FINITE QUANTUM FIELD THEORIES:
CLIFFORD ALGEBRAS FOR YUKAWA COUPLINGS?

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Abstract

By imposing on the most general renormalizable quantum field theory the requirement of the absence of ultraviolet-divergent renormalizations of the physical parameters (masses and coupling constants) of the theory, finite quantum field theories in four space-time dimensions may be constructed. Famous “prototypes” of these form certain well-known classes of supersymmetric finite quantum field theories. Within a perturbative evaluation of the quantum field theories under consideration, the starting point of all such investigations is represented by the conditions for one- and two-loop finiteness of the gauge couplings as well as for one-loop finiteness of the Yukawa couplings. Particularly attractive solutions of the one-loop Yukawa finiteness condition involve Yukawa couplings which are equivalent to generators of Clifford algebras with identity element. However, our closer inspection shows, at least for all simple gauge groups up to and including rank 8, that Clifford-like solutions prove to be inconsistent with the requirements of one- and two-loop finiteness of the gauge coupling and of absence of gauge anomalies.

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1 Introduction

The standard theory of elementary particle physics, in spite of its enormous success in describing the strong and electroweak interactions, exhibits a very unpleasant feature, which it shares with almost all quantum field theories: the appearance of “ultraviolet divergences,” order by order in the perturbative loop expansion. Of course, within the subset of renormalizable theories these divergences may be dealt with by application of the so-called renormalization programme. Nevertheless, the ultimate goal here should be an understanding of nature in terms of a finite theory, i.e., a theory without any divergence.

Every additional symmetry is potentially able to improve the high-energy behaviour of a quantum field theory—as may be seen by increasing gradually the number $N$ of supersymmetries of the theory:

- All one-loop finite $N = 1$ supersymmetric theories are (at least) two-loop finite [1], even if this $N = 1$ supersymmetry is softly broken (in a well-defined way) [2]. Under certain circumstances, $N = 1$ supersymmetric theories may be finite to all orders of their perturbative expansion [3].
- All $N = 2$ supersymmetric theories satisfying merely one single “finiteness condition” are finite to all orders of the perturbative expansion [4], even if one or both supersymmetries are softly broken (in a well-defined way) [5]; these theories have been classified under various aspects [6].
- In the case of the $N = 4$ supersymmetric Yang–Mills theory, that “$N = 2$ finiteness condition” is trivially fulfilled by the particle content of this theory enforced by $N = 4$ supersymmetry [7].

The next target must be non-supersymmetric finite quantum field theories [8, 9]: Is supersymmetry a necessary prerequisite for finiteness? Do there exist non-supersymmetric finite quantum field theories? A fundamental result specifying the particle content of finite quantum field theories in four space-time dimensions has immediately been found [8, 9, 10, 11]: Any non-trivial finite quantum field theory must necessarily comprise vector bosons related to a non-Abelian gauge group, fermions, and scalar bosons. However, the analysis of specific (classes of) models revealed, for instance, that models being finite in dimensional regularization, at least up to some loop order, may be plagued by quadratic divergences in cut-off regularization [12, 13].

In all searches for non-supersymmetric finite quantum field theories, the first genuine hurdle to be taken is the condition for one-loop finiteness of the Yukawa couplings necessarily present in the theory. A particular class of solutions of this one-loop Yukawa finiteness condition is characterized by Yukawa couplings which are equivalent to the generators of some Clifford algebra $\mathbb{C}$ with identity element [14]. It has been speculated [15] that finite theories involving these Clifford-like Yukawa couplings might be constructed. The intention of the present analysis is to scrutinize systematically the relevance of these Clifford-like Yukawa solutions for the construction of new, i.e., non-supersymmetric, finite quantum field theories on a rather general basis. Details of this investigation may be found in Refs. [16, 17, 18]. The $C$ package developed in order to perform the numerical scan through possible candidates for finite theories is extensively described in Ref. [19].

2 Finiteness Conditions in General Quantum Field Theories

Let us start from the most general [20] renormalizable quantum field theory (for particles up to spin 1 $h$) invariant with respect to gauge transformations forming some compact simple Lie group $G$ with corresponding Lie algebra $\mathfrak{g}$. The particle content of this theory is described by

- (gauge) vector-boson fields $A_\mu(x) = (A^\mu_a(x)) \in \mathfrak{g}$ in the adjoint representation $R_{\text{ad}}$ of the gauge group $G$, of dimension $d_\mathfrak{g} := \dim \mathfrak{g}$;
- two-component (Weyl) fermion fields $\psi(x) = (\bar{\psi}^i(x)) \in V_F$ in some arbitrary representation $R_F$ of $G$, of dimension $d_F := \dim V_F$; and
- Hermitean scalar boson fields $\phi(x) = (\phi^\alpha)(x) \in V_B$ in some arbitrary real representation $R_B$ of $G$, of dimension $d_B := \dim V_B$.

