Dynamical Symmetry Breaking in Curved Spacetime
– Four-Fermion Interactions –

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Abstract
This review deals with the theory of four-fermion interactions in curved spacetime. Starting with the D-dimensional Minkowski spacetime \(2 \leq D \leq 4\) the effective potential in the leading order of \(1/N\)-expansion is calculated and the phase structure of the theory is investigated. Using the same technique the effective potential for composite operator \(\bar{\psi}\psi\) in four-fermion models is calculated under the following circumstances:

a) \(D\)-dimensional weakly curved spacetime (in linear curvature approximation),
b) \(D\)-dimensional de Sitter and anti-de Sitter universe,
c) \(D\)-dimensional Einstein universe.

The phase structure of the theory is investigated analytically as well as numerically. Curvature induced phase transitions are discussed where fermion masses are dynamically generated.

As an extension of four-fermion models we consider the gauged Nambu-Jona-Lasinio (NJL) model, higher derivative NJL model and supersymmetric NJL model in weakly curved spacetime where the effective potential is analytically evaluated. The phase structure of the models is again analyzed and the condition for the chiral symmetry breaking in the gauged NJL model is given in an analytical form.

Finally the influence of two external effects (non-zero temperature and gravitational field, non-trivial topology and gravitational field as well as magnetic and gravitational field) to the phase structure of four-fermion models is analyzed. The possibility of curvature and temperature-induced or curvature- and topology-induced phase transitions is discussed. It is also argued that the chiral symmetry broken by a weak magnetic field may be restored due to the presence of gravitational field. Some applications of four-fermion models in quantum gravity are also briefly investigated.

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1 Introduction

An idea of dynamical symmetry breaking (DSB) was introduced by Nambu and Jona-Lasinio in quantum field theory in Ref. [1] (for modern reviews see, e.g., Ref. [2]). The original Nambu-Jona-Lasinio (NJL) model and its generalizations (all of which are somehow related to the theory of superconductivity a la Bardeen, Cooper and Schriffer [3]) are extremely useful toy models in the study of composite states in various circumstances. To discuss the dynamical symmetry breaking nonperturbative treatments are essential. One of the simplest nonperturbative techniques is the 1/N-expansion scheme which works well in the four-fermion models.

In the simplest version of the NJL model [1] (or four-fermion model) the chiral symmetry of the theory is spontaneously broken according to the emergence of a vacuum condensate of the composite field of fermions $\psi : \langle \bar{\psi} \psi \rangle \neq 0$. At the same time the non-vanishing fermion mass is generated. In the spectrum one finds a mode of the composite scalar field $\sigma \sim \bar{\psi} \psi$, which may correspond to the Higgs field in the theory with the elementary Higgs scalar field. Clearly the model may be regarded as a composite Higgs model based on the mechanism of dynamical symmetry breaking.

Four-fermion models (gauged four-fermion models) have been often considered as an effective low energy theory for quantum chromodynamics (QCD, see Ref. [4] for an introduction). The chiral symmetry of such a theory is broken by the creation of quark-anti-quark condensates and quark masses are dynamically generated. Such a theory is extremely useful in dealing with non-perturbative phenomena of QCD.

Moreover the (gauged) four-fermion models may be well applied to the description of the dynamical symmetry breaking in standard model (SM). At the electroweak scale the fundamental fermion condensate as in the technicolor model.[5] The similar mechanism could work in the symmetry breaking at the GUT era in early universe.

The description of elementary particle theories (QCD, SM, GUT, ···) in terms of the effective four-fermion model is particularly convenient when the theories are considered under some external conditions (finite temperature, finite density, non-trivial topology, external gauge and gravitational fields, etc.). As an extreme external condition one immediately considers the early universe where the gravity is strong and the temperature is high. Since the dynamical symmetry breaking mechanism replaces the spontaneous symmetry breaking caused by the elementary Higgs scalar field, it is natural for us to employ the composite field $\sigma$ as an inflaton in the inflationary universe.[6] Through renormalization group arguments [7] one finds the necessity of the scalar-gravitational non-minimal coupling $\xi$ [7] whose value is governed by the renormalization group running. The value of $\xi$ varies in different theories and is crucial to have a successful inflation.

