Duality of Quasilocal Black Hole Thermodynamics

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Abstract

We consider T-duality of the quasilocal black hole thermodynamics for the three-dimensional low energy effective string theory. Quasilocal thermodynamic variables in the first law are explicitly calculated on a general axisymmetric three-dimensional black hole solution and corresponding dual one. Physical meaning of the dual invariance of the black hole entropy is considered in terms of the Euclidean path integral formulation.

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I. INTRODUCTION

T-duality implies the equivalence of two apparently very different non-linear $\sigma$-models and their respective corresponding low energy effective theories [1]. This symmetry is described in the context of toroidal compactifications: for $d$-dimensional compactifications the T-dual transformation is an element of an infinite order discrete symmetry group $O(d, d; \mathbb{Z})$ [2]. It has been also interpreted in terms of the gauging of an isometry symmetry [3]. For the simplest case of a single compactified dimension of radius $R$ on the flat spacetime background, the entire physics is left unchanged under the replacement $R \to \alpha'/R$ provided one also transforms the dilaton field $\phi \to \phi - \log(R/\sqrt{\alpha'})$. Thus, geometric properties and/or topology of the dual\(^1\) solution can in general be quite different from those of the original solution. For example, it has been shown [4] that the dual solution of the BTZ black hole [5] is the three-dimensional charged black string solution [6]. The BTZ black hole does not have a curvature singularity, while the black string has a timelike singularity. Moreover, the former is asymptotically anti-de Sitter spacetime, while the latter is asymptotically flat\(^2\).

In this respect it is of great interest to study behavior of physical quantities depending on geometry of a given spacetime under the dual transformation. Horne et al. [8] have shown that on asymptotically flat solutions, the duality of the conserved quantities defined on asymptotic region is given in such a way that mass is unchanged, while the axion charge and angular momentum are interchanged each other. It has been also shown that the Hawking temperature and horizon area (viz. entropy) of black hole solutions to the low energy effective string theory are dual invariant [9]. It is a surprising observation because they are explicitly associated with a given spacetime geometry.

As mentioned above, asymptotic behavior of a spacetime may be changed by the dual transformation. Thus, asymptotic flatness is in general not an appropriate assumption for

\(^1\)Hereafter, we use the word ‘dual’ as only ‘T-dual’

\(^2\)About the change of asymptotic geometry due to dualities, see also [7].
the study of the duality of the physical quantities defined on a given spacetime. To avoid this difficulty, we shall introduce a finite spatial boundary and study the duality of quasilocal quantities irrespective of asymptotic behavior of a given spacetime. Quasilocal boundary has been considered on several studies of gravitational systems [10 - 18]. Especially, it has been shown that under the consideration of the quasilocal boundary, a black hole can be consistently interpreted within the framework of the ordinary thermodynamics, i.e., the black hole partition function is well-defined and positivity of the heat capacity is recovered [12, 13].

In this paper, we shall consider the duality of the quasilocal black hole thermodynamics, explicitly the quasilocal black hole thermodynamic first law, for the three-dimensional low energy effective string theory. For the purpose, we study the duality of quasilocal thermodynamic quantities as well as the black hole entropy on a general axisymmetric three-dimensional spacetime, say a minisuperspace model [19]. We shall also consider the physical meaning of the dual invariance of the black hole entropy in terms of the path integral formulation [15, 20].

In Sect. II, the quasilocal black hole thermodynamics is briefly recapitulated. Then, we shall study the duality of the quasilocal thermodynamic first law in Sect. III. Physical meaning of the dual invariance of the black hole entropy is also considered in terms of the Euclidean path integral formulation. Summary and discussions are given in Sect. IV.

II. QUASILOCAL BLACK HOLE THERMODYNAMICS

In this section, we briefly recapitulate the quasilocal black hole thermodynamics for the three-dimensional low energy effective string theory [13, 14]. Quasilocality means that the three-dimensional manifold $M$ has a finite timelike spatial boundary $\Sigma^r$ as well as two spacelike boundaries (initial and final ones denoted by $\Sigma'_\nu$ and $\Sigma''_{\nu'}$, respectively). The boundary of $\Sigma_t$, which is denoted by $S'_{\Sigma_t}$, is given by an intersection space of $\Sigma_t$ and $\Sigma^r$. We assume that $\Sigma_t$ is orthogonal to $\Sigma^r$. The orthogonality means that on the boundary $\Sigma^r$, the
timelike unit normal $u^a$ to $\Sigma_t$ and the spacelike unit normal $n^a$ to $\Sigma_r$ satisfy the relation $u^a n_a|_{\Sigma_r} = 0$. The induced metric forms defined on $\Sigma_t$, $\Sigma_r$ and $S^r_t$ are denoted by $h_{ab}$, $\gamma_{ab}$ and $\sigma_{ab}$, respectively.

