2 Hologization of a Quark-DiQuark Toy Model

The problem of confinement in QCD has been a major challenge in particle physics. Our approach is based on the holographic principle, which posits that the dynamics of quantum gravity in higher-dimensional spacetime can be represented by a lower-dimensional field theory. This allows us to study the strong interactions that underlie the confinement of QCD using techniques from quantum field theory.

In this section, we will demonstrate how this can be done by holographic means.

I Introduction

We will now introduce the holographic model and discuss its key features. This model is based on the duality between string theory and QCD, which allows us to translate the strong interactions of QCD into a more tractable field theory. In the holographic approach, the confinement of quarks is understood as a consequence of the higher-dimensional geometry of spacetime.

After the method of partial hologization is applied to a local gauge-field theory, we will be able to relate the confinement properties of QCD to the geometric properties of the higher-dimensional space.
\[ \mathcal{L}^{\text{MD}} = \mathcal{L}^{\text{M}}_N + D^\dag \left( -\Box - M_D^2 \right) D + \hat{G} \left( \chi D^\dag \right) \left( D\chi \right). \]  

Here \( D(x) \) is the field of an (elementary) scalar isoscalar diquark of mass \( M_D \), and \( \hat{G} \) is a coupling constant.

Analogously to the introduction of collective meson fields into the NJL model (Cf. (4) of I) let us now introduce collective baryon (nucleon) fields \( B \) by using the identity

\[ e^{i \int d^4x \hat{G} \left( \chi D^\dag \right) D\chi} = \mathcal{N} \int DB \tilde{D} e^{i \int d^4x \left( -\frac{1}{G} B - \chi D^\dag B - B D\chi \right)}. \]

The “hadronization” of the generating functional of the Lagrangian (1) will now be performed step by step by integrating over the microscopic quark and diquark fields. We obtain

\[ \mathcal{Z} = \mathcal{N}_1 \int \mathcal{D} \mu \left( \bar{\sigma}, \phi, B, D \right) e^{i \int d^4x \left[ -\text{tr} \ln S^{-1} - \frac{1}{G} \bar{B} B \right]} \times \]

\[ \int d^4x d^4y \left[ -\text{tr} \ln \left( \Delta^{-1} B S_B \right) \right] \]

\[ \mathcal{Z} = \mathcal{N}_2 \int \mathcal{D} \mu \left( \bar{\sigma}, \phi, B \right) \exp \left\{ i \int d^4x \left[ -\text{tr} \ln S^{-1} - \frac{1}{G} \bar{B} B + \ln \left( 1 - \bar{B} S_B \Delta B \right) \right] \right\}, \]

where the trace \( \text{tr} \) runs over Dirac, isospin, and colour indices, \( \mathcal{D} \mu \) denotes the integration measure of fields, \( \Delta^{-1} = -\Box - M_B^2 \) is the inverse diquark propagator, and \( S_B \) is the quark propagator defined by (Cf. (28) of I)

\[ S_B^{-1} = iD - m - \bar{\sigma} - A_{\gamma 5}. \]

Expanding now the logarithms in power series at the one-loop level (see Fig. 1) and performing a low-momentum expansion of Feynman diagrams (corresponding to a derivative expansion in configuration space), one describes both the generation of kinetic and mass terms of the composite baryon field \( B \). This yields the expression

\[ \int d^4x d^4y \bar{B}(x) \left[ -\left( \frac{1}{G} + Z_1^{-1} \overline{\nabla} + g_A \frac{\tau}{2} A_{\gamma 5} \right) \delta^4(x - y) - \Sigma(x - y) \right] B(y). \]

The vertex renormalization constant \( Z_1^{-1} \) and the axial coupling \( g_A \) arise from the low-momentum (low derivative) expansion of the vertex diagrams of Fig. 1b) and c), respectively. The nucleon self-energy \( \Sigma \) has in momentum space the decomposition \( \Sigma(p) = p \overline{\Sigma_n} (p^2) + \overline{\Sigma}_s (p^3) \). Its low momentum expansion generates a kinetic term \( \sim \frac{1}{G} \) and, together with the constant term \( -1/G \), a mass \( M_B \) given by the equation

\[ -\frac{1}{G} B - \chi D^\dag B - B D\chi = 0. \]
Fig. 1. Baryon self-energy diagram $\Sigma$ (a) and vertex diagrams (b,c) arising from the loop expansion of the logarithms in Eq. (4).