One realizes an importance of studying the phase structure of the four-fermion model in curved background and finds the cosmological motivation to deal with this model.

The main purpose of the present article is to present what is and is not established in the study of the four-fermion models in curved spacetime in full detail. We do not consider in the present article any cosmological applications. Instead we give a review of the phase structure of the different kinds of the four-fermion model with different gravitational backgrounds. Hence the classical external gravity is considered here as a probe of quantum four-fermion models and is regarded as a source of the curvature-induced phase transition in these models.

The paper is organized as following. In §2 fundamental properties of four-fermion models are briefly reviewed. To investigate the phase structure of the four-fermion theory the effective potential is introduced in curved spacetime. We adopt the 1/N expansion method as a nonperturbative approach and calculate the effective potential in the leading order of the 1/N expansion. As an instructive rehearsal we work in the Minkowski spacetime and evaluate the effective potential analytically as well as numerically. On the basis of the effective potential we discuss the phase structure of the four-fermion model in the Minkowski spacetime.

In §3 the weakly curved spacetime is considered. We keep only terms independent of and linear in the curvature. Evaluating the effective potential in the leading order of 1/N expansion the phase structure
of the theory is clarified and the curvature-induced phase transition is discussed in arbitrary dimensions \((2 \leq D \leq 4)\).

In §4 we consider the spacetimes with the constant curvature (de Sitter background, anti-de Sitter background and Einstein universe background). In these spacetimes we can evaluate the effective potential of the four-fermion model without making any approximation with regard to the curvature. With the aim of justifying the validity of the weak curvature approximation we investigate the phase structure in the constant curvature spacetime. We then compare the result with the one in the weakly curved spacetime.

Section 5 is devoted to the study of the gauged NJL model, supersymmetric NJL model and higher derivative four-fermion model in the weakly curved spacetime. We evaluate the effective potential in the gauged NJL model with finite cut-off and let the cut-off tend to infinity. In order to do so we use the renormalization group method and appeal to the equivalence with the gauge Higgs-Yukawa model. The analytical form of the condition for the chiral symmetry breaking is found. For the four-fermion model with higher derivative terms we also study the effective potential and the phase structure of the model. The phase structure of supersymmetric NJL model is also discussed in detail.

In §6 the combined effect of curvature and temperature is discussed in curved spacetime. To introduce temperature we suppose the thermal equilibrium and we restrict ourselves to the Einstein universe background. The effective potential is evaluated at finite temperature in spacetimes with positive or negative curvature. For sufficiently high temperature we show that the broken chiral symmetry is restored and the symmetric phase can be realized even in the spacetime with negative curvature.

In §7 we discuss the influence of the non-trivial topology to the chiral symmetry breaking in the weakly curved spacetime. We start with the calculation of the effective potential in the four-fermion model in the flat torus-compactified spacetime. We find the same quantity as in the weakly curved spacetime with the torus-compactified dimension. The possibility of curvature and topology-induced phase transition is explicitly shown.

Section 8 is devoted to the investigation of chiral symmetry breaking in the weakly curved spacetime under the influence of the external magnetic field. The effective potential is found and its phase structure is studied. The chiral symmetry is shown to be broken in the NJL model under the external magnetic field. However, the inclusion of the external gravitational field with positive curvature acts in opposite direction and it may restore chiral symmetry.

In §9 some applications of four-fermion models in quantum gravity are presented. In particular, we discuss semiclassical approach in 2D Gross-Neveu-dilaton model. The conformal factor dynamics based on four-fermion theory is studied in 4D quantum gravity. The effective potential for composite gravitino in \(N = 1\) supergravity on de Sitter background is also calculated.

In §10 we give an outlook and prospects for future researches.