Consider the three-dimensional low energy effective string action [6] given by

$$I = \frac{1}{2\pi} \int_M d^3x \sqrt{-g} \Phi \left[ R + \Phi^{-2} (\nabla \Phi)^2 - \frac{1}{12} H^2 + \frac{4}{l^2} \right]$$

$$- \frac{1}{\pi} \int_{\Sigma_t'} d^2x \sqrt{h} \Phi K - \frac{1}{\pi} \int_{\Sigma_r} d^2x \sqrt{-\gamma} \Phi \Theta ,$$

(1)

where $(-1/2) \ln \Phi$ is the dilaton field and $H$ denotes the three-form field strength of the antisymmetric two-form field $B$. Comparing with the three-dimensional general relativity, $-l^{-2}$ can be interpreted as the negative cosmological constant $\Lambda$ [4]. In boundary terms of eq.(1), $K$ and $\Theta$ are traces of extrinsic curvatures of $\Sigma_t$ and $\Sigma_r$ as embedded in the three-dimensional spacetime $M$, $K_{ab} = -h^c_a \nabla_c u_b$ and $\Theta_{ab} = -\gamma^c_a \nabla_c n_b$, respectively. The boundary terms are involved such that when one applies a solution of equations motion into the action and requires the boundary condition that field variables be fixed on the boundaries, the action has an extremum value.

The canonical form of the action (1) becomes

$$I = \int_M d^3x \left[ P_{ab} \dot{h}_{ab} + P_\Phi \dot{\Phi} + P_{\mathcal{B}} \dot{B}_{ab} - N \mathcal{H} - N^a \mathcal{H}_a - B_{at} \psi^a \right]$$

$$+ \int_{\Sigma_r} dt d\phi \left[ -\mathcal{E} N + \mathcal{J}_a N^a - Q^a B_{at} \right] ,$$

(2)

where $N$, $N^a$ denote the lapse function and shift vector, respectively, and $\mathcal{H}$, $\mathcal{H}_a$, and $\psi^a$ are the Hamiltonian, momentum, and Gauss constraints, respectively. Conjugate momenta $P$’s are given by

$$P_{ab} = \frac{\delta L}{\delta \dot{h}_{ab}} = \sqrt{h} \left[ \Phi (h^{ab} K - K^{ab}) - \frac{1}{N} \Phi h^{ab} + \frac{1}{N} N^c \partial_c \Phi h^{ab} \right] ,$$

$$P_\Phi = \frac{\delta L}{\delta \dot{\Phi}} = \sqrt{h} \left[ \frac{2}{N} \Phi \Phi^{-1} + \frac{2}{N} \Phi^{-1} N^c \partial_c \Phi \right] ,$$

$$P_{\mathcal{B}} = \frac{\delta L}{\delta \dot{B}_{ab}} = \sqrt{h} \frac{4}{4\pi N} \Phi h^{ac} h^{bd} \left[ \dot{B}_{cd} + \partial_c B_{dt} + \partial_d B_{tc} \right] .$$

The integrands of boundary term in eq.(2) are interpreted as the quasilocal surface energy density $\mathcal{E}$, momentum density $\mathcal{J}_a$, and axion charge density $Q^a$ [10, 13, 14] given by
\[ E = -\frac{\sqrt{\sigma}}{\pi} (n^a \nabla_a \Phi - \Phi k), \]
\[ J_a = \frac{2\sqrt{\sigma}}{\sqrt{h}} n_c \sigma_{ad} P^{cd}, \]
\[ Q^a = \frac{2\sqrt{\sigma}}{\sqrt{h}} P^{ab} n_b, \]

where \( k \) is the trace of the extrinsic curvature as embedded in \( \Sigma_t \), \( k_{ab} = -\sigma^c_a D_c n_b \), and \( D_c \) is the covariant derivative on \( \Sigma_t \). These quantities are called extensive variables which are composed by intensive variables, e.g., the lapse function and shift vector.

According to the Hamilton-Jacobi type analysis, the informations of a gravitational system with a spatial boundary are encoded on the boundary and variations of the boundary variables determine a generating functional. Varying the action (2), the \( \Sigma^r \) boundary terms are given by

\[
\delta I|_{\Sigma^r} = \int_{\Sigma^r} dtd\phi \left[-E\delta N + J_a \delta N^a - Q^a \delta B_{at} + N \left((s^{ab}/2)\delta \sigma_{ab} + \mathcal{Y} \delta \Phi\right)\right],
\]

where \( s^{ab}, \mathcal{Y} \) are interpreted as the quasilocal surface stress density and dilaton pressure density \([10, 13, 14]\), respectively, defined by

\[
s^{ab} = \frac{\sqrt{\sigma}}{\pi} \left[\sigma^{ab} n^c \nabla_c \Phi + \Phi [k^{ab} - \sigma^{ab} (k - n^c a_c)]\right],
\]
\[
\mathcal{Y} = \frac{\sqrt{\sigma}}{\pi} \left[\Phi^{-1} n^c \nabla_c \Phi - (k - n^c a_c)\right]
\]

where \( a^c = u^a \nabla_a u^c = N^{-1} h^{ac} \nabla_a N \) is the acceleration of the timelike unit normal \( u^c \). Note that since our considering manifold is not the four (or higher) dimensional spacetime, but the three-dimensional spacetime, the term of the electromotive force is not included in eq.(4) in contrast to the case of Ref.[14]. These densities (5) appear due to the fact that we choose the finite spatial boundary, and become zero at asymptotic region for the asymptotically flat case.

