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GENERAL SOLUTION FOR A RELATIVISTIC STAR

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In a spherically symmetric spacetime we find the general solution for a relativistic star in hydrostatic equilibrium having the spheroidal geometry for the 3-space embedded in 4-Euclidean space. The parameter λ is the measure of spheroidal character and it determines the physical properties of the star. It has a lower bound $> 3/17$; stars with smaller mass to radius ratio can occur for all allowed λ while ultra compact stars, having the ratio between $1/3$ and $1/2$, will have $\lambda \gtrsim 0.9$. It turns out that a wide range of values of λ can represent objects with the fluid density of a neutron star. The model is shown to possess all the desirable physical features.

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Relativistic stellar models have been considered ever since the first solution of the Einstein equation was obtained by Schwarzschild for the interior of a star in hydrostatic equilibrium. Recently Vaidya and Tikekar [1] have obtained a solution for a superdense star by proposing an ansatz for the geometry of the 3-surface embedded in a 4-Euclidean space. The ansatz prescribes a spheroidal geometry for the 3-surface, described by two parameters λ and R ; $\lambda = 0$ gives spherical while $\lambda = -1$ corresponds to flat space. Their solution corresponds to $\lambda = 2$, which was also obtained independently by Durgapal and Bannerji [2] without reference to the spheroidal geometry of the 3-space. Tikekar [3] obtained the solution for $\lambda = 7$ and made a detailed study of the model. Recently Maharaj and Leach [4] have obtained more solutions for a set of discrete values of λ . The solutions are either polynomials or products of polynomials and algebraic functions. Since the parameter λ describes a 3-space geometry it is desirable to look for an analytic solution valid for all possible values of λ . In this paper we first obtain the exact analytic solution for all admissible values of $\lambda > 3/17$. This permits us to bring out the crucial role played by this geometric parameter in the context of compactness and other physical properties of the model.

We begin with the spherically symmetric metric,

$$ds^2 = -e^{2\nu(r)} dt^2 + e^{2\mu(r)} dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2) \quad (1)$$

where the ansatz mentioned above determines

$$e^{2\mu(r)} = \frac{1 + \lambda r^2/R^2}{1 - r^2/R^2}. \quad (2)$$

For the perfect fluid matter distribution, we have ($8\pi G = c = 1$)

$$\rho = \frac{(1 + \lambda)(3 + \lambda r^2/R^2)}{R^2(1 + \lambda r^2/R^2)^2}, \quad (3)$$

$$p = \frac{(2\nu'/r + 1/r^2)(1 - r^2/R^2)}{1 + \lambda r^2/R^2} - \frac{1}{r^2} \quad (4)$$

and the pressure isotropy condition

$$(1 + \lambda - \lambda x^2)\psi''(x) + \lambda x\psi'(x) + \lambda(\lambda + 1)\psi = 0 \quad (5)$$

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where $\psi = e^\nu$, $z^2 = 1 - r^2/R^2$ and a prime denotes a differentiation w.r.t. the argument. The eqn. (6) is the only equation to be solved for the metric function ν . Tikekar [3] obtained a solution of eqn. (6) for $\lambda = 7$ and Maharaj and Leach [4] for a set of values of λ , $\lambda = n^2 - 2$, where n is an integer. To obtain the general solution, we define a new variable, $z = [\lambda/(\lambda + 1)]^{1/2}x$, and rewrite eqn. (6) as

$$(1 - z^2)\psi''(z) + z\psi'(z) + (\lambda + 1)\psi(z) = 0. \quad (6)$$

This admits the general solution

$$\psi = \psi_1 + \psi_2 = A_1 T_{n+1}^{-3/2}(z) + A_2 (1 - z^2)^{3/2} T_{n-2}^{3/2}(z), \quad (7)$$

where $n = (\lambda + 2)^{1/2}$ and T_α^β is a Gegenbauer function [5]. Also note that

$$\psi'(z) = A_1 T_n^{-1/2}(z) + A_2 (n + 1)(n + 3)(1 - z^2)^{1/2} T_{n-1}^{1/2}(z). \quad (8)$$

