{5.34}
\]

\[
\tilde{Z}_M[J] = \int D\tilde{C} \exp \left\{ -i \int_M \left( 2e^2 \left( \frac{k'}{8\pi} \right)^2 \tilde{C} \wedge \star \tilde{C} + \frac{k'}{8\pi} \tilde{C} \wedge F(\tilde{C}) + \tilde{C} \wedge \star J \right) \right\} \tag{5.35}
\]

are the generating functionals for the topologically massive and Chern-Simons-Proca gauge theories, and

\[
e^{iJ[C]} = \exp \left\{ -\frac{2\pi i}{k'} \int_{M(x)} \int_{M(y)} J(x) \wedge \frac{d}{\Box} (x - y) J(y) \right\} \tag{5.36}
\]

is a non-local phase factor, which, when \( J \) is the current for a particle with closed worldline \( \mathcal{C} \) and charge \( Q \), gives the topological self-linking number

\[
\mathcal{J}[\mathcal{C}] = \frac{4\pi Q^2}{k'} \oint_{\mathcal{C}(x)} \int_{\Sigma(\mathcal{C}(y))} \delta_{\mathcal{M}}^{(3)}(x - y) \tag{5.37}
\]

of the loop \( \mathcal{C} \) in the 3-manifold \( \mathcal{M} \) (the double integral in (5.37) is taken with the DeRham current of \( \mathcal{C} \) over a surface \( \Sigma(\mathcal{C}) \) spanned by the loop).

The “dual” of the topologically massive gauge theory (4.17) is therefore the Chern-Simons theory plus an additional Proca mass term for the gauge field, which has topological mass \( k'e^2/4\pi \). This is true modulo the non-local self-linking contribution (5.36), which we recall also appeared in the Chern-Simons wavefunctions upon application of the mirror map (compare (5.28) and (5.25)). Here the self-linking contribution is crucial to the global properties of this duality map. As discussed in [20], a charged particle interacting with the Chern-Simons-Proca field carries no induced magnetic flux, so that the linking term (5.36) (with linking number 1) alters this property so as to yield the charge-flux relationship (1.6) of the topologically massive gauge theory.

The integral (5.37) diverges for \( x \sim y \) and must be regularized. For this, we define a framing of the loop \( \mathcal{C} \) in terms of a unit vector \( \hat{n}(t) \) normal to it, where \( t \in [0, 1] \) is the parameter of the loop. We then construct a second loop \( \mathcal{C}' \) by deforming \( \mathcal{C} \) infinitesimally along this framing, and compute (5.37) as the linking number of \( \mathcal{C}' \) and \( \mathcal{C} \). This yields the torsion of the curve \( \mathcal{C} \) [12, 32],

\[
\mathcal{J}[\mathcal{C}] = \frac{4Q^2}{k'} \oint_{\mathcal{C}} \mathbf{d}\hat{x} \cdot \left( \hat{n} \times \dot{\hat{n}} \right) \tag{5.38}
\]

Thus the addition of the non-local phase factor to the dual theory amounts to the addition of the phase

\[
\mathcal{J}[\mathcal{C}] = \frac{4Q^2}{k'} \int_0^1 dt \hat{\theta}(t) \tag{5.39}
\]

where \( \hat{\theta} = \dot{x} \cdot (\hat{n} \times \dot{\hat{n}}) \) is the normal connection (or torsion form) of the curve \( \mathcal{C} \). The phase factor (5.38) appears in the form of the charged particle propagating along a Wilson loop.
with non-zero point-like flux. In fact, it can be shown [32] that $\dot{\theta}(t)$ is essentially a Dirac potential on the sphere $(\vec{x})^2 = 1$ in $\mathcal{M}$ with a magnetic monopole located at its center. This shows that the transformation between the topologically massive gauge theory (4.17) and its dual is also due to some non-perturbative, topological process induced by the interactions between the charged particles and a magnetic monopole in the 3-manifold $\mathcal{M}$, and furthermore that this duality is achieved by a deformation in the three-dimensional language as described in subsection 5.1 above.