Note that the first three terms eq.(4) involve variations of intensive variables with extensive coefficients. In this paper, it is appropriate for our purpose to choose the microcanonical boundary condition in which the thermodynamical extensive variables are fixed.
on the boundary [12, 14]. The microcanonical action can be obtained from the action via
the Laplace transformation as follows

\[
I_{\text{micro}} = I + \int_{\Sigma} dt d\phi \left[ \mathcal{E} N - \mathcal{J}_a N^a + Q^a B_{at} \right]
\]
\[
= \int_M d^3x \left[ P^{ab} h_{ab} + P_\Phi \dot{\Phi} + P_B^B \dot{B}_{ab} - N H - N^a H_a - B_{at} \psi^a \right].
\]  

(6)

Then, varying the microcanonical action (6), the \( \Sigma^r \) boundary terms are

\[
\delta I_{\text{micro}}|_{\Sigma^r} = \int_{\Sigma^r} dt d\phi N \left[ \delta \mathcal{E} - \omega^a \delta \mathcal{J}_a + V_a \delta Q^a + (s^{ab}/2) \delta \sigma_{ab} + \mathcal{Y} \delta \Phi \right],
\]  

(7)

where \( N \omega^a = N^a \) and \( N V_a = B_{at} \).

From the path integral point of view, physical states are labeled by the boundary variables \( \mathcal{E}, \mathcal{J}^a \) and \( Q_a \), and parameterized by the boundary metric \( \sigma \) and the dilaton field \( \Phi \) [15]. Thus, the entropy is statistically defined as the logarithm of the density of states \( \nu(\mathcal{E}, \mathcal{J}, Q; \sigma, \Phi) \), which is a functional of the boundary variables defined on the timelike spatial boundary \( \Sigma^r \). The density of states is given by tracing the microcanonical density matrix \( \rho_m(h_{v'}, \Phi_{v'}, B_{v'}; h_v, \Phi_v, B_v; \mathcal{E}, \mathcal{J}, Q; \sigma, \Phi) \) from the initial spacelike boundary \( \Sigma_v \) to the final one \( \Sigma_{v'} \) as follows

\[
\nu(\mathcal{E}, \mathcal{J}, Q; \sigma, \Phi) = \int \mathcal{D}[h, \Phi, B] \rho_m(h_{v'}, \Phi_{v'}, B_{v'}; h_v, \Phi_v, B_v; \mathcal{E}, \mathcal{J}, Q; \sigma, \Phi),
\]  

(8)

where the subscript ‘t’ of fields denotes that they are not the field variables on the timelike boundary \( \Sigma^r \), but on the spacelike hypersurface \( \Sigma_t \). The microcanonical density matrix \( \rho_m \) is given by

\[
\rho_m(h_{v'}, \Phi_{v'}, B_{v'}; h_v, \Phi_v, B_v; \mathcal{E}, \mathcal{J}, Q; \sigma, \Phi) = \int \mathcal{D}[g, \Phi, B] e^{-I_{\text{micro}}^{E}[g, \Phi, B]},
\]  

(9)

where \( I_{\text{micro}}^{E} \) is the microcanonical Euclidean action obtained from the microcanonical action (6) via the Wick rotation \( t \rightarrow \tau = it \). Then the entropy of a gravitational system is given by up to zeroth-order

\[
S \approx \ln \nu(\mathcal{E}, \mathcal{J}, Q; \sigma, \Phi) \approx -I_{\text{micro}}^{E}|_d,
\]  

(10)
where the subscript $\text{cl}$ means that a solution to equations of motion is plugged into the action. Assuming stationarity of the solution, the entropy (10) becomes zero. However, if it is a black hole solution, the entropy does not vanish.

For an observer who lives on asymptotic region, all informations are accreted to the event horizon, and the horizon can be considered as another boundary, say inner boundary. However, it must be emphasized that the inner boundary is not a system boundary on which the thermodynamic data must be specified [14, 15]. Then the Euclidean microcanonical action has another boundary term as follows

$$I^E_{\text{micro}, bh} = I^E_{\text{micro}} + \frac{1}{\pi} \int_{\Sigma H} d\tau d\phi N \sqrt{\sigma} [\Phi \Theta - n^a \partial_a \Phi],$$

(11)

where $r_H$ denotes the horizon. As a result, the black hole entropy becomes

$$S_{BH} \approx -I^E_{\text{micro}, cl} = -\frac{1}{\pi} \int_{\Sigma H} d\tau d\phi N \sqrt{\sigma} [\Phi \Theta - n^a \partial_a \Phi]_{cl},$$

(12)

where it was assumed that the black hole solution is stationary, i.e., $I^E_{\text{micro}}|_{cl} = 0$. Note that if $\Phi = 1$, then the expression for the black hole entropy (12) is equal to that of Ref.[21], in which authors have evaluated the black hole entropy on the action conformally transformed into the Einstein-Hilbert form.