The Lagrangian defining this theory is given by

$$
\mathcal{L} = -\frac{1}{4} F_{\mu
u}^a F^{\mu\nu}_a - i \bar{\psi}_i \gamma^\mu (D_\mu \psi)_i + \frac{1}{2} (\bar{\psi}_i (D_\mu \psi)_i)^2 + \frac{1}{2} (D_\mu \phi^\alpha)^\alpha - (D_B \phi)^\alpha \phi_\alpha + \text{mass terms} + \text{cubic scalar-boson self-interactions} + \text{gauge-fixing and ghost terms}.
$$

Where $F_{\mu\nu}^a$ is the field strength of the gauge field $A_\mu^a$, $D_\mu \psi = \partial_\mu \psi + i A_\mu \psi$, $D_\mu \phi = \partial_\mu \phi + i A_\mu \phi$, and $D_B \phi = \partial_\mu \phi + i A_\mu \phi$.

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The $d_g$ Hermitean generators $T^a_R$, $a = 1, 2, \ldots, d_g$, of the Lie group $G$ in an arbitrary, maybe reducible representation $R$ satisfy the commutation relations

$$[T^a_R, T^b_R] = i f^{ab}_{\ c} T^c_R ,$$

with the structure constants $f^{ab}_{\ c}$, $a, b, c = 1, 2, \ldots, d_g$, defining the Lie algebra $\mathfrak{g}$ under consideration. The gauge coupling constant is denoted by $g$. The gauge-covariant field strength tensor $F^a_{\mu\nu}$ is of the usual form,

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^{abc} A^b_\mu A^c_\nu .$$

The gauge-covariant derivatives $D_\mu$ acting on the representation spaces $V_F$ and $V_B$, respectively, read

$$(D_\mu)_R := \partial_\mu - ig T^a_R A^a_\mu \quad \text{for} \quad R = F, B .$$

The four $2 \times 2$ matrices $\sigma^\mu$ in the kinetic term for the Weyl fermion fields are defined in terms of the $2 \times 2$ unit matrix, $1_2$, and the three Pauli matrices, $\sigma$, by $\sigma^\mu = (1_2, -\sigma)$. Without loss of generality, all the Yukawa couplings $Y_{\alpha ij}$ may be assumed to be completely symmetric in their fermionic indices $i$ and $j$, and all the quartic scalar-boson self-couplings $V_{\alpha\beta\gamma\delta}$ may be taken to be completely symmetric under an arbitrary permutation of their indices. The group invariants for an arbitrary representation $R$ of $G$ are defined in terms of the generators $T^a_R$ of $\mathfrak{g}$ in this representation $R$ as usual: the quadratic Casimir operator $C_R$ is given by

$$C_R := \sum_{a=1}^{d_g} T^a_R T^a_R ,$$

and the second-order Dynkin index $S_R$ is obtained from

$$S_R g^{ab} := \text{Tr} \left( T^a_R T^b_R \right) .$$

In the adjoint representation $R_{ad}$, the Casimir eigenvalue $c_G$ equals the Dynkin index $S_G$, i.e., $c_G = S_G$.

According to our understanding of “finiteness” of a general renormalizable quantum field theory, finiteness is tantamount to the vanishing of the beta functions of all physical parameters of this theory, in case of perturbative evaluation of this quantum field theory order by order in its loop expansion. By application of the standard renormalization procedure with the help of dimensional regularization in the minimal-subtraction scheme, the relevant finiteness conditions may be easily extracted [21, 22], see also Refs. [8, 9]:

- The condition for one-loop finiteness of the gauge coupling constant $g$ reads

$$22 c_g - 4 S_F - S_B = 0 . \quad (1)$$

- Adopting this result, the condition for two-loop finiteness of the gauge coupling constant $g$ reads

$$\text{Tr}_F \left( C_F \sum_{\beta=1}^{d_F} Y^{1\beta}_\beta Y_\beta \right) - 2 g^2 \left[ \text{Tr}(C_F)^2 + \text{Tr}(C_B)^2 + d_g c_g \left( S_F - 2 c_g \right) \right] = 0 , \quad (2)$$

where by $\text{Tr}_F$ we indicate the partial trace over the fermionic indices only.

