The microscopic spectrum of the QCD Dirac operator with finite quark masses

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We compute the microscopic spectrum of the QCD Dirac operator in the presence of dynamical fermions in the framework of random-matrix theory for the chiral Gaussian unitary ensemble. We obtain results for the microscopic spectral correlators, the microscopic spectral density, and the distribution of the smallest eigenvalue for an arbitrary number of flavors, arbitrary quark masses, and arbitrary topological charge.

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I. INTRODUCTION

The low-lying eigenvalues of the QCD Dirac operator are of great importance for the understanding of the spontaneous breaking of chiral symmetry in QCD. In Euclidean space, the Dirac operator reads $i\slashed{D} = i\slashed{\partial} + g(\lambda^\alpha/2)A^\alpha$, where $g$ is the coupling constant, $\lambda^\alpha$ are the generators of SU($N$)-color, and $A^\alpha_a$ are the gauge fields. The corresponding eigenvalue equation is $i\slashed{D}\gamma_5 \psi_i = \lambda_i \psi_i$ with real eigenvalues $\lambda_i$. Because the Dirac operator anticommutes with $\gamma_5$, the eigenvalues either occur in pairs $\pm\lambda_i$ or are zero. Let $\rho(\lambda) = \langle \sum_i \delta(\lambda - \lambda_i) \rangle$ be the eigenvalue density of the Dirac operator averaged over gauge field configurations. The Banks-Casher formula, $\langle \bar{\psi}\psi \rangle = \pi\rho(0)/V$ [1], relates the order parameter of chiral symmetry breaking, $\langle \bar{\psi}\psi \rangle$, to the density of eigenvalues of the Dirac operator near zero virtuality. Here, it is essential that the thermodynamic limit be taken before the chiral limit. Note also that $\rho(0)$ is normalized to the space-time volume $V$. In the real world chiral symmetry is spontaneously broken and $\langle \bar{\psi}\psi \rangle \neq 0$. This implies that the low-lying Dirac eigenvalues must be spaced like $\sim 1/V$. Therefore, it is natural to magnify this region of the spectrum by a factor of $V$. This was first suggested by Shuryak and Verbaarschot [2] who introduced the so-called microscopic spectral density defined by

$$
\rho_\mu(z) = \lim_{V \to \infty} \frac{1}{V^2} \rho\left( \frac{z}{\Sigma} \right),
$$

where $\Sigma$ is the absolute value of the chiral condensate. Based on an analysis of sum rules derived by Leutwyler and Smilga [3] for the inverse powers of the Dirac eigenvalues in a finite volume, Shuryak and Verbaarschot conjectured that the microscopic spectral density should be a universal quantity which depends only on global symmetries, the number of flavors $N_f$, and the topological charge $\nu$. What is the meaning of universality in this statement? In principle, one could compute $\rho_\mu$ directly from full QCD. In practice, of course, the complexity of the QCD Lagrangian makes this impossible. However, if $\rho_\mu$ is universal, one can try to compute it in another theory which is simpler than QCD but has the same symmetries. The simplest theory in which such a calculation is possible is random-matrix theory (RMT). Whether the results obtained in RMT correctly describe the microscopic spectrum of full QCD is a question that can only be answered by empirical evidence, most importantly by large-scale lattice QCD simulations.

Let us briefly recall how a chiral random-matrix model is constructed. In a chiral basis in Euclidean space, the Dirac operator is replaced by a random matrix $W$ with suitable symmetry properties,

$$
\begin{pmatrix}
\text{im} & i\slashed{D}
\end{pmatrix}
\begin{pmatrix}
i\slashed{D}^\dagger & \text{im}
\end{pmatrix}
\rightarrow
\begin{pmatrix}
\text{im} & W
\end{pmatrix}
\begin{pmatrix}
W^\dagger & \text{im}
\end{pmatrix},
$$

where we have also added a quark mass $m$. If $W$ has $N + \nu$ rows and $N$ columns, then the Dirac matrix in (2) (without the mass term) has $N$ positive and $N$ negative eigenvalues, respectively, and $\nu$ zero modes (we assume $\nu \geq 0$ and $\nu \ll N$). We can identify $\nu$ with the topological charge and $2N + \nu \approx 2N$ with the volume $V$. In full QCD with $N_f$ flavors, the weight function used in averaging over gauge field configurations contains the gluonic action in the form $\exp(-S_\text{G})$ and $N_f$ fermion determinants. In the random-matrix model, the gluonic part of the weight function is replaced by a convenient distribution of the random matrix $W$, usually a Gaussian distribution of the individual matrix elements (which are uncorrelated). Since the fermion determinants can be expressed as products over the Dirac eigenvalues they can also be taken into account in the random-matrix model. The symmetries of $W$ are determined
by the anti-unitary symmetries of the Dirac operator. They were classified by Verbaarschot in Ref. [4]. Depending on
the number of colors and the representation of the fermions the matrix elements of $W$ are real, complex, or quaternion
real. The corresponding random-matrix ensembles are called chiral Gaussian orthogonal (chGOE), unitary (chGUE),
and symplectic (chGSE) ensemble, respectively. The microscopic spectral density has been computed analytically in
all three ensembles for all $N_f$ and $\nu$ in the chiral limit [5–7].

Evidence in support of the conjecture that $\rho_s$ is a universal function has been accumulated in a number of recent
studies which we list here. The moments of $\rho_s$ generate the Leutwyler-Smilga sum rules [5]. The result for $\rho_s$
is insensitive to the probability distribution of the random matrix [8,9]. Lattice data for the valence quark mass
dependence of the chiral condensate could be understood using the analytical expression for $\rho_s$ [10,11]. The functional
form of $\rho_s$ does not change at finite temperature [12]. The analytical result for $\rho_s$ is found in the Hofstadter
model for universal conductance fluctuations [13]. For an instanton liquid, $\rho_s$ shows quite good agreement with the random-
matrix result [14]. Recently, a high-statistics lattice calculation directly demonstrated agreement between the random-
matrix results and lattice data, not only for $\rho_s$ but also for the distribution of the smallest eigenvalue and for the
microscopic spectral two-point correlator [15]. (This study was done for gauge group SU(2) with staggered fermions
in the quenched approximation.) We believe that the universality of $\rho_s$ is now firmly established.