Finally, for the case that a black hole is embedded into a finite cavity, the desired thermodynamic first law can be obtained via variation of the entropy (12) [13, 14] as follows

$$\delta S_{BH} = \int_{\Sigma^r} d\phi \beta \left[ \delta \mathcal{E} - \omega^a \delta \mathcal{J}_a + V_a \delta Q^a + (s^{ab}/2) \delta \sigma_{ab} + \mathcal{V} \delta \Phi \right],$$

(13)

where $\beta = \int N d\tau$ is the inverse temperature defined on the finite spatial boundary $\Sigma^r$.

III. DUALITY OF QUASILOCAL THERMODYNAMIC VARIABLES

We are now ready to study the duality of the first law of the quasilocal black hole thermodynamics (13). First of all, consider a general axisymmetric solution to the equations of motion of the action (1) [19], which is the form as follows
\[ ds^2 = -N^2(r)dt^2 + f^{-2}(r)dr^2 + r^2 \left[ d\phi + N^\phi(r)dt \right]^2, \]
\[ \Phi = \Phi(r), \quad B_{\phi t} = B_{\phi t}(r), \quad (14) \]

where all other components of the antisymmetric field \( B_{ab} \) are zero. The horizon \( r_H \) satisfies the relation \( N^2(r_H) = 0 \). In the metric (14), we require the regularity of the antisymmetric field and vanishing shift vector at the horizon as \( N^\phi(r_H) = B_{\phi t}(r_H) = 0 \). The requirement of vanishing shift vector at horizon is actually the same with doing a coordinate transformation \( \phi \to \phi - \Omega_H t \) on a metric including non-vanishing shift vector at horizon, where \( \Omega_H \) is the angular velocity of the horizon [9]. This sustains the regularity of the antisymmetric field at horizon under the dual transformation.

Since the metric (14) and the fields are independent of the coordinate \( \phi \), the solution has a translational symmetry in the direction \( \phi \). Thus there exists a corresponding dual solution by means of the dual transformation [1]

\[ g^d_{\phi\phi} = g^{-1}_{\phi\phi}, \quad g^d_{\phi\alpha} = B_{\phi\alpha} g^{-1}_{\phi\phi}, \]
\[ g^d_{\alpha\beta} = g_{\alpha\beta} - (g_{\phi\alpha} g_{\phi\beta} - B_{\phi\alpha} B_{\phi\beta}) g^{-1}_{\phi\phi}, \]
\[ B^d_{\phi\alpha} = g_{\phi\alpha} g^{-1}_{\phi\phi}, \quad B^d_{\alpha\beta} = B_{\alpha\beta} - 2 g_{\phi\alpha} B_{\beta\phi} g^{-1}_{\phi\phi}, \]
\[ \Phi^d = g_{\phi\phi} \Phi, \quad (15) \]

where \( \alpha, \beta \) run over all directions except \( \phi \). After performing the dual transformation on the solution (14), the dual one is obtained by

\[ ds^2_d = -N^2_d(r)dt^2 + f^{-2}_d(r)dr^2 + \frac{1}{r^2} \left[ d\phi + N^\phi_d(r)dt \right]^2, \]
\[ \Phi^d = g_{\phi\phi} \Phi, \quad B^d_{\phi t} = N^\phi, \quad (16) \]

where \( N^\phi_d(r) = B_{\phi t}(r) \). Note that the lapse function and \( rr \)-component of the metric are unchanged under the dual transformation. Thus the position of the horizon and the original unit normals \( u^a \) and \( n^a \) are unchanged under the dual transformation. On the other hand, the shift vector and antisymmetric field are interchanged each other. It gives us an impression for the duality of the quasilocal thermodynamic variables, i.e., the energy density is unchanged,
while the axion charge density and the momentum density are interchanged each other. Note that here, we set the coordinate $\phi$ to a compact one such as $\phi \sim \phi + 2\pi$. Then dual solutions represent the same conformal field theories [8].

Now, we shall consider the duality of the respective thermodynamic variables in the following subsections.

**A. Black Hole Entropy and Temperature**

It has already been shown in Ref.[9] that the black hole entropy and temperature are dual invariant for $n$-dimensional black string solutions. In this subsection, we shall reconfirm their results with direct calculation of eq.(12) for the case of the solution (14).