On a product manifold $\mathcal{M} = \Sigma \times \mathbb{R}$, the expression (5.39) coincides with the angle function term in the wavefunction (5.22). Thus the monopole-instanton induced process in the topologically massive gauge theory is identical to that for its ground state in the canonical formalism in which the self-linking numbers of the particle trajectories (see (5.28)) play a crucial role. Since the framing vectors of $\mathcal{C}$ define a basis in the tangent space, $\dot{\theta}$ is actually a connection on the tangent bundle of the Riemann surface $\Sigma$. As the particle encircles the monopole-instanton at $z_0 \in \Sigma$ once, using the Gauss-Bonnet theorem we find the self-linking contribution

$$J[\mathcal{C}] = \frac{16\pi g^2}{k^2} (1 - g)$$

(5.40)

in the transformation law (5.28). The magnetic monopoles therefore lead to highly non-trivial dynamical effects in the vacuum sector of the topologically massive gauge theory. In addition, by taking the pure Chern-Simons limit $e^2 \to \infty$, the discussion of this subsection illustrates the origin of the mirror map as a particle winding process, as well as the appearance of the magnetic flux symmetry from the induced phases (5.40) in the wavefunctions (5.28), around a monopole-instanton from a purely three-dimensional perspective.

6. Mirror symmetry in quantum Hall systems

In this section we shall present a concrete application of the more formal results that we have developed thus far in this paper to the quantum Hall effect, for which Chern-Simons gauge theory provides an effective mean field description [14]. We will see that the mirror map for the class of toroidal target spaces defined by the linear sigma-model implies an intriguing transformation between various quantum Hall systems, and, in certain cases, a non-trivial map between quantum Hall filling fractions. The toroidal compactifications constitute the simplest examples of Calabi-Yau spaces in which the mirror transformation can be represented in a completely transparent form.

6.1. Mirror maps in linear sigma-models

The mirror map for quantum Hall systems originates from the simplest example of mirror symmetry, namely the two-dimensional linear sigma-model. We consider a $U(1)^p$
Chern-Simons gauge theory with fields $A^I$, $I = 1, \ldots, p$, that have pure gauge degrees of freedom $\theta^I$. The bulk action is

$$K \cdot S_{\text{CS}}^{U(1)^p}[A] = \sum_{I,J} \frac{K_{IJ}}{8\pi} \int_M A^I \wedge dA^J$$

(6.1)

where now $K_{IJ}$ is some real-valued matrix. This action induces on the boundary $\Sigma = \partial M$ the $c = p$ conformal field theory [20, 25]

$$S_{XY}[\theta] = \frac{1}{8\pi} \int_\Sigma d^2z K_{IJ} \partial_x \theta^I \partial_{\bar{z}} \theta^J$$

(6.2)

From a string theoretic point of view, the matrix $K_{IJ}$ is related to the background matrix of the $p$-dimensional target space by

$$K_{IJ} = 4 (g_{IJ} + \beta_{IJ}) / \alpha'$$

(6.3)

where $g_{IJ}$ is the metric tensor and $\beta_{IJ}$ the torsion form of the spacetime. The compactification $\theta^I \in [0, 2\pi)$ of the fields of (6.2) yields the $XY$ model which is well-known to possess non-trivial duality properties [20, 33]. Henceforth we assume that we are dealing with a compact $U(1)^p$ Chern-Simons gauge theory, or equivalently that the target space of the induced conformal field theory (6.2) is a flat $p$-torus $T^p \cong \mathbb{R}^p / 2\pi \Gamma$, where $\Gamma$ is a compactification lattice of rank $p$. The notion of a mirror symmetry on such a string background is especially well-understood and can be given explicitly in terms of a matrix transformation law for $K_{IJ}$ [34]

$$K \to \tilde{K} \equiv \left[ (I_p - E_p)K + \frac{4}{\alpha'} E_p \right] \left[ \frac{\alpha'}{4} E_p K + (I_p - E_p) \right]^{-1}$$

(6.4)

where $I_p$ is the $p \times p$ identity matrix and $(E_p)_{IJ} = \delta_{Ip} \delta_{Jp}$ is the step operator in the $p$-th direction of $T^p$. The mirror transformation (6.4) of the Chern-Simons coefficient matrix $K$ is actually one of the $p$ factorized duality maps of compactified bosonic string theory. It can be viewed as a $T$-duality transformation along the $p$-th direction of $T^p$ by choosing a particular compactification lattice $\Gamma$ that splits the $p$-torus into the Cartesian product of a circle of radius $R_p$ and a $(p-1)$-dimensional background $T^{p-1}$. Then the transformation (6.4) maps $R_p \to \alpha'/R_p$ leaving $T^{p-1}$ unchanged [34]. However, we stress again that this equivalence with $T$-duality only occurs at a special point of the moduli space of toroidal compactifications.

