On the Evaluation of Compton Scattering Amplitudes in String Theory

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**Abstract:** We consider the Compton amplitude for the scattering of a photon and a (massless) “electron/positron” at one loop (i.e. genus one) in a four-dimensional fermionic heterotic string model. Starting from the bosonization of the world-sheet fermions needed to explicitly construct the spin-fields representing the space-time fermions, we present all the steps of the computation which leads to the explicit form of the amplitude as an integral of modular forms over the moduli space.
1 Introduction and Summary

The computation of scattering amplitudes is one of the most powerful tools we have to study the general features of first-quantized (perturbative) string theories. These kinds of computations are indeed necessary for a deeper understanding of the analyticity properties of string amplitudes, their divergences and associated renormalizations [1]—[8].

Even if not of direct phenomenological interest, string amplitudes can also be useful for our understanding of field theory. In the low energy limit of string theory ($\alpha' \to 0$), the gravitational and non-local effects are negligible and we get an ordinary field theory. It is then possible to use the field-theory limit of string scattering amplitudes to reproduce known results of field theory in an alternative way, which can lead to the discovery of new features of field theory itself (see for instance [9, 10, 11]).

Up to now these results (such as new Feynman-like rules for pure Yang-Mills at one-loop) have been derived using amplitude involving space-time bosons as external states. Very few one-loop amplitudes having space-time fermions as external states have appeared in the literature (see for example ref. [12]), mostly because of some technical issues appearing in the explicit evaluations of these amplitudes, as discussed for example in refs. [13, 14]. On the other hand, computations of this type could be very useful if one wishes to get similar results for QCD.

In this paper we present one of the simplest four-point one-loop scattering amplitude involving external space-time fermions, that is the Compton scattering of an “electron/positron” and a photon. Here we call “electron” (or “positron”) a massless space-time fermion charged under a $U(1)$ component of the total gauge group. One can easily extend the results we present to the case of the scattering of a massless “quark” on a gluon, since this requires only some simple modifications of the left-moving part of our equations.

We choose to work in a specific four-dimensional heterotic string model, which has the properties that its space-time spectrum depends on a set of parameters and, as described in ref. [14], only for some values of these parameters is supersymmetric. Of course the details of the model chosen for the computation affect the particular features of the scattering amplitude. However we expect that the general properties of string scattering amplitudes are independent of the specific string model, in particular when one takes the field-theory limit.
The paper is organized as follows. In the section 2, we review the KLT-formalism for constructing four-dimensional heterotic string models with free world-sheet fermions and we describe the particular model we choose to work with. In section 3, we list the tools necessary for the computation of the one-loop amplitude. By bosonizing the world-sheet fermions, we introduce the spin-fields with which we build the vertex operators for the external fermionic states. We then define the gamma matrices and the charge conjugation matrix. In section 4 we illustrate the steps of the computation leading to the “off-shell” four-point scattering amplitude of two photons and two massless chiral fermions. We discuss the role of the PCO operators, the evaluation of the (world-sheet) correlators, the appearance of the identities in theta-functions necessary to have a Lorentz covariant result, and the use of the GSO projection conditions. In section 5 we use the Dirac equation and the other on-shell conditions to obtain the final on-shell amplitude. Then we discuss the general properties of such amplitude and we show its independence on the point of insertion of the PCO. Finally we briefly discuss the relation of our string scattering amplitude with the analogous result in field theory.

2 4d Free Fermion Heterotic String Models

In this section we briefly review the main lines of the construction of four-dimensional heterotic string models built with free world-sheet fermions following the conventions of Kawai-Lewellen-Tye [15] and we describe the particular model we have chosen for the computation of the one-loop Compton scattering amplitude. Our notations differ somewhat from those of ref. [15]. In particular we choose to work in the Lorentz-covariant formulation, rather than in the light-cone gauge. Moreover we perform all computations directly in Minkowski space-time, with metric $\eta_{\mu\nu} = (-1, 1, 1, 1)$.

2.1 The KLT Formalism

The four-dimensional heterotic string models we consider [15] (see also [16, 17]) are constructed by fermionizing all the two-dimensional degrees of freedom other than those associated with the four-dimensional space-time coordinates and by treating them as free world-sheet fermions. In the Lorentz-covariant formulation, these models are built with the four space-time coordinate fields $X^\mu(z, \bar{z})$, twenty-two left-moving complex fermions $\psi_{[l]}(z)$ (with $l = 1, \ldots, 22$), eleven right-moving complex fermions
\( \psi_{(l)}(z) \) (with \( l = 23, \ldots, 33 \)), right-moving superghosts \( \beta, \gamma \), left- and right-moving reparametrization ghosts \( b, c \). The \( N = 1 \) world-sheet supersymmetry of the right movers \([16]\) is generated by the supercurrent

\[
T_F = T_F^{[X, \psi]} - \partial \beta - \frac{3}{2}(\partial \gamma) \beta + \frac{1}{2} \gamma b, \tag{2.1}
\]

where the orbital part is given by

\[
T_F^{[X, \psi]} = -\frac{i}{2} \partial X \cdot \psi - \frac{i}{2} \sum_{m=1}^{2} (\psi_{(23)}^m \psi_{(24)}^m \psi_{(25)}^m + \psi_{(26)}^m \psi_{(27)}^m \psi_{(28)}^m + \psi_{(29)}^m \psi_{(30)}^m \psi_{(31)}^m). \tag{2.2}
\]

In eq. (2.2), \( \psi^\mu \) are four real Majorana fermions related to the two complex fermions \( \psi_{(32)} \) and \( \psi_{(33)} \) by:

\[
\begin{align*}
\psi^0 &= \frac{1}{\sqrt{2}} (\psi_{33} - \psi_{33}^*), & \psi^2 &= \frac{1}{i \sqrt{2}} (\psi_{32} - \psi_{32}^*), \\
\psi^1 &= \frac{1}{\sqrt{2}} (\psi_{33} + \psi_{33}^*), & \psi^3 &= \frac{1}{\sqrt{2}} (\psi_{32} + \psi_{32}^*). \tag{2.3}
\end{align*}
\]

They transform as a space-time vector and are the world-sheet superpartners of the space-time coordinate fields. Analogously, the real \( \text{internal fermions} \) \( \psi_{(l)}^m(z) \) are defined from the nine complex right-moving fermions \( \psi_{(l)}(z) \) \( (l = 23, \ldots, 31) \) by

\[
\psi_{(l)}^m = \left\{ \frac{1}{\sqrt{2}} (\psi_{(l)} + \psi_{(l)}^*), \frac{1}{i \sqrt{2}} (\psi_{(l)} - \psi_{(l)}^*) \right\}, \quad m = 1, 2. \tag{2.4}
\]

They correspond to the compactified dimensions and provide internal symmetry indices to the states of the string. Moreover, we introduce another right-moving fermion which is needed to fermionize the superghosts in the usual way, \( \beta = \partial \xi \psi_{(34)}^* \) and \( \gamma = \psi_{(34)/\eta} \).

After fermionization, any KLT model is specified by the set of possible boundary conditions (spin structures) for the 33 world-sheet fermions and the superghost. On the cylinder, parametrized by a complex coordinate \( z \), the boundary conditions for the fermions assume the form:

\[
\begin{align*}
\bar{\psi}_{(l)}(e^{-2\pi i z}) &= e^{-2\pi i (\frac{1}{2} - \alpha_l)} \bar{\psi}_{(l)}(z) & l = 1, \ldots, 22 \\
\psi_{(l)}(e^{2\pi i z}) &= e^{2\pi i (\frac{1}{2} - \alpha_l)} \psi_{(l)}(z) & l = 23, \ldots, 33, \tag{2.5}
\end{align*}
\]

where \( \alpha_l \) \( (l = 1, \ldots, 22) \) and \( \alpha_l \) \( (l = 23, \ldots, 33) \) are real numbers. The Ramond (R) and Neveu-Schwarz (NS) boundary conditions correspond to \( \alpha_l = 0 \) and \( \alpha_l = \frac{1}{2} \) respectively.
World-sheet supersymmetry, modular invariance of the one and multi-loop partition function [15, 18] constraint the possible choices for the fermion boundary conditions. As a consequence, any KLT model is completely specified by a certain number of basis vectors $W_i$, giving the set of possible boundary conditions (spin structures) for the fermions, and by a set of parameters $k_{ij}$ defining the GSO projections.

For example, for the supercurrent (2.1) to have well-defined boundary conditions, the fermions $\psi_{(32)}$ and $\psi_{(33)}$ associated with space-time coordinates, the superghosts, and the products of triplets of internal fermions must all carry the same spin structure: for $\sum_{l=23}^{25} \alpha_{l} \mod 1 = \sum_{l=26}^{28} \alpha_{l} \mod 1 = \sum_{l=29}^{31} \alpha_{l} \mod 1 = \alpha_{32}$,
\[ \alpha_{32} = \alpha_{33} = \alpha_{34}. \] (2.6)

Since fermions associated with space-time coordinates can only have periodic or anti-periodic boundary conditions, it follows that all the right-moving fermions are restricted to have only R or NS boundary conditions. For the left-movers, boundary conditions other than R or NS are possible. It is this freedom in the choice of the boundary conditions that allows these models to have interesting gauge groups.

Define $\alpha$ to be a 32-dimensional vector of components $\bar{\alpha}_l$ ($l = 1, \ldots, 22$) and $\alpha_l$ ($l = 23, \ldots, 32$). Each vector $\alpha$ then specifies a choice of boundary conditions for the free world-sheet fermions as for eq. (2.5). All vectors compatible with the constraints just described, can be expressed as linear combinations of a set of basis vectors $W_i$ as [15]
\[ \alpha = \sum_{i=0,1,\ldots} m_i W_i \equiv m W, \] (2.7)
where the integers $m_i$ take values in $\{0, \ldots, M_i - 1\}$, $M_i$ being the smallest integer such that $M_i W_i$ ($i$ not summed) is a vector of integer numbers. The set of basis vectors always includes the vector [15]
\[ W_0 = \left( \frac{1}{2} \right)^{22} \left( \frac{1}{2} \right)^{11} \left( \frac{1}{2} \right)^{3} \left( \frac{1}{2} \right)^{1}, \] (2.8)
which describes the NS boundary conditions for all fermions. (Since the fermions $\psi_{(32)}$ and $\psi_{(33)}$ and the superghosts have the same spin structure we do not need to extend $W_i$ to 34-dimensional vectors by adding $\alpha_{33} = \alpha_{34}$.)

Each distinct choice of boundary conditions (each vector $\alpha$) defines a sector in the spectrum of string states. There are $\prod_i M_i$ such sectors. All states belonging
to a given sector have the same space-time properties: they are space-time bosons (fermions) depending on whether the last right-moving component $\alpha_{32}$ of the boundary vector (which specifies the boundary conditions for the supercurrent) takes value $1/2$ ($0$) MOD 1. We will refer to such a sector as a bosonic or fermionic sector respectively.

In each sector, the set of all possible string states is constructed by acting on the vacuum with the creation operators. Generally one considers states in the superghost vacuum with charge $0$ including the supercurrent superghost vacua. In the Lorentz-covariant formulation the GSO projections assume the form [14]

$$W_i \cdot N[\alpha] - s_i(N_{[\alpha_{32}]}^{(0)} - N_{[\alpha_{32}]}^{(\beta_7)}) \equiv \text{MOD} \sum_j k_{ij} m_j + s_i + k_{i0} - W_i \cdot [\alpha].$$ (2.9)

Here the inner-product of two vectors, such as $W_i \cdot N$, includes a factor of $(-1)$ for right-moving components. Also, for any real number $\alpha$ we define $[\alpha] \equiv \alpha - \Delta$, where $0 \leq [\alpha] < 1$ and $\Delta \in \mathbb{Z}$; $s_i = \alpha_{32}$ is the last entry of $W_i$. $N_{[\alpha]}$ is the number operator for the “longitudinal” complex fermion $\psi_{[\alpha]}$ and $N_{[\alpha_{32}]}$ is the superghost number operator

$$N_{[\alpha]}^{(l)} = \sum_{q=1}^{\infty} \left[ \psi_{l-q+[\alpha]} \psi_{l+q+[\alpha]} - \psi_{l+q+[\alpha]} \psi_{l-q+[\alpha]} \right],$$ (2.10)

and

$$N_{[\alpha_{32}]}^{(\beta_7)} = \sum_{q=1}^{\infty} \left[ \beta_{-q+[\alpha_{32}]} \gamma_{q-[\alpha_{32}]} + \gamma_{-q+1+[\alpha_{32}]} \beta_{q-1+[\alpha_{32}]} \right],$$ (2.11)

where we introduced the mode expansions

$$\psi_{q}(z) = \sum_{q \in \mathbb{Z}} \psi_{l-q+[\alpha]} z^{-q+[\alpha]-1/2},$$

$$\beta(z) = \sum_{q \in \mathbb{Z}} \beta_{-q+[\alpha_{32}]} z^{-q+[\alpha_{32}]-3/2},$$

$$\gamma(z) = \sum_{q \in \mathbb{Z}} \gamma_{-q+[\alpha_{32}]} z^{-q+[\alpha_{32}]+1/2}. $$ (2.12)

Consistency at one loop level constrains the quantities $k_{ij}$ parameterizing the GSO projections eq. (2.9) and the vectors $W_i$ to satisfy the following conditions

$$k_{ij} + k_{ji} \equiv \text{MOD} (W_i \cdot W_j)$$
\[
M_j k_{ij} \equiv 0
\]
\[
k_{ii} + k_{i0} + s_i - \frac{1}{2} \mathbf{W}_i \cdot \mathbf{W}_i \equiv 0.
\]  \hspace{1cm} (2.13)

More precisely, these constraints follow from the requirement of modular invariance of the 1-loop partition function

\[
Z = \sum_{m_i, n_j} C_{\alpha \beta}^\alpha \int \frac{d^2 \tau}{(\text{Im} \tau)^2} \left( \eta(\tau) \right)^{-12} \prod_{l=1}^{32} \Theta^\alpha \left[ \delta^{\alpha \beta}_{l \bar{l}} \right] (0 | \bar{\tau}) \times (\eta(\tau))^{-12} \prod_{l=23}^{32} \Theta^\alpha \left[ \alpha^{\bar{l}}_l \right] (0 | \bar{\tau}) \frac{1}{\text{Im} \tau}, \]  \hspace{1cm} (2.14)

where the summation coefficients are given by [14, 20]

\[
C_{\alpha \beta}^\alpha = \frac{1}{\prod_i M_i} \exp \left\{ -2\pi i \left[ \sum_i (n_i + \delta_{i0}) \left( \sum_j k_{ij} m_j + s_i - k_{i0} \right) + \sum_i m_i s_i + \frac{1}{2} \right] \right\}.
\]  \hspace{1cm} (2.15)

At one loop it is necessary to specify the boundary conditions of the fermions around the two non-contractible loops of the torus. The spin structure \([\alpha^\beta_i]\) of the fermion \((l)\) is thus parametrized by the two sets of integers, \(m_i\) and \(n_i\), each taking values in \(\{0, \ldots, M_i - 1\}\):

\[
\alpha = \sum_{i=0,1,\ldots} m_i \mathbf{W}_i
\]
\[
\beta = \sum_{i=0,1,\ldots} n_i \mathbf{W}_i.
\]  \hspace{1cm} (2.16)

The \(m_i\) specify the sector of states being propagated in the loop. The summation over the \(n_j\) in eq. (2.14) enforces the GSO projection on the states in the \(m_i\)'th sector. Therefore the sum over all spin structures gives the sum over the full spectrum of GSO projected states circulating in the loop. The coefficients \(C_{\alpha \beta}^\alpha\) of eq. (2.15) are chosen so that all the states in the GSO-projected spectrum describing space-time bosons (fermions) contribute to the partition function with weight +1 (−1). \(^1\)

2.2 Our Model

The model we consider has been proposed in ref. [15] and has been extensively analyzed in ref. [14]. It has two main features: the gauge group contains a \(U(1)\) and the

\(^1\)Our expression (2.15) for the summation coefficients is somewhat simpler than that given in ref. [15], thanks to certain phases being absorbed into the definition of the \(\Theta\) functions (see Appendix A for conventions).
spectrum can be $N = 1$ space-time supersymmetric or not depending on the values of the parameters $k_{ij}$.