One of the most interesting and most important problems of present-day lattice simulations is the inclusion of
dynamical fermions. This is essential for a realistic description of hadronic properties. Unfortunately, it is computa-
tionally expensive to include the fermion determinants in a Monte-Carlo simulation. Moreover, the computational
increase with lower quark mass, thus it is extremely difficult to take the chiral limit on the lattice. Any analytical
insight into the distribution of the Dirac eigenvalues in the presence of massive dynamical quarks will therefore be
helpful. It is the purpose of this work to extend the random-matrix results obtained in the chiral limit to the general
case of massive quarks to be able to compare with actual lattice calculations. This issue was first discussed in Ref. [16].
The most interesting case of QCD with 3 colors and fermions in the fundamental representation corresponds to the
chGUE of RMT on which we concentrate here. In what range of the quark mass can we expect the random-matrix
results to be in agreement with full QCD? Gasser and Leutwyler [17] have shown that in the range $1/A < L < 1/m_x$
the dependence of the finite-volume QCD partition function on the quark mass $m$ is determined solely by global
symmetries ($A$ is a typical hadronic scale, $L$ is the linear size of the Euclidean box, and $m_x \sim \sqrt{mA}$ is the pion mass).
The partition function was computed for equal quark masses using an effective Lagrangian in Ref. [3] and for the
general case of different quark masses in the framework of chiral RMT in Ref. [18]. The condition $L < 1/m_x$ sets an
upper limit of $1/\sqrt{V \Lambda^2}$ on the quark mass. A lower limit of $1/(V \Lambda^3)$ on the quark mass is set by the requirement
that the chiral condensate be non-zero in a finite volume. To obtain non-trivial results for the dependence of the various
quantities we compute on $m$, the quark mass has to be rescaled by the same factor $1/(V \Sigma)$ as the eigenvalues,
cf. Eq. (1). For large quark masses, our results reduce to those obtained in the quenched approximation.

This paper is organized as follows. In Sec. II we discuss the random-matrix model and construct the orthogonal
polynomials and the partition function which are needed in the course of the calculation. Physical observables such as
the microscopic spectral density and the distribution of the smallest eigenvalue are computed in Sec. III. Section IV
is a short summary.

After completion of this work we learned that very recently a similar calculation was performed simultaneously and
independently by P.H. Damgaard and S. Nishigaki [19]. In this work, the authors also give a universality proof for
more general distributions of the random matrix and construct spectral sum rules for the massive Dirac operator (see
also Ref. [20]) but do not address the distribution of the smallest eigenvalue. Wherever the two manuscripts overlap
the results are identical.

II. RANDOM-MATRIX THEORY FOR FINITE QUARK MASSES

In Sec. II A, we formulate the chiral random-matrix model by introducing the probability density function and the
partition function in the presence of finite quark masses. In Sec. II B, we construct the orthogonal polynomials of
the model. An alternative construction is presented in Sec. II C which is based on a certain duality between models in
different matrix spaces.

A. Formulation of the model

The statistical properties of the random-matrix model for the massive Dirac operator with $N_f$ flavors and dynamical
quark masses $M_f$ ($f = 1, \ldots, N_f$) which was introduced above are given by the probability density function
\( P^{(N_f)}_N(W, M) = \frac{1}{Z^{(N_f)}_N(M)} \prod_{f=1}^{N_f} \det(W W^\dagger + M_f^2) \exp(-N \Sigma^2 \text{tr} W W^\dagger) \) (3)

with \( M = (M_1, \ldots, M_{N_f}) \). Here, \( W \) is a complex matrix of dimension \( N \). Apart from a Gaussian, this distribution contains \( N_f \) fermionic determinants. Eventually, we are mainly interested in the microscopic limit. Hence, it suffices to take into account a Gaussian in the total distribution (3) since various authors [8, 9] have given proofs of universality in related problems. Note that even for nonzero topological charge \( \nu \) it is sufficient to consider only square matrices \( W \) in the calculation because of the duality between topology and flavor [5, 4]. A nonzero \( \nu \) is obtained by introducing \( \nu \) additional massless flavors at the end of the calculation. The normalization constant as discussed in the Introduction.

With the help of the Christoffel-Darboux formula the kernel can be expressed as

\[
\tilde{Z}^{(N_f)}_N(M) = \int d[W] \prod_{f=1}^{N_f} \det(W W^\dagger + M_f^2) \exp(-N \Sigma^2 \text{tr} W W^\dagger)
\]

(4)
depends on the quark masses and plays the role of a partition function. Since both of the above expressions depend on the matrices \( W \) only through invariant functions of \( W W^\dagger \), they can be reduced to integrals over radial coordinates. We write \( W = UAV \) with \( U \in U(N) \), \( V \in U(N)/U(N)^{(1)} \), and \( \Lambda = \text{diag}(\lambda_1, \ldots, \lambda_N) \), where the radial coordinates \( \lambda_i \) are restricted to the positive real axis. Thus, we have \( WW^\dagger = U \Lambda^2 U^\dagger \) suggesting the introduction of new variables \( x_i = \lambda_i^2 \). The measure transforms as \( d[W] = \Delta_N^2(X) d[x] d\mu(U) d\mu(V) \), where \( X = \text{diag}(x_1, \ldots, x_N) \), \( \Delta_N(X) = \prod_{x,y} (x - y) \) is the Vandermonde determinant, and \( d\mu \) is the Haar measure. After integrating over the unitary groups, we obtain

\[
P^{(N_f)}_N(X, M) = \frac{1}{Z^{(N_f)}_N(M)} \Delta_N^2(X) \prod_{i=1}^N w^{(N_f)}(x_i, M)
\]

(5)
for the probability density and

\[
Z^{(N_f)}_N(M) = \int d[X] \Delta_N^2(X) \prod_{i=1}^N w^{(N_f)}(x_i, M)
\]

(6)
for the partition function \( Z^{(N_f)}_N(M) \) which differs from \( \tilde{Z}^{(N_f)}_N(M) \) by the group volumes. The expression

\[
w^{(N_f)}(x, M) = \exp(-N \Sigma^2 x) \prod_{f=1}^{N_f} (x + M_f^2)
\]

(7)
will be referred to as the weight function.