For the three-dimensional description of this mirror map, we couple the allowed charges

$$Q_I = q_I - \frac{1}{2} \sum_J K_{IJ} n^J$$

(6.5)

to the $I$-th Chern-Simons gauge field $A^I$, whose associated complex line bundle $L^{(I)}_\Sigma \to \Sigma$ has monopole number $n^I$. Then the mirror map is precisely the process (5.8) which exchanges the particle winding numbers $q_p$ with the monopole numbers $n^p$ in the $p$-th component of $U(1)^p$, leaving all of the other $p-1$ components unchanged. Note that
the factorized duality map (6.4) is defined for any \( p \), even when \( p \) is odd (and hence for compactifications which are not complex manifolds).

When \( p \) is even, as explained in section 2 the mirror map is implemented in the linear sigma-model by considering an appropriate Kähler deformation, leaving the shape (i.e. the angles between the homology cycles) of \( T^p \) fixed but changing its volume, and also a complex structure deformation, which leaves the volume fixed but changes the shape \[6\]. It exchanges the complex and Kähler structures of \( T^p \). To see this, consider the case of the 2-torus \( T^2 \). The metric and torsion form can then be written explicitly as

\[
[g_{IJ}] = \frac{R_2}{\text{Im} \, \tau} \begin{pmatrix} 1 & \text{Re} \, \tau \\text{Re} \, \tau & |\tau|^2 \end{pmatrix}, \quad [\beta_{IJ}] = \begin{pmatrix} 0 & R_1 \\ -R_1 & 0 \end{pmatrix}
\] (6.6)

where \( R_1 \in \mathbb{R}, R_2 \in \mathbb{R}^+, \) and \( \tau \in \mathbb{C}^+ \) specifies a complex structure on \( T^2 \). Written in terms of complex coordinates \( z = x^1 + \tau x^2 \) we have

\[
g = \frac{1}{2} \, \frac{R_2}{\text{Im} \, \tau} \, dz \otimes d\bar{z}, \quad \beta = \frac{i}{2} \, \frac{R_1}{\text{Im} \, \tau} \, dz \wedge d\bar{z}
\] (6.7)

The Kähler 2-form on \( T^2 \) is

\[
\omega \equiv ig_{\bar{z}z} \, dz \wedge d\bar{z} = \frac{i}{2} \, \frac{R_2}{\text{Im} \, \tau} \, dz \wedge d\bar{z}
\] (6.8)

with \( d\omega = 0 \). The moduli space of \( T^2 \) is described by two complex numbers – the modular parameter \( \tau \) which specifies the complex structure on \( T^2 \) and also the parameter

\[
\sigma = R_1 + iR_2 = \int_{T^2} (\beta + i\omega)
\] (6.9)

which describes the (deformed) Kähler structure on \( T^2 \). Mirror symmetry interchanges the complex and Kähler structures, \( \tau \leftrightarrow \sigma \), or equivalently

\[
\text{Re} \, \tau \leftrightarrow R_1, \quad \text{Im} \, \tau \leftrightarrow R_2
\] (6.10)

In the corresponding Chern-Simons gauge theory, the coefficient matrix (6.3) can be written in terms of geometrical parameters as (here we set \( \alpha' = 1 \) for simplicity)

\[
K = \frac{4R_2}{\text{Im} \, \tau} \begin{pmatrix} 1 & \text{Re} \, \tau \, |\tau|^2 \\ \text{Re} \, \tau - \frac{R_1}{R_2} \text{Im} \, \tau & \text{Re} \, \tau + \frac{R_1}{R_2} \text{Im} \, \tau \end{pmatrix}
\] (6.11)

In the mirror theory, this matrix becomes

\[
\tilde{K} = 4 \, \frac{\text{Im} \, \tau}{R_2} \begin{pmatrix} 1 & \text{Re} \, \tau + \frac{R_1}{R_2} \text{Im} \, \tau \\ \text{Re} \, \tau - \frac{R_1}{R_2} \text{Im} \, \tau & R_1 - \frac{\text{Re} \, \tau}{\text{Im} \, \tau} \end{pmatrix}
\] (6.12)

which can easily be checked to coincide with the general formula (6.4). The matrix (6.12) can likewise be written as in (6.3) in terms of a mirror metric tensor and torsion form.
6.2. Composite fermions and the Jain hierarchy