The model is specified by the following boundary vectors

\[ W_0 = \left( \frac{1}{2} \right)^{22} \left( \frac{1}{2} \right)^3 \left( \frac{1}{2} \right) \]

\[ W_1 = \left( \frac{1}{2} \right)^{22} \left( \frac{1}{2} \right)^3 (0) \]

\[ W_2 = \left( \frac{1}{2} \right)^{14} (0)^8 \left( \frac{1}{2} \right)^2 (0) \]

\[ W_3 = \left( \frac{1}{2} \right)^7 (0)^7 \left( \frac{1}{2} \right)^3 (0)^5 \left( \frac{1}{2} \right)^2 (0) \]

\[ W_4 = \left( 0 \right)^7 (0)^7 \left( \frac{1}{2} \right)^2 (0) (0)^5 \left( \frac{1}{2} \right)^2 (0) (0) \]

(2.17)

Since all components of the vectors $W_i$ are 0 or 1/2, the integers $M_i$ and $m_i$ assume the values

$M_i = 2$ and $m_i, n_j = 0, 1$ with $i, j = 0, \ldots, 4$. (2.18)

The constraints (2.13) are satisfied by any set of $k_{ij}$ of the form

\[
\begin{pmatrix}
  k_{00} & k_{01} & k_{02} & k_{03} & k_{04} \\
  k_{10} & k_{11} & k_{12} & k_{13} & k_{14} \\
  k_{20} & k_{21} & k_{22} & k_{23} & k_{24} \\
  k_{30} & k_{31} & k_{32} & k_{33} & k_{34} \\
  k_{40} & k_{41} & k_{42} & k_{43} & k_{44}
\end{pmatrix}
\]

\[ \text{MOD} \frac{1}{2} = \begin{pmatrix}
  k_{00} & k_{01} & k_{02} & k_{03} & k_{04} \\
  k_{01} & k_{02} & k_{03} & k_{14} \\
  k_{02} & k_{03} & k_{02} & k_{04} \\
  k_{03} & k_{13} & k_{23} & k_{34} \\
  k_{04} & k_{14} & k_{24} & k_{34} & k_{04}
\end{pmatrix},
\]

(2.19)

where all the unspecified $k_{ij}$ take values 0, $\frac{1}{2}$, and we choose as independent $k_{00}$ and $k_{ij}$ with $i < j$.

The set of basis vectors (2.17) generates a total of 32 sectors in the spectrum according to eq.(2.7). To indicate such sectors we introduce the shorthand notation

\[ \alpha = \sum_{i=0}^{4} m_i W_i \equiv W_{\text{subscript}}, \]

(2.20)

where “subscript” is the list of those $i$ for which $m_i = 1$. The only exception is the sector for which all the $m_i$ are zero which we will just denote by $\alpha = 0$.

From the left-moving part of the vectors (2.17), it follows that the world-sheet fermions are grouped together according to $W_4$: For example, the first seven left-moving complex fermions always have the same spin structure; from the corresponding
14 real fermions we may build up the Kac-Moody algebra of $SO(14)$. Therefore the
gauge group of the model is

$$SO(14) \otimes SO(14) \otimes SO(4) \otimes U(1) \otimes SO(10),$$

(2.21)

where the $U(1)$ is actually realized as an $SO(2)$. Indeed, as we will show presently,
the gauge bosons in the physical spectrum exactly fill out the adjoint representation
of this group.

In ref. [14] the reader can find the list of the vacuum energies of all the 32 sectors
in the model and of the excited states up to mass level $\alpha' M^2 = 1$, as well as the
truncation of the spectrum due to the GSO projections. Here we just review the
main features of the spectrum which will interest us.

In each sector, the vacuum state is given by the tensor product of the vacua
Corresponding to every single fermion. Since the vectors $W_i$ have all entries 0 and
1/2, each fermion can only have Ramond or Neveu-Schwarz boundary conditions. In
the first case, the vacuum is the conformal one, $|0\rangle$, while in the second it is given by
the two-fold degenerate Ramond vacua $|\alpha\rangle$, where $\alpha = \pm 1/2$.

For a set of several Ramond vacua we introduce the shorthand notation
$|\alpha_{21,28,29}, \alpha\rangle \equiv |\alpha_{21}, \alpha_{28}, \alpha_{29}, \alpha\rangle$, where $\alpha \equiv (\alpha_{32}, \alpha_{33})$ is a space-time spinor index.

The spectrum of excited states is constructed, sector by sector, by acting on the
vacuum with the creation operators. We restrict ourselves to states in the (super)
ghost vacuum with superghost charge $q = -1$ ($q = -1/2$) for bosonic (fermionic)
sectors. Only states satisfying the level-matching condition $L_0 = L_0$ can propagate,
and their masses are given by

$$\frac{\alpha'}{4} M^2 = L_0 - \frac{\alpha'}{4} p^2 = L_0 - \frac{\alpha'}{4} p^2. \quad (2.22)$$

In terms of the oscillator operators for the various excitation modes, the mass of a
state in a sector $\alpha$ is [15, 14]

$$\frac{\alpha'}{4} M^2 = \sum_{i=23}^{33} \left\{ E_{[\alpha_i]} + \sum_{q=1}^{\infty} \left( (q + [\alpha_i]) - 1 \right) n_{q+|\alpha_i|}^{(l)} \right\} + \sum_{q=1}^{\infty} q a_{-q} \cdot a_q - 1 + E_{[\alpha_{32}]}^{(\beta_7)}, \quad (2.23)$$

where $a_{q}^{\alpha}$ are the (right-moving) modes of $X^\mu(z, \bar{z})$, $n^{(l)}$ and $n^{(l)\ast}$ are the fermion and
antifermion mode occupation numbers defined by

$$n_{q+|\alpha_i|}^{(l)} = \frac{\psi_{q+|\alpha_i|+1}^{(l)}}{\psi_{q+|\alpha_i|-1}^{(l)}}, \quad n_{q-|\alpha_i|}^{(l)\ast} = \frac{\psi_{q-|\alpha_i|-1}^{(l)\ast}}{\psi_{q-|\alpha_i|+1}^{(l)\ast}}. \quad (2.24)$$
$E_{[\psi_{-1/2}(\ell)]}$ is the superghost vacuum energy, which equals $+1/2$ ($+3/8$) in a bosonic (fermionic) sector, while the contribution of minus one represents the reparametrization ghost vacuum energy. Finally $E_{[\psi_{-1/2}]}$ is the vacuum energy of the $l$'th complex fermion (relative to the conformal vacuum)

$$E_{[\psi_{-1/2}]} = \frac{1}{2} \left( \left[ \alpha_{0} \right] - \frac{1}{2} \right)^{2}.$$  (2.25)

A similar formula holds for the left-movers, without the superghost vacuum energy, and with right-movers replaced by left-movers.

Let us consider first the bosonic sector $W_{0}$. Before the GSO projections, it contains the standard charged tachyon of mass squared $\alpha' M^2 = -2$, and some massless states: the graviton, dilaton and axion

$$\bar{a}_{-1}^{\mu} |0\rangle_{L} \otimes \psi^{-}_{-1/2} |0\rangle_{R},$$

a set of charged vectors

$$\bar{\psi}^{-}_{-1/2,(\ell)} \bar{\psi}^{-}_{-1/2,(k)} |0\rangle_{L} \otimes \psi^{\mu}_{-1/2} |0\rangle_{R} \quad \ell, k = 1, \ldots, 22, \ell \neq k,$$

$$\bar{a}_{-1}^{\mu} |0\rangle_{L} \otimes \psi_{-1/2,(j)}^{m} |0\rangle_{R} \quad j = 23, \ldots, 31,$$

and a set of charged scalars

$$\bar{\psi}^{-}_{-1/2,(\ell)} \bar{\psi}^{-}_{-1/2,(k)} |0\rangle_{L} \otimes \psi_{-1/2,(j)}^{m} |0\rangle_{R} \quad \ell, k = 1, \ldots, 22, \ell \neq k, j = 23, \ldots, 31.2$$

The GSO projections eliminate from the physical spectrum the tachyon, all scalars, as well as the spin-1 states where the vector index is carried by the oscillators $\bar{a}_{-1}^{\mu}$. Moreover the massless vectors fill out the adjoint representation of the gauge group $SO(14) \otimes SO(14) \otimes SO(4) \otimes U(1) \otimes SO(10)$, and therefore are the gauge bosons of the model.

In the sector $W_{1}$ we find 8 massless spin-1/2 states

$$\bar{\psi}^{-}_{-1/2,(\ell)} \bar{\psi}^{-}_{-1/2,(k)} |0\rangle_{L} \otimes |a_{23,26,29}, \alpha\rangle_{R} \quad \ell, k = 1, \ldots, 22, \ell \neq k$$

and 8 massless spin-3/2

$$\bar{a}_{-1}^{\mu} |0\rangle_{L} \otimes |a_{23,26,29}, \alpha\rangle_{R},$$

$\text{2m, n = 1, 2 as in eq.(2.4)}$
which represent the gauginos and gravitinos respectively. Imposing the GSO projections, one finds that only a single gravitino survives, if and only if

\[ k_{02} + k_{12} \equiv k_{04} + k_{14}. \]  

This is then the condition for the model to be \( N = 1 \) supersymmetric. The same analysis applies to the gauginos, leading consistently to the same condition for space-time supersymmetry.

If the model is supersymmetric (i.e., equation (2.26) holds), then given a state in the sector \( mW \), the superpartner resides in the sector \( W_0 + W_1 + mW \) [14]. Notice that \( W_{\text{SUSY}} = W_0 + W_1 \) also exchanges the boundary conditions of the internal world-sheet fermions \( \psi_{(23)} \), \( \psi_{(26)} \) and \( \psi_{(29)} \). The associated degrees of freedom are not family indices for the states and should be considered instead as enumerative indices for the elements of the space-time supermultiplets.

We end this section by considering in some details the GSO projection conditions for the ground states of the sector \( W_{134} \). In this sector, the ground state is given by a set of massless fermions charged under the \( U(1) \) and the first \( SO(14) \) components of the gauge group: \( \{|a_1,...,7,17 \rangle_L \otimes |a_{24,28,29}, \alpha \rangle_R \} \). These are the fermions which we will use in the Compton amplitude. On such states the GSO conditions, eq.(2.9), become (considering only zero-mode excitations)

\[
\begin{align*}
\frac{1}{2} \left[ \sum_{l=1}^{7} N_{0}^{(l)} + N_{0}^{(17)} - \sum_{l=24,28,29} N_{0}^{(l)} \right] & \equiv \frac{1}{2} + k_{00} + k_{01} + k_{03} + k_{04}, \\
\frac{1}{2} \left[ \sum_{l=1}^{7} N_{0}^{(l)} + N_{0}^{(17)} - \sum_{l=24,28} N_{0}^{(l)} \right] & \equiv - \frac{1}{2} + k_{13} + k_{14}, \\
\frac{1}{2} \left[ \sum_{l=1}^{7} N_{0}^{(l)} - \sum_{l=24,28,29} N_{0}^{(l)} \right] & \equiv \frac{1}{2} + k_{02} + k_{12} + k_{23} + k_{24}, \\
\frac{1}{2} \left[ \sum_{l=1}^{7} N_{0}^{(l)} + N_{0}^{(17)} - \sum_{l=28,29} N_{0}^{(l)} \right] & \equiv \frac{1}{2} + k_{13} + k_{34}, \\
\frac{1}{2} \left[ - \sum_{l=24,29} N_{0}^{(l)} \right] & \equiv k_{14} + k_{34}. 
\end{align*}
\]

It is convenient to rewrite the GSO conditions for the Ramond zero modes in term
of Pauli matrices. Consider the generic projection condition

$$\frac{1}{2} \left( \sum_{i \in I} N_0^{(l)} \right) \mod 1 = r,$$

(2.32)

where \( r \in \{0, 1/2\} \) and the left-hand side involves a total of \( m \) number operators. Then, since \( N_0^{(l)} = 1/2 (1 + \sigma_3^{(l)}) \) for zero-mode excitations, this projection condition can be rewritten as

$$\bigotimes_{i \in (I,I)} \sigma_3^{(l)} = \exp \left\{ 2\pi i \frac{r}{2} \right\} \quad \text{for } \ m \ \text{odd}$$

$$\bigotimes_{i \in (I,I)} \sigma_3^{(l)} = \exp \{ 2\pi i r \} \quad \text{for } \ m \ \text{even}. \quad (2.33)$$

Then the projection conditions eqns (2.27-2.31) become

$$\sigma_3^{(T)} \otimes \gamma^5 = \exp \left\{ 2\pi i \left[ k_{00} + k_{01} + k_{02} + k_{03} + k_{04} + k_{12} + k_{23} + k_{24} + \frac{1}{2} \right] \right\}$$

$$\sigma_3^{(24)} \otimes \gamma^5 = \exp \left\{ 2\pi i \left[ k_{00} + k_{01} + k_{03} + k_{04} + k_{13} + k_{34} + \frac{1}{2} \right] \right\}$$

$$\sigma_3^{(29)} \otimes \gamma^5 = \exp \left\{ 2\pi i \left[ k_{00} + k_{01} + k_{03} + k_{04} + k_{13} + k_{14} + \frac{1}{2} \right] \right\}$$

$$\gamma_{SO(14)} \otimes \sigma_3^{(28)} = \exp \left\{ 2\pi i \left[ k_{02} + k_{12} + k_{14} + k_{23} + k_{24} + k_{34} + \frac{1}{2} \right] \right\}, \quad (2.34)$$

where we introduced the space-time chirality operator \( \gamma^5 \equiv \sigma_3^{(32)} \otimes \sigma_3^{(33)} \) and the chirality matrix in the spinor representation of the gauge group \( SO(14) \), \( \gamma_{SO(14)} = \bigotimes_{i=1}^{7} \sigma_3^{(l)} \).

3 Amplitudes, Vertex Operators and Cocycles

In this section we introduce the tools necessary for the computation of the one-loop Compton scattering amplitude in the context of a KLT four-dimensional model in a Minkowski background following refs. \([14, 20, 21]\).