Because of a main result of RMT [21], all spectral correlation functions \( R^{(N_f)}_k(x_1, \ldots, x_k, M) \) of our model can readily be written in terms of the polynomials \( p^{(N_f)}_n(x, M) \) which are orthogonal with respect to the weight function \( w^{(N_f)}(x, M) \),

\[
\int_0^\infty \, dx \, w^{(N_f)}(x, M) p^{(N_f)}_n(x, M) p^{(N_f)}_m(x, M) = \frac{Z^{(N_f)}_{n+1}(M)}{(n + 1) Z^{(N_f)}_n(M)} \delta_{nm}.
\]

(8)
The correlation functions are given as the determinant

\[
R^{(N_f)}_k(x_1, \ldots, x_k, M) = \det \left[ K^{(N_f)}_N(x_i, x_j, M) \right]_{i,j=1,\ldots,k}
\]

(9)
with a kernel given by

\[
K^{(N_f)}_N(x, y, M) = \sqrt{w^{(N_f)}(x, M) w^{(N_f)}(y, M)} \sum_{n=0}^{N-1} \frac{(n + 1) Z^{(N_f)}_n(M)}{Z^{(N_f)}_{n+1}(M)} p^{(N_f)}_n(x, M) p^{(N_f)}_n(y, M)
\]

(10)

With the help of the Christoffel-Darboux formula the kernel can be expressed as

\[
K^{(N_f)}_N(x, y, M) = N \frac{Z^{(N_f)}_{N-1}}{Z^{(N_f)}_N} \sqrt{w^{(N_f)}(x, M) w^{(N_f)}(y, M)}
\]

\[
\times \frac{p^{(N_f)}_N(x, M) p^{(N_f)}_{N-1}(y, M) - p^{(N_f)}_{N-1}(x, M) p^{(N_f)}_N(y, M)}{x - y},
\]

(11)
provided that the coefficient of the power \( x^n \) in \( p^{(N_f)}_n(x, M) \) is unity. Eventually, we will take the microscopic limit as discussed in the Introduction.
B. Construction of orthogonal polynomials

The polynomials $p_{n}^{(N_f)}(x, M)$ can be constructed by applying a standard formula [22] in the theory of orthogonal polynomials. In our case, it reads

$$p_{n}^{(N_f)}(x, M) = \frac{1}{Z_{n}^{(N_f)}(M)} \int d[X] \Delta_{n}^{2}(X) \prod_{i=1}^{n} w^{(N_f)}(x_i, M)(x - x_i).$$

(12)

This is an integral over $n$ variables $x_i$ ($i = 1, \ldots, n$). We notice that $n$ replaces the dimension $N$ of the matrices $W$ in the partition function $Z_{n}^{(N_f)}(M)$ and in the Vandermonde determinant $\Delta_{n}(X)$. Moreover, by construction, the coefficient in front of $x^n$ is unity.

Fortunately, there is a further result, the Christoffel formula [22], which allows us to immediately write down the polynomials explicitly: The weight function $w^{(N_f)}(x, M)$ is a product of the weight function $w^{(0)}(x)$ and a polynomial whose zeros are the negative squared quark masses $-M_f^2$ and which is nonnegative on the real axis. Therefore, the polynomials $p_{n}^{(N_f)}(x, M)$ can be written in terms of the polynomials which are orthogonal with respect to the weight $w^{(0)}(x) = \exp(-N\Sigma^2 x)$. These are precisely the Laguerre polynomials $L_{n}^{(0)}(N\Sigma^2 x)$. The application of the Christoffel formula then yields

$$p_{n}^{(N_f)}(x, M) = \frac{1}{C_{n}^{(N_f)}(M)} \prod_{j=1}^{N_f} (x + M_j^2)^{n + N_f} \frac{(n + N_f)!}{(N\Sigma^2)^{n + N_f}}$$

$$\times \begin{vmatrix} L_{n}^{(0)}(N\Sigma^2 x) & L_{n+1}^{(0)}(N\Sigma^2 x) & \cdots & L_{n+N_f}^{(0)}(N\Sigma^2 x) \\ L_{n}^{(0)}(-N\Sigma^2 M_{f_1}^2) & L_{n+1}^{(0)}(-N\Sigma^2 M_{f_1}^2) & \cdots & L_{n+N_f}^{(0)}(-N\Sigma^2 M_{f_1}^2) \\ \vdots & \vdots & \ddots & \vdots \\ L_{n}^{(0)}(-N\Sigma^2 M_{N_f}^2) & L_{n+1}^{(0)}(-N\Sigma^2 M_{N_f}^2) & \cdots & L_{n+N_f}^{(0)}(-N\Sigma^2 M_{N_f}^2) \end{vmatrix}$$

(13)

with a function $C_{n}^{(N_f)}(M)$ yet to be determined. In Sec. II C, we will present a very direct and highly convenient computation of the polynomials (13) including all normalization factors. Here we proceed as follows: Since the polynomial $p_{n}^{(N_f)}(x, M)$ has to be invariant under permutations of the masses $M_f$ and must have a finite limit when all quark masses are degenerate, a reasonable guess for the normalization function is

$$C_{n}^{(N_f)}(M) = \text{det} \left[ L_{n+f-1}^{(0)}(-N\Sigma^2 M_{f}^2) \right]_{f,f'=1,\ldots,N_f},$$