Our notation is basically in conformity with the \((-,-,-)\) convention in the book by Misner, Thorne and Wheeler.[8]

2 Four-fermion models in Minkowski spacetime

The four-fermion model is one of the prototype models of the dynamical symmetry breaking. In the present section we briefly review the fundamental properties of the four-fermion models. A four-fermion model and its effective potential are introduced in curved spacetime. We adopt the \(1/N\) expansion as a non-perturbative approach to investigate the dynamical symmetry breaking in the model. The phase structure of the theory is discussed in Minkowski spacetime as an instructive example of our method.
2.1 Gross-Neveu type model

Here we consider the Gross-Neveu type model in curved spacetime and discuss the symmetry and the symmetry breaking in arbitrary dimensions \(2 \leq D \leq 4\). It is defined by the action

\[
S = \int \sqrt{-g} d^D x \left[ \sum_{k=1}^{N} \bar{\psi}_k i \gamma^\mu \nabla_\mu \psi_k + \frac{\lambda_0}{2N} \left( \sum_{k=1}^{N} \bar{\psi}_k \psi_k \right)^2 \right],
\]

(1)

where index \(k\) represents the flavors of the fermion field \(\psi\), \(N\) is the number of fermion species, \(g\) the determinant of the metric tensor \(g_{\mu\nu}\), \(\gamma^\mu\) the Dirac matrix in curved spacetime and \(\nabla_\mu \psi\) the covariant derivative of the fermion field \(\psi\). In two spacetime dimensions the theory is nothing but the Gross-Neveu model.[9] For simplicity, we neglect the flavor index below.

The action (1) is invariant under the discrete transformation

\[
\begin{align*}
\bar{\psi} \psi &\rightarrow -\bar{\psi} \psi, \\
\bar{\psi} \gamma_\mu \psi &\rightarrow \bar{\psi} \gamma_\mu \psi. 
\end{align*}
\]

(2)

In two or four spacetime dimensions this transformation is realized by the discrete chiral transformation

\[
\psi \rightarrow \gamma_5 \psi.
\]

(3)

Below we call this \(Z_2\) symmetry the discrete chiral symmetry.

The discrete chiral symmetry prohibits the fermion mass term. If the composite operator constructed by the fermion and anti-fermion develops the non-vanishing vacuum expectation value

\[
\langle \bar{\psi} \psi \rangle \neq 0,
\]

(4)

fermion mass term appears in the four-fermion interaction term and the chiral symmetry is broken down dynamically.

We consider the theory with \(N\) flavors of the fermion fields. Thus the theory also has \(SU(N)\) flavor symmetry

\[
\psi \rightarrow e^{i \sum_a g_a T^a} \psi,
\]

(5)

where \(T^a\) are generators of the \(SU(N)\) symmetry. Under the circumstance of the global \(SU(N)\) flavor symmetry we may work in a scheme of the \(1/N\) expansion.

For practical calculations in four-fermion theories it is more convenient to introduce auxiliary field \(\sigma\).[9] The generating functional is given by

\[
Z = \int [d\bar{\psi}] [d\tilde{\psi}] e^{iS},
\]

(6)

where we choose the path-integral measure in terms of the field variables

\[
\begin{align*}
\tilde{\psi} &= \sqrt{-g} \psi, \\
\bar{\tilde{\psi}} &= \sqrt{-g} \bar{\psi},
\end{align*}
\]

(7)

so that the covariance under the general coordinate transformation \([10]\) is guaranteed.

We consider the Gaussian integral

\[
C = \int [d\sigma'] \exp i \int \sqrt{-g} d^D x \left( - \frac{N}{2\lambda_0} \sigma'^2 \right).
\]

(8)

Obviously \(C\) is a constant. The path integral measure is invariant under the redefinition of the parameter \(\sigma'\)

\[
\sigma' \rightarrow \sigma = \sigma' - \frac{\lambda_0}{N} \bar{\psi} \psi.
\]

(9)
After the redefinition, Eq. (8) becomes

\[ C = \int [d\sigma] \exp i \int \sqrt{-g} d^p x \left[ -\frac{N}{2\lambda_0} \left( \sigma + \frac{\lambda_0}{N} \bar{\psi} \psi \right)^2 \right]. \] (10)