- The condition for one-loop finiteness of the Yukawa couplings $Y_{\alpha ij}$ reads

$$\sum_{\beta=1}^{d_F} \left\{ 4 Y_{\beta} Y^{1\alpha} Y_\beta + Y_{\alpha} Y^{1\beta} Y_\beta + Y_{\beta} Y^{1\beta} Y_\alpha + Y_\beta \text{Tr}_F \left( Y^{1\alpha} Y_\beta + Y^{1\beta} Y_\alpha \right) \right\}$$

$$- 6 g^2 \left[ Y_{\alpha} C_F + (C_F)^T Y_\alpha \right] = 0 ; \quad (3)$$

we call this (cubic and thus troublesome) relation, for brevity, the “Yukawa finiteness condition” (YFC).

These three (lowest-order) finiteness conditions for the gauge and Yukawa couplings, Eqs. (1), (2), and (3), have been identified as the central part of the whole set of finiteness conditions: any investigation of (perturbative) finiteness of quantum field theories should start from this set of equations [15]. (The very first term in Eq. (2) constitutes the link between the two-loop gauge-coupling finiteness condition (2) and the relation one obtains when multiplying the YFC (3) by $Y^{1\alpha}$, performing the sum over all $\alpha = 1, \ldots, d_F$, and taking the trace of the resulting expression with respect to the fermionic indices.)

\footnote{For a more detailed discussion of our notion of “finiteness” of arbitrary renormalizable quantum field theories, consult, for instance, Refs. [8, 9, 16].}
3 The Standard Form of the Yukawa Finiteness Condition

The YFC (3) is obviously invariant under arbitrary $U(d_F) \otimes O(d_B)$ transformations [15]. Luckily, this invariance and the gauge invariance of the Yukawa couplings $Y_{\alpha ij}$ enforced by the gauge invariance of our Lagrangian conspire to render possible the simultaneous diagonalization of the Casimir operators $C_F$ and $C_B$, which results in

\[
(C_F)^i_j = \delta^i_j C_F^i, \\
(C_B)^{\alpha \beta} = \delta^{\alpha \beta} C_B^{\alpha \beta},
\]

and certain sesquilinear products of the Yukawa couplings $Y$, namely, the bosonic and fermionic traces

\[
\sum_{\beta=1}^{d_B} (Y^{\dagger \beta} Y_\beta)^i_j = \delta^i_j y_F^i, \\
\text{Tr}_F \left( Y^{\dagger \alpha} Y_\beta + Y^{\dagger \beta} Y_\alpha \right) = 2 \delta^{\alpha \beta} y_B^\beta.
\]

With all these eigenvalues, the YFC (3) simplifies to what may be regarded its standard form [15, 16]:

\[
4 \sum_{\beta=1}^{d_B} (Y^{\dagger \beta} Y_\beta)_{ij} + Y_{\alpha ij} \left( 2 y_B^\alpha + y_F^i + y_F^j - 6 g^2 C_F^i - 6 g^2 C_B^\alpha \right) = 0. 
\]

In order to explore the implications of gauge invariance for the Yukawa couplings, we decompose the bosonic index $\alpha$ and the fermionic index $i$ into pairs of indices, $\alpha = (A, \alpha_A)$ and $i = (I, i_I)$, where the indices $A$ and $I$ distinguish the irreducible representations $R^A_B \subset R_B$ and $R^I_F \subset R_F$, respectively, while the indices $\alpha_A = 1, \ldots, d_A$ and $i_I = 1, \ldots, d_I$ label the components of $R^A_B$ and $R^I_F$, respectively. If and only if the product $R^A_B \otimes R^I_F$ of any three irreducible representations $R^A_B \subset R_B$, $R^I_F \subset R_F$, and $R^J_C \subset R_G$ of $G$ contains the trivial representation, 1, $N^{(A,I,J)}$ times, there exist $N^{(A,I,J)}$ invariant tensors $(\Lambda^{(k)})_{\alpha A iI jJ}$. In terms of these tensors, the gauge-covariant expansion of $Y$, with coefficients $p^{(k)}_{AIJ} \in \mathbb{C}$, reads

\[
Y_{\alpha ij} = Y_{(A,\alpha_A)(I,i_I)(J,j_J)} = \sum_{k=1}^{N^{(A,I,J)}} p^{(k)}_{AIJ} \left( \Lambda^{(k)} \right)_{\alpha A iI jJ}. 
\]