(14)

see Sec. II C below, in particular Eq. (24). With this ansatz, the partition function $Z_{n}^{(N_f)}(M)$ is obtained by studying the case of $N_f + 1$ flavors. On the one hand, Eqs. (6) and (12) yield $p_{n}^{(N_f)}(-M_{N_f+1}^2, M) = -Z_{n}^{(N_f+1)}(M, M_{N_f+1})/Z_{n}^{(N_f)}(M)$, which, on the other hand, has to be equal to the right-hand side of Eq. (13) for $x = -M_{N_f+1}^2$. Thus, we arrive at the recursion relation

$$\frac{Z_{n}^{(N_f+1)}(M, M_{N_f+1})}{C_{n}^{(N_f+1)}(M, M_{N_f+1})} = \frac{Z_{n}^{(N_f)}(M)}{C_{n}^{(N_f)}(M)} \frac{(n + N_f)!}{(N\Sigma^2)^{n + N_f}} \frac{1}{\prod_{j=1}^{N_f} (M_j^2 - M_{N_f+1}^2)}$$

(15)

which implies

$$Z_{n}^{(N_f)}(M) = \frac{N_f}{\Delta_{N_f}(-M^2)} \prod_{j=1}^{N_f} \frac{(n + f - 1)!}{(N\Sigma^2)^{n + f - 1}} C_{n}^{(N_f)}(M).$$

(16)

The correctness of these results can be verified using the orthogonality relation (8). In the microscopic limit, the partition function agrees with the result obtained in Ref. [18], see Sec. III, Eq. (27). (Note that in Ref. [18], the result is expressed in terms of derivatives of modified Bessel functions. Using the recurrence relations for Bessel functions [23] and the properties of the determinant, one can show that the results are identical.)
C. A duality of matrix ensembles

In this section, we present an alternative derivation of the orthogonal polynomials \( p_n^{(N_f)}(x, M) \) and of the partition function \( Z_n^{(N_f)}(M) \). It employs a certain kind of duality between matrix ensembles and is therefore of conceptual interest. On the practical side, this method advantageously yields all normalizations, particularly the function \( C_n^{(N_f)}(M) \) and the partition function \( Z_n^{(N_f)}(M) \), in a very direct and convenient way. We consider the integral

\[
X_N^{(K)}(a) = \int d[W] \exp(-\alpha tr WW^\dagger) \prod_{j=1}^K \det \begin{bmatrix} -ia & W \\ W^\dagger & -ia \end{bmatrix}
\]

(17)
depending on \( K \) parameters \( a_f \) (\( f = 1, \ldots, K \)). For \( \alpha = N\Sigma^2 \), \( K = N_f \), and \( a_f = M_f \), the integral (17) is just the partition function (4). For \( N = n, K = N_f + 1, a_f = M_f \) (\( f = 1, \ldots, N_f \)), and \( a_{N_f+1}^2 = -x \), the integral (17) is, up to the group volumes, equal to \( Z_n^{(N_f)}(M)p_n^{(N_f)}(x, M) \) as defined in Eq. (12). Thus, the function \( X_N^{(K)}(a) \) yields all desired quantities. We define \( K \) 2N-component vectors \( \zeta_f \) (\( f = 1, \ldots, K \)) with complex Grassmannian entries \( \zeta_i \) (\( i = 1, \ldots, 2N \)) as well as the 2NK-component combination \( \zeta = (\zeta_1, \ldots, \zeta_K)^T \). The determinants in the integrand of Eq. (17) are represented as the integral

\[
\prod_{f=1}^K \det \begin{bmatrix} -ia_f & W \\ W^\dagger & -ia_f \end{bmatrix} = (2\pi)^{2NK} \int d[\zeta] \exp \left( \zeta^\dagger \left( \mathbb{1}_K \otimes \begin{bmatrix} 0 & W \\ W^\dagger & 0 \end{bmatrix} - ia \otimes \mathbb{1}_N \right) \zeta \right)
\]

(18)

with \( a = \text{diag}(a_1, \ldots, a_K) \). In this form, the Gaussian average over the matrices \( W \) can easily be performed. We arrive at a four-fermion interaction which is then decoupled by a Hubbard-Stratonovich transformation. In Ref. [24], a very similar route was taken. In this study, however, a problem involving fermionic and bosonic determinants was solved which led to the introduction of supervectors of commuting and anticommuting variables. In the present case, the vectors \( \zeta \) have only anticommuting entries. Fortunately, this difference has little influence on the structure of the results. Instead of the supermatrices which were used in Ref. [24] for the Hubbard-Stratonovich transformation, we have to employ ordinary matrices here. We arrive at

\[
X_N^{(K)}(a) = \left( \frac{\alpha}{\pi} \right)^{K^2} \int d[\sigma] \exp(-\alpha tr (\sigma - a)(\sigma^\dagger - a)) \det^N \sigma \sigma^\dagger,
\]

(19)

where \( \sigma \) is an ordinary complex \( K \times K \) matrix without further symmetries. Thus, we have mapped the original model of Sec. II A in the form (17) in the space of the \( N \times N \) matrices \( W \) onto a dual model in the space of the \( K \times K \) matrices \( \sigma \). This is highly convenient because, eventually, we wish to take the limit \( N \to \infty \). Remarkably, these dual models differ with respect to rotation invariance. The model of Sec. II A is invariant under rotations in the sense that the integrand of the partition functions in Eqs. (4) and (6) depends only on the radial coordinates of the matrices \( W \), i.e., on the variables \( x_i \). This is not true for the model (19). The presence of the matrices \( a \) breaks this kind of rotation invariance. In Refs. [25] and [24], a procedure was developed to deal with models precisely of this kind. We introduce radial coordinates by \( \sigma = us\bar{v} \) with \( u \in U(K) \), \( \bar{v} \in U(K)/U(K) \), and \( s = \text{diag}(s_1, \ldots, s_K) \). Again, the radial coordinates \( s_f \) are restricted to the positive real axis. The measure transforms as \( d[\sigma] = \Delta_K(s^2) \prod_{f=1}^K s_f ds_f d\mu(u)d\mu(\bar{v}) \). The integral over the unitary groups is now nontrivial. The solution can be found in Ref. [25],

\[
\left( \frac{\alpha}{\pi} \right)^{K^2} \int d\mu(u) \int d\mu(\bar{v}) \exp(-\alpha (us\bar{v} - a)(\bar{v}^\dagger su^\dagger - a)) = \frac{1}{K!} \Delta_K(s^2) \Delta_K(a^2) \cdot
\]