As Eq. (10) is a constant, we are free to insert it in the right-hand side of Eq. (6) without any change. We obtain

\[ Z = \frac{1}{C} \int [d\bar{\psi}] [d\bar{\psi}] [d\sigma] e^{i S_y}, \] (11)

where action \( S_y \) is given by

\[ S_y = \int \sqrt{-g} d^p x \left( \bar{\psi} i \gamma_\mu \nabla^\mu \psi - \frac{N}{2\lambda_0} \sigma^2 - \bar{\psi} \sigma \psi \right). \] (12)

The four-fermion interaction term in Eq. (1) has been canceled out by introducing the Gaussian integral (10).

The action \( S_y \) is equivalent to the original action (1).[11] If the non-vanishing vacuum expectation value is assigned to the auxiliary field \( \sigma \)

\[ \langle \sigma \rangle = m \neq 0, \] (13)

there appears a mass term for the fermion field \( \psi \) and the discrete chiral symmetry (the \( Z_2 \) symmetry) is eventually broken.

\[ \psi \) propagator:

\[ \sigma \) propagator:

\[ \bar{\psi} \gamma_\mu \nabla^\mu \psi \) vertex:

\[ \psi \) loop:

In Fig. 1 we present the Feynman rules for the action (12) in Minkowski spacetime. These Feynman rules are useful to make quantum calculations in the Gross-Neveu type model. The Feynman rule for the \( \sigma \) propagator contains the factor \( 1/N \). Thus the radiative corrections including the \( \sigma \) propagator are suppressed at the large \( N \) limit. Internal \( \sigma \) lines have no contributions in the leading order of the \( 1/N \)
Therefore we easily calculate the radiative correction in the Gross-Neveu type model at the leading order of the $1/N$ expansion. For example the effective potential in Gross-Neveu type model is calculated from the two-point function of massive free fermions (see §2.2).

2.2 Effective potential

We would like to find a ground state of the system described by the four-fermion model. For this purpose we evaluate an effective potential for the field $\sigma$. The ground state is determined by observing the minimum of the effective potential. If the auxiliary field $\sigma$ develops the non-vanishing vacuum expectation value, the ground state is no longer invariant under the discrete chiral transformation and then the chiral symmetry is broken down dynamically. In the present subsection we briefly review how to calculate the effective potential in general situation, i.e., curved spacetime. The case of the Minkowski space is included as a special case.

We start with the generating functional

$$Z[J] = \int [d\bar{\psi}] [d\psi] [d\sigma] \exp \left( iS_y + iN \int \sqrt{-g} d^D x \sigma(x) J(x) \right), \quad (14)$$

where $J(x)$ is the source function for the field $\sigma(x)$. Performing the integration over the fermion fields $\psi$ and $\bar{\psi}$, we get

$$Z[J] = \int [d\sigma] \exp iN \left[ \int \sqrt{-g} d^D x \left( -\frac{\sigma^2}{2\lambda_0} + \sigma J \right) - i \ln \det (i\gamma^\mu \nabla_\mu - \sigma) \right]. \quad (15)$$

We divide the field $\sigma$ into two parts

$$\sigma = \sigma_0 + \tilde{\sigma}, \quad (16)$$

where $\sigma_0$ is a classical background which satisfies the classical equation of motion and $\tilde{\sigma}$ is a quantum fluctuation. In terms of $\sigma_0$ and $\tilde{\sigma}$ Eq. (15) is rewritten as

$$Z[J] = \exp iN \left[ \int \sqrt{-g} d^D x \left( -\frac{\sigma_0^2}{2\lambda_0} + \sigma_0 J \right) - i \ln \det (i\gamma^\mu \nabla_\mu - \sigma_0) \right]$$

$$\times \int [d\tilde{\sigma}] \exp iN \left[ \int \sqrt{-g} d^D x \left( -\frac{2\sigma_0 \tilde{\sigma} + \tilde{\sigma}^2}{2\lambda_0} + \tilde{\sigma} J \right) - i \ln \det \frac{\tilde{\sigma}}{(i\gamma^\mu \nabla_\mu - \sigma_0)} \right]. \quad (17)$$