Following Ref. [15], we introduce a certain—and, upon application of the two-loop gauge-coupling finiteness condition (2), purely group-theoretic—quantity $F^2$, by defining

\[
F^2 := \frac{\text{Tr}_F \left( \sum_{\beta=1}^{d_B} Y^{\dagger \beta} Y_\beta \right)}{6 g^2 \text{Tr}(C_F)^2} = \frac{\text{Tr}(C_F)^2 + \text{Tr}(C_B)^2 + d_g c_g (S_F - 2 c_g)}{3 \text{Tr}(C_F)^2}. 
\]

Remarkably, each theory which satisfies the central part of finiteness conditions, Eqs. (1), (2), and (3), also satisfies the inequality $F^2 \leq 1$. In particular, the extremum $F^2 = 1$ seems to play a decisive rôle in the analysis of these finiteness conditions [15]: For $F^2 = 1$ and only for this case, the cubic YFC (3) simplifies to a merely quadratic system, which is fulfilled by every $N = 1$ supersymmetric (two-loop-) finite theory. Numerical investigations [15] revealed that for finite quantum field theories the value of $F$ is close to $F^2 = 1$. These findings led to conjecture [15] that all finite theories might satisfy $F^2 = 1$.

We call a quantum field theory “potentially finite” if its particle content fulfills both the finiteness condition (1) and the inequalities $0 < F^2 \leq 1$ for the quantity $F^2$ as defined by Eq. (6), if the anomaly index of its fermionic representation, $R_F$, vanishes, if its bosonic representation, $R_B$, is real, $R_B \simeq R^*_B$, and if, at least, one fundamental invariant tensor, required for the decomposition (5) of $Y_{\alpha ij}$, exists.

4 ($\Omega$-Fold) Reducibility of the Yukawa Finiteness Condition

We re-order both the fermionic indices $i$ and the bosonic indices $\alpha$ such that the first $n \leq d_F$ fermionic indices and the first $m \leq d_B$ bosonic indices cover precisely those subsets of $R_F$ and $R_B$, respectively, which have non-vanishing Yukawa couplings. This ordering is then equivalent to the requirement [16]

\[
y_F^i \neq 0 \iff i \in \{1, 2, \ldots, n \leq d_F\}, \\
y_B^\alpha \neq 0 \iff \alpha \in \{1, 2, \ldots, m \leq d_B\}.
\]
Let us call any two sets $M_q = \{(R^{\mu r}, R^{i s}, R^{j t})\}$, $q = 1, 2$, of combinations of real bosonic blocks\(^2\) $R^{\mu r} \subset R_B$ and irreducible fermionic representations $R^{i s} \subset R_F$ appearing in $Y_{(\mu,\alpha,\beta)}(i, j, t)$ to be disjoint if and only if $\{R^{\mu r}\} \cap \{R^{i s}\} = \{R^{i s}\} \cap \{R^{j t}\} = \emptyset$. With this, we define [17]: Let $M = \{R^{\mu r}, R^{i s}, R^{j t}\} R^{\mu r} \subset R_B, R^{i s} \subset R_F$ be the set of all combinations of real bosonic blocks and irreducible fermionic representations in the YFC (4). If $M$ is the union of $\Omega \geq 1$ pairwise disjoint non-empty subsets $M_\omega$, $\omega = 1, 2, \ldots, \Omega$, that is,

$$M = \bigcup_{\omega=1}^\Omega M_\omega,$$

we call this YFC $\Omega$-fold reducible. Rather trivially, an only 1-fold reducible YFC is called irreducible.\(^3\) $\Omega$-fold reducibility of the YFC splits both the “fermionic” representation space $V_F$ and the “bosonic” representation space $V_B$ into direct sums of subspaces, with corresponding “fermionic” and “bosonic” dimensions $n_\omega$ and $m_\omega$, respectively; each of these subspaces is related to some irreducible component $M_\omega$. For every of these irreducible components $M_\omega$, the YFC (4) assumes its reduced standard form:

$$4 \sum_{\beta=1}^{m_\omega} (Y_\beta Y^{i \alpha} Y_\beta)_{ij} + Y_{\alpha ij} \left(2 y_{\beta} F + y_{\beta} F - 6 g^2 C^i_F - 6 g^2 C^j_F\right) = 0 \quad (7)$$

for $1 \leq i, j \leq n_\omega$ and $1 \leq \alpha \leq m_\omega$.