(20)
The entries of the determinant are given by

\[
\gamma(s_f, a_{f'}) = 2\alpha \exp(-\alpha(s_f^2 + a_{f'}^2)) I_0(2\alpha s_fa_{f'}) ,
\]

(21)

where \( I_0 \) is the modified Bessel function of the first kind and zeroth order. Collecting everything, we obtain

\[
X_N^{(K)}(a) = \frac{1}{K!} \frac{1}{\Delta_K(a^2)} \int_0^\infty ds_f s_f^{2N+1} \Delta_K(s^2) \det \left[ \gamma(s_f, a_{f'}) \right]_{f,f'=1,...,K} .
\]

(22)

This integral can be solved in a straightforward way. We write \( \Delta_K(s^2) = \det[s_f^{2(\alpha'f'-1)}]_{f,f'=1,...,K} \) and integrate this determinant row by row. Using the representation
\[
2\alpha \int_{0}^{\infty} dw \, w^{2n+2m+1} \exp \left( -\alpha(z^2 + w^2) \right) I_0(2\alpha w z) = \frac{(n+m)!}{\alpha^{n+m}} f_{n+m}^{(0)}(-\alpha z^2)
\]  

for the Laguerre polynomials, we find

\[
X_N^{[K]}(\alpha) = \frac{1}{\Delta_K(a^2)} \prod_{f=0}^{K-1} \frac{(N+f)!}{\alpha^{N+f}} \det \left[ L_{N+f-1}^{(0)}(-\alpha a_f^2) \right]_{f,f'=1,...,K}.
\]

This result immediately yields the orthogonal polynomials \( p_n^{(N_f)}(x, M) \) and the partition function \( Z_N^{(N_f)}(M) \).

### III. CALCULATION OF OBSERVABLES

The microscopic spectral correlators, in particular the microscopic spectral density, are computed in Sec. III A. In Sec. III B, we derive the distribution of the smallest eigenvalue.

#### A. Microscopic spectral correlators and density

To calculate the spectral correlations, we insert the polynomials \( p_n^{(N_f)}(x, M) \) in Eq. (11) for the kernel \( K_N^{(N_f)}(x, y, M) \). The microscopic limit is obtained according to Eq. (1) by rescaling energies and quark masses by \( 2N \Sigma \). We define \( \lambda = 2N \Sigma \sqrt{2} \) and \( m_f = 2N \Sigma \). In the limit \( N \to \infty \), we want both \( \lambda \) and \( m_f \) to be of order unity. The quenched approximation then corresponds to taking \( m_f \to \infty \) (or, of course, setting \( N_f = 0 \)). In the limit \( m_f \to 0 \), our results must reduce to those obtained earlier in the chiral limit \([5]\). After rescaling the parameters one uses the asymptotic relation \( \lim_{N \to \infty} L_N^{(0)}(z/N)/N^\alpha = z^{-\alpha/2} I_{\alpha/2}(2\sqrt{z}) \), where \( I_{\alpha/2} \) denotes the Bessel function of the first kind and order \( \alpha \). To obtain the correct large-\( N \) limit one has to use the recurrence relations for the Laguerre polynomials \([23]\) in Eq. (13). The properties of the determinant can be used to simplify the expression. After summing up one uses

\[
\left( \frac{z^2}{N} \right)^\alpha L_N^{(\alpha)} \left( \frac{z^2}{N} \right) = \frac{N}{N+\alpha} \left( \frac{z^2}{N} \right)^\alpha L_N^{(\alpha)} \left( \frac{z^2}{N} \right)
\]

Note that the first, second, and third term on the right-hand side is of \( O(1) \), \( O(1/N) \), and \( O(1/N^2) \), respectively. Using the properties of the determinant and collecting the terms of \( O(1) \) carefully, one obtains the microscopic limit of (11). We arrive at

\[
K^{(N_f)}(\lambda, \lambda', m) = \lim_{N \to \infty} \frac{2\sqrt{\lambda \lambda'}}{(2N \Sigma)^2} K_N^{(N_f)} \left( (\lambda/2N \Sigma)^2, (\lambda'/2N \Sigma)^2, m/2N \Sigma \right)
\]

\[
= \frac{\sqrt{\lambda \lambda'}}{\lambda^2 - \lambda'^2} B^{(N_f)}(\lambda, m) \sum_{f=0}^{N_f} \tilde{B}_f^{(N_f)}(\lambda', m) - B^{(N_f)}(\lambda', m) \sum_{f=0}^{N_f} \tilde{B}_f^{(N_f)}(\lambda, m),
\]

where the normalization \( C^{(N_f)}(m) \) is the microscopic limit of the normalization \( C_N^{(N_f)}(M) \) of Eq. (14),

\[
C^{(N_f)}(m) = \lim_{N \to \infty} C_N^{(N_f)}(m/2N \Sigma) = \det \left[ m_f^{-1} I_{f-1}(-m_f) \right]_{f,f'=1,...,N_f}.
\]