The Hawking temperature $T_H = \kappa_H/2\pi$ can be obtained from the relation

$$\kappa_H^2 = -\frac{1}{2} \nabla^a \chi^b \nabla_a \chi_b \Bigg|_{r=r_H},$$

where $\kappa_H$ is the surface gravity and $\chi^a = \xi^a + \Omega_H \zeta^a$ is the Killing vector normal to the horizon. $\xi^a$, $\zeta^a$ are the timelike and spacelike Killing vectors, respectively. Since we have required that the shift vector at the horizon $N^\phi(r_H)$ vanish, so does the angular velocity of the horizon $\Omega_H$. The Killing vector $\chi^a$ is then equal to the timelike Killing vector $\xi^a$. Plugging the metric (14) and its dual one (16) into (17), it can be shown that the surface gravity is dual invariant

$$\kappa_H = \kappa_H^d = f \frac{f}{2N(N^2)'} \Bigg|_{r=r_H},$$

where $'$ denotes differentiation with respect to the radial coordinate $r$. On the other hand, since the lapse function $N$ is dual invariant, Tolman temperature $T_H/N(r) = T(r)$ [22], which is red-shifted temperature from the horizon to the finite spatial boundary, as well as the Hawking one is also dual invariant, $T(r) = T(r)^d$.

Now, consider the black hole entropy (12). As mentioned above, the expression for the black hole entropy (12) is a generalization of that of the Einstein gravity, which is obtained
from the eq.(12) as we set $\Phi = 1$. It can be also shown that the black hole entropy (12) satisfies the perimeter law, which is the 2+1 dimensional version of the area law [5, 19], as follows; the extrinsic curvature $\Theta$ in the metric (14) becomes

$$\Theta\big|_{r=r_H} = -f \left( \frac{N'}{N} + \frac{1}{r} \right)\big|_{r=r_H} = - \left( \frac{2\pi}{\beta_H N} + \frac{f}{r} \right)\big|_{r=r_H}. \quad (19)$$

And the divergence of the dilaton field is $n^a \partial_a \Phi = f \Phi'$. Then, the black hole entropy becomes

$$S_{BH} \approx -\frac{1}{\pi} \int_0^{2\pi} d\phi \sqrt{\sigma} \left[ -2\pi \Phi - N f \beta_H \left( \frac{\Phi}{r} + \Phi' \right) \right]_{r=r_H}. \quad (20)$$

Since the second term in eq.(20) becomes zero because of $N(r_H) = 0$, we finally obtain the desired black hole entropy satisfying the perimeter law for the non-minimally coupled gravity version [23]

$$S_{BH} = 2 \cdot \int_0^{2\pi} d\phi \sqrt{\sigma} \Phi_{r=r_H}. \quad (21)$$

For the dual solution (16), through similar calculation with above one, we obtain the dual black hole entropy

$$S^d_{BH} = 2 \cdot \int_0^{2\pi} d\phi \sqrt{\sigma} \Phi^d_{r=r_H}. \quad (22)$$

Thus the same form of the perimeter law is still satisfied after the dual transformation. Furthermore, it can be shown that the black hole entropy is dual invariant as

$$S_{BH} = S^d_{BH} = 2 \cdot 2\pi r_H \Phi(r_H), \quad (23)$$

where we used the relations $\Phi^d = g_{\Phi\Phi} \Phi = r^2 \Phi$, $\sqrt{\sigma^d} = 1/\sqrt{\sigma} = 1/r$. This is always available to all solutions with types of the minisuperspace model (14). Note that for the case of the three-dimensional general relativity, BTZ black hole with $\Phi(r) = 1$, the entropy (23) satisfies the ordinary perimeter law $S_{BH} = 2 \cdot 2\pi r_H$ [5, 19, 24].
B. Quasilocal Energy Density, Momentum Density, and Axion Charge Density

Horne et al. [8] have shown that for asymptotically flat solutions, the duality of the conserved quantities defined on asymptotic region is given in such a way that mass is unchanged, while the axion charge and angular momentum are interchanged each other. Let us now examine the duality of these corresponding quasilocal densities for the minisuperspace model (14).

From eqs. (3), (14), and (16), the quasilocal surface energy density $E$, momentum density $J_\phi$, and axion charge density $Q_\phi$ and dual ones are given by

$$E = E^d = -\frac{f}{\pi} (r\Phi' + \Phi),$$

$$J_\phi = -Q_\phi^d = -\frac{r^3 f}{2\pi N} \Phi (N^\phi)' ,$$

$$Q_\phi = -J_\phi^d = \frac{f}{2\pi r N} \Phi (B_{\phi t})'. $$

Thus the duality of the quasilocal densities defined on the finite spatial boundary is equal to the result of Ref. [6], i.e., the quasilocal surface energy density is unchanged, while the momentum density and axion charge density are interchanged each other.

Note that the quasilocal energy is different from the quasilocal mass $M(r)$ [16, 17], which is a conserved charge on evolution of spacelike surfaces, given by

$$M(r) = \int_{S^t_r} dx^{n-2} N E = - \int_{S^t_r} dx^{n-2} \sqrt{\sigma} \frac{N}{\pi} (n^a \nabla_a \Phi - \Phi k).$$

From eq. (25), according to the fact that the lapse function is unchanged under the dual transformation, it is easily checked that the quasilocal mass is also dual invariant.