As may be seen through Fig. 1 radiative corrections in $\tilde{\sigma}$ are higher order corrections in the $1/N$ expansion. In the leading order of the $1/N$ expansion the generating functional $W[J]$ for connected Green functions is given by

$$NW[J] = -i \ln Z[J].$$
\[
\Gamma[\sigma_c] = W[J] - \int \sqrt{-g} d^p x \sigma_c(x) J(x),
\]
where the new variable \(\sigma_c\) is given by
\[
\sigma_c = \frac{\delta W[J]}{\delta J} = \sigma_0 + O\left(\frac{1}{N}\right),
\]
which corresponds to the vacuum expectation value of \(\sigma\) in the presence of the source \(J\). We substitute Eqs. (18) and (20) into Eq. (19) to get
\[
\Gamma[J] = -\int \sqrt{-g} d^p x \frac{\sigma_c^2}{2\lambda_0} - i \ln \text{Det}(i\gamma^\mu \nabla_\mu - \sigma_c) + O\left(\frac{1}{N}\right).
\]

The effective potential \(V(\sigma)\) (with \(N\) factored out) is defined by
\[
V(\sigma) = -\frac{\Gamma(\sigma)}{\int \sqrt{-g} d^p x},
\]
where we put \(\sigma_c(x) = \sigma\), a constant independent of the coordinate \(x\). The effective potential gives the energy density under the constant background \(\sigma\). The vacuum state should minimize the effective potential. Thus the effective potential is useful to determine the vacuum state in a static and homogeneous universe where the assumption \(\sigma_c(x) = \sigma\), a constant, is adopted. According to the Schwinger proper time method [12] the second term of the right-hand side of Eq. (21) for constant \(\sigma\) reads
\[
\ln \text{Det}(i\gamma^\mu \nabla_\mu - \sigma) = \text{tr} \int d^p x \ln(i\gamma^\mu \nabla_\mu - \sigma) - \text{tr} \int d^p x \sqrt{-g} \int_0^\sigma ds S(x,x;s) + \text{const},
\]
where const is a constant number which is independent of the field \(\sigma\) and \(S(x,x;s)\) is the spinor two-point function which satisfies the Dirac equation
\[
(i\gamma^\mu \nabla_\mu - s)S(x,y;s) = \frac{1}{\sqrt{-g}} \delta^D(x,y).
\]
Thus the effective potential (22) reads
\[
V(\sigma) = \frac{1}{2\lambda_0} \sigma^2 - i \text{tr} \int_0^\sigma ds S(x,x;s) + O\left(\frac{1}{N}\right).
\]

It should be noted that the effective potential is normalized so that \(V(0) = 0\). From Eq. (25) we recognize that the effective potential is described by the two-point function \(S(x,x;s)\) of the free fermion with mass \(s\).
We turn our attention to the theory governed by the action
\[ S = \int \sqrt{-g} d^Dx \left\{ \sum_{k=1}^{N} \bar{\psi}_k i\gamma^\mu \nabla_\mu \psi_k \right. \\
+ \lambda_0 \left[ \left( \sum_{k=1}^{N} \bar{\psi}_k \psi_k \right)^2 + \left( \sum_{k=1}^{N} \bar{\psi}_k i\gamma^5 \psi_k \right)^2 \right] \right\}. \tag{26} \]

In four dimensions the action (26) describes the NJL (Nambu-Jona-Lasinio) model.\[1\] The action is invariant under the chiral \( U(1) \) transformation in even dimensions,
\[ \psi \rightarrow e^{i\theta \gamma^5} \psi. \tag{27} \]