The constraint $F^2 = 1$ may also be expressed by requiring $y_{\beta} F = 6 g^2 C^i_F$ for all $i \in \{1, \ldots, n = d_F\}$. More generally, we may demand $y_{\beta} F = 6 g^2 C^i_F$ for all $i \in \{1, \ldots, n < d_F\}$ and, of course, $y_{\beta} F = 0$ for all $i \in \{n + 1, \ldots, d_F\}$.

We then arrive at a situation similar to the case $F^2 = 1$ but now with $F^2 < 1$: the cubic YFC (4) simplifies also in this case to a quadratic system. With the help of our C package [19], the existence of potentially finite theories solving the one-loop gauge-coupling finiteness condition (1) and the latter quadratic system may be shown numerically [23]. Hence, the advantages brought about by the particular value $F^2 = 1$ are shared by models with values of $F^2$ different from this special case.

5 Symmetries of the Yukawa Finiteness Condition

Since the term which causes all these troubles in the analysis of the YFC is the first (cubic) expression on the left-hand side of Eq. (4), we focus our attention to the investigation of a quantity $x$ defined by

$$2 x^{i \alpha} j_\beta := \left(Y^{i \alpha} Y_\beta + Y^{j \beta} Y_\alpha\right)_{ij}.$$

Our basic building blocks are the irreducible components $M_\omega$: for any index $\alpha$ or $i$ corresponding to a given $M_\omega$, we simply write $\alpha \in M_\omega$ and $i \in M_\omega$, respectively. Every component $M_\omega$ is invariant under all $U(n_\omega) \otimes O(m_\omega)$ transformations. Let’s assume that the corresponding irreducible component $x_\omega$ of $x$ is diagonalizable by some transformation $S_\omega$ of this kind (for a detailed discussion, consult Ref. [18]):

$$(x_\omega)^{i \alpha} j_\beta U(n_\omega) \otimes O(m_\omega) \rightarrow (S_\omega x_\omega S_\omega^\dagger)^{i \alpha} j_\beta = \delta^i_j \delta^{\alpha} \beta x_\omega^{j \beta} \quad \forall i, j, \alpha, \beta \in M_\omega.$$

This transformation enables us to cast the YFC (7) into a form quasi-linear in the Yukawa couplings:

$$Y_{\alpha ij} \left(4 x^{i \alpha} + 4 x^{j \beta} + 2 y_{\beta} F - 2 y_{\beta} F - 6 g^2 C^i_F - 6 g^2 C^j_F\right) = 0 \quad \forall \alpha, i, j \in M_\omega.$$

A particularly important subset of $S_\omega$-diagonalizable quantities $x_\omega$ is given by $x_\omega$ being the tensor product of a factor, $u_\omega$, carrying only bosonic indices and a factor, $v_\omega$, carrying only fermionic indices: $x_\omega = u_\omega \otimes v_\omega$. For $x_\omega$ tensorial, our situation simplifies drastically: one easily proves the relation [16]

$$(x_\omega)^{i \alpha} j_\beta = \delta^i_j \delta^{\alpha} \beta x_\omega^{j \beta} = \delta^i_j \delta^{\alpha} \beta \sum_{k \in M_\omega} y_{\beta} y_{\beta} F \quad \forall i, j, \alpha, \beta \in M_\omega,$$

from which one learns that every irreducible component $x_\omega$ of $x$ is invertible, and the commutator [16]

$$Y_{\alpha ij} \left(y_{\beta} F - y_{\beta} F\right) = 0 \quad \forall \alpha, i, j \in M_\omega.$$

\(^2\) Since the bosonic representation $R_B$ must be real, every non-orthogonal irreducible representation $R_B^c \subset R_B$ has to find a mutually contragredient companion $(R_B^c)^c \subset R_B^c$ in order to be able to form a real orthogonal block: $R_B^c \cong R_B^c \otimes (R_B^c)^c$.