In the first law of the quasilocal black hole thermodynamics (13), since the lapse function is unchanged, while the shift vector and antisymmetric field are interchanged under the dual transformation in eq. (16), the quantities $\omega_\phi = N^\phi N^{-1}$ and $V_\phi = B_{\phi t} N^{-1}$ are also interchanged under the transformation. As a result, the first three terms in the first law (13) are invariant under the dual transformation,

$$\delta E - \omega_\phi \delta J_\phi + V_\phi \delta Q_\phi = \delta E^d - \omega_\phi^d \delta J_\phi^d + V_\phi^d \delta Q_\phi^d .$$

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For a noncompact spacetime, e.g., the BTZ black hole, the quasilocal quantities in (3) are divergent at asymptotic region. Thus, for the case, one should introduce a background spacetime in order to obtain the well-defined quasilocal quantities [10, 11, 13, 17]. The background spacetime can be chosen such that if there is not a black hole, the quasilocal quantities vanish. We shall give a comment about that and in Sect. IV.

C. Quasilocal Surface Stress Density and Dilaton Pressure Density

In this subsection, we study the duality of the quasilocal surface stress density $s_{ab}$ and dilaton pressure density $Y$. Existence of these quantities are originated from the fact that we have concerned with the finite spatial boundary $\Sigma^r$. Actually, in the three-dimensional case, the quasilocal surface stress density is just the ‘pressure’ on the boundary conjugate to the perimeter $2\pi r$ [10, 13, 14].

For the case of the minisuperspace model (14), pressure terms in (13) become

$$
\frac{(s^{ab}/2)\delta\sigma_{ab} + Y\delta\Phi}{2\pi r N} = \frac{f N'}{2\pi^2 N} \delta\Phi(r^2) + \frac{f}{\pi} \left(1 + \frac{r N'}{N}\right) \delta\Phi
$$

$$
= \frac{f N'}{2\pi^2 N} \Phi(2\pi r) + \frac{f}{\pi} \left(1 + \frac{r N'}{N}\right) \Phi
$$

$$
= P\delta(\text{perimeter}) + P\Phi\delta\Phi.
$$

(27)

On the other hand, from the dual metric (16), their dual forms are given by

$$
\frac{(s^d_{ab}/2)\delta\sigma^d_{ab} + Y^d\delta\Phi^d}{2\pi^2} = \frac{r^2 f}{2\pi^2} \Phi(2 + \frac{r N'}{N}) \delta \left(\frac{1}{r^2}\right) + \frac{f}{\pi r^2} \left(1 + \frac{r N'}{N}\right) \delta(\Phi r^2)
$$

$$
= \frac{rf}{2\pi^2} \Phi(2 + \frac{r N'}{N}) \delta \left(2\pi \frac{1}{r}\right) + \frac{f}{\pi r^2} \left(1 + \frac{r N'}{N}\right) \delta(\Phi r^2)
$$

$$
= P^d\delta(\text{perimeter})^d + P_{\Phi}^d\delta\Phi^d.
$$

(28)

In eqs.(27) and (28), there seems to be no conspicuous relationship between the (dilaton) pressure ($P_{\Phi}$) $P$ and the dual one ($P_{\Phi}^d$) $P^d$. However, after simple calculation, it can be easily shown that the work terms, which are the surface pressure density times the perimeter and the dilaton pressure density times the dilaton field, are not separately invariant under duality, but are invariant only in the above combination,
Thus, from the point of view of the duality, the dilaton pressure density $P_\Phi$ can be indeed interpreted as a ‘pressure’ one.

As a result, from eqs. (18), (23), (26) and (29), the quasilocal black hole thermodynamic first law (13) is invariant under the dual transformation (15). Specially, the dual invariance of the red-shifted black hole temperature and the black hole entropy tells us that the zeroth and the second laws of the black hole thermodynamics are not spoiled by the dual transformation. Thus, we can say that though the thermodynamic variables are closely related to geometry of a considered spacetime, if two different geometric spacetimes are related by the dual transformation (15), they are described by the same black hole thermodynamics.

Before ending this section, it seems to be appropriate to comment about the physical meaning of the dual invariance of the black hole entropy. This is the task of the next subsection.

### D. Duality on the Path Integral Formulation

Following Brown’s path integral interpretation of the black hole entropy [15], the entropy is derived from a sum over boundary states, which are labeled by $E$, $J_a$ and $Q^a$, and parameterized by the boundary metric $\sigma$ and the dilaton field $\Phi$. Thus, from this viewpoint, the dual invariance of the black hole entropy is the invariance according to ‘coordinate’ interchange, $J_a \leftrightarrow Q^a$, i.e., $(E, J_a, Q^a) \rightarrow (E^d, J^d_a, Q^a_d) = (E, Q^a, J_a)$, and reparametrization, $(\sigma, \Phi) \rightarrow (\sigma^d, \Phi^d)$ in phase space of the boundary states. Now, we represent this comment with the language of the path integral formulation of [15].