We define the \( T \)-matrix element as the connected \( S \)-matrix element with certain normalization factors removed

$$\frac{\langle \lambda_1, \ldots, \lambda_{N_{\text{fund}}} | S_c | \lambda_{N_{\text{fund}} + 1}, \ldots, \lambda_{N_{\text{fund}} + N_{\text{max}}} \rangle}{\prod_{i=1}^{N_{\text{fund}}} \langle \langle \lambda_i | \lambda_i \rangle \rangle^{1/2}} =$$
\[ i(2\pi)^4\delta^4(p_1 + \ldots + p_{N_{\text{out}}}, \ldots, p_{N_{\text{out}}+1} - \ldots - p_{N_{\text{tot}}}) \prod_{i=1}^{N_{\text{tot}}} (2p_i^0 V)^{-1/2} \times \]
\[ T(\lambda_1, \ldots, \lambda_{N_{\text{out}}}, \lambda_{N_{\text{out}}+1}, \ldots, \lambda_{N_{\text{out}}+N_{\text{in}}}), \]  

where \( N_{\text{tot}} = N_{\text{in}} + N_{\text{out}} \) is the total number of external states, \( p_i \) is the momentum of the \( i \)th string state (\( p_i^0 > 0 \) for all states) and \( V \) is the usual volume-of-the-world factor.

Corresponding to each state \( |\lambda\rangle \) we have a vertex operator \( \mathcal{V}_{|\lambda\rangle}(\bar{z}, z) \) and the 1-loop contribution to the \( T \)-matrix element is given by the operator formula

\[ T^{1-\text{loop}}(\lambda_1, \ldots, \lambda_{N_{\text{out}}}, \lambda_{N_{\text{out}}+1}, \ldots, \lambda_{N_{\text{out}}+N_{\text{in}}}) = \]
\[ C_{g=1} \int \prod_{I=1}^{N_{\text{tot}}} \text{d}^2 m^I \sum_{m_i, n_j} C_\alpha^\beta \left\langle \prod_{I=1}^{N_{\text{tot}}} (\eta_I | b) \prod_{i=1}^{N_{\text{tot}}} c(z_i) \right\rangle^2 \]
\[ \prod_{A=1}^{N_{B}+N_{FP}} \Pi(w_A) \mathcal{V}_{|\lambda_A\rangle}(\bar{z}_1, z_1) \ldots \mathcal{V}_{|\lambda_{N_{\text{tot}}}}\rangle(\bar{z}_{N_{\text{tot}}}, z_{N_{\text{tot}}}) \rangle. \]  

(3.2)

Here \( C_{g=1} = 1/(2\pi \alpha')^2 \) is a constant giving the correct normalization of the vacuum amplitude [20, 22]. The coefficients \( C_\alpha^\beta \) of the summation over spin structures are given in eq. (2.15). \( m^I \) is a modular parameter, \( \eta_I \) is the corresponding Beltrami differential [23], and the overlap \( (\eta_I | b) \) with the antighost field \( b \) is given explicitly in ref. [24]. The integral is over one fundamental domain of \( N \)-punctured genus-one moduli space. By definition the correlator \( \langle \langle \ldots \rangle \rangle \) includes the partition function (more details on our conventions for the partition function, spin structure and field operators can be found in Appendix A, see also ref [14, 20, 21]).

In eq. (3.2), the ghost factors present in the BRST invariant version of the vertex operator, \( \mathcal{W}_{|\lambda\rangle}(\bar{z}, z) = \bar{c}(\bar{z}) c(z) \mathcal{V}_{|\lambda\rangle}(\bar{z}, z) \), have been factored out. One generically takes all space-time fermionic vertex operators to have the superghost charge \( q = -1/2 \) and all the bosonic ones to have the superghost charge \( q = -1 \). In an amplitude involving \( N_B \) space-time bosons and \( 2N_{FP} \) space-time fermions, in order to compensate the superghost vacuum charge, we insert \( N_B + N_{FP} \) Picture Changing Operators (PCO) \( \Pi(w_A) \), at arbitrary points \( w_A \) on the Riemann surface [19, 23]. In practical calculations it is more convenient to insert one PCO at each of the vertex operators describing the space-time bosons. This leaves \( N_{FP} \) PCO’s at arbitrary points and the boson vertex operators in their picture changed version with superghost charge \( q = 0 \).
An explicit expressions of the vertex operators and of the PCO can be given
bosonizing all the fermionic degrees of freedom. We use the prescription for bosoniza-
tion of world-sheet fermions in Minkowski space-time proposed in ref. [21]. Bosoniza-
tion in Minkowski space-time differs from the one in Euclidean space-time because
the world-sheet fermions \( \psi_{(33)} \) and \( \psi_{(33)}^* \), which are related to the time direction in
space-time, have different hermiticity properties compared to all other fermions. All
left- and right-moving complex fermions are bosonized according to

\[
\begin{align*}
\psi_{(l)}(z) &= e^{\phi_{(l)}(z)} c_{(l)} \\
\bar{\psi}_{(l)}(\bar{z}) &= e^{\bar{\phi}_{(l)}(\bar{z})} c_{(l)}^* \\
\psi_{(\bar{l})}^*(\bar{z}) &= e^{-\phi_{(\bar{l})}^*(\bar{z})} c_{(\bar{l})}^* \\
\bar{\psi}_{(\bar{l})}^*(\bar{z}) &= e^{-\bar{\phi}_{(\bar{l})}^*(\bar{z})} c_{(\bar{l})}^* 
\end{align*}
\]

where the scalar field \( \phi_{(l)} \) has operator product expansion (OPE)

\[
\phi_{(l)}(z)\phi_{(k)}(w) = +\delta_{l,k} \log(z - w) + \ldots .
\]

The factors \( c_{(l)} \) are cocycles needed to guarantee the correct anti-commutation relations between different fermions. We will return to them in the next subsection.

As shown in [21], because of the Minkowski metric in the OPE,

\[
\psi^\mu(z)\psi^\nu(w) \ \text{OPE} \ \frac{g_{\mu\nu}}{z - w} + \cdots,
\]

the fermion \( \psi^0 \) (see eq. (2.3)) and the associated scalar field \( \phi_{(33)} \) are hermitian, while all other fermions, with fields \( \phi_{(l)} \) (\( l = 1, \ldots, 22 \) and \( l = 23, \ldots, 32 \)), are anti-
hermitian.

The superghosts are bosonized in the standard way

\[
\beta = \partial \xi \ e^{-\phi}(c_{(34)})^{-1}, \quad \gamma = e^{\phi} c_{(34)} \eta,
\]

where \( c_{(34)} \) is their cocycle factor. The scalar field \( \phi \) in (3.6) is hermitian and has the “wrong” metric

\[
\phi(z)\phi(w) = -\log(z - w) + \ldots ,
\]

and the PCO is given by

\[
\Pi = 2 \epsilon \partial \xi + 2 e^{\phi} c_{(34)} T^{[X,\psi]} - \frac{1}{2} \partial (e^{2\phi}(c_{(34)})^2 \eta \bar{b}) - \frac{1}{2} e^{2\phi}(c_{(34)})^2 (\partial \eta) b.
\]

The spin field operator which creates from the conformal vacuum the Ramond ground
state associated with the \( \ell \)th fermion can be written as

\[
S_{\ell q}^{(l)}(z) = e^{\alpha \phi_{(l)}(z)} (c_{(l)})^{\alpha},
\]

(3.9)
where \( a_l = \pm \frac{1}{2} \) is related to the spin structure \( \alpha_l \) by \( a_l = \frac{1}{2} - \alpha_l \) MOD 1. A similar expression holds for the left-movers.

If we define the scalar field in (3.6) as \( \phi \equiv \phi_{(34)} \), the superghost part of any physical state vertex operator can be expressed by eq.(3.9) with \( l = 34 \) and

\[
\alpha_{34} = -\frac{1}{2} - \left\lfloor \alpha_{32} \right\rfloor = \begin{cases} 
-1 & \text{in bosonic sector} \\
-1/2 & \text{in fermionic sector}
\end{cases}
\]  

Then, in any given sector \( \alpha \) the vertex operator describing the ground state of momentum \( p \) has the form

\[
\mathcal{V} = \mathcal{N} \cdot \prod_{l=1}^{22} S_{\alpha_l}^{(l)} \prod_{l=23}^{34} S_{\alpha_l}^{(l)} \cdot e^{ik \cdot X} \equiv \mathcal{N} \cdot S_A \cdot e^{ik \cdot X},
\]

where \( \mathcal{N} = \kappa / \pi \) [20], \( A \equiv (A; a_{34}) \equiv (\tilde{a}_1, \ldots, \tilde{a}_{22}; a_{23}, \ldots; a_{34}) \) and we introduced the dimensionless momentum \( k_\mu = \sqrt{\alpha' / \pi} p_\mu \).

### 3.1 Cocycles

As already mentioned, the cocycle factors are necessary to ensure that different fermions anti-commute when they are written in the bosonized form. We write the cocycle operators as follows [14, 21]

\[
\begin{align*}
c^{(l)}_{c_0} &= c^{(l)}_{gh} \cdot \exp \left\{ i\pi \sum_{j=1}^{22} Y_{ij} J^{(j)}_0 \right\} \quad \text{for} \quad l = 1, \ldots, 22 \\
c^{(l)}_{c_0} &= c^{(l)}_{gh} \cdot \exp \left\{ i\pi \left( \sum_{j=1}^{22} Y_{ij} J^{(j)}_0 + \sum_{j=23}^{34} Y_{ij} J^{(j)}_0 \right) \right\} \quad \text{for} \quad l = 23, \ldots, 34,
\end{align*}
\]

with

\[
\begin{align*}
c^{(l)}_{gh} &= \exp \left\{ -i\pi \varepsilon^{(l)} N_{(\eta, \xi)} \right\} \exp \left\{ i\pi \varepsilon^{(l)} (N_{(b, c)} - N_{(b, c)}) \right\} \\
c^{(l)}_{gh} &= \exp \left\{ -i\pi \varepsilon^{(l)} N_{(\eta, \xi)} \right\} \exp \left\{ i\pi \varepsilon^{(l)} (N_{(b, c)} - N_{(b, c)}) \right\}.
\end{align*}
\]

Here all the parameters \( Y \), as well as the \( \varepsilon \), take values +1 or −1. \( N_{(b, c)}, N_{(b, c)} \) and \( N_{(\eta, \xi)} \) are the number operators of the \( (b, c), (\tilde{b}, \tilde{c}) \) and \( (\eta, \xi) \) systems respectively, while \( J^{(j)}_0 \) and \( J^{(j)} \) are the number operators for the world-sheet fermions

\[
J^{(l)}_0 = \int_0^1 \frac{dz}{2\pi i} \partial \phi^{(l)}(z)
\]  

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where $\partial \phi_{(t)} = -\psi_{(l)}^\dagger \psi_{(l)} = -i \psi_{(l)}^\dagger \psi_{(l)}^2$.

A priori there are many possible different choices of the cocycle operators, which reflect themselves in the arbitrariness in the signs $Y$ and $\varepsilon$. The possible choices are however restricted by a certain number of consistency conditions [14, 25]: Cocycles have to be chosen in such a way that the left- and right-moving BRST currents have well-defined statistics with respect to all vertex operators, otherwise, a product of BRST invariant vertex operators would not necessarily be BRST invariant. Likewise, all Kac-Moody currents must satisfy Bose statistics with respect to all vertex operators, otherwise, a product of vertex operators $\mathcal{V}_i$ transforming in various representations $D_i$ of the gauge group would not necessarily transform in the tensor representation $\otimes_i D_i$. It is also necessary that the PCO eq. (3.8) obeys Bose statistics with respect to all vertex operators. These consistency conditions have been discussed in the case of Euclidean space-time in ref. [14], where it is also shown how to construct an explicit solution to these constraints for our specific model. The same conditions hold for Minkowski space-time [21], thus the discussion in ref. [14] applies also to our case.

In the computation that follows, we will not use any specific choice of the set of cocycles since, as follows from ref. [14], a choice of cocycles which satisfies all consistency conditions exists and all choices of cocycles that satisfy the consistency conditions give rise to the same scattering amplitude. In other words, the consistency conditions guarantee that the scattering amplitude does not depend on the choice of cocycles (see also the discussion in ref. [26]).

Finally, notice that the ordering chosen for the fermions in eq. (3.11) is not accidental, it is a consequence of the introduction of cocycles and of the fact that in Minkowski space-time the number operator related to fermion $\psi^0$ is anti-hermitian, rather than hermitian as the other ones. Therefore to retain the correct hermiticity properties of the bosonized fermions, it is necessary to assign labels 33 and 34 to the fermion associated with the time direction in space-time and to the superghost respectively [21].
3.2 Gamma Matrices

We define a set of four-dimensional gamma matrices by means of the OPE between the real space-time fermions $\psi^\mu$ and the spin field $S_A \equiv S_A S_{a34}^{(34)}$:

$$\psi^\mu(z)S_A(w) = \frac{1}{\sqrt{2}} (\Gamma^\mu)_A^B S_B(w) \frac{1}{\sqrt{z-w}}. \quad (3.15)$$

The explicit representation of these matrices depends on the choice of the cocycle signs $Y$. In terms of the usual Pauli matrices we have ($\sigma_0$ is the $2 \times 2$ unit matrix)

$$\Gamma^0 = \left( -i \prod_{j=1}^{33} Y_{33,j} \right) \otimes \tilde{\Gamma}_l^{(7)} \sigma_3 \otimes \sigma_3^{(17)} \otimes \eta_{24,28,29}^{(32)} \otimes \sigma_2^{(33)} \otimes \sigma_0^{(34)}$$

$$\Gamma^1 = \left( \prod_{j=1}^{33} Y_{33,j} \right) \otimes \tilde{\Gamma}_l^{(7)} \sigma_3 \otimes \sigma_3^{(17)} \otimes \eta_{24,28,29}^{(32)} \otimes \sigma_1^{(33)} \otimes \sigma_0^{(34)}$$

$$\Gamma^2 = \left( -\prod_{j=1}^{33} Y_{32,j} \right) \otimes \tilde{\Gamma}_l^{(7)} \sigma_3 \otimes \sigma_3^{(17)} \otimes \eta_{24,28,29}^{(32)} \otimes \sigma_2^{(33)} \otimes \sigma_0^{(34)}$$

$$\Gamma^3 = \left( -\prod_{j=1}^{33} Y_{32,j} \right) \otimes \tilde{\Gamma}_l^{(7)} \sigma_3 \otimes \sigma_3^{(17)} \otimes \eta_{24,28,29}^{(32)} \otimes \sigma_1^{(33)} \otimes \sigma_0^{(34)}. \quad (3.16)$$

For a generic ground state eq. (3.11) it is also convenient to define a “generalized charge conjugation matrix” $C$ by

$$S_A(z, \bar{z})S_B(w, \bar{w})^{\text{OPE}} = C_{AB} \delta_{a34, b34} \frac{1}{(z-w)^p} \frac{1}{(\bar{z}-\bar{w})^{\bar{p}}}, \quad (3.17)$$

where $\bar{p} = \sum_{l=1}^{33} (\bar{a}_l)^2$ and $p = \sum_{l=1}^{33} (a_l)^2$. This matrix is related to the choice of cocycles by

$$C_{AB} = e^{iA \cdot Y \cdot B} \left( \prod_{L=1}^{33} \delta_{A_L+B_L} \right) \delta_{a34, b34}, \quad (3.18)$$

where $a_{34} = b_{34}$ are given by eq. (3.10).