Let us now turn to some basic ideas in the quantum Hall effect. Blok and Wen [35] showed that the hierarchy scheme of states proposed by Jain [36] for the fractional quantum Hall effect can be viewed in terms of an effective theory of composite fermions. The basic idea of this construction is to consider a two-dimensional gas of electrons in an external magnetic field \( b \) at filling fraction

\[
\nu_J = \frac{p}{2mp+1}
\]  

where \( m \) and \( p \) are integers, and attach \( 2m \) flux tubes to each electron. The resulting gas of “composite” fermions (i.e. bound states of fermions and flux tubes) carries an effective flux (and hence filling fraction) \( 1/p = 2m + 1/\nu \). In this way, the \( \nu = p/(2mp + 1) \) fractional quantum Hall states are related to the \( \nu = p \) integer quantum Hall states of the composite object.

The effective action which describes the binding of \( 2m \) units of flux to each electron is

\[
S_{\text{QH}} = \int_M d^3x \left( \psi^\dagger i (\partial_0 - iA_0 - iea_0) \psi + \frac{1}{2m_e} \psi^\dagger (\nabla - iA - iea)^2 \psi \right) + \frac{1}{m} S_{\text{CS}}^{(U(1))}[A]
\]  

where \( \psi \) is the fermionic field for the composite object, \( a \) is the external electromagnetic vector potential and \( A \) is a “fictitious” abelian Chern-Simons gauge field. Here \( e \) is the electron charge and \( m_e \) the electron mass. The equation of motion for \( A_0 \),

\[
n_e \equiv \langle \psi^\dagger \psi \rangle = -\frac{1}{2\pi} \frac{1}{2m} \langle B \rangle
\]  

implies that each electron carries \( 2m \) units of flux and so the fermions see (on average) an effective magnetic field \( B_{\text{eff}} = b + \langle B \rangle \). Since the filling fraction specifies the number of filled Landau levels, \( \nu \propto n_e/B \), this implies that the effective filling fraction of the composite object moving in the effective magnetic field is

\[
\frac{1}{\nu_{\text{eff}}} = \frac{2mp+1}{p} - 2m = \frac{1}{p}
\]  

The most general fractional quantum Hall states are described by the effective theory

\[
S[A,a] = 2K \cdot S_{\text{CS}}^{(U(1)p)}[A] + \sum_I \int_M \left( A^I \wedge \star J_I + \frac{eQ_I}{2\pi} a \wedge dA^I \right)
\]  

(6.17)

The state described by (6.17) contains rank(\( K \)) quasi-particle excitations which have current densities \( J_I \). The index \( I \) labels different levels of the quasi-particles. The electromagnetic field \( a \) minimally couples to each of the topological currents \( \star dA^I \) with charge strengths \( eQ_I \). By integrating over \( A^I \) one can prove that the filling fraction for this state is given by

\[
\nu = \sum_{I,J} Q_I (K^{-1})^{IJ} Q_J
\]  

(6.18)
Different hierarchy schemes of quantum Hall states correspond to different forms of the Chern-Simons coefficient matrix $K$ and the charges $Q_I$. For the case of the Jain hierarchy, the filling fractions (6.13) are obtained by taking $Q_I = 1$ $\forall I = 1, \ldots, p$ and the $p \times p$ matrix

$$K^{(j)} = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix} + 2m \begin{pmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1 \end{pmatrix} \quad (6.19)$$

It is important to realize though that there can be many different fractional quantum Hall states corresponding to the same filling fraction $\nu$. Such states do, however, exhibit some degree of topological order which is intimately related to the properties of the edge theory [37].

6.3. The mirror Jain hierarchy

We now consider the effects of mirror symmetry of the edge theory as a map on quantum Hall systems in the bulk. In terms of the $XY$ model the mirror map exchanges spin wave and magnetic vortex degrees of freedom along the $p$-th direction of $\Gamma$ [33]. A similar monopole-instanton induced process for multi-layer quantum Hall systems has been described in [38], where it was shown that when an electron hops from one layer to another the corresponding currents are no longer separately conserved but change in a manner similar to the mirror map described in this paper (indicating the presence of a monopole-instanton). The interlayer electron hopping corresponds to the instanton described by a monopole. The monopoles exchange the fluxes associated with each layer, and they have also been shown to convert an anyon superfluid into a normal fluid [38]. The role of duality in the quantum Hall effect has also been discussed in [39] where it was shown to be the transformation $\nu \rightarrow 1 - \nu$ that corresponds to particle-hole conjugation with respect to the completely filled state. In [40] it was shown how to use elements of the duality group of toroidally compactified string theory to map some of the well-known hierarchies of quantum Hall states into one another.