Consider a stationary, non-extreme black hole which is embedded within a finite cavity, and its event horizon is the bifurcate Killing horizon satisfying the relation

$$\beta|_B = \beta_\omega^a|_B = 0, \quad (30)$$
where ‘B’ denotes the bifurcate Killing horizon. It must be noted that the density of states \( \nu[\mathcal{E}, \mathcal{J}, Q; \sigma, \Phi] \) in eq.(8) is a functional of the extrinsic variables and the fields on the outer boundary not the inner boundary, i.e., the Killing horizon. In addition, only the data on the outer boundary are specified. Thus the density of outer boundary states is given by the path integral

\[
\nu[\mathcal{E}, \mathcal{J}, Q; \sigma, \Phi] = \int \mathcal{D}[\mathcal{E}_B, \mathcal{J}_B, Q_B, \sigma_B, \Phi_B] \mu[\sigma_B, \Phi_B] \nu[\mathcal{E}, \mathcal{J}, Q; \sigma, \Phi|\mathcal{E}_B, \mathcal{J}_B, Q_B; \sigma_B, \Phi_B],
\]

(31)

where \( \mu[\sigma_B, \Phi_B] \) is the measure for the parameters of the inner boundary states. In the steepest descents approximation, eq.(31) yields the condition that variation of the Euclidean microcanonical action, with by the additional term \( \ln \mu \) with respect to \( \mathcal{E}^B, \mathcal{J}^a_B \) and \( Q^n_B \) must vanish

\[
\delta I_{\text{micro}}^{E,B} \bigg|_{B} = - \int_0^{2\pi} d\phi \beta \left( \delta \mathcal{E} - \omega^a \delta \mathcal{J}_a + V_a \delta Q^a + \left( s^{ab} / 2 \right) \delta \sigma_{ab} + \mathcal{Y} \delta \Phi \right) \bigg|_{B} 
+ \left( \frac{\delta \ln \mu}{\delta \sigma_{ab}} \delta \sigma_{ab} + \frac{\delta \ln \mu}{\delta \Phi} \delta \Phi \right)_{B} = 0.
\]

(32)

The first two terms of the integrands in eq.(32) vanish due to the relation (30). If we again require that the antisymmetric field \( B_{ab} \) be regular at the bifurcate Killing horizon, \( B_{ab} |_{B} = 0 \), the third term of the integrands vanishes. Finally, the last two terms should be eliminated by variation terms of the measure factor. Then we obtain the relations

\[
\frac{\delta \ln \mu}{\delta \sigma_{ab}} \bigg|_{B} = 2\pi \beta (s^{ab} / 2) \bigg|_{B} = \sqrt{\sigma} \Phi a^{ab} n^c \partial_c \beta \bigg|_{B} \approx 2\pi \sqrt{\sigma} \Phi a^{ab} \bigg|_{B},
\]

\[
\frac{\delta \ln \mu}{\delta \Phi} \bigg|_{B} = 2\pi \beta \mathcal{Y} \bigg|_{B} = 2\sqrt{\sigma} n^c \partial_c \beta \bigg|_{B} \approx 4\pi \sqrt{\sigma} \bigg|_{B},
\]

(33)

where we used eqs.(5) and (18). From these relations (33), we can read off the measure factor as

\[
\ln \mu[\sigma_B, \Phi_B] \approx 2 \int_0^{2\pi} d\phi \sqrt{\sigma} \Phi \bigg|_{B}.
\]

(34)

The black hole entropy is then

\[
S_{BH} \approx \ln \nu[\mathcal{E}, \mathcal{J}, Q; \sigma, \Phi] \approx \ln \mu[\sigma_B, \Phi_B] \approx 2 \int_0^{2\pi} d\phi \sqrt{\sigma} \Phi \bigg|_{B}.
\]

(35)
In the above equation, we can see that the perimeter law (21) is well recovered.

Now, consider the dual transformation of the black hole entropy. Since eq.(30) and the regularity condition of the antisymmetric field are still satisfied in the dual transformation (15) and the first three terms of the integrands in eq.(32) is dual invariant (see eq.(26)), the relations for the measure factor (33) are again recovered in the dual transformation. Thus the dual measure factor becomes

$$\ln \mu^d[\sigma^d_B, \Phi^d_B] \approx 2 \int_0^{2\pi} d\phi \sqrt{\sigma^d} \Phi^d \bigg|_B.$$  (36)

Finally, it can be shown that the dual entropy is still satisfied the perimeter law and equal to the original entropy for the case of the minisuperspace model (14)

$$S^d_{BH} \approx \ln \mu^d[\sigma^d_B, \Phi^d_B] \approx 2 \int_0^{2\pi} d\phi \sqrt{\sigma^d} \Phi^d \bigg|_B \approx S_{BH}.$$  (37)

As a result, from the path integral viewpoint, the dual invariance of the black hole entropy means that the measure factor of the inner boundary states for the parameters $\sigma_B, \Phi_B$ is invariant under just reparametrization, $(\sigma_B, \Phi_B) \rightarrow (\sigma^d_B, \Phi^d_B)$. Actually, the reparametrization invariance is a common event. Thus, according to the above interpretation, the dual invariance of black hole entropy seems to be quite natural.