\(^3\) Interestingly, for supersymmetric theories the YFC (4) is always irreducible [23]. An example for a reducible YFC has been constructed by considering some non-supersymmetric particle content [24].
The solution of the quasilinear YFC for $x^\alpha_\omega$ then involves some sort of average $C^i_\omega = (C^i_F + C^i_B)/2$ of fermionic Casimir eigenvalues (where, as indicated by our notation, the fermionic index $j$ depends on $i$ but not on $\alpha$):

$$ (4 - m_\omega) x^\alpha_\omega = 6 g^2 \left( C^i_\omega - \frac{1}{4 - m_\omega + n_\omega} \sum_{k \in M_\omega} C^i_k \right) \quad \forall i, \alpha \in M_\omega . \quad (9) $$

Consequently, for a tensorial $x_\omega$, the corresponding eigenvalues are independent of $\alpha \in M_\omega$: $x^\alpha_\omega = x^\omega$.

Now, what about Clifford algebras in this context? It is rather straightforward to prove [14, 16, 23] that any set of matrices $Y_\alpha$ satisfying the relations

$$ (Y^{\alpha}_\alpha + Y^{\beta}_\beta Y^\alpha) \gamma = 2 \delta^{\alpha} \delta^\gamma \delta^\beta x^\beta \quad \forall \alpha, \beta, \gamma \in M_\omega $$

is equivalent to the union of the $n_\omega \times n_\omega$ unit matrix $1_{n_\omega}$ and the subset

$$ \mathfrak{B}_\omega = \{ N_\alpha | \{ N_\alpha, N_\beta \} = 2 \delta_{\alpha\beta}, N_{\alpha i j} \in \mathbb{R}, \alpha = 1, \ldots, m_\omega - 1 \} $$

of real, symmetric, and anticommuting elements $N_\alpha$ of a representation of some Clifford algebra $\xi$. This Clifford algebra structure restricts, for any irreducible component $M_\omega$, the possible ranges of the respective bosonic and fermionic dimensions $m_\omega$ and $n_\omega$: The rank $p_i$ of some Clifford algebra $\xi$ may be either even, $p_i = 2 \nu_i$, or odd, $p_i = 2 \nu_i + 1$, with $\nu_i \in \mathbb{N}$. In both cases, any matrix representation of $\xi$ is built from $2^{\nu_i}$-dimensional blocks, and there exist precisely $q_i = \nu_i + 1$ symmetric anticommuting elements. Demanding to have enough of these elements at one’s disposal translates into the inequality [16]

$$ n_\omega \geq 2^{m_\omega - 2} . \quad (10) $$

### 6 Representations of Clifford Algebras for $F^2 = 1$ Theories

The restrictivity of Inequality (10) may be demonstrated by applying it to the class of $F^2 = 1$ theories:

**Theorem 1:** Let the YFC be $\Omega$-fold reducible and assume $x_\omega = u_\omega \otimes v_\omega$, $1 \leq \omega \leq \Omega$; then there does not exist any $F^2 = 1$ solution of the YFC obeying the following criteria:

1. The fermionic representation $R_F$ has vanishing anomaly index.

2. The bosonic representation $R_B$ is real (orthogonal).

3. The beta function for the gauge coupling $g$ vanishes in one-loop approximation.

In order to prove the above statement, we employ the subroutine constraint of our C package [19] to implement the requirements $F^2 = 1$ and

$$ n = \sum_{\omega=1}^\Omega n_\omega = d_F . $$

The bosonic and fermionic dimensions of every irreducible component $M_\omega$ of the YFC are related by

$$ 4 + n_\omega = 2 m_\omega , \quad (11) $$

indicating that any fermionic dimension $n_\omega$ must be even. This relation may then be used to eliminate the bosonic dimension $m_\omega$ from the inequality (10), with the result

$$ n_\omega \geq 2^{m_\omega - 2} = 2^{n_\omega/2} , $$

from which we deduce that the fermionic dimension $n_\omega$ is necessarily restricted to one of three values: $n_\omega = 2, 3, 4$. Consequently, for potentially finite $F^2 = 1$ theories with Clifford-type Yukawa couplings, there are no more than two options for the dimensions of any irreducible component $M_\omega$ of the YFC: $(n_\omega = 2, m_\omega = 3$) or $(n_\omega = 4, m_\omega = 4)$. Needless to say, every irreducible component $M_\omega$ of the YFC has to embrace both complete irreducible fermionic representations $R^i_F$ and complete real orthogonal bosonic blocks $R^i_B$ of representations of the Lie algebra $\mathfrak{sl}$, coupling invariantly within the component $M_\omega$. Direct inspection of all simple Lie algebras $\mathfrak{sl}$ shows that only the four Lie algebras $A_1$, $A_2$, $A_3$, and $B_2$ possess irreducible representations of sufficiently low dimension for use in the fermionic sector. The following case-by-case examination of all potentially finite theories extracted in this manner then allows us to claim that there are no potentially finite $F^2 = 1$ solutions of the quasi-linear irreducible YFC (8) obeying simultaneously Inequality (10) for corresponding bosonic and fermionic dimensions. (For the purpose of these analyses, let us denote any $d$-dimensional irreducible representation by $[d]$.)
6.1 The Lie Algebra $A_1$