When α is zero or positive integer n , T_α^β is a finite polynomial in z given by

$$T_\alpha^\beta(z) = D_{\beta n} (1 - z^2)^\beta \frac{d^n}{dz^n} (1 - z^2)^{n+\beta}, \quad (9)$$

where $D_{\beta n}$ is given in terms of β and n . All the solutions obtained earlier can be recovered from eqns. (8) and (10) for integral values of $(\lambda + 2)^{1/2}$. For example, for $\lambda = 7$, we have

$$\psi_1 = A_1 T_4^{-3/2}(z) = c_1 (1 - z^2)^{-3/2} \frac{d^4}{dz^4} (1 - z^2)^{5/2} \quad (10)$$

$$\psi_2 = A_2 (1 - z^2)^{3/2} T_1^{3/2}(z) = c_2 \frac{d}{dz} (1 - z^2)^{5/2}, \quad (11)$$

which give the Tikekar [3] solution.

We note that $n T_n^{-1/2}(z)$ and $(1 - z^2)^{1/2} T_{n-1}^{1/2}(z)$ are actually Tschebyscheff polynomials. Thus for a real z with $0 < z \leq [\lambda/(\lambda + 1)]^{1/2}$,

$$n T_n^{-1/2}(z) = (2/\pi)^{1/2} \cos(n \cos^{-1} z), \quad (12)$$

$$(1 - z^2)^{1/2} T_{n-1}^{1/2}(z) = (2/\pi)^{1/2} \sin(n \cos^{-1} z). \quad (13)$$

Substituting these in eqn. (9) and integrating we can obtain the general solution ψ as

$$\psi = A \left[\frac{\cos[(n + 1)\zeta + \gamma]}{n + 1} - \frac{\cos[(n - 1)\zeta + \gamma]}{n - 1} \right], \quad (14)$$

where $\zeta = \cos^{-1} z$, and A and γ are constants. Although the solution is obtained by considering integral values of n , it is now valid for all $\lambda > -2$.

The simple form of the solution permits us to study physical features of a spheroidal star in a general way, which are outlined as follows.

- 1) In this model, ρ and p are given by

$$\rho = \frac{1}{R^2(1 - z^2)} \left[1 + \frac{2}{(\lambda + 1)(1 - z^2)} \right] \quad (15)$$

$$p = -\frac{1}{R^2(1 - z^2)} \left[1 + \frac{2z\psi'}{(\lambda + 1)\psi} \right]. \quad (16)$$

We note that ρ is obviously positive for $\lambda > -1$, while $p \geq 0$ requires $z\psi'/\psi \leq -\frac{1}{2}(\lambda + 1)$.

- 2) The radius of the star, b , is determined by the condition $p = 0$ at $r = b$. This gives

$$z_b \frac{\psi'(z_b)}{\psi(z_b)} = -\frac{1}{2}(1 + \lambda) \quad (17)$$

where $z_b = [\lambda/(\lambda + 1)]^{1/2}(1 - b^2/R^2)^{1/2}$. It will determine the constant γ for a given radius b of the star.

- 3) For p to remain finite ψ should not have a zero in the range $z_b \leq z \leq z_0$, where $z_0 = [\lambda/(\lambda + 1)]^{1/2}$. We also require that both $d\rho/dz$ and dp/dz should increase monotonically as z increases from z_b to z_0 . In Fig. 1 p is plotted against ρ which clearly shows that there is no singular behaviour for p in the prescribed range. The constant A in eqn. (15) is determined by matching the solution to the Schwarzschild exterior solution at $r = b$.
- 4) The most stringent constraint in a stellar model comes from the requirement that the sound propagation be causal; i.e. $\frac{dp}{d\rho} < 1$. This implies

$$\frac{1}{(1 - z^2)} \left(\frac{1}{2z} - D \right) \leq \frac{\psi'}{\psi} \leq \frac{1}{(1 - z^2)} \left(\frac{1}{2z} + D \right), \quad (18)$$

where

$$D = [4 + 1/4z^2 + (1 + \lambda)(1 - z^2)]^{1/2}. \quad (19)$$

Again $p \geq 0$ gives

$$\frac{\psi'}{\psi} \leq -\frac{\lambda + 1}{2z} \quad (20)$$

Combining the two constraints, we get the effective bounds

$$\frac{1}{1 - z^2} \left(\frac{1}{2z} - D \right) \leq \frac{\psi'}{\psi} \leq -\frac{\lambda + 1}{2z}, \quad (21)$$

for a realistic model. Eqn. (22) readily leads to a lower bound

$$\lambda > \frac{3}{17}. \quad (22)$$

This clearly excludes the flat space ($\lambda = -1$) and the Schwarzschild interior solution ($\lambda = 0$). This happens because the condition $\frac{dp}{d\rho} < 1$ cannot be satisfied in these cases.