In the computation of the amplitude it will be useful to introduce also another set of gamma matrices, which are defined by means of the OPE between the real space-time fermions $\psi^\mu$ and the four-dimensional spin field operator $S_\alpha \equiv S_\alpha^{(32)} S_\alpha^{(33)}$:

$$\psi^\mu(z)S_\alpha(w) = \frac{1}{\sqrt{2}} (\gamma^\mu)_\alpha^\beta S_\beta(w) \frac{1}{\sqrt{z-w}}. \quad (3.19)$$
As for the matrices in eq. (3.16), the explicit representation of this new set of gamma matrices depends on the value of $Y_{33,32}$

$$
\begin{align*}
\gamma^0 &= (iY_{33,32}) \sigma_3^{(32)} \otimes \sigma_1^{(33)} \\
\gamma^1 &= (Y_{33,32}) \sigma_3^{(32)} \otimes \sigma_2^{(33)} \\
\gamma^2 &= -\sigma_2^{(32)} \otimes \sigma_0^{(33)} \\
\gamma^3 &= \sigma_1^{(32)} \otimes \sigma_0^{(33)}.
\end{align*}
$$

(3.20)

### 3.3 Vertex Operators in Our Model

We are now ready to introduce the explicit vertex operators necessary for the computation of the amplitude we are interested in. The process we want to study is the Compton scattering of a photon and a massless fermion. The photons are the $U(1)$ gauge bosons belonging to the sector $\mathbf{W}_0$. The vertex operator describing a $U(1)$ gauge boson with polarization $\epsilon$ and momentum $k$ in the superghost picture $q = -1$ is [19]

$$
\gamma_{\text{photon}}^{(-1)}(z, \bar{z}; k; \epsilon) = \frac{K}{\pi} \bar{\psi}_{[17]} \bar{\phi}_{[17]}(\bar{z}) \epsilon \cdot \psi(z) e^{-\phi(z)} (c_{(31)})^{-1} e^{ik \cdot X(z, \bar{z})},
$$

(3.21)

where the gravitational coupling $\kappa$ is related to Newton's constant by $\kappa^2 = 8\pi G_N$, $\epsilon \cdot \epsilon = 1$, $k^2 = \epsilon \cdot k = 0$. The picture changed version of the same vertex is

$$
\gamma_{\text{photon}}^{(0)}(z, \bar{z}; k; \epsilon) = -i \frac{K}{\pi} \bar{\psi}_{[17]} \bar{\phi}_{[17]}(\bar{z}) \left( \epsilon \cdot \partial_z X(z) - ik \cdot \psi(z) \epsilon \cdot \psi(z) \right) e^{ik \cdot X(z, \bar{z})}.
$$

(3.22)

As space-time fermions, we choose the massless fermions which form the ground state of the sector $\mathbf{W}_{134}$. They have $U(1)$ charge $\pm 1/2$ and they are described by the vertex operator

$$
\gamma^{(-1/2)}(z, \bar{z}; k; \mathbf{V}) = \frac{K}{\pi} \mathbf{V}^A S_A(z, \bar{z}) e^{-\frac{1}{2} \phi(z)} (c_{(31)})^{-1/2} e^{ik \cdot X(z, \bar{z})},
$$

(3.23)

where $k$ is the momentum and the label $-1/2$ indicates the superghost vacuum charge. $S_A$ is the spin field which creates the Ramond ground state from the conformal vacuum

$$
S_A = \left( \prod_{l=1}^7 S_{\delta_l}^{(\bar{\alpha})}(\bar{z}) \right) S_{\alpha l}^{(17)}(\bar{z}) \left( \prod_{l=21,28,29} S_{\alpha l}^{(l)}(z) \right) S_{\alpha l}^{(l)}(z).
$$

(3.24)

The "spinor" $\mathbf{V}$ decomposes accordingly

$$
\mathbf{V}^A = \left( \prod_{l=1}^7 S_{\delta l}^{(\bar{\alpha})}(\bar{z}) \right) \Delta_{\mathbf{U}(1)}^{(\bar{\alpha}) l} \left. \frac{S_{\delta l}^{(17)}(\bar{z})}{U^{a l}(l)} \right) \mathbf{V}^a.
$$

(3.25)
The left-moving spinor indices $\bar{a} = (\bar{a}_1, \ldots, \bar{a}_7)$ all take values $\pm 1/2$ and indicate that the fermion transforms in the spinor representation of the first $SO(14)$. $\bar{a}_{17} = \pm \frac{1}{2}$ is the $U(1)$ charge. The right-moving spinor indices also take values $\pm 1/2$: $\alpha = (a_3, a_{33})$ is the four-dimensional space-time spinor index, while the others are just enumerative family indices.

In order to describe physical external states, the vertex operators $\bar{c}c\mathcal{V}$, with $\mathcal{V}$ as in eqs. (3.22) and (3.23), must be BRST invariant [19, 14]. The BRST currents are given by

$$ j_{\text{BRST}} = cT_B^{[X,\psi,\beta,\gamma]} - cb\partial c - T_F^{[X,\psi]} e^{\phi(c_{(34)})} \eta - \frac{1}{4} e^{2\phi(c_{(34)})^2} \eta(\partial \eta) b $$

$$ \bar{j}_{\text{BRST}} = \bar{c}\bar{T}_B^{[X,\bar{\psi}]} - \bar{c}b\partial \bar{c}, $$

(3.26)

where $T_B$ and $\bar{T}_B$ are the energy-momentum tensors. The first-order pole in the OPE of $\bar{j}_{\text{BRST}}$ with $\bar{c}c\mathcal{V}$, as well as the first order pole in the OPE of the first two terms of $j_{\text{BRST}}$ with $\bar{c}c\mathcal{V}$, vanish merely by imposing that the vertex operator $\mathcal{V}$ is a primary conformal field of dimension one. The last term in $j_{\text{BRST}}$ has a non-singular OPE with $\bar{c}c\mathcal{V}$ for any operator $\mathcal{V}$ whose superghost part is given by $e^{-\phi}$ or $e^{-\phi/2}$. Therefore the BRST-invariance is reduced to the requirement that the first-order pole in the OPE $e^{\phi(c_{(34)})}T_F^{[X,\psi]}(w) \bar{c}c\mathcal{V}(z, \bar{z})$ should vanish. For a gauge boson, this equation becomes the transversality condition

$$ \epsilon \cdot k = 0, $$

(3.27)

whereas for the space-time fermion it becomes the “Dirac equation” [14]

$$ \mathcal{V}^T(k)\Gamma^\mu k_\mu = 0, $$

(3.28)

with $\Gamma^\mu$ given by eqs. (3.16).

### 3.4 Incoming and Outgoing Vertex Operators

In the formula for the 1-loop $T$-matrix element eq. (3.2) we have quoted an explicit value for the normalization coefficient $C_{g=1}$. Obviously this value is meaningful only when one specifies at the same time also the normalization of all other ingredients of the formula. In particular we need to normalize consistently all the vertex operators. The general issue can be briefly summarized as follows. In string theory we compute the connected part of a transition amplitude $<\lambda_1, \ldots; in|S|\ldots, \lambda_{N_f}; in>$ by the “master” formula eq. (3.1) (and eq. (3.2)), where each single-string state —whether
appearing as a bra or a ket—is represented by a vertex operator \( W \). Now suppose we are given a vertex operator \( W_{\lambda in} > \) representing a single-string ket-state, what is the vertex operator \( W_{\lambda in<} \) representing the same single-string state but now as a bra? The problem of course has to do with the hermiticity and unitarity properties of the scattering amplitude, and must be solved at the same time as solving the problem of the normalization of the vertex operators. The general discussion can be found in ref. [20], here we just report the final result for the vertex operators of interest to us. First of all, for the photons the bra and ket vertex operators are identical and normalized as in eqs. (3.21) and (3.22).

The situation is different for the fermion vertex operator in eq. (3.23). Let us assume that this vertex operator describes a ket-state. Then the analysis of ref. [20] implies that the “spinor” \( V_A \) satisfies the following normalization condition

\[
V^\dagger(k) V(k) = \sqrt{2} |k^0|.
\]

Moreover if we denote the vertex operator describing the same state but outgoing by

\[
Y^{(-1/2)}(z, \bar{z}; k; W) = \frac{\kappa}{\pi} W_A S_A(z, \bar{z}) \, e^{-\frac{1}{2} \phi(z)} (c_{(34)})^{-1/2} \, e^{-i k \cdot X(z, \bar{z})},
\]

then the vector \( W_A \) is related to the “spinor” \( V_A \) now describing the outgoing fermion by

\[
W^T = -i Y_{33,34} V^\dagger_{(33)} C.
\]

An explicit example of how these vertex operators are used in the computation of a scattering amplitude can be found in section 8 of ref. [20] (see also section 2 of ref. [27]).

Finally, the gauge coupling constant \( e \) is expressed in terms of the constant \( \kappa \) by the relation

\[
e^2 = \kappa^2/(2 \alpha')
\]
as it has been described in ref. [27] by the comparison of the tree level (genus zero) Compton scattering amplitude and the field theory result.

4 The Explicit Computation of the Amplitude

Having introduced all the ingredients, we can now describe the explicit computation of the 1-loop Compton scattering amplitude in our chosen four-dimensional heterotic
string model. Being a “Compton” scattering amplitude, the incoming/outgoing states are a photon and an “electron” (or “positron”), which in a string model are represented by a massless space-time fermionic state charged under the $U(1)$ component of the gauge group. Notice that in string theory we usually cannot find a state which is only charged under the $U(1)$, and indeed our fermion is also charged under the first $SO(14)$ and carries some other enumerative indices, as discussed in the previous section.

We will label the incoming particles by the indices $2$ and $4$, and the outgoing with $1$ and $3$. Moreover we choose all momenta as incoming (not as in eq. (3.30)). For what concerns the $N_B + N_{FP} = 3$ PCO’s, we insert one at each of the photon vertex operators, changing them from the $q = -1$ to the $q = 0$ picture. Thus we remain with one PCO inserted at an arbitrary point $w$ and the fermionic vertex operators are in the $q = -1/2$ picture. This choice is the most convenient from the following point of view. First of all, it is more convenient to have the photon vertex operators in the zero picture than to have to deal with two other PCO’s inserted at arbitrary points on the world-sheet; in this way the number of terms to be computed decreases and no complication is added. On the other hand, it is technically not convenient to absorb the last PCO in the vertex operator of a fermion since this will then be in the $+1/2$ picture, and the expressions to be computed will become much more cumbersome. Moreover, having one free PCO on the world-sheet will give us a powerful tool for checking the correctness of the computation, since the final scattering amplitude must be independent on the position of the insertion of any PCO.

The form of the amplitude which we start from is then

$$T_{g=1} \left( e^{\pm} + \gamma \to e^{\pm} + \gamma \right) =$$

$$C_{g=1} \int d^2\tau d^2z_1 d^2z_2 d^2z_3 \prod_{m_i, n_j} C^{\alpha}_{\beta} \left( \langle \langle \eta_1 \mid b \rangle (\eta_{z_2} \mid b \rangle (\eta_{z_3} \mid b \rangle (\eta_{z_4} \mid b \rangle \left( c(z_1) c(z_2) c(z_3) c(z_4) \right)^2 \Pi(w) \right) V^{(0)}_{\text{photon}}(z_1, z_1; k; \epsilon_1) V^{(0)}_{\text{photon}}(z_2, z_2; k; \epsilon_2)$$

$$\times V^{(-1/2)}(z_3, z_3; k_3; W_3) V^{(-1/2)}(z_4, z_4; k_4; V_4) \right),$$

(4.1)

where we used the translation invariance of the torus to fix the position $z_1$ of the outgoing photon vertex operator to an arbitrary value.

If we substitute in eq.(4.1) the explicit expression for the PCO (eq.(3.8)) and the
vertex operators (eqs. (3.22) and (3.23)), we obtain

\[ T^{1\text{-loop}} = -i \left( \frac{\hbar^2}{4\pi^3}\alpha' \right)^2 \mathcal{W}_3^{A\mathbf{Y}B} \int d^2\tau d^2z_2 d^2z_3 d^2z_4 \sum_{m_i,n_j} C^\alpha_{\beta} e^{i\mathbf{X}\cdot \mathbf{Y}\cdot \mathbf{B}} \langle (|\eta_\tau|b)(\eta_{z_2}|b)(\eta_{z_3}|b)(\eta_{z_4}|b) c(z_1)c(z_2)c(z_3)c(z_4)^2 \rangle \times T_L \times T_R \]  

(4.2)

where

\[ T_L = \langle \prod_{l=1}^{7} \left( \tilde{S}_{d_l}^{(1)}(z_3) \tilde{S}_{b_i}^{(1)}(z_4) \right) \tilde{\partial}\tilde{\phi}_{(T\bar{T})}(z_1) \tilde{\partial}\tilde{\phi}_{(T\bar{T})}(z_2) \tilde{S}_{\bar{a}_{17}}^{(T\bar{T})}(z_3) \tilde{S}_{\bar{b}_{17}}^{(T\bar{T})}(z_4) \rangle \]  

(4.3)

and

\[ T_R = \langle \prod_{l=1}^{3} \left( \tilde{S}_{a_l}^{(1)}(z_3) \tilde{S}_{b_l}^{(1)}(z_4) \right) \left[ \psi(z_1) - i k_1 \cdot \psi(z_1) \psi(z_1) \right] \times \left[ \psi_2 - i k_2 \cdot \psi(z_2) \psi(z_2) \right] \left( \partial X(w) \cdot \psi(w) \right) \times \left\{ e^{i k_3 \cdot X(z_3,z_4)} e^{i k_4 \cdot X(z_4,z_4)} e^{i k_1 \cdot X(z_1,z_1)} e^{i k_2 \cdot X(z_2,z_2)} e^{-\frac{i}{2}\phi(z_3)} e^{-\frac{i}{2}\phi(z_4)} e^{\phi(w)} \right\} \]  

(4.4)

In \( T_L \) and \( T_R \), we rearranged the operators in such a way to group all fermionic right-movers and all fermionic left-movers together (but notice that \( T_R \) depends on \( \bar{z} \) due to the presence of the \( X_\mu \)). To do this we moved the fermions \( \psi^\mu \) across the superghosts, and the spin fields across each other. In doing so, the cocycles give rise to phases which combine in the overall factor \( e^{\pi \mathbf{A}\cdot \mathbf{Y}\cdot \mathbf{B}} \) in eq.(4.2) [14]. Notice also that, because of fermion number conservation, only the first term of the PCO operator gives a non-zero contribution to the correlation function, so that effectively \( \Pi \simeq -ie^{\phi}\epsilon_{(z;4)}\partial X \cdot \psi \).