The effective edge theory for the quantum Hall system described by the bulk action (6.17) is identical to the action (6.2) for bosonic string theory compactified on a $p$-dimensional torus. The mirror edge theory itself can then be viewed as originating from some mirror three-dimensional Chern-Simons gauge theory with coefficient matrix $\tilde{K}$. The mirror filling fraction for this new quantum Hall system is given by

$$\tilde{\nu} = \sum_{I,J} Q_I (\tilde{K}_{\text{sym}}^{-1})^{IJ} Q_J \quad (6.20)$$

Note that the mirror matrix $\tilde{K}$ in (6.4) will in general contain an anti-symmetric part. This piece corresponds to a torsion form on the boundary, and acts as a topological instanton term affecting only the global, topological properties of the edge theory. Upon integrating (6.1) by parts, we see that only the symmetric part of $\tilde{K}$ contributes to the
bulk dynamics and hence to local observables such as the Hall conductivity $\sigma_H$. Therefore, before inversion, we must symmetrize $\tilde{K}$ in order to calculate the mirror filling fraction using (6.20).

To see the effects of mirror symmetry on states of the Jain hierarchy, we use the same charges $Q_I = 1$, i.e. all the Chern-Simons fields carry the charge of an electron. Using (6.4) and (6.19) we find after some algebra that

$$\tilde{K}^{(j)}_{\text{sym}} = \frac{1}{2m+1} \begin{pmatrix}
4m+1 & 2m & \ldots & 2m & 0 \\
2m & 4m+1 & \ldots & 2m & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
2m & 2m & \ldots & 4m+1 & 0 \\
0 & 0 & \ldots & 0 & 1
\end{pmatrix}
= \begin{pmatrix}
1 & 0 & \ldots & 0 & 0 \\
0 & 1 & \ldots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \ldots & 1 & 0 \\
0 & 0 & \ldots & 0 & \frac{1}{2m+1}
\end{pmatrix} + \frac{2m}{2m+1} \begin{pmatrix}
1 & 1 & \ldots & 1 & 0 \\
1 & 1 & \ldots & 1 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
1 & 1 & \ldots & 1 & 0 \\
0 & 0 & \ldots & 0 & 0
\end{pmatrix}\text{(6.21)}$$

Inverting (6.21) we find that the mirror transformation maps the filling fractions as

$$\nu_j = \frac{p}{2mp+1} \rightarrow \tilde{\nu}_j = 2(m+1) + \frac{p - 2(m+1)}{2mp+1} = (2m+1)^2 \nu_j \text{(6.22)}$$

Hence the mirror quantum Hall system can be interpreted as having $2(m+1)$ completely filled Landau levels plus $p - 2(m+1)$ partially filled levels. Since the number of independent degrees of freedom in the Jain picture can be calculated as the sum of the number of filled Landau levels and the number of partially filled levels, we see that both the original and the mirror quantum Hall systems have $p = \text{rank}(K)$ degrees of freedom.

The mirror system defined by the coefficient matrix (6.21) represents a new hierarchy of states of the quantum Hall effect. Note that for $m = -1$, the mirror map leaves the filling fraction (6.22) invariant. Thus the lowest lying composites of the Jain hierarchy are mapped into the same (class of) quantum Hall systems under mirror symmetry, with a new hierarchical description determined by (6.21). For generic $m$, it is intriguing to interpret the mirror filling fraction using a composite fermion picture by writing

$$\frac{1}{\tilde{\nu}_j} = \frac{1}{p(2m+1)^2} + \frac{2m}{(2m+1)^2} \text{(6.23)}$$