IV. SUMMARY AND DISCUSSIONS

We have considered the duality of the quasilocal black hole thermodynamics, specifically the first law of the quasilocal black hole thermodynamics (13). Motivation of this work is originated from the fact that the T-dual transformation (15) may change the asymptotic behavior of a spacetime. Thus the assumption of the asymptotic flatness is unavailable in the context of the dual transformation. To avoid this difficulty, it is appropriate for the study of the duality of the black hole thermodynamics to introduce the quasilocal boundary.

It has been shown that the black hole entropy is obtained in terms of the extrinsic curvature in a form similar to that in [21], in which the authors used the conformally
transformed Einstein-Hilbert action. This expression satisfies a version of the area law that is relevant to the three-dimensional, non-minimally coupled gravity. It has been also shown that the Hawking temperature and the black hole entropy are dual invariant, respectively. In addition, the Tolman temperature, which is redshifted temperature from the horizon to the spatial boundary is also dual invariant.

On the other hand, the duality of the quasilocal surface energy density $\mathcal{E}$, momentum density $\mathcal{J}_a$, and axion charge density $Q^a$, which are the quasilocal thermodynamic extensive variables in the thermodynamic first law, is equal to the result of Ref.[8]; the energy density is unchanged, while the momentum density and charge density are interchanged each other under the dual transformation.

Since we have considered a quasilocal boundary, the work terms have appeared in the first law (13), which are given by the quasilocal pressure densities times the variations of corresponding conjugate variables; in our case, the surface pressure density times the variation of perimeter and the dilaton pressure density times the variation of dilaton field. It has been shown that the pressure densities do not have any well-defined dual behavior, however, the combination of the work terms is still dual invariant.

The above observations turn out that the first law of the quasilocal black hole thermodynamics is dual invariant. In addition, according to the fact that the Tolman temperature and black hole entropy are dual invariant, the zeroth and the second laws are also dual invariant. As a result, though the thermodynamic variables are closely related to geometry of a spacetime, if two different geometric spacetimes are related by the dual transformation, they are described by the same black hole thermodynamics. Furthermore, this dual invariance is irrelevant of the asymptotic behavior of the spacetimes, e.g., one is asymptotically flat and the other asymptotically anti-de Sitter.

On the other hand, we have also considered the dual invariance of the black hole entropy in terms of the Euclidean path integral formulation. Through this analysis, one can deeply understand a physical meaning of the dual invariance as follows; the dual invariance of the black hole entropy is just the invariance of the ‘coordinate’ interchange, $\mathcal{J}_a \leftrightarrow Q^a$, i.e.,
\((\mathcal{E}, \mathcal{J}_a, Q^a) \rightarrow (\mathcal{E}^d, \mathcal{J}_a^d, Q^a_d) = (\mathcal{E}, Q^a, \mathcal{J}_a)\) and the reparametrization, \((\sigma, \Phi) \rightarrow (\sigma^d, \Phi^d)\), in the phase space of the boundary states. In short, the measure factor of the inner boundary states for the parameters \(\sigma_B, \Phi_B\) is invariant under reparametrization. Thus, in the context of the path integral formalism, the dual invariance of the black hole entropy is quite natural.

For a noncompact geometry, the quasilocal quantities in eqs. (3) and (5) are not well defined in the limit \(r \rightarrow \infty\), i.e., divergent. However, the unexpected divergence can be naturally eliminated by introducing a reference background spacetime with an action \(I_0\) and being defined a physical action as \(I_p \equiv I - I_0\) [10, 11]. Then, the physical quasilocal thermodynamic variables are given as subtracting those of the reference background from the prescribed ones. In the case, the duality of the physical asymptotic thermodynamic variables as well as quasilocal ones is the same figure with that of our consideration in this paper. Thus it can be confirmed that the black hole thermodynamics in asymptotic region on the noncompact geometry is still dual invariant [18].

Recently, constant curvature black holes, \(n\)-dimensional generalizations of the BTZ black hole, and its thermodynamics has been carefully studied [25, 26]. Particularly, the black hole has an unusual property in the black hole thermodynamics such as the black hole entropy is not given by the outer horizon instead inner horizon [25] or depends on the size of the quasilocal surface [26]. As mentioned above, the duality transformation drastically changes the geometry of a spacetime. Thus, one may inquire as to how the entropy behaves under such a transformation. It would be interesting to study the duality of black hole thermodynamics for \(n\)-dimensional constant curvature black holes.

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