### 4.1 Computation of Correlators

We now turn our attention to the computation of the correlators appearing in eqs. (4.3) and (4.4). The correlators involving the bosonic coordinate fields \( X^\mu \) can be computed using the Wick theorem and the contraction given by the bosonic Green function (see Appendix A) and will not be reported explicitly here. We instead give the correlators involving the world-sheet fermions, the spin fields, the ghosts and superghosts. As already mentioned, the notation \( \langle \ldots \rangle \) indicates correlators including their complete contribution to the partition function. For each complex fermion, it is convenient to introduce a correlator function \( \langle \ldots \rangle \) where the non-zero
mode part of the partition function has been removed:

$$\langle \{ \mathcal{O}_i(z_1) \ldots \mathcal{O}_N(z_N) \} \rangle_{(l)} = \prod_{n=1}^{\infty} (1 - k^n)^{-1} \langle \mathcal{O}_1(z_1) \ldots \mathcal{O}_N(z_N) \rangle_{(q)}.$$

(4.5)

The subscript \((l)\) indicates that the correlator depends on the spin structure. Moreover, notice that

$$\langle 1 \rangle_{(l)} = \Theta \left[ \frac{a_l}{\beta_l} \right] (0|\tau),$$

(4.6)

which vanishes when the spin structure is odd.

All the correlators involving spin fields are obtained from the fundamental one \(\langle \prod_{i=1}^{N} e^{i\phi(z_i)} \rangle\) (given in Appendix A), and correlators involving \(\partial \phi\) can be obtained from this by differentiation. The basic spin field correlator is

$$\langle S_{a_l}^{(l)}(z_3) S_{b_l}^{(l)}(z_4) \rangle_{(l)} = \left(\sigma_3 \right)_{a_l,b_l}^{(l)} \langle S_+(z_3) S-(z_4) \rangle_{(l)},$$

(4.7)

where we introduced

$$S_l \equiv (1 - 2a_l)(1 + 2\beta_l),$$

(4.8)

which is 01(MOD2) depending on whether the spin structure \(\left[ a_l/\beta_l \right]\) is even (odd). Notice that the correlator (4.7) develops a dependence on the sign of the charge \(a_l\) whenever the spin structure is odd. The correlator \(\langle S_+(z_3) S-(z_4) \rangle\) is given by

$$\langle S_+(z_3) S-(z_4) \rangle = \left( e^{i\phi(z_3)} e^{-i\phi(z_4)} \right) = \left( E(z_3, z_4) \right)^{-1/2} \Theta \left[ \frac{a_l}{\beta_l} \right] \left( \frac{1}{2} \nu_{34}|\tau \right).$$

(4.9)

The other spin field correlators in equations (4.3) and (4.4) are

$$\langle \delta \phi(\tau_1) S_{a_1}^{(l)}(z_1) \delta \phi(\tau_2) S_{a_2}^{(l)}(z_2) \rangle = \left( \sigma_3 \right)_{a_1,a_2}^{(l)} \langle S_+(z_1) S_-(z_2) \rangle_{(l)}$$

(4.10)

$$\times \left\{ \begin{array}{l}
\partial_z \partial_{\bar{z}} \log \frac{E(z_1, \bar{z}_2)}{E(z_1, \bar{z}_1)} + \frac{1}{4} \partial_z \log \frac{E(z_1, \bar{z}_2)}{E(z_1, \bar{z}_1)} \partial_{\bar{z}} \log \frac{E(z_2, \bar{z}_3)}{E(z_2, \bar{z}_4)} \\
+ \frac{1}{2 \pi i} \log \frac{E(z_1, \bar{z}_2)}{E(z_1, \bar{z}_1)} \partial_z \log \Theta \left[ \frac{a_1}{\beta_1} \right] (\nu|\bar{\nu}) \big|_{\nu = \frac{1}{2} \nu_{34}} \\
+ \frac{1}{2 \pi i} \log \frac{E(z_1, \bar{z}_2)}{E(z_1, \bar{z}_1)} \partial_{\bar{z}} \log \Theta \left[ \frac{a_2}{\beta_2} \right] (\nu|\bar{\nu}) \big|_{\nu = \frac{1}{2} \nu_{34}} \\
+ \frac{\omega_1(\bar{z}_1) \omega_2(\bar{z}_2)}{2 \pi i} \left( \Theta \left[ \frac{a_1}{\beta_1} \right] \left( \frac{1}{2 \nu_{34}} |\bar{\nu} \right) \right)^{-1} \partial_{\nu} \Theta (\nu |\bar{\nu}) \big|_{\nu = \frac{1}{2} \nu_{34}} \end{array} \right\}.$$
\begin{align}
\langle S_{s_{22}}^{(2)} \rangle_{b_{32}} (z_3) S_{b_{32}}^{(2)} (z_4) S_{\alpha_{33}}^{(32)} (z_1) S_{b_{33}}^{(32)} (z_4) \psi^\tau (w) \rangle &= \frac{1}{\sqrt{2}} \left( (S_+ (z_3) S_- (z_4))^{(2)} \right) \mathcal{I}^{[\alpha_{32}]}_{[\beta_{33}]} (z_3, z_4, w) \left( \gamma^\tau (\gamma^5) S C \right)_{\alpha \beta} e^{-i\pi \alpha_3 \lambda_{333} b_{33}}. \tag{4.11} \\
\langle S_{s_{32}}^{(2)} \rangle_{b_{32}} (z_3) S_{b_{32}}^{(2)} (z_4) S_{\alpha_{33}}^{(33)} (z_3) S_{b_{33}}^{(33)} (z_4) \psi^\mu \psi^\nu (z_1) \psi^\tau (w) \rangle &= \frac{1}{\sqrt{2}} \left( (S_+ (z_3) S_- (z_4))^{(32)} \right) \mathcal{I}^{[\alpha_{32}]}_{[\beta_{33}]} (z_1, z_3, z_4) \\
&\quad \times \left\{ G^+ \left[ \alpha_{32}^{[\beta_{33}] / \lambda_{32}} \right] (z_1, z_3, z_4, w) \left( \gamma^{\mu \nu \rho} \gamma^\mu (\gamma^5) S C \right)_{\alpha \beta} \\
&\quad + G^+ \left[ \alpha_{32}^{[\beta_{33}] / \lambda_{32}} \right] (z_1, z_3, z_4, w) \left( (\gamma^{\mu \nu \rho} \gamma^\mu \gamma^\nu \gamma^\rho) (\gamma^5) S C \right)_{\alpha \beta} \right\} e^{-i\pi \alpha_3 \lambda_{333} b_{33}}. \tag{4.12} \\
\langle S_{s_{32}}^{(2)} \rangle_{b_{32}} (z_3) S_{b_{32}}^{(2)} (z_4) S_{\alpha_{33}}^{(33)} (z_3) S_{b_{33}}^{(33)} (z_4) \psi^\mu \psi^\nu (z_1) \psi^\rho \psi^\sigma (z_2) \psi^\tau (w) \rangle &= \frac{1}{\sqrt{2}} \left( (S_+ (z_3) S_- (z_4))^{(33)} \right) \mathcal{I}^{[\alpha_{32}]}_{[\beta_{33}]} (z_3, z_4, w) \\
&\quad \times \left\{ (G^+ \left[ \alpha_{32}^{[\beta_{33}] / \lambda_{32}} \right] (z_1, z_3, z_4, w) \left( \gamma^{\mu \nu \rho \sigma} \gamma^\mu (\gamma^5) S C \right)_{\alpha \beta} \\
&\quad + G^+ \left[ \alpha_{32}^{[\beta_{33}] / \lambda_{32}} \right] (z_1, z_3, z_4, w) \left( (\gamma^{\mu \nu \rho \sigma} \gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma) (\gamma^5) S C \right)_{\alpha \beta} \right\} e^{-i\pi \alpha_3 \lambda_{333} b_{33}}. \tag{4.13} 
\end{align}

In these equations \( \gamma^\mu \) and \( \gamma^{\mu \nu \rho} \) are the four-dimensional gamma matrices defined in eq. (3.19) and their antisymmetrized products. We also introduced the shorthand notations \( S = S_{32} \) and

\[ C_{\alpha \beta} \equiv \delta_{\alpha_{32} + b_{32}} \delta_{\alpha_{33} + b_{33}} e^{i\pi \alpha_3 \lambda_{333} b_{33}} \tag{4.14} \]

and defined the function of the world-sheet coordinates

\[ \mathcal{I}^{[\beta]}_{[\alpha]} (z; z_3, z_4) = \frac{E(z_3, z_4)}{E(z, z_3) E(z, z_4)} \frac{\Theta^{[\beta]}_{[\gamma]} (\theta_{z_3} | \tau)}{\Theta^{[\gamma]}_{[\gamma]} (\theta_{z_4} | \tau)} \]

\[ \mathcal{I}^{[\alpha]}_{[\beta]} (z; z_3, z_4) = \partial_z \log \frac{E(z, z_3)}{E(z, z_4)} + 2 \frac{\omega(z)}{2\pi i} \partial_\tau \log \Theta^{[\alpha]}_{[\beta]} (\nu | \tau) \bigg|_{\nu = \frac{1}{2} \tau_{34}}, \]
where an explicit example of this kind of computation has been given. We do not show here the steps leading to expressions \[ G^\pm \frac{\beta}{\beta} (z, w; z_3, z_4) \]

\[
= \frac{1}{2E(z, w)} \left\{ \frac{\Theta \frac{\beta}{\beta} (\rho_{w,z} | \tau)}{\Theta \frac{\beta}{\beta} (1/2 \nu_{34} | \tau)} \sqrt{E(z, z_3)E(w, z_4)} \right. \\
+ \frac{\Theta \frac{\beta}{\beta} (\rho_{w,z} | \tau)}{\Theta \frac{\beta}{\beta} (1/2 \nu_{34} | \tau)} \sqrt{E(w, z_3)E(z, w)} \\
\left. \right\}
\]

\[
= \frac{I \frac{\beta}{\beta} (w, z_3, z_4)}{I \frac{\beta}{\beta} (z, z_3, z_4)} \left\{ \partial_z \log \frac{E(z, w)}{\sqrt{E(z, z_3)E(w, z_4)}} \\
+ \frac{\omega(z)}{2\pi i} \partial_z \log \Theta \frac{\beta}{\beta} (| \nu | \tau) \big|_{\nu = \mu_w} \right\},
\]

\[
G^- \frac{\beta}{\beta} (z, w; z_3, z_4) = \frac{1}{2E(z, w)} \left\{ \frac{\Theta \frac{\beta}{\beta} (\rho_{w,z} | \tau)}{\Theta \frac{\beta}{\beta} (1/2 \nu_{34} | \tau)} \sqrt{E(z, z_3)E(w, z_4)} \\
- \frac{\Theta \frac{\beta}{\beta} (\rho_{w,z} | \tau)}{\Theta \frac{\beta}{\beta} (1/2 \nu_{34} | \tau)} \sqrt{E(w, z_3)E(z, w)} \\
\right\} \\
= \frac{1}{2} I \frac{\beta}{\beta} (z, z_3, z_4) I \frac{\beta}{\beta} (w, z_3, z_4),
\]

\[
(4.15)
\]

where

\[
\begin{align*}
\nu_{12} &\equiv \int_{z_4}^{z_3} \frac{\omega}{2\pi i} \\
\mu_z &\equiv \int_{z_4}^{z_3} \frac{\omega}{2\pi i} - \frac{1}{2} \int_{z_3}^{z_4} \frac{\omega}{2\pi i} - \frac{1}{2} \int_{z_4}^{z_2} \frac{\omega}{2\pi i} \\
\rho_{z,w} &\equiv \int_{z_4}^{z_3} \frac{\omega}{2\pi i} + \frac{1}{2} \int_{z_3}^{z_4} \frac{\omega}{2\pi i}.
\end{align*}
\]

We do not show here the steps leading to expressions (4.10-4.13), but we refer to [14], where an explicit example of this kind of computation has been given. The crucial point is the recovering of the explicit Lorentz covariance of the correlators, which is lost once the fermions have been bosonized. First of all it is necessary to reconstruct the four-dimensional gamma matrix algebra out of the phases coming from the fundamental correlator defined in Appendix A (eq. (A.16)). Moreover it turns out that correlators (4.12) and (4.13) involving several space-time fermions \( \psi_{\mu} \) have different expressions in terms of theta-functions for different values of the Lorentz indices. For instance, the two different expressions we give for the functions \( G^\pm \frac{\beta}{\beta} \) appear when we compute the correlator (4.12) for different values of the indices \( \mu, \nu, \tau \). These two expressions, as well as similar ones coming from the other correlators, must be identical (up to different phases due to the gamma matrices), otherwise the amplitude

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would not be Lorentz covariant. It is therefore necessary to prove some identities involving theta-functions, mostly of the form of the trisecant identity [28]. In Appendix B we give the list of the identities we needed to prove in the case of our four particle amplitude, while in Appendix E of ref. [14] it is sketched the proof of one of these identities.

Finally there are the correlators for the reparametrization ghosts and the superghosts. They can be computed using the general expressions eqs. (A.18), (A.20) of Appendix A and are given by

\[
\langle \langle e^{-\frac{1}{2} \phi(z_3)} e^{-\frac{1}{2} \phi(z_4)} e^{\phi(w)} \rangle \rangle = (-1)^{S_3} k^{1/2} \prod_{n=1}^{\infty} (1 - k^n) \tag{4.17}
\]

\[
\times \frac{(\omega(z_3) \omega(z_4))^{1/2}}{\omega(w)} \left( \langle S_+ (z_3) S_- (z_4) \rangle_{\mathbb{Z}_2} T \left[ \alpha_3 \gamma_3 \right] (z_3, z_4, w) \right)^{-1},
\]

\[
\langle \langle [\eta_{\beta}] [\eta_{z_2}] [\eta_{z_4}] [\eta_{\beta}] c(z_1) c(z_2) c(z_3) c(z_4) \rangle \rangle = \frac{1}{\omega(z_1)} \prod_{n=1}^{\infty} |1 - k^n| \tag{4.18}
\]