This shows that in the mirror theory there are $2m/(2m+1)^2$ fractional units of flux attached to each electron and so the quasi-particles become anyons which see an effective magnetic field $\tilde{B}_{\text{eff}} = B_{\text{eff}}/(2m+1)^2$. Thus we have an explicit physical realization of the anyonic magnetic flux symmetry responsible for the mirror map. Since the number of filled Landau levels is inversely proportional to the effective magnetic field seen by the electrons, this yields a physical reason as to why mirror symmetry maps a quantum Hall system with a partially-filled Landau level to another quantum Hall system with more than one filled Landau level.
The mirror map is not a symmetry of the edge theory induced by the bulk action (6.17), because of the gauging of the matter-coupled Chern-Simons terms with respect to the electromagnetic field. However, the spectra of the edge excitations are identical for the Hall conductivity $\sigma_H \propto \nu$ and its mirror. For the Jain hierarchy, the only true symmetry appears at the point $m = -1$ in the moduli space. What we have here is a non-perturbative, topology changing mechanism that maps a given quantum Hall system onto another, thus determining new hierarchies of states. Combinations of these mappings could eventually leave the filling fractions determined as in (6.22) invariant, leading to new sorts of topological symmetries of the quantum Hall effect. In this way the role of the monopole-instantons seems to affect the quasi-particle excitations in some non-trivial way, for example turning composite fermions into anyons, and interchanging winding modes of the quasi-particles with magnetic monopole configurations of the Chern-Simons gauge fields both of which interact with the electromagnetic field. It remains to determine precisely how all of these topological effects affect the global and local properties of quantum Hall systems.

7. Conclusions

In this paper we have described mirror transformations in three theories. Beginning with an algebraic isomorphism of $N = 2$ superconformal field theories, we described how two topologically distinct Calabi-Yau manifolds were identical from the point of view of quantum string theory. We then showed how two topologically inequivalent topological membranes induce the same string theory, and how all of the basic topological symmetries of the target space are realized as discrete geometry altering transformations of a 3-manifold $\mathcal{M}$. The mirror map in this case is non-perturbative in character and it interchanges particle winding numbers with monopole numbers. This suggests that mirror symmetry in this description is a sort of $S$-duality. We also showed that, in the three-dimensional picture, there is an intimate connection between mirror symmetry and $T$-duality. This confirms other expectations about the realization of the mirror transformation from a $T$-dual perspective, such as the proposal of [41] that mirror symmetry can be thought of as $T$-duality ($\star d \leftrightarrow d$) on toroidal fibers which are supersymmetric 3-cycles of certain Calabi-Yau spaces, or the more recent algebraic realization of the action of mirror symmetry as a Poisson-Lie $T$-duality transformation of the $N = 2$ superconformal algebra [42]. It is intriguing to note that the major role played by monopole-instantons in the three-dimensional picture is reminescent of the role of instantons (complex curves) in Witten’s linear sigma-model approach to quantum geometry [16]. Moreover, the map between particle winding modes and monopole-instantons could be a link between topological membrane theory and 11-dimensional M Theory, in light of the recent realization that the interchange of momentum modes with instantons is a symmetry of Matrix Theory (see [43] and references therein).
One important aspect that we have not described in this paper is how to explicitly obtain other conventional realizations of the $N = 2$ superconformal algebra, such as those provided by Landau-Ginsburg orbifold models. These models are important for the description of flows on the moduli space of $N = 2$ superconformal field theories, and hence for more complete descriptions of mirror manifolds. These potentially complicated processes are presumably described by charge deformations arising from the coupling of Chern-Simons gauge theories to dynamical charged matter fields. The study of these more involved dynamical models could relate the geometric operation of the mirror map in terms of topology changing processes on principal fiber bundles to those predicted from algebraic geometry, such as the ‘flop’ transition in which the area of a rational curve is shrunk down to zero size and then expanded back to positive volume in a transverse direction [3, 6], and also from the combinatorical ideas of toric geometry [6]. This could also lead to three-dimensional realizations of spacetime topology change from conifold singularities in which physically smooth transitions between Calabi-Yau manifolds with different Hodge numbers occur [44]. This would link the properties of superstrings, black holes and $D$-branes to the topological membrane approach to string theory.

The third model in which we described mirror symmetry was the mean field theory for the quantum Hall effect. We showed how it implies a non-trivial hierarchical changing process in these systems, although the (physical) interpretation of the mirror map in these cases is far less clear. It would be interesting to analyse more carefully the physics associated with the mirror Hall systems, such as the direct interpretation of the magnetic monopole-instantons in the corresponding physical states, and to determine if such systems are indeed experimentally observable. This would then imply the existence of an experimental laboratory in which one could study the physics of mirror symmetry.

Acknowledgements: We thank A. Lopez for many helpful discussions concerning the quantum Hall effect. L.C. gratefully acknowledges financial support from the University of Canterbury, New Zealand. The work of I.I.K. and R.J.S. was supported in part by the Particle Physics and Astronomy Research Council (U.K.).
References


[23] B.R. Greene, in [5].