The determinant \( B^{(N_f)}(\lambda, m) \) in Eq. (26) is given by

\[
B^{(N_f)}(\lambda, m) = \begin{vmatrix}
J_0(\lambda) & I_0(-m_1) & \cdots & I_0(-m_{N_f}) \\
\lambda J_1(\lambda) & m_1 I_1(-m_1) & \cdots & m_{N_f} I_1(-m_{N_f}) \\
\vdots & \vdots & \ddots & \vdots \\
\lambda^{N_f} J_{N_f}(\lambda) & m_1^{N_f} I_{N_f}(-m_1) & \cdots & m_{N_f}^{N_f} I_{N_f}(-m_{N_f})
\end{vmatrix}
\]
The determinant \( \tilde{B}^{(N_f)}(\lambda, m) \) is constructed from \( B^{(N_f)}(\lambda, m) \) by replacing the first column by \((\lambda J_1(\lambda), \ldots, \lambda^{N_f+1}J_{N_f+1}(\lambda))^{T} \) if \( f = 0 \) and by replacing the \((f+1)\)-th column by \((m_f J_1(-m_f), \ldots, m_f^{N_f+1}J_{N_f+1}(-m_f))^{T} \) if \( f = 1, \ldots, N_f \). All \( k \)-point spectral correlations can be obtained from the kernel (26) in analogy with Eq. (9). Most importantly, the microscopic spectral density is obtained by taking the limit \( \lambda' = \lambda = z \) in (26). This yields

\[
\rho_s^{(N_f)}(z, m) = \frac{z}{2} \frac{B^{(N_f)}(z, m) \sum_{f=0}^{N_f} \tilde{D}_f^{(N_f)}(z, m) - D^{(N_f)}(z, m) \sum_{f=0}^{N_f} \tilde{B}_f^{(N_f)}(z, m)}{[C^{(N_f)}(m)]^2 \prod_{f=1}^{N_f} (z^2 + m_f^2)},
\]

where the determinant \( D^{(N_f)}(z, m) \) is constructed from \( B^{(N_f)}(z, m) \) by replacing the first column in (28) by \((z^{-1}J_{-1}(z), \ldots, z^{N_f}J_{N_f+1}(z))^{T}, \tilde{D}_f^{(N_f)}(z, m) \) is constructed from \( D^{(N_f)}(z, m) \) by replacing the \((f+1)\)-th column by \((m_f J_1(-m_f), \ldots, m_f^{N_f+1}J_{N_f+1}(-m_f))^{T} \) if \( f = 1, \ldots, N_f \), and \( \tilde{B}_0^{(N_f)}(z, m) = B^{(N_f)}(z, m) \).

We have verified that our results (26) and (29) reproduce the known results for the chiral limit and the quenched approximation if \( m_f \to 0 \) and \( m_f \to \infty \), respectively. In Refs. [24,26], the correlations of the Dirac operator at finite temperature were studied using the supersymmetry method. These investigations suggest that the expression (26) for the kernel can be simplified further and expressed in form of a single determinant. Indeed, such a more compact form was very recently obtained in Ref. [19]. We write the result in the form

\[
K^{(N_f)}(\lambda, \lambda', m) = \frac{\sqrt{\lambda' \lambda}}{\lambda'^2 - \lambda^2} \frac{1}{C^{(N_f)}(m) \prod_{f=1}^{N_f} \sqrt{(\lambda^2 + m_f^2)(\lambda'^2 + m_f^2)}} \times \begin{vmatrix}
J_0(\lambda) & J_0(\lambda') & I_0(-m_1) & \cdots & I_0(-m_{N_f}) \\
\lambda J_1(\lambda) & \lambda' J_1(\lambda') & m_1 I_1(-m_1) & \cdots & m_N I_1(-m_{N_f}) \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\lambda^{N_f+1} J_{N_f+1}(\lambda) & \lambda^{N_f+1} J_{N_f+1}(\lambda') & m_1^{N_f+1} I_{N_f+1}(-m_1) & \cdots & m_N^{N_f+1} I_{N_f+1}(-m_{N_f})
\end{vmatrix}.
\]

The microscopic spectral density then reads

\[
\rho_s^{(N_f)}(z, m) = -\frac{z}{2} \frac{1}{C^{(N_f)}(m) \prod_{f=1}^{N_f} (z^2 + m_f^2)} \times \begin{vmatrix}
z^{-1}J_{-1}(z) & J_0(z) & I_0(-m_1) & \cdots & I_0(-m_{N_f}) \\
J_0(z) & z J_1(z) & m_1 I_1(-m_1) & \cdots & m_N I_1(-m_{N_f}) \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
z^{N_f} J_{N_f}(z) & z^{N_f+1} J_{N_f+1}(z) & m_1^{N_f+1} I_{N_f+1}(-m_1) & \cdots & m_N^{N_f+1} I_{N_f+1}(-m_{N_f})
\end{vmatrix}.
\]

We have verified by direct calculation that the expressions (26), (30) and (29), (31) are identical, respectively. A number of interesting special cases can be derived from (29) or (31). (For \( N_f = 1 \) and \( \nu = 0 \), \( \rho_s \) was first computed in Ref. [16]. However, like the authors of [19], we fail to agree with the result given in Eq. (80) of Ref. [16] which does not reproduce the quenched result in the limit \( m \to \infty \).) As an example, we give the result for \( \rho_s \) for two degenerate flavors of mass \( m \) and \( \nu = 0 \),

\[
\rho_s^{(2)}(z, m) = \frac{z}{2} \left( J_0^2(z) + J_f^2(z) \right) = \frac{2z}{(z^2 + m^2)^2} \frac{[z J_1(z) I_0(m) + m J_0(z) I_1(m)]^2}{I_0^2(m) - I_1^2(m)}.
\]