In Fig. 1, the slope of the p versus ρ curve is always less than that of $\rho = p$, which shows that $dp/d\rho < 1$ always. Further it lies below and never intersects the $\rho = 3p$ curve indicating $\rho > 3p$ and hence satisfying the strong energy condition.

- 5) The constraint (22) leads to an upper bound on possible values of $\bar{b} = b/R$, which reads as

$$(1 - \bar{b}^2) \geq \frac{\lambda^2 + 5\lambda + 12 - (17\lambda^2 + 82\lambda + 129)^{1/2}}{\lambda(5 + \lambda)}. \quad (23)$$

In Fig. 2, the upper bound on $\bar{b}, \bar{b}_u$ is plotted against $\log_{10} \lambda$ and it has the maximum value, 0.72 for $\lambda \approx 1.2$. This shows that for a given $\bar{b}_u$, there correspond two values of λ (for example, $\bar{b}_u = 0.29$ for $\lambda = 0.2$ and 45).

- 6) The mass contained inside a radius r is given by

$$M(r) = \frac{1}{2} \int_0^r r'^2 \rho(r') dr' \quad (24)$$

which on integration for $r = b$ yields

$$\frac{M(b)}{b} = \frac{(1 + \lambda)b^2/R^2}{2(1 + \lambda b^2/R^2)}. \quad (25)$$

- 7) Eqns. (24) and (26) also lead to an upper bound on M/b , which measures the compactness of the star. The upper bound approaches the Schwarzschild limit $1/2$ asymptotically as λ tends to infinity. We note that objects with lower values of M/b can occur for any $\lambda > 3/17$, but for ultra compact objects ($1/2 > M/b \geq 1/3$) [6], $\lambda \gtrsim 0.9$. Further the upper bound on M/b increases monotonically from 0 to $1/2$ as λ is increased from its lowest value $3/17$ as shown in Fig. 2. For a given radius b , stars with larger λ will have bigger mass. It may be noted that the parameter R can be determined by giving the value of central density, $\rho_0 = 3(\lambda + 1)/R^2$. Assuming the central density to be $\sim 7 \times 10^{14} \text{ gm cm}^{-3}$, typical of a neutron star, we find that the upper bound on radii will be 14, 22, 18 and 3.5 km for $\lambda = 0.5, 1.2, 7$ and 100 respectively. Thus to represent an object with the neutron star density, λ has a wide range. Note that the upper bound on the neutron star radius occurs at $\lambda \approx 1.2$.
- 8) Fig. 1 shows: (i) p is well-behaved in the prescribed range for z and the solid curve p versus ρ indicates the implicit equation of state for the fluid, (ii) $dp/d\rho < 1$ always as slope of the solid curve is less than that of $\rho = p$ curve, and (iii) both $\rho > p$ and $\rho > 3p$ (the weak and strong energy conditions) are satisfied as the solid curve lies below and never intersects the other two curves. Thus all the physical conditions are satisfied.

To conclude we have found an exact general solution of the Einstein equations for a spheroidal star in hydrostatic equilibrium satisfying all physical constraints. Since our solution is valid for all $\lambda > 3/17$, we have been able to study the model in a general way. In particular the special role played by λ , and hence the geometry of the 3-space, in determining the physical features of the star can be demonstrated. For a spherically symmetric distribution, it is well-known that $e^{2\mu(r)} = 1 - \frac{2M(r)}{r}$, where $M(r)$ is defined in eqn. (25). The ansatz (2) then relates mass function and the geometric parameter λ leading to a clear association of a specific geometric property (curvature of 3-space) with the physical properties of the star. Although the assumption of hydrostatic equilibrium normally applies to a superdense star, the model allows for arbitrarily small M/b . In particular a wide range of values of λ can sustain neutron star like objects. The details of the model and its application to realistic compact objects will be reported elsewhere.

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Fig.1

FIG. 1. The solid curve represents p versus ρ while $\rho = p$ and $\rho = 3p$ curves are as labelled. The parameters λ , $\bar{b}$ and γ are as indicated.

FIG. 2. The parameters $\bar{b}_u$ and M/b are plotted against $\log_{10} \lambda$ and the curves are labelled.