$$\frac{1}{G} + M_B \Sigma_N (M_B^2) + \Sigma_3 (M_B^2) = 0. \quad (7)$$

In terms of renormalized fields defined by $B = Z \hat{B} r$, where $Z$ is the wave function renormalization constant satisfying the Ward identity $Z = Z_1$, we obtain from (6) the effective chiral meson-baryon Lagrangian [10]

$$\mathcal{L}_{\text{eff.}}^{MB} = \bar{B} \left( i D - M_B \right) B - g^A_{\mu} B r_{\gamma \mu} \gamma_5 r_{\gamma} B r_\mu A^\mu. \quad (8)$$

with $\hat{D} \equiv \hat{\partial} + i \hat{\gamma} \bar{\nu}$ and $g^A_{\mu} = Z g_A$ being the renormalized axial coupling constant. Note that expression (8) completely coincides in structure with the famous phenomenological chiral Lagrangians introduced at the end of the Sixties when considering nonlinear realizations of chiral symmetry [6,7]. However, in our case these Lagrangians are not obtained on the basis of symmetry arguments alone, but derived from an underlying microscopic quark-diquark picture which allows us to estimate masses and coupling constants of composite hadrons. Note that, due to $A^\mu = \frac{1}{F_\pi} \partial^\mu \varphi + \cdots$, the second term in (8) leads to a derivative coupling of the pion field $\varphi$ with the axial-vector baryon current. In order to get rid of the derivative and to reproduce the standard $\gamma_5$ coupling, it is convenient to redefine the baryon field $B_r \rightarrow \tilde{B}$ by

$$B_r = e^{-ig_{\mu} A_{\mu} r_\mu} \tilde{B}. \quad (9)$$

Inserting (9) into (8) and performing a power series expansion in $\varphi$ leads to the expression

$$\mathcal{L}_{\text{eff.}}^{MB} = \bar{\tilde{B}} \left( i \vartheta - M_B \right) \tilde{B} + g^A_{\mu} \frac{M_B}{F_\pi} B r_{\gamma \gamma} B r_{\gamma} \varphi + O (\varphi^2). \quad (10)$$

Obviously, we have to identify the factor in front of the interaction term as the pion-nucleon coupling constant $g_{BB\varphi}$,

$$g_{BB\varphi} = g^A_{\mu} \frac{M_B}{F_\pi}. \quad (11)$$
which is nothing else than the Goldberger-Treiman relation of the composite nucleon. Finally, by combining the Lagrangian (8) with the effective chiral meson Lagrangian of the nonlinear $\sigma$ model (Cf. (31) of I), we obtain the complete meson-baryon Lagrangian

$$L_{\text{eff. tot.}}^{\text{MB}} = L_{\text{spin}}^{\text{MB}} + L_{\text{eff.}}^{\text{MB}}.$$  \hspace{1cm} (12)

It is further possible to estimate magnetic moments as well as electric and magnetic radii of composite protons and neutrons by introducing electromagnetic interactions into the toy Lagrangian (1). The obtained pattern of predicted low-energy characteristics of nucleons has been shown to describe data at best qualitatively [11]. Obviously, in order to get better agreement with data, it is necessary to consider more realistic, but also more complicated models with non-local quark-diquark interactions containing both scalar and axial-vector diquarks [3-5, 12].

3 Further Extensions

3.1 Heavy Baryons with Scalar and Axial-Vector Diquarks

The above considerations can be easily generalized to heavy-light baryons $B \sim (Qqq)$, where $Q = c, b$ is a heavy quark and the light quarks $q = u, d, s$ form scalar ($D$) and axial-vector diquarks ($F^a_\mu$) of the flavour group $SU(3)_F$. The quantum numbers of the light diquarks with respect to the spin, flavour, and colour follow here from the decomposition

spin : $\frac{1}{2} \times \frac{1}{2} = 0_s + 1_s$

flavour : $3_F \times 3_F = 5_{F, a} + 6_{F, s}$

colour : $3_c \times 3_c = 3_{c, a} (+ 6_{c, s})$,

where the indices $s, a$ refer to symmetry, antisymmetry of the respective wave functions under interchange of quark indices. According to the Pauli principle the fields of the scalar and axial-vector diquarks must then form a flavour (anti) triplet or sextet, respectively: $D_{F, a}^\mu$, $F_{F, a}^\mu$. The baryons as bound states of diquarks with a quark $Q$ are evidently colourless, since the product representation $3_c \times 3_c$ contains the colour singlet $1$. In Ref. [8] we have studied an extended quark-diquark model given by the Lagrangian