### 4.2 Using the GSO Projections

The next step in the computation of the amplitude, after substituting all the correlators into eq. (4.2), is to simplify the factors of gamma and sigma matrices appearing in the equation. Indeed in each term of the amplitude there appears a factor of the form

\[
e^{i\pi A \cdot Y \cdot B} \prod_{l=1}^{7} \left( \sigma_3^{(l)} \sigma_1^{(l)} \right)_{a_l b_l} \left( \sigma_3^{(l)} \sigma_1^{(l)} \right)_{a_l b_l} \times
\]

\[
\prod_{l=2, 3, 4} \left( \sigma_3^{(l)} \sigma_1^{(l)} \right)_{a_l b_l} \left( \sigma_3^{(l)} \sigma_1^{(l)} \right)_{a_l b_l} \left( \gamma_\alpha \left( \gamma_\alpha \right) S_{32} \right)_{\alpha \beta} e^{-i\pi \alpha_3 Y_{33, 32} \beta_{2}}, \tag{4.19}
\]

where \( \gamma_\alpha \) denotes either \( \gamma^\mu \) or \( \gamma^{\mu \nu} \). This product can be rearranged in such a way to reconstruct the charge conjugation matrix \( C_{AB} \) and the complete gamma matrices (eq. (3.16)). This can be accomplished by moving all the \( \sigma_1^{(l)} \) to the left and by rewriting the phase factor in (4.19) as \( e^{i\pi A \cdot Y \cdot B} = e^{i\pi B' \cdot Y \cdot B} e^{i\pi (A - B') \cdot Y \cdot B}. \) The first factor in this expression, together with \( \sigma_1^{(l)} \), are what we need to reconstruct the matrix \( C_{AB} \), while the second can be rewritten as a product of \( \sigma_3 \) matrices acting directly on the spinor \( W_3 \) and contributes to give the complete gamma matrices. In
this way we obtain the following relation:

\[
e^{i\pi A \cdot Y \cdot B} \prod_{l=1}^{7} \left( (\sigma_3^{(l)})^{\tilde{S}_1} \tilde{\sigma}_1^{(l)} \right)_{\tilde{a}_l} \left( (\sigma_3^{(7)})^{\tilde{S}_1} \tilde{\sigma}_1^{(7)} \right)_{\tilde{a}_1} \times \prod_{l=24,28,29} \left( (\sigma_3^{(l)})^{\tilde{S}_1} \tilde{\sigma}_1^{(l)} \right)_{\tilde{a}_l} \left( \gamma_5^{\tilde{S}_1} \tilde{\sigma}_1^{(l)} \right)_{\tilde{a}_{l1}} e^{-i\pi Q_{Y_33,33} Y_0 Z_{02}} = \left( \prod_{l=1}^{7} (\sigma_3^{(l)})^{\tilde{S}_1} \tilde{\sigma}_1^{(l)} \right)_{\tilde{a}_l} \prod_{l=24,28,29} (\sigma_3^{(l)})^{\tilde{S}_1} \Gamma_i (\Gamma^5)^{S_32} C \right)_{AB} ,
\]

(4.20)

with \( \Gamma_i \) now denoting either \( \Gamma^\mu \) or \( \Gamma^{\mu\nu\rho} \). Since these structures are sandwiched between the spinors \( W_3 \) and \( \mathbf{V}_4 \), we can use the GSO projection conditions for the sector \( W_{134} \), which these spinors belong to, to further simplify expression (4.20):

\[
\left( \prod_{l=1}^{7} (\sigma_3^{(l)})^{\tilde{S}_1} \tilde{\sigma}_1^{(l)} \right)_{\tilde{a}_l} \prod_{l=24,28,29} (\sigma_3^{(l)})^{\tilde{S}_1} \Gamma_i (\Gamma^5)^{S_32} C \right)_{AB} = \exp \{ 2\pi i K_{GSO} \} \left( \Gamma, \Gamma^5 C \right)_{AB} ,
\]

(4.21)

where \( \tilde{S} = S_{17} + S_{24} + S_{29} + S_{32} \) and

\[
K_{GSO} = (k_{02} + k_{12} + k_{14} + k_{23} + k_{24} + k_{31} + 1/2) S_1 +
+ (k_{00} + k_{01} + k_{02} + k_{03} + k_{04} + k_{12} + k_{23} + k_{24} + 1/2) S_17 +
+ (k_{00} + k_{01} + k_{03} + k_{04} + k_{13} + k_{31} + 1/2) S_{24} +
+ (k_{00} + k_{01} + k_{03} + k_{04} + k_{13} + k_{14} + 1/2) S_{29} .
\]

(4.22)

At this point the amplitude is given by the following equation

\[
T_{g=1} = \left( \frac{e^4}{\pi^6} \right) \sum_{m_{i,j}} C^{i,j}_{B^4} e^{2\pi i k_{GSO} W_3 A^4 V_4 B^4 \mu \nu} \int \frac{d^2 \tau}{(2\pi)^2} \left| \tilde{\eta}(\tau) \right|^{-24} (\eta(\tau))^{-12} \times
\]

\[
\int d^2 z_1 d^2 z_2 d^2 z_3 d^2 z_4 \frac{\sqrt{\omega(z_3) \omega(z_4)}}{\omega(z_1) \omega(z_1) \omega(w)} \exp \left[ \sum_{i<j} (k_i k_j) G_B(z_i, z_j) \right]
\]

\[
\times T_L \left[ \frac{\tilde{a}}{\beta} \right] (z_1, z_2, z_3, z_4, w) \times T_R \left[ \frac{\tilde{a}}{\beta} \right] (z_1, z_2, z_3, z_4, w) ,
\]

(4.23)

where

\[
T_L = \prod_{l=1,7,11} \Theta \left[ \frac{\tilde{a}}{\beta} \right] (1/2) \tilde{E}_{34} | \tilde{\eta} \rangle \times \prod_{l=8,18} \Theta \left[ \frac{\tilde{a}}{\beta} \right] (0 | \tilde{\eta} \rangle \times (\tilde{E}(\bar{z}_3, \bar{z}_4))^{-2}
\]

\[
\times \left\{ \partial_{\bar{z}_1} \partial_{\bar{z}_2} \log \tilde{E}(\bar{z}_1, \bar{z}_2) + \frac{1}{4} \partial_{\bar{z}_1} \log \frac{\tilde{E}(\bar{z}_1, \bar{z}_3)}{\tilde{E}(\bar{z}_1, \bar{z}_4)} \partial_{\bar{z}_2} \log \frac{\tilde{E}(\bar{z}_2, \bar{z}_3)}{\tilde{E}(\bar{z}_2, \bar{z}_4)} \right\}
\]

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\[
\begin{align*}
+ & \frac{1}{2} \frac{\omega(z_1)}{2\pi i} \partial_{z_1} \log \frac{\bar{E}(z_2, z_3)}{E(z_2, z_1)} \partial_{\nu} \log \hat{\Theta} \left[ \frac{\hat{\alpha}_1}{\hat{\beta}_1} \right] (\nu | \bar{\tau}) |_{\nu = \frac{1}{2} \hat{\tau}_{34}} \\
+ & \frac{1}{2} \frac{\omega(z_2)}{2\pi i} \partial_{z_2} \log \frac{\bar{E}(z_1, z_3)}{E(z_1, z_2)} \partial_{\nu} \log \hat{\Theta} \left[ \frac{\hat{\alpha}_1}{\hat{\beta}_1} \right] (\nu | \bar{\tau}) |_{\nu = \frac{1}{2} \hat{\tau}_{34}} \\
+ & \frac{\omega(z_1) \omega(z_2)}{2\pi i} \left( \hat{\Theta} \left[ \frac{\hat{\alpha}_1}{\hat{\beta}_1} \right] \left( \frac{1}{2} \hat{\tau}_{34} | \bar{\tau} \right) \right)^{-1} \partial_{\nu} \hat{\Theta} (\nu | \bar{\tau}) |_{\nu = \frac{1}{2} \hat{\tau}_{34}} \\
& \right), \quad (4.24) \\
\end{align*}
\]

and

\[
T_R = \prod_{l=24,28,32} \Theta \left[ \frac{\tau}{\hat{\beta}_l} \right] \left( \frac{1}{2} \hat{\tau}_{34} | \tau \right) \prod_{l=23,35,36,27,30,31} \Theta \left[ \frac{\tau}{\hat{\beta}_l} \right] (0 | \tau) \frac{(-1)^{s_{33}}}{\sqrt{2} E(z_3, z_4)}
\]

\[
\times \left\{ \sum_{l=1}^{4} \sum_{n \neq 1} \sum_{p \neq 2} k_j^l k_j^p \partial_w G_B(z_j, w) \partial_1 G_B(z_n, z_1) \partial_2 G_B(z_p, z_2) + g^{\mu \nu} \partial_1 \partial_2 G_B(z_1, z_2) \sum_{j=1}^{4} k_j^l \partial_w G_B(z_j, w) \\
+ g^{\rho \sigma} \partial_1 \partial_2 G_B(z_1, w) \sum_{j \neq 2} k_j^l \partial_2 G_B(z_j, z_2) + g^{\rho \sigma} \partial_2 \partial_w G_B(z_2, w) \sum_{j \neq 1} k_j^l \partial_1 G_B(z_j, z_1) \right\}
\]

\[
\times \left( \mathcal{I}(w) \right)^{-1} \mathcal{I}(z_1) \left[ G^{-}(z_1, w) \left( (\Gamma^\mu \Gamma^\nu - g^{\mu \nu} \Gamma^\nu ) (\Gamma^5)^S \mathcal{C} \right)_{AB} \\
+ \left( G^{-}(z_1, w) + G^{+}(z_1, w) \right) \left( g^{\rho \sigma} \Gamma^\rho - g^{\sigma \rho} \Gamma^\sigma \right) (\Gamma^5)^S \mathcal{C} \right)_{AB} \\
\right] \\
\]
\[ + (g^{\sigma\tau} g^{\rho\sigma} - g^{\mu\tau} g^{\nu\rho}) \Gamma^\rho - (g^{\mu\nu} g^{\rho\nu} - g^{\mu\rho} g^{\nu\sigma}) \Gamma^\sigma (\Gamma^5) \hat{S} \] \[ \times \left( \left( (g^{\alpha\beta} g^{\nu\sigma} - g^{\nu\alpha} g^{\beta\sigma}) \Gamma^\nu - (g^{\mu\nu} g^{\beta\nu} - g^{\mu\beta} g^{\nu\sigma}) \Gamma^\sigma \right) (\Gamma^5) \hat{S} \right) \] \[ - (I(w))^{-1} I(z_2) G^+(z_1, z_2) G^-(z_1, w) \] \[ \times \left( \left( (g^{\alpha\beta} g^{\nu\sigma} - g^{\nu\alpha} g^{\beta\sigma}) \Gamma^\nu - (g^{\mu\nu} g^{\beta\nu} - g^{\mu\beta} g^{\nu\sigma}) \Gamma^\sigma \right) (\Gamma^5) \hat{S} \right) \] \[ - (I(w))^{-1} I(z_2) G^+(z_1, z_2) G^-(z_1, w) \] \[ \times \left( \left( (g^{\alpha\beta} g^{\nu\sigma} - g^{\nu\alpha} g^{\beta\sigma}) \Gamma^\nu - (g^{\mu\nu} g^{\beta\nu} - g^{\mu\beta} g^{\nu\sigma}) \Gamma^\sigma \right) (\Gamma^5) \hat{S} \right) \] \[ - (I(w))^{-1} I(z_1) G^+(z_1, z_2) G^-(z_2, w) \] \[ \times \left( \left( (g^{\alpha\beta} g^{\nu\sigma} - g^{\nu\alpha} g^{\beta\sigma}) \Gamma^\nu - (g^{\mu\nu} g^{\beta\nu} - g^{\mu\beta} g^{\nu\sigma}) \Gamma^\sigma \right) (\Gamma^5) \hat{S} \right) \] \[ - (I(w))^{-1} I(z_1) G^+(z_1, z_2) G^-(z_2, w) \] \[ \times \left( \left( (g^{\alpha\beta} g^{\nu\sigma} - g^{\nu\alpha} g^{\beta\sigma}) \Gamma^\nu - (g^{\mu\nu} g^{\beta\nu} - g^{\mu\beta} g^{\nu\sigma}) \Gamma^\sigma \right) (\Gamma^5) \hat{S} \right) \] \[ \left\{ \right\} \] (4.25)

As already mentioned in Section 3, BRST-invariance of the vertex operators describing physical states implies on-shell conditions for the external states, as well as the transversality condition for the photons and the Dirac equation for the fermions:

\[
k_1^2 = k_2^2 = k_3^2 = k_4^2 = 0,
\]

\[
\epsilon \cdot k = 0,
\]

\[
W_3^T \delta_3^\epsilon = \delta_4^\epsilon C V_4 = 0.
\] (4.26)

These constraints were used to derive the explicit form of the vertex operators (3.22) and (3.23), but we were careful not to explicitly use them in the computation leading to eq. (4.23). Apart from technical advantages in doing the computations, the reasons for this choice are that this expression for the amplitude is somewhat more compact than the final one we will show in the next section and, being somehow “off-shell”, it could turn out to be useful as a starting point for the analysis of the field theory limit of the scattering amplitude, where one usually faces the problem of “going off-shell” (see for instance [9, 10, 11]). Of course, not being on-shell, the amplitude (4.23) is not gauge nor conformal invariant or independent on the position of insertion of the PCO operator.
5 The Final Form of the Amplitude

The momentum conservation $k_1 + k_2 + k_3 + k_4 = 0$ and on-shell conditions (4.26) allow us to considerably simplify the Lorentz structures of the amplitude (4.23). Expanding all the products, several terms in expression (4.23) drop out because of the on-shell and transversality conditions. Then we eliminate $k_2$ using momentum conservation, and we rearrange all the products of gamma matrices contracted with the momenta $k_3$ and $k_4$ in such a way to have the factor $\gamma_3$ ($\gamma_4$) always on the left (right). At this point all terms involving $\gamma_3$ and $\gamma_4$ vanish because of Dirac equations, and we obtain the final form of the on-shell amplitude:

$$T_g=1 \left( e^{\pm} + \gamma \rightarrow e^{\pm} + \gamma \right) =
\left( \frac{e^{A_3}}{\pi^6} \sum_{m_i,n_j} C_{\alpha K G \phi} e^{2\pi i K G \phi} \int \frac{d^2 \tau}{(\text{Im} \tau)^2} (\bar{\eta}(\tau))^{-24}(\eta(\tau))^{-12} \times
\right.
\int d^2 z_3 d^2 z_4 \frac{\sqrt{\omega(z_3)\omega(z_4)}}{\omega(z_1)\omega(z_2)} e^{x \sum_{i<j} (k_i k_j) G_B(z_i, z_i, z_j, z_j)}
\left. \times T_L \left[ \frac{\alpha}{\beta} \right](z_1, z_2, z_3, z_4, w) \times T_R \left[ \frac{\alpha}{\beta} \right](z_1, z_2, z_3, z_4, w). \right)$$

where

$$T_L \left( z_1, z_2, z_3, z_4, w \right) = \prod_{l=1-7,17} \Theta \left[ \frac{\alpha}{\beta} \right] \left( \frac{1}{2} \nu_{\lambda_34} | \nu \right) \prod_{l=8,18} \prod_{l=1-22} \Theta \left[ \frac{\alpha}{\beta} \right] (0 | \nu) \times (E(z_1, z_2))^{-2} \times
\left\{ \partial_{z_1} \partial_{z_2} \log E(z_1, z_2) + \frac{1}{4} \partial_{z_1} \log \frac{E(z_1, z_3)}{E(z_1, z_4)} \partial_{z_2} \log \frac{E(z_2, z_3)}{E(z_2, z_4)}
+ \frac{1}{2} \frac{\omega(z_1)}{2 \pi i} \partial_{z_2} \log \frac{E(z_2, z_3)}{E(z_2, z_4)} \partial_{z_1} \log \Theta \left[ \frac{\alpha}{\beta} \right] (\nu | \nu) \big| \nu = \frac{1}{2} \nu_{34} \right.
+ \frac{1}{2} \frac{\omega(z_2)}{2 \pi i} \partial_{z_1} \log \frac{E(z_1, z_3)}{E(z_1, z_4)} \partial_{z_2} \log \Theta \left[ \frac{\alpha}{\beta} \right] (\nu | \nu) \big| \nu = \frac{1}{2} \nu_{34}
+ \frac{\omega(z_1)}{2 \pi i} \frac{\omega(z_2)}{2 \pi i} \Theta \left[ \frac{\alpha}{\beta} \right] \left( \frac{1}{2} \nu_{34} | \nu \right)^{-1} \partial^2 \Theta(\nu | \nu) \big| \nu = \frac{1}{2} \nu_{34} \right\}, \right.$$ 

$$T_R \left( z_1, z_2, z_3, z_4, w \right) = \prod_{l=24,28,29,32} \Theta \left[ \frac{\alpha}{\beta} \right] \left( \frac{1}{2} \nu_{34} | \nu \right) \prod_{l=23,25,26,27,30,31} \Theta \left[ \frac{\alpha}{\beta} \right] (0 | \nu) \times
\left\{ \frac{(-1)^{33}}{\sqrt{2}} (E(z_3, z_4))^{-1} \left[ \frac{W^T_3 h_1 h_2}{(\Gamma^5)^S} \right] \left[ \frac{C^4 A}{z_1, z_2, z_3, z_4, w} \right] \right\}.$$
The factors $A$, multiplying the gamma matrices, are functions of the world-sheet coordinates, kinematics invariants and polarization vectors. They are given explicitly in Appendix C.