The chiral \( U(1) \) symmetry prevents the action from having mass terms. Using the auxiliary field method and applying the similar method mentioned above, we obtain the effective potential for the NJL model:
\[ V(\sigma) = \frac{1}{2\lambda_0} \sigma^2 - i \text{tr} \int_0^{s'} ds S(x, x; s) + O \left( \frac{1}{N} \right), \tag{28} \]
\[ \sigma' = \sqrt{\sigma^2 + \pi^2}. \tag{29} \]

We need, corresponding to the two four-fermion interaction terms in the action (26), two kinds of auxiliary fields \( \sigma \) and \( \pi \). If \( \sigma' \) develops the non-vanishing vacuum expectation value the chiral \( U(1) \) symmetry is broken down and then Nambu-Goldstone (NG) mode appears \[13\] (i.e., \( \pi \) becomes massless\[4\]). In two dimensions it is well-known that the \( \pi \) loop has an infrared divergence and we cannot neglect the next-to-leading order terms of the \( 1/N \) expansion.\[14\] Therefore our present analysis is not sufficient for dealing with the action (26) in two dimensions.

The effective potentials (25) and (28) have the same form in the leading order of the \( 1/N \) expansion. Thus, using the effective potential (25), we can discuss the phase structure of both models except in two dimensions.

### 2.3 Effective potential in Minkowski spacetime

An instructive example to study the phase structure of the four-fermion model may be found in an infinite volume flat spacetime (i.e., Minkowski spacetime). In the present subsection we calculate the effective potential \( V(\sigma) \) and show the phase structure of the Gross-Neveu type model in Minkowski spacetime.

The two-point function \( S(x, x; s) \) in Minkowski spacetime is given by
\[ S(x, x; s) = \int \frac{d^Dk}{(2\pi)^D} \frac{1}{\sqrt{-g} - s}, \tag{30} \]
where \( \sqrt{-g} = 1 \). Inserting Eq. (30) into Eq. (25) and performing the integration, we obtain the effective potential in Minkowski spacetime,
\[ V_0(\sigma) = \frac{1}{2\lambda_0} \sigma^2 - \frac{\text{tr}1}{(4\pi)^{D/2}D!} \Gamma \left( 1 - \frac{D}{2} \right) \sigma^D, \tag{31} \]
where \( \text{tr}1 \) is the trace of the unit Dirac matrix, the suffix 0 for \( V_0(\sigma) \) is introduced to keep the memory that \( R = 0 \) and \( L \rightarrow \infty \) with \( R \) the spacetime curvature and \( L \) the size of a space or time component.

\[ ^4\] In Gross-Neveu type model the symmetry under the transformation (3) is not a continuous symmetry. Thus no NG boson appears. Our analysis is valid for Gross-Neveu type model in two dimensions.
The effective potential (31) is divergent in two and four dimensions. It happens to be finite in three dimensions in the leading order of the $1/N$ expansion. As is well-known, the four-fermion model is renormalizable in the two-dimensional Minkowski spacetime. Therefore the potential (31) is made finite at $D = 2$ by the usual renormalization procedure. For $D = 3$, four-fermion model is known to be renormalizable in the sense of the $1/N$ expansion.[15] The effective potential (31) happens to be finite in three dimensions in the leading order of the $1/N$ expansion.\(^5\) In four dimensions the four-fermion model is not renormalizable and the finite effective potential cannot be defined. In the present section we regard the effective potential for $D = 4 - \epsilon$ with $\epsilon$ sufficiently small positive as a regularization of the one in four dimensions and consider the theory for $D = 4 - \epsilon$ as a low energy effective theory stemming from more fundamental theories.