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_{\text{int.}}.$$  \hspace{1cm} (13)

$$\mathcal{L}_0 = \bar{q} \left( i \partial - m_0 \right) q + \tilde{Q} \mu \nu \partial^\mu Q_{\nu} + \text{tr} \left[ \partial_{\mu} D^\dagger \partial^\mu D - M_0 D^\dagger D \right]$$
\[ -\frac{1}{4} \text{tr} \ F_{\mu}^I F^{\mu \nu} \ + \text{tr} \ M_2^2 \ F_{\mu}^I F^{\mu} , \]  
\[ \mathcal{L}_{\text{int.}} = \tilde{G}_1 \text{tr} \left( \bar{Q}_1 \ D^I \right) \left( DQ_1 \right) - \tilde{G}_2 \text{tr} \left( \bar{Q}_2 \ F^{I \mu} \right) P_{\mu, \nu}^I \left( F^{\nu} Q_2 \right) , \]

where \( D^I \), \( F^{I \mu} \) are antisymmetric/symmetric \( 3 \times 3 \) flavour matrices, \( Q_1 \) is a heavy quark spinor of 4-velocity \( v_{\mu} \) (using the notation of Heavy Quark Effective Theory [13]), \( P_{\mu, \nu}^I \) is a transverse projector, and \( \tilde{G}_1, \tilde{G}_2 \) are coupling constants.

In order to bilinearize the interaction term (15) one now needs two types of baryon fields

\[ T_\nu \left( \frac{1}{2}^{+} \right) \sim Q_\nu D \quad S_{\nu, \mu} \left( \frac{1}{2}^{+} \cdot \frac{3}{2}^{+} \right) \sim Q_\nu F_{\mu} \]

where \( S_{\nu, \mu} \) is a superfield of spin-symmetry partners \( B_{\nu} \left( \frac{1}{2}^{+} \right), B_{\nu, \mu}^{\ast} \left( \frac{3}{2}^{+} \right) \) admitting the decomposition

\[ S_{\nu, \mu} = \frac{1}{\sqrt{3}} \gamma_5 \left( \gamma_{\mu} - v_{\mu} \right) B_{\nu} + B_{\nu, \mu}^{\ast} . \]

The interaction term (15) can then be rewritten as a sum of terms bilinear in the baryon field and a Yukawa interaction term. Integrating successively over the quark fields \( q, Q \) and then over the diquark fields \( D, F_{\mu} \) leads to determinants containing the baryon fields \( T \) and \( S \). Finally, by employing again a loop expansion and taking into account only lowest order derivative terms, leads to the free effective baryon Lagrangian of heavy flavour type

\[ \mathcal{L}_{\text{eff.}} = \text{tr} \bar{T}_\nu \left( i v \cdot \partial - \Delta M_T \right) T_\nu - \text{tr} \bar{S}_{\nu, \mu} \left( i v \cdot \partial - \Delta M_S \right) S_{\mu}^{\nu} , \]

where the mass differences \( \Delta M_{T,S} \equiv M_{T,S} - m_Q \) are calculable. Moreover, taking into account vertex diagrams analogously to those shown in Fig.1 \( (b,c) \) leads to the inclusion of interactions with the \( SU(3)_F \)-octet of light pseudoscalar mesons \( \phi_i \) [8].

### 3.2 Composite Diquarks

For simplicity, we have considered up to now only models containing diquarks as elementary fields. Clearly, it is desirable to treat diquarks on the same footing as composite mesons as composite particles. This has been done in Ref. [9] by considering an extended NJL model for light \( (q = u, d, s) \) and heavy quarks \( (Q = c, b) \) containing 2-body interactions of diquark-type \((qq)\) and \((qQ)\)

\[ \mathcal{L}^{\text{NJL}}_{\text{int.}} = G_1 \left( \bar{q} \Gamma_\alpha q \right) \left( \bar{q} \Gamma_\alpha q \right) + G_2 \left( \bar{q} \Gamma_\alpha q \right) \left( \bar{q} \Gamma_\alpha q \right) , \]

Here \( q^c = C \bar{q} \gamma^7 \) denotes a charged-conjugated quark field, and \( \Gamma_\alpha, \Gamma_\alpha^c \) are flavour and Dirac (spin) matrices. The interaction term (18) can again be rewritten in
terms of a Yukawa coupling of light composite scalar and axial-vector diquark fields $D(0^+, 1^+)$ or heavy diquarks $D_c (J^P = 0^+, 1^+)$ to quarks. Notice that these models lead to a nonlocal quark-diquark interaction mediated by quark exchange shown in Fig. 2.

![Quark-exchange diagram]

**Fig. 2.** Quark-exchange diagram leading to a nonlocal quark-diquark interaction with composite diquarks.

As shown in a series of papers [3-5, 9], the baryon spectrum can then be found by solving Faddeev-type equations for quark-diquark bound states. Finally, the above method of path integral hadronization can also be applied to investigate 3-body interactions of the type $\mathcal{L} = \bar{f}f(gqg)$ [5].

I hope that these few examples are sufficient to show you that path integral hadronization is indeed a powerful nonperturbative method in particle physics.

**References**