Notice that, except for the factors $\exp \sum_{i<j} (k_i k_j) G_B (z_i, \bar{z}_i, z_j, \bar{z}_j)$, all the dependence on the external momenta, spinors and polarizations is in the function $T_R$. As expected, after all Lorentz algebra has been done, only four independent gamma matrix structures remain.

The amplitude is formally hermitian and presents divergences in the integration over the moduli $[1, 2]$. It becomes absolutely convergent only for purely imaginary values of the Mandelstam variables $s, t$ and $u [1, 3, 4]$. These divergences are typical of string amplitudes in the Lorentz covariant formulation. The physical interpretation of such divergences has been discussed for instance in refs. $[20, 30, 31, 1, 3, 5] - [8]$, and is related to the unitarity of the scattering amplitudes. Indeed the two problems of these amplitudes, divergences and formal hermiticity instead of unitarity, are strictly related and can be cured at the same time. One can regularize the integrals by an analytical continuation in the external momenta so that at the same time the correct poles and branch cuts required by unitarity appear. In refs. $[1, 3, 8]$ one can find examples of such analytical continuation procedure (the stringy version of the Feynman $+ i \epsilon$).

The amplitude presents also infrared divergences due to the presence of massless states. As in field theory, these divergences correspond to the emission/absorption of soft photons/electrons, and can be removed by introducing an infrared cut-off.

On the other hand, as any string amplitude, it is free from ultraviolet divergences and automatically supplies a regulator for the chiral massless fermions. Notice however that these divergences will reappear in the field theory limit as divergences in $\alpha' \rightarrow 0$.

The dependence of our result on the PCO variable $w$ deserves some comments. It is well known that the amplitude should not depend on a PCO insertion point $w$. Therefore its derivative with respect to $w$ must be zero. In general this comes about only after the integration over the moduli space has been performed, since the differ-
entiation with respect to \( w \) gives rise to a total derivative in the integrand. However, in this case, it is possible to show that the integrand itself is independent on the PCO’s insertion point. Consider the amplitude (4.1), where we absorbed two PCO’s in the photon vertices, and write the remaining PCO as \( \Pi(w) = 2\{Q_{BRST}, \xi(w)\} \). Then moving the BRST commutator onto the other operators, we get

\[
\partial_w T^{1\text{-loop}} = 2C_{g=1} \int d^2\tau d^2z_2 d^2z_3 d^2z_4 \sum_{m, n, j} C_{\alpha}^\beta \sum_{m' = \tau, z_2, z_3, z_4} \frac{\partial}{\partial \bar{m}'} \langle \langle (\bar{\eta}_2 \bar{b})(\bar{\eta}_3 \bar{b})(\bar{\eta}_4 \bar{b}) \frac{\partial}{\partial (\eta_{m'})} \rangle \rangle ((\eta_{z_2} | b)(\eta_{z_3} | b)(\eta_{z_4} | b)) \\
\times |c(z_1) c(z_2) c(z_3)|^2 \partial_w \xi(w) V^{(0)}_{\text{photon}}(z_1, z_2; k_1; \epsilon_1) V^{(0)}_{\text{photon}}(z_2, z_3; k_2; \epsilon_2) \\
\times V^{(-1/2)}(z_3, z_4; k_4; W_3) V^{(-1/2)}(z_4, z_1; k_4; V_4) \rangle \rangle, \tag{5.4}
\]

where we used the fact that \( \bar{c} \mathcal{V} \) is BRST invariant and that [32]

\[
\langle \langle \{Q_{BRST}, (\eta | b) \} \ldots \rangle \rangle = \langle \langle (\eta | T_H) \ldots \rangle \rangle = \frac{\partial}{\partial \eta} \langle \langle \ldots \rangle \rangle. \tag{5.5}
\]

Then superghost charge conservation forces the derivative (5.4) to vanish altogether, meaning that the integrand must be explicitly independent of \( w \).

As a check of the correctness of our computation, we can then verify that the integrand eq.(5.3) is indeed independent of \( w \). We could compute its derivative with respect to \( w \) and see if it gives zero. However, in the case at hand, it is not so straightforward to prove this statement explicitly due to the identities in theta-functions we would need to prove. Instead we proved that the quantity \( \frac{1}{w(w)} T_R(w) \), which is the only factor in the amplitude depending on \( w \), is a meromorphic function of \( w \) on the torus, that it does not have zeros and that the residues at poles vanish. Thus it is a constant (as a function of \( w \)) and hence independent of \( w \).

Finally we can try to compare the string scattering amplitude with the corresponding one in field theory. The amplitude (5.1) corresponds to a field theory amplitude where the integral over loop momenta and the Lorentz algebra have been already done. The integrals over the moduli correspond to the integrals over the Schwinger proper-times. At first glance, one notices that the string amplitude contains more kinematical structures than the field theory one (see for example [29]). However a precise comparison between them would require the explicit evaluation of the integrals over the Schwinger parameters in field theory and the integrals over moduli as well as the sum over spin-structures in string theory. While in field theory the integrals
can be done, the same is usually not true in string theory. This prevents us from making any particular claim about the properties of the string-theory before making any suitable approximation or taking the field theory limit.

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Appendices

A Notations, Conventions and Useful Formulae

In this appendix we will give our conventions for the operator fields, partition functions and correlation functions on the torus. First of all we state our conventions for the Dedekind \( \eta \)-function, the theta functions and the prime form. The Dedekind \( \eta \)-function is given by

\[
\eta(\tau) = k^{1/24} \prod_{n=1}^{\infty} (1 - k^n), \quad k = e^{2\pi i \tau},
\]

and our conventions for the theta functions are

\[
\Theta \left[ \frac{a}{b} \right] (\nu | \tau) = e^{i \pi (\frac{1}{2} - \alpha)^2} e^{2\pi i (\frac{1}{2} + \beta)(\frac{1}{2} - \alpha)} e^{2\pi i (\frac{1}{2} - \alpha) \nu} \times \prod_{n=1}^{\infty} (1 - k^n) (1 - k^{n+\alpha-1} e^{-2\pi i (\beta + \nu)}) (1 - k^{n-\alpha} e^{2\pi i (\beta + \nu)})
\]

\[
= \sum_{r \in \mathbb{Z}} e^{i \pi (\frac{1}{2} - \alpha)^2 r^2 + 2\pi i (\frac{1}{2} - \alpha) (\nu + \beta + \frac{1}{2})} \Theta_1 \equiv \Theta \left[ \frac{0}{0} \right], \quad \Theta_2 \equiv \Theta \left[ \frac{0}{1/2} \right], \quad \Theta_3 \equiv \Theta \left[ \frac{1/2}{1/2} \right], \quad \Theta_4 \equiv \Theta \left[ \frac{1/2}{0} \right].
\]
The standard Riemann identity is
\[ \sum_{\alpha, \beta} e^{2\pi i (\alpha + \beta)} \prod_{i=1}^{4} \Theta [\beta] (x_i | \tau) = 0, \]  
where \( \alpha, \beta = \{0, \frac{1}{2}\} \) and one of the following equations must hold
\[ x_1 + x_2 + x_3 + x_4 = 0 \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad x_1 - x_2 - x_3 + x_4 = 0 \]
\[ x_1 - x_2 + x_3 - x_4 = 0 \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad x_1 + x_2 - x_3 - x_4 = 0. \]  
\[ \text{(A.5)} \]

The prime form is
\[ E(z, w) = \frac{2\pi i \Theta_1(\nu_{zw} | \tau)}{\sqrt{\omega(z) \omega(w) \Theta'(0|\tau)}}, \quad \nu_{zw} = \int_{w}^{z} \frac{\omega}{2\pi i} \]  
where \( \omega(z) \) is the holomorphic 1-form on the torus, normalized to have period \( 2\pi i \) around the \( \alpha \)-cycle. In the parametrization where \( \omega(z) = 1/z \) the prime form (A.6) becomes
\[ E(z, w) = (z - w) \prod_{n=1}^{\infty} \frac{(1 - \frac{z}{w} k^n)(1 - \frac{w}{z} k^n)}{(1 - k^n)^2}. \]  
\[ \text{(A.7)} \]

The space-time coordinate fields \( X^\mu \) have mode expansion
\[ X^\mu(z, \bar{z}) = q^\mu - i k^\mu \log(z, \bar{z}) + i \sum_{n \neq 0} \frac{\phi^\mu_n}{n} z^{-n} + i \sum_{n \neq 0} \frac{\bar{\phi}^\mu_n}{n} \bar{z}^{-n} \]  
and satisfy the OPE
\[ X^\mu(z, \bar{z}) X^\nu(w, \bar{w}) \overset{\text{OPE}}{=} -g^\mu \nu \log(z - w) + \log(\bar{z} - \bar{w})) + \ldots. \]  
\[ \text{(A.8)} \]

Their one-loop partition function is given by
\[ Z_X = \prod_{n=1}^{\infty} (1 - k^n)^{-8(2\pi \Im \tau)^{-2}}, \]  
\[ \text{(A.10)} \]
and the genus one correlator is
\[ \langle \langle X^\mu(z, \bar{z}) X^\nu(w, \bar{w}) \rangle \rangle = -g^\mu \nu \ G_B(z, \bar{z}; w, \bar{w}) \ Z_X. \]  
\[ \text{(A.11)} \]

Here \( G_B \) is the bosonic Green function on the torus where the non-zero mode part of the partition function has been removed:
\[ G_B(z, \bar{z}; w, \bar{w}) = 2 \left[ \log |E(z, w)| - \frac{1}{2} \Re \left( \int_{w}^{z} \omega \right)^2 \frac{1}{2\pi \Im \tau} \right] \]  
\[ \text{(A.12)} \]
The real world-sheet fermions associated with the space-time coordinates (eq. (2.3)) have mode expansion

$$\psi^\mu(z) = \sum_n \psi_n^\mu z^{-n-1/2}, \quad \{\psi^\mu, \psi^\nu\} = g^\mu\nu \delta_{m+n,0}$$

(A.13)

with \(n\) integer (half-integer) for Ramond (Neveu-Schwarz) boundary conditions. Their OPE is

$$\psi^\mu(z)\psi^\nu(w) \text{ OPE} = \frac{g^\mu\nu}{z - w} + \ldots$$

(A.14)

The real internal world-sheet fermions (eq. (2.4)) have a similar mode expansion and their OPE is given by

$$\psi_{i_0}^{m}(z)\psi_{i_0}^{n}(w) \text{ OPE} = \frac{\delta^{mn}\delta_{i_0}}{z - w} + \ldots$$

(A.15)

Correlations functions are defined according to eq. (4.5) and are computed bosonizing all complex fermions according to eq. (3.3). The fundamental genus one correlator [33] is

$$\langle \prod_{i=1}^{N} e^{q_i \phi(z_i)} \rangle [\alpha]_\beta = \delta_{\sum_i q_i,0} \prod_{i<j} [E(z_i, z_j)]^{q_{ij}} \Theta [\alpha]_\beta \left( \sum_{i=1}^{N} q_i \int^{z_i} \frac{\omega}{2\pi i} \right),$$

where we have explicitly displayed the spin structure dependence of the correlator, whereas in the paper we often adopt the following shorthand notation

$$\langle \prod_{i=1}^{N} e^{q_i \phi(z_i)} \rangle (l) = \langle \prod_{i=1}^{N} e^{q_i \phi(z_i)} \rangle [\alpha]_{\beta_l}.$$  

(A.17)

For the reparametrization ghosts we follow the conventions of ref. [19]. The normalization of the partition function is the standard one and the explicit expression can be found in refs. [24, 33]. The correlator relevant for our 1-loop scattering amplitude involving \(N\) physical external states is

$$\frac{d^2 k}{k^2 k^2} \prod_{i=1}^{N-1} d^2 z_i \langle \left( \sum_{i=1}^{N-1} c(z_i) \right)^2 \rangle$$

(A.18)

$$= \frac{d^2 k}{k^2 k^2} \prod_{i=1}^{N-1} d^2 z_i \frac{1}{\omega(z_N)} \prod_{n=1}^{\infty} [1 - k^n].$$

(A.19)

For the superghosts mode expansions, OPE and bosonization we follow the standard conventions of ref. [19] (see also eqq. (2.12), (3.6)). We always remain inside the
little algebra; with our definition of the theta functions, the partition function for the
superghost is \([14, 20]\]

\[
\langle \prod_{i=1}^{N} e^{q \phi(z_i)} \rangle = (-1)^S k^{1/2} \prod_{n=1}^{N} (1 - k^n) \prod_{i=1}^{N} (\omega(z_i))^{-q_i} \times \\
\prod_{i<j} (E(z_i, z_j))^{-q_{ij}} \left[ \Theta [\alpha_{ij}] \left( \sum_{j=1}^{N} q_j \int_{z_0}^{z_j} \frac{\omega}{2\pi i} \tau \right) \right]^{-1}.
\]

(A.20)

### B Theta Functions Identities

In this section we give the list of the identities in the Theta functions which arise in computing the correlators of Section 3.