For the Lie algebra $A_1$, the only irreducible representations of dimensions less than or equal to 4 are the two-, three-, and four-dimensional representations $[2]$, $[3]$, $[4]$. All potentially finite $F^2 = 1$ theories based on $A_1$ with fermionic representations $R_F$ containing only these three irreducible representations are listed (consecutively numbered) in Table 1. The appearance of (any number of) three-dimensional irreducible representations in the fermionic representation $R_F$ of a potentially finite theory is certainly incompatible with either of the two conceivable values, $n_\omega = 2$ or $n_\omega = 4$, of the fermionic dimension $n_\omega$ of any irreducible component $M_\omega$ of our YFC. Inspection of Table 1 leaves us with two candidates:

- Theory no. 1 is consistent with the requirement $d_F \leq 4$, valid for an irreducible YFC. Invariant tensors to construct gauge-invariant Yukawa couplings exist only for $[3] \otimes [4] \otimes [4]$. Therefore, $m$ may take values in $\{3, 6, 9, \ldots, 21\}$, whereas Eq. (11) for $d_F = 4$ implies $m = 4$.

- Theory no. 4 involves the four-dimensional irreducible fermion representation $[4]$, to be covered by an irreducible component $M_\omega$ of fermionic dimension $n_\omega = 4$, which, in turn, implies $m_\omega = 4$ for its bosonic dimension. However, since $[4] \otimes [4] \nsubseteq [2]$, there is no suitable invariant tensor $\Lambda^{(k)}$.

Thus there remains no candidate for a Clifford-type finite $F^2 = 1$ theory based on the Lie algebra $A_1$.

6.2 The Lie Algebra $A_2$

For the Lie algebra $A_2$, the only irreducible representation of dimension less than or equal to 4 is the three-dimensional fundamental representation $[3]$. Obviously, it is not possible to construct invariant $n_\omega = 2$ or $n_\omega = 4$ blocks from three-dimensional representations only. This fact rules out any theory based on $A_2$.

6.3 The Lie Algebra $A_3$

For the Lie algebra $A_3$, the only irreducible representation of dimension less than or equal to 4 is the (four-dimensional) fundamental representation $[4]$. However, there exists no potentially finite $F^2 = 1$ theory with a fermionic representation $R_F$ which involves only this representation $[4]$; in other words, every fermionic representation $R_F$ in potentially finite $F^2 = 1$ theories based on $A_3$ contains at least one irreducible representation of dimension greater than 4. This circumstance rules out every theory based on $A_3$. 

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</table>
The only quantity in Eq. (8) which does not depend on the Yukawa couplings $Y$ is algebra $B$. Consequently, no appropriate gauge-invariant tensors may be constructed. We conclude that the Lie representations $[4], [5], [10]$ of $B$ Table 2: Potentially finite quantum field theory. Clearly, all the above sets of solutions are consistent with the general result (9).

Then, for an $\Omega$-fold reducible YFC with tensorial $x^\omega$ for all irreducible components $M_\omega$, $\omega = 1, \ldots, \Omega$, the solutions for any given component $M_\omega$ may be classified according to the following characteristics:

A: For $a_\omega = 0$, only one common value for all $y^\omega_F$, proportional to some average $C_m^\omega := (C^i_F + C^i_{\bar{F}})/2$ of Casimir eigenvalues, is conceivable: $y^\omega_F \equiv y_\omega = 6 g^2 m_\omega \frac{m_\omega}{4 - m_\omega + n_\omega} C_m^\omega \forall i \in M_\omega$.

B: For $a_\omega \neq (4 - m_\omega)^{-1}$, only one fermionic Casimir eigenvalue $C_\omega$ is admissible, $(C_F)^i_j = \delta^i_j C_\omega$, and only one common value $y_\omega$ for all $y^i_F$ is allowed: $y^i_F \equiv y_\omega = 6 g^2 \frac{m_\omega}{4 - m_\omega + n_\omega} C_\omega \forall i \in M_\omega$.