This expression is plotted in Fig. 1 compared with the results obtained in the chiral limit and in the quenched approximation. Clearly, the presence of light flavors leads to a suppression of small eigenvalues as expected. We emphasize again that while we have worked with \( \nu = 0 \), the general case \( \nu > 0 \) can easily be obtained by introducing \( \nu \) additional massless flavors. For completeness, we state the result for the microscopic spectral density for general \( \nu \) which is obtained by expanding the determinants in (31),

\[
\rho_s^{(N_f,\nu)}(z, m) = -\frac{z}{2} \frac{1}{\tilde{C}^{(N_f,\nu)}(m) z^{2\nu} \prod_{f=1}^{N_f} (z^2 + m_f^2)} \times \begin{vmatrix}
z^{-1}J_{-1}(z) & J_0(z) & I_0(-m_1) & \cdots & I_0(-m_{N_f}) \\
J_0(z) & z J_1(z) & m_1 I_1(-m_1) & \cdots & m_N I_1(-m_{N_f}) \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
z^{N_f} J_{N_f}(z) & z^{N_f+1} J_{N_f+1}(z) & m_1^{N_f+1} I_{N_f+1}(-m_1) & \cdots & m_N^{N_f+1} I_{N_f+1}(-m_{N_f})
\end{vmatrix}.
\]
constant defined in Eqs. (14) and (27). On the other hand, we have

\[ 27. \]

Hence, the distribution of the smallest eigenvalue (\( \lambda_{\text{min}} \)) and masses by 2

\[ 26. \]

has been computed so far in the literature applies to the quenched approximation where

\[ P \]

identical with that of full QCD under the same conditions as the microscopic spectral density. The only result that

\[ \text{P} \]

eigenvalue. It is therefore of interest to compute an analytical result for the distribution of the smallest eigenvalue,

\[ \text{as required} [5]. \]

B. Distribution of the smallest eigenvalue

In lattice simulations, the performance of an algorithm is frequently determined by the magnitude of the smallest
eigenvalue. It is therefore of interest to compute an analytical result for the distribution of the smallest eigenvalue,

\[ P^{(N)}(\lambda_{\text{min}}, m), \]

in the framework of RMT. This is the subject of this section. The RMT result should be universal and

\[ \text{identical with } \text{that of full QCD under the same conditions as the microscopic spectral density. The only result that has been computed so far in the literature applies to the quenched approximation where } P^{(0)}(\lambda) = \lambda \exp(-\lambda^2/2) [27]. \]

A well-known quantity in RMT is the so-called “hole probability” \( E(s_1, s_2) \) which is the probability that the interval

\[ (s_1, s_2) \]

is free of eigenvalues. For the smallest eigenvalue, we again move to the microscopic scale by rescaling energies

\[ \text{and masses by } 2N \Sigma. \]

For the interval \((0, s)\) at the spectrum edge, we then have

\[ E(0, s) = \int_s^\infty dx_1 \cdots dx_N \tilde{P}^{(N)}_N(x_1, \ldots, x_N; m), \]

where the probability density function on the microscopic scale (with \( x_i = z_i^2 \)) is given by

\[ \tilde{P}^{(N)}_N(x_1, \ldots, x_N; m) = c^{(N)}_N(m) \Delta_N^2(x) \prod_{i=1}^N e^{-x_i/4N} \prod_{f=1}^{N_f} (x_i + m_f^2), \]

cf. Eqs. (5) and (7). The factor \( c^{(N)}_N(m) \) ensures that \( E(0, 0) = 1 \) and should not be confused with the normalization constant defined in Eqs. (14) and (27). On the other hand, we have

\[ E(0, s) = \int_s^\infty dx_{\text{min}} P^{(N)}(x_{\text{min}}, m). \]

Hence, the distribution of the smallest eigenvalue \((x_{\text{min}} = \lambda_{\text{min}}^2)\) is given by

\[ P^{(N)}(\lambda_{\text{min}}, m) = 2\lambda_{\text{min}} P^{(N)}(x_{\text{min}}, m) = -2\lambda_{\text{min}} \frac{dE(0, s)}{ds} \bigg|_{s=\lambda_{\text{min}}^2}. \]

Performing the derivatives, we obtain

\[ -E'(0, s) = c^{(N)}_N(m) N e^{-s/4N} \prod_{f=1}^{N_f} (s + m_f^2) \]

\[ \times \int_s^\infty dx_1 \cdots dx_{N-1} \Delta_{N-1}^2(x) \prod_{i=1}^{N-1} e^{-x_i/4N} (x_i - s) \prod_{f=1}^{N_f} (x_i + m_f^2). \]

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We now shift $x_i \rightarrow x_i + s$ and finally obtain from (39)

$$P^{(N_f)}(\lambda, m) = 2c^{(N_f)}(m)N\lambda e^{-\lambda^2/4} \prod_{f=1}^{N_f} (\lambda^2 + m_f^2)$$

$$\times \int_0^\infty dx_1 \cdots dx_{N_f-1} \Delta_{N_f-1}^2(x) \prod_{f=1}^{N_f-1} \int_0^\infty dx_i e^{-x_i/4N} \prod_{f=1}^{N_f} (x_i + \lambda^2 + m_f^2),$$

(41)

where we have written $\lambda$ instead of $\lambda_{\min}$ for brevity. The integrals in Eq. (41) can be done by comparison with the case of the microscopic spectral density. In terms of the probability density function on the microscopic scale, $\rho_\nu$ is given by

$$\rho^{(N_f)}_\nu(z, m) = 2z \cdot N \int_0^\infty dx_1 \cdots dx_{N_f-1} \tilde{\rho}^{(N_f)}_\nu(x_1, \ldots, x_{N_f-1}, z^2; m)$$

$$= 2c^{(N_f)}(m)Nz \prod_{f=1}^{N_f} (z^2 + m_f^2)$$

$$\times \int_0^\infty dx_1 \cdots dx_{N_f-1} \Delta_{N_f-1}^2(x) \prod_{f=1}^{N_f-1} \int_0^\infty dx_i e^{-x_i/4N} \prod_{f=1}^{N_f} (x_i + z^2 + m_f^2)$$