\[
\mathcal{A} [\gamma] (z_1, z_2, w) \mathcal{I} [\alpha_{ij}] (w) = G^+ [\gamma] (z_1, z_2) \mathcal{D}_+ [\gamma] (z_1, z_2, w),
\]

\[
\mathcal{B} [\alpha_{ij}] (z_1, z_2) \mathcal{I} [\gamma] (w) = G^+ [\alpha_{ij}] (z_1, z_2) \mathcal{E}_2 [\alpha_{ij}] (z_1, z_2, w),
\]

\[
\mathcal{C}_- [\alpha_{ij}] (z_1, z_2, w) = -G^+ [\alpha_{ij}] (z_1, z_2) G^+ [\alpha_{ij}] (z_1, w),
\]

\[
\mathcal{C}_+ [\gamma] (z_1, z_2, w) \mathcal{I} [\gamma] (z_2) = G^- [\gamma] (z_1, z_2) \mathcal{D}_- [\gamma] (z_1, z_2, w),
\]

\[
\mathcal{D}_- [\gamma] (z_1, z_2, w) = \mathcal{I} [\gamma] (z_2) G^+ [\gamma] (z_1, w),
\]

\[
\mathcal{D}_+ [\alpha_{ij}] (z_1, z_2, w) = \mathcal{E}_1 [\alpha_{ij}] (z_1, z_2, w) - \mathcal{D}_- [\alpha_{ij}] (z_1, z_2, w) + \mathcal{E}_2 [\alpha_{ij}] (z_1, z_2, w)
\]

where the functions \(G^\pm\) and \(\mathcal{I}\) are defined in Section 3, eq.(4.15), and \(3\)

\[
\mathcal{A} [\gamma] (z_1, z_2, w) = \left\{ \partial_{z_1} \partial_{z_2} \log E(z_1, z_2) + \frac{\omega(z_1)}{2\pi i} \frac{\omega(z_2)}{2\pi i} \partial_{\nu} \log \Theta(\nu | \tau) \right|_{\nu = \mu w} \right\} \frac{E(z_1, w)}{\sqrt{E(z_1, z_3) E(z_1, z_4)}} \right. \\
+ \left. \left( \partial_{z_1} \log \frac{E(z_1, w)}{\sqrt{E(z_1, z_3) E(z_1, z_4)}} \right) + \frac{\omega(z_1)}{2\pi i} \partial_{\nu} \log \Theta [\alpha_{ij}] (\nu | \tau) \right|_{\nu = \mu w} \\
\times \left. \left( \partial_{z_2} \log \frac{E(z_2, w)}{\sqrt{E(z_2, z_3) E(z_2, z_4)}} \right) + \frac{\omega(z_2)}{2\pi i} \partial_{\nu} \log \Theta [\alpha_{ij}] (\nu | \tau) \right|_{\nu = \mu w} \right\},
\]

\[
\mathcal{B} [\alpha_{ij}] (z_1, z_2) = \left\{ \partial_{z_1} \partial_{z_2} \log E(z_1, z_2) + \frac{1}{4} \partial_{z_1} \log \frac{E(z_1, z_3)}{E(z_1, z_4)} \partial_{z_2} \log \frac{E(z_2, z_3)}{E(z_2, z_4)} \right\}
\]

\(3\) Even if not explicitly written, all functions listed below and in the following Appendix depend also on the world-sheet coordinates \(z_3, z_4\). For definitions of theta functions and prime form see Appendix A.
$$
\begin{align*}
\frac{1}{2} \omega(z_1) \frac{1}{2} \partial_{z_1} \log \frac{E(z_1, z_3)}{E(z_1, z_4)} \partial_v \log \Theta [\beta] \left( \nu | \tau \right)_{\nu = \frac{1}{2} v_{34}} \\
+ \frac{1}{2} \omega(z_2) \frac{1}{2} \partial_{z_2} \log \frac{E(z_2, z_3)}{E(z_2, z_4)} \partial_v \log \Theta [\beta] \left( \nu | \tau \right)_{\nu = \frac{1}{2} v_{34}} \\
+ \omega(z_1) \omega(z_2) \frac{1}{2} \partial_{\nu} \Theta (\nu | \tau)_{\nu = \frac{1}{2} v_{34}} \bigg) ^{-1} \partial_{\nu} \Theta (\nu | \tau)_{\nu = \frac{1}{2} v_{34}} \bigg) ,
\end{align*}
$$

$$
C_{\pm [\beta]} (z_1, z_2, w) = \frac{1}{2} (E(z_2, w))^{-1} \left\{ \sqrt{\frac{E(z_3, w) E(z_3, z_4)}{E(z_2, z_3) E(z_3, w)}} \Theta [\beta] \left( \rho_{w,z} | \tau \right) + \frac{\omega(z_1)}{2 \pi i} \partial_v \log \Theta [\beta] \left( \nu | \tau \right)_{\nu = \rho_{w,z}} \right\} ,
$$

$$
D_{\pm [\beta]} (z_1, z_2, w) = \frac{1}{2} \sqrt{E(z_3, z_4)} \left( \Theta [\beta] \left( \frac{1}{2} v_{34} | \tau \right) \right) ^{-1} \left\{ \sqrt{\frac{E(z_2, z_3) E(z_2, z_4)}{E(z_1, z_3) E(z_1, z_4) E(w, z_3) E(w, z_4)}} \Theta [\beta] \left( \mu_{z,w} | \tau \right) \right\} ,
$$

$$
E_{1,2 [\beta]} (z_1, z_2, w) = \frac{1}{2} \sqrt{E(z_3, z_4)} \left( \Theta [\beta] \left( \frac{1}{2} v_{34} | \tau \right) \right) ^{-1} \left\{ \sqrt{\frac{E(z_2, z_3) E(z_2, z_4)}{E(z_1, z_3) E(z_1, z_4) E(w, z_3) E(w, z_4)}} \Theta [\beta] \left( \mu_{z,w} | \tau \right) \right\} ,
$$
C Lorentz Functions of the On-Shell Amplitude

In this appendix we display the explicit expressions for the coefficients $\mathcal{A}_k$ which appear in the amplitude $\mathcal{T}_R$ of eq.(5.3) They are functions of the external momenta, polarization vectors and world sheet coordinates.$^4$

\[
\mathcal{A}_1(z_1, z_2, w) = (k_1k_2) \left[ \partial_w G_B(w, z_1) - \partial_w G_B(w, z_2) \right] G^+(z_1, z_2) G^-(z_1, z_2)
+ \frac{1}{2} \sum_{j=1}^{4} (k_1k_j) \partial_{z_1} G_B(z_j, z_1) \partial_w G_B(z_1, w) I(z_2) + \\
+ \frac{1}{2} \sum_{j=1}^{4} (k_2k_j) \partial_{z_2} G_B(z_j, z_2) \partial_w G_B(z_2, w) I(z_1) + \\
+ \sum_{j=1}^{4} \partial_w G_B(w, z_j) \left[ (k_1k_j) I(z_2) G^+(z_1, w) + (k_2k_j) I(z_1) G^+(z_2, w) \right] \frac{G^-(z_1, z_2)}{I(w)},
\]

\[
\mathcal{A}_2(z_1, z_2, w) = \left[ \partial_w G_B(z_1, w) - \partial_w G_B(z_2, w) \right] \left[ (\epsilon_1 \epsilon_2) \partial_{z_1} \partial_{z_2} G_B(z_1, z_2) + \\
+ \sum_{i,j=1}^{4} (\epsilon_1 k_i)(\epsilon_2 k_j) \partial_{z_1} G_B(z_i, z_1) \partial_{z_2} G_B(z_j, z_2) \right] + \\
+ (\epsilon_1 k_3) \sum_{j=1}^{4} (\epsilon_2 k_j) \partial_{z_2} G_B(z_j, z_2) I(z_1) \left[ \partial_w G_B(z_2, w) - \partial_w G_B(z_3, w) \right] + \\
- (\epsilon_2 k_3) \sum_{j=1}^{4} (\epsilon_1 k_j) \partial_{z_1} G_B(z_j, z_1) I(z_2) \left[ \partial_w G_B(z_1, w) - \partial_w G_B(z_3, w) \right] + \\
- (\epsilon_1 \epsilon_2) \sum_{j=1}^{4} \left[ (k_2k_j) \partial_{z_2} G_B(z_j, z_2) \partial_w G_B(z_2, w) B^+(z_1, w) + \\
- (k_1k_j) \partial_{z_1} G_B(z_j, z_1) \partial_w G_B(z_1, w) B^-(z_2, w) \right] + \\
- \sum_{i,j=1}^{4} (\epsilon_1 k_i)(\epsilon_2 k_j) \left[ \partial_{z_1} G_B(z_i, z_1) \partial_w G_B(z_j, w) B^+(z_2, w) + \\
- \partial_{z_2} G_B(z_j, z_2) \partial_w G_B(z_i, w) B^+(z_1, w) \right] + \\
+ \sum_{j=1}^{4} \partial_w G_B(z_j, w) \left\{ [ (\epsilon_2 k_2)(\epsilon_2 k_j) - (\epsilon_1 \epsilon_2)(k_2k_j) ] C_2(z_1, z_2, w) + \\
+ [ (\epsilon_2 k_1)(\epsilon_1 k_j) - (\epsilon_1 \epsilon_2)(k_1k_j) ] C_1(z_1, z_2, w) \right\} + \\
\]

$^4$All functions listed here depend also on the coordinates $z_3, z_4$. 

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\[
+ 2 \sum_{j=1}^{4} \partial_{u} G_{B}(z_{j}, w) \left[ (e_{k_{3}}) (e_{k_{j}}) D_{2}(z_{1}, z_{2}, w) + (e_{k_{4}}) (e_{k_{j}}) D_{1}(z_{1}, z_{2}, w) \right] + \\
+ \left[ (e_{k_{2}}) (e_{k_{1}}) - (e_{k_{2}}) (k_{k_{2}}) \right] \left[ \partial_{u} G_{B}(z_{1}, w) - \partial_{u} G_{B}(z_{2}, w) \right] \times \left[ G^{+}(z_{1}, z_{2})^{2} + 2G^{+}(z_{1}, z_{2}) G^{-}(z_{1}, z_{2}) \right] + \\
+ 2G^{+}(z_{1}, z_{2}) G^{-}(z_{1}, z_{2}) \left\{ (\sum_{j=1}^{3} (e_{k_{j}}) \left[ \partial_{u} G_{B}(z_{2}, w) - \partial_{u} G_{B}(z_{j}, w) \right] + \\
+ \left[ (e_{k_{2}}) (k_{k_{2}}) + (e_{k_{3}}) (e_{k_{1}}) \right] \left[ \partial_{u} G_{B}(z_{1}, w) - \partial_{u} G_{B}(z_{3}, w) \right] \right\},
\]

\[
A_{3}(z_{1}, z_{2}, w) = \partial_{u} G_{B}(z_{1}, w) \left[ (e_{k_{3}}) \partial_{z_{1}} G_{B}(z_{1}, z_{2}) \sum_{j=1}^{4} (k_{k_{j}}) \partial_{z_{2}} G_{B}(z_{2}, z_{j}) \right] + \\
- \sum_{j,k=1}^{4} (e_{k_{j}}) (e_{k_{1}}) \partial_{z_{1}} G_{B}(z_{1}, z_{2}) \partial_{z_{2}} G_{B}(z_{j}, z_{2}) + \\
- (e_{k_{3}}) \sum_{j=1}^{4} (e_{k_{j}}) \partial_{z_{2}} G_{B}(z_{j}, z_{2}) \left[ \partial_{u} G_{B}(z_{2}, w) - \partial_{u} G_{B}(z_{3}, w) \right] I(z_{1}) + \\
- (e_{k_{4}}) \partial_{u} G_{B}(z_{1}, w) \sum_{j=1}^{4} (k_{k_{j}}) \partial_{z_{1}} G_{B}(z_{j}, z_{1}) I(z_{2}) + \\
- \sum_{j=1}^{4} \left[ (e_{k_{2}}) (e_{k_{j}}) - (e_{k_{2}}) (k_{k_{2}}) \right] \partial_{u} G_{B}(z_{j}, w) G(z_{1}, z_{2}, w) + \\
+ B^{+}(z_{1}, w) \sum_{j=1}^{4} \partial_{z_{2}} G_{B}(z_{j}, z_{2}) \left[ (e_{k_{2}}) (e_{k_{1}}) \partial_{u} G_{B}(z_{2}, w) + \\
- \sum_{i=1}^{4} (e_{k_{j}}) (e_{k_{1}}) \partial_{u} G_{B}(z_{1}, w) \right] + \\
+ 2G^{+}(z_{1}, z_{2}) G^{-}(z_{1}, z_{2}) \left\{ (e_{k_{1}}) (k_{k_{3}}) \left[ \partial_{u} G_{B}(z_{3}, w) - \partial_{u} G_{B}(z_{1}, w) \right] + \\
- (e_{k_{4}}) (k_{k_{2}}) \left[ \partial_{u} G_{B}(z_{3}, w) - \partial_{u} G_{B}(z_{2}, w) \right] \right\} + \\
- 2 \sum_{j=1}^{4} \partial_{u} G_{B}(z_{j}, w) \left[ (e_{k_{4}}) (k_{k_{2}}) D_{1}(z_{1}, z_{2}, w) + (k_{k_{3}}) (e_{k_{j}}) D_{2}(z_{1}, z_{2}, w) \right],
\]

\[
A_{4}(z_{1}, z_{2}, w) = \partial_{u} G_{B}(z_{2}, w) \left[ (e_{k_{2}}) \partial_{z_{2}} G_{B}(z_{1}, z_{2}) \sum_{j=1}^{4} (k_{k_{j}}) \partial_{z_{1}} G_{B}(z_{1}, z_{j}) + \\
- \sum_{j,k=1}^{4} (e_{k_{j}}) (k_{k_{1}}) \partial_{z_{1}} G_{B}(z_{j}, z_{1}) \partial_{z_{2}} G_{B}(z_{1}, z_{2}) \right] + \\
\]
\]
\begin{align*}
&- (k_2 k_3) \sum_{j=1}^{4} (\epsilon_1 k_j) \partial_z G_B(z_j, z_1) \left[ \partial_w G_B(z_1, w) - \partial_w G_B(z_3, w) \right] I(z_2) + \\
&+ (\epsilon_1 k_4) \partial_w G_B(z_1, w) \sum_{j=1}^{4} (k_1 k_j) \partial_z G_B(z_j, z_1) I(z_2) + \\
&+ \sum_{j=1}^{4} \left[ (k_1 k_2)(\epsilon_1 k_j) - (\epsilon_1 k_2)(k_1 k_j) \right] \partial_w G_B(z_j, z_1) C_1(z_1, z_2, w) + \\
&- B^+(z_2, w) \sum_{j=1}^{4} \partial_2 G_B(z_j, z_1) \left[ \sum_{i=1}^{4} (\epsilon_1 k_j)(k_2 k_i) \partial_w G_B(z_i, w) + \\
&- (k_1 k_j)(\epsilon_1 k_2) \partial_w G_B(z_1, w) \right] \\
&+ 2G^+(z_1, z_2) G^-(z_1, z_2) \left[ (\epsilon_1 k_3)(k_1 k_2) - (\epsilon_1 k_2)(k_1 k_3) \right] \times \\
&\times \left[ \partial_w G_B(z_3, w) - \partial_w G_B(z_2, w) \right] + \\
&+ 2 \sum_{j=1}^{4} \partial_w G_B(z_j, w) \left[ (\epsilon_1 k_j)(k_1 k_3) - (k_1 k_j)(\epsilon_1 k_3) \right] D_1(z_1, z_2, w),
\end{align*}

and finally

\begin{align}
B^\pm \left[ \gamma \right] (z_i, w) &= \frac{I(z_i)}{I(w)} \left[ G^+(z_i, w) \pm G^-(z_i, w) \right], \\
D_{1,2} \left[ \gamma \right] (z_1, z_2, w) &= \frac{I(z_{2,1})}{I(w)} G^+(z_{1,2}, w) G^-(z_1, z_2), \\
C_{1,2} \left[ \gamma \right] (z_1, z_2, w) &= \left[ G^+(z_1, z_2) G^- (z_1, z_2) + \\
&+ \frac{I(z_{2,1})}{I(w)} G^+(z_{1,2}, w) \left( G^+(z_1, z_2) + G^- (z_1, z_2) \right) \right].
\end{align}

References