C: For $a_\omega = (4 - m_\omega)^{-1}$, different values for $y^i_F$ are possible: $(4 - m_\omega) y^i_F = 6 g^2 m_\omega \left( C^i_F - \frac{1}{4 - m_\omega + n_\omega} \sum_{k \in M_\omega} C^i_k \right) \forall i \in M_\omega$.

Of course, every Clifford-like solution of the YFC deduced in this way has to be subjected to, at least, the additional requirements of one- and two-loop finiteness of the gauge coupling $g$ as well as anomaly freedom of the theory, in order to be considered a serious candidate for a (non-supersymmetric) finite quantum field theory. Clearly, all the above sets of solutions are consistent with the general result (9).
In order to list all candidate theories of interest for us, we employ again our C package [19], which provides us with all potentially finite theories for any given simple Lie algebra $A$. We confine ourselves to theories where all irreducible representations able to evolve invariant tensors for Yukawa couplings, together with their respective partners, if necessary, indeed contribute. The systematic search [16, 17] for finite quantum field theories with Clifford-like Yukawa couplings may be carried out numerically. We investigated all simple Lie algebras up to and including rank 8. Our findings may be summarized by some sort of no-go theorem [16, 17]:

**Theorem 2:** Consider a simple Lie algebra $A \in \{ A_r, B_r, C_r, D_r, E_6, E_7, E_8, F_4, G_2 \mid r = \text{rank} A \leq 8 \}$. Let the YFC be $\Omega$-fold reducible and let every irreducible component $M_\omega$ of the YFC be tensorial, i.e., assume that, for $1 \leq \omega \leq \Omega$, the irreducible components $x_\omega$ of $x$ are of the tensor form $x_\omega = u_\omega \otimes v_\omega$, with $u_\omega$ carrying only bosonic indices and $v_\omega$ carrying only fermionic indices. Let

$$x^{i\alpha}_\omega = x^i = 6 g^2 a_\omega C^i_F + b_\omega \text{ for } a_\omega, b_\omega \in \mathbb{C}, \ 1 \leq \omega \leq \Omega, \ i, \alpha \in M_\omega.$$

Then, for arbitrary $a_\omega$, there does not exist any solution if the YFC is irreducible (i.e., for $\Omega = 1$) and, for $a_\omega \neq 0$, there does not exist any solution if the YFC is arbitrarily $\Omega$-fold reducible, provided these solutions are subject to the following requirements:

1. The fermionic representation $R_F$ has vanishing anomaly index.
2. The bosonic representation $R_B$ is real (orthogonal).
3. The beta function for the gauge coupling $g$ vanishes in one- and two-loop approximation.
4. Irreducible blocks $R^\mu_B \subset R_B$ and $R^I_F \subset R_F$, with multiplicities $b_\mu, f_I$, and $f_J$, respectively, which allow for invariant couplings, i.e., $R^\mu_B \otimes R^I_F \otimes R^J_F \supset 1$, contribute to the YFC such that

$$y^F|_{1 \times f_I, R^I_F} \neq 0,$$

and

$$y^B|_{1 \times b_\mu, R^\mu_B} \neq 0.$$  

8 Summary and Conclusions

Motivated by the (recent) conjecture [14, 15] that some particular set of solutions of the condition for one-loop finiteness of the Yukawa couplings in general renormalizable quantum field theories which is characterized by the fact that the resulting Yukawa coupling matrices are equivalent to the generators of a Clifford algebra with identity element might allow to construct new finite quantum field theories, we checked the possibility to find among all these models with Clifford-like Yukawa couplings theories which solve, in addition, the conditions for one- and two-loop finiteness of the gauge coupling constant as well as for absence of gauge anomalies. Very surprisingly, for all gauge groups with rank less than or equal to 8, we did not succeed to find any candidate for a finite quantum field theory with Clifford-like Yukawa couplings (under few reasonable assumptions about the structure of these Yukawa solutions). A Clifford structure of the Yukawa couplings, despite perhaps ideally suited for solving the one-loop Yukawa finiteness condition, appears to be incompatible already with finiteness of the gauge coupling.

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4 In the case of the Lie algebra $A_3$, that is, for the Lie group SU(4), with representations involving the antisymmetric irreducible representation, the required computer resources become rather large.
References