(42)

in the large-$N$ limit. Apart from prefactors in front of the integrals, we note that $p^{(N_f)}(\lambda, m)$ can be obtained by setting $z = 0$ in the integrand of Eq. (42) and by replacing $m_f$ by $(\lambda^2 + m^2_f)^{1/2}$ in the result for $\rho_\nu$ (but not in the normalization factors). Setting $z = 0$ in the numerator determinant of Eq. (31) makes the first two columns trivial, and the dimension of the matrix can be reduced to $N_f$ by expanding the determinant. Including the prefactors and the proper normalization, we obtain

$$p^{(N_f)}(\lambda, m) = \frac{\lambda}{2} e^{-\lambda^2/4} \frac{\det \left[ \left( \lambda^2 + m^2_f \right)^{(f+1)/2} I_{f+1} \left( \left( \lambda^2 + m^2_f \right)^{1/2} \right) \right]_{f,f'=1,\ldots,N_f}}{C^{(N_f)}(m) \prod_{f=1}^{N_f} (\lambda^2 + m_f^2)}$$

(43)

with $C^{(N_f)}(m)$ given in Eq. (27). For $N_f = 0$, Eq. (43) reduces to the quenched result trivially. We also reproduce the quenched result by taking the limit $m_f \rightarrow \infty$. The simplest nontrivial case is $N_f = 1$, $\nu = 0$ where

$$p^{(1)}(\lambda, m) = \frac{\lambda}{2} e^{-\lambda^2/4} \frac{I_2 \left( \sqrt{\lambda^2 + m^2} \right)}{I_0(m)} .$$

(44)

For two degenerate flavors of mass $m$ and $\nu = 0$, we obtain

$$p^{(2)}(\lambda, m) = \frac{\lambda}{2} e^{-\lambda^2/4} \frac{1}{I_0^2(m)} \frac{I_2^2(t) - I_1(t)I_3(t)}{mI_1^2(t)} \bigg|_{t = \sqrt{\lambda^2 + m^2}} .$$

(45)

In Fig. 2, we plot the distribution of the smallest eigenvalue for the same parameters as in Fig. 1. We observe the same suppression of small eigenvalues in the presence of light dynamical quarks. Other cases can readily be derived from (43). The most general case where $\nu > 0$ can again be obtained by introducing $\nu$ additional massless flavors and expanding the determinants in (43). As an example, we state the result for one massive flavor and $\nu = 1$,

$$p^{(N_f=1,\nu=1)}(\lambda, m) = \frac{\lambda}{2} e^{-\lambda^2/4} \frac{1}{mI_1(m)} \frac{I_2^2(\lambda)I_3(t) - I_2(t)I_3(\lambda)}{mI_1^2(t)} \bigg|_{t = \sqrt{\lambda^2 + m^2}} .$$

(46)

**IV. DISCUSSION**

We have computed the microscopic spectrum of the massive QCD Dirac operator, including the microscopic spectral density and the distribution of the smallest eigenvalue, for an arbitrary number of flavors, arbitrary quark masses, and arbitrary topological charge. Our results generalize the previously known results for the microscopic correlations in the
chiral limit [5]. The distribution of the smallest eigenvalue was previously known only in the quenched approximation. Wherever there is an overlap, our results agree with those obtained in the recent preprint by Damgaard and Nishigaki [19]. It would be very interesting to obtain the result (30) using the supersymmetry method as in Refs. [24,26]. Work in this direction is in progress.

The quantities we have computed in this paper are conjectured to be universal, i.e., identical to those of QCD, for quark masses of the order of $1/V$. This conjecture can be verified straightforwardly in lattice calculations. One computes the eigenvalues of the Dirac matrix and constructs the microscopic spectral density and the distribution of the smallest eigenvalue by averaging over a sufficient number of independent gauge field configurations. The lattice data can then be compared to the analytical predictions of RMT. For pure SU(2) gauge theory without dynamical quarks, this was already done in Ref. [15], and the agreement was excellent.

Our present calculation applies to full QCD with 3 colors in the continuum limit. In the context of lattice QCD, it applies to SU(3) with dynamical staggered fermions. The necessary lattice algorithms are available, e.g., the hybrid Monte-Carlo algorithm for generating unquenched configurations and the Lanczos algorithm for computing eigenvalues of large sparse matrices. Therefore, we are confident that the results obtained in this paper will be verified in the very near future.

Given that the analytical results obtained in this paper really describe QCD, one may ask what their relevance is in practice. We list here a few possible applications. (i) The parameter which appears in the microscopic quantities contains the absolute value of the chiral condensate, $\Sigma$. As we have already seen in quenched SU(2) [28], the analytical information available from RMT can be used to determine $\Sigma$ and to extract its thermodynamic limit from very small lattices. Since otherwise one would have to perform an extrapolation, this seems to offer an interesting technological advantage. (ii) Lattice simulations with light dynamical quarks are very expensive, and extrapolations to the chiral limit are difficult. In RMT, we have computed exact analytical expressions which apply for very small quark masses, and the chiral limit can be taken analytically. One would, therefore, hope that lattice practitioners can use the analytical information in their extrapolations to the chiral limit for those quantities that depend on the small eigenvalues. (iii) A very interesting issue is topology. As one approaches the continuum limit on the lattice, one expects zero modes to appear which give rise to non-zero topological charge $\nu$. The microscopic quantities we consider are sensitive to topology, and we have computed them for arbitrary $\nu$. Hence, one can analyze lattice results in different sectors of topological charge. Whether or not topology is seen on present-day lattices is still a somewhat controversial issue, and for quenched SU(2) the lattice results of Ref. [15] were consistent with $\nu = 0$. We hope that our results will prove useful in future analyses of topology. (iv) The performance of many lattice algorithms depends on the magnitude of the smallest eigenvalue. While RMT cannot predict this magnitude, it predicts the distribution of $\lambda_{\text{min}}$ over the ensemble of gauge field configurations. It would be interesting to see if experts in algorithm development can make use of the analytical information available from RMT.

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FIG. 1. Microscopic spectral density for two degenerate flavors of rescaled mass $m = 5$ (solid line), in the chiral limit (short dashes), and in the quenched approximation (long dashes).
FIG. 2. Distribution of the smallest eigenvalue for two degenerate flavors of rescaled mass $m = 5$ (solid line), in the chiral limit (short dashes), and in the quenched approximation (long dashes).