We introduce the renormalization procedure by imposing the renormalization condition,

$$\left. \frac{\partial^2 V_0(\sigma)}{\partial \sigma^2} \right|_{\sigma = \mu} = \frac{\mu^{D-2}}{\lambda},$$

(32)

where $\mu$ is the renormalization scale. From the renormalization condition (32) we get the renormalized coupling $\lambda$,

$$\frac{1}{\lambda_0} = \frac{\mu^{D-2}}{\lambda} + \frac{\text{tr} 1}{(4\pi)^{D/2}} (D-1) \Gamma \left( 1 - \frac{D}{2} \right) \mu^{D-2}.$$

(33)

Replacing the bare coupling constant $\lambda_0$ with the renormalized one $\lambda$ we obtain the renormalized effective potential

$$V_0(\sigma) = \frac{1}{2\lambda} \sigma^2 \mu^{D-2} + \frac{\text{tr} 1}{(4\pi)^{D/2}} (D-1) \Gamma \left( 1 - \frac{D}{2} \right) \sigma^2 \mu^{D-2} - \frac{\text{tr} 1}{(4\pi)^{D/2}} \Gamma \left( 1 - \frac{D}{2} \right) \sigma^D.$$

(34)

In Minkowski spacetime the renormalized effective potential $V_0(\sigma)$ is no longer divergent in the whole range of the spacetime dimensions considered here, i.e. for $2 \leq D < 4$.

We consider the two-, three- and four-dimensional limit of the effective potential (34). Taking the two dimensional limit, $D \to 2$, we get

$$\frac{V_0^{D=2}(\sigma)}{\mu^2} = \frac{1}{2\lambda} \left( \frac{\sigma}{\mu} \right)^2 + \frac{\text{tr} 1}{8\pi} \left[ -3 + \ln \left( \frac{\sigma}{\mu} \right) \right] \left( \frac{\sigma}{\mu} \right)^2.$$

(35)

Taking the three dimensional limit $D \to 3$, we find

$$\frac{V_0^{D=3}(\sigma)}{\mu^3} = \frac{1}{2\lambda} \left( \frac{\sigma}{\mu} \right)^2 - \frac{\text{tr} 1}{4\pi} \left[ \left( \frac{\sigma}{\mu} \right)^2 - \frac{1}{3} \left( \frac{\sigma}{\mu} \right)^3 \right].$$

(36)

Equations (35) and (36) are not divergent for finite $\sigma$ confirming that the divergent part in the effective potential (31) is removed by the renormalization procedure for $2 \leq D < 4$.

If we take the four-dimensional limit $D \to 4$, the effective potential (34) reads

$$\frac{V_0^{D=4}(\sigma)}{\mu^4} = \frac{1}{2\lambda} \left( \frac{\sigma}{\mu} \right)^2 - \frac{\text{tr} 1}{4(4\pi)^2} \left[ 6 \left( C_{\text{div}} - \frac{2}{3} \right) \left( \frac{\sigma}{\mu} \right)^2 - \left[ C_{\text{div}} + \frac{1}{2} - \ln \left( \frac{\sigma}{\mu} \right) \right] \left( \frac{\sigma}{\mu} \right)^4 \right].$$

(37)

\(^5\)If we perform the momentum integration by using the cut-off regularization, linear divergence appears in three dimensions. The divergence is dropped in Eq. (31) by the analytic continuation of $\Gamma$ function. After renormalization both regularization methods give the same results.
where we express the divergent part by
\[ C_{\text{div}} = \frac{2}{4-D} - \gamma + \ln 4\pi + 1. \] (38)

On the other hand we can calculate the effective potential for \( D = 4 \) by using the cut-off regularization. Inserting Eq. (30) into Eq. (25) at \( D = 4 \) we obtain the effective potential for \( D = 4 \),
\[ V_{0}^{D=4}(\sigma) = \frac{1}{2\lambda_0} \sigma^2 - \text{tr} \int \sigma \int \frac{d^4k}{(2\pi)^4} \frac{1}{k - s}. \] (39)

We cut off the higher momentum region of the divergent integral in Eq. (39) and get
\[ V_{0}^{D=4}(\sigma) = \frac{1}{2\lambda_0} \sigma^2 - \frac{\text{tr} \Gamma(1-D/2)}{(4\pi)^{D/2}} \left[ 2\sigma^2 \Lambda^2 - \frac{1}{2} \sigma^4 - \sigma^4 \ln \left( \frac{\Lambda}{\sigma} \right) \right] + O \left( \frac{\sigma^2}{\Lambda^2} \right), \] (40)

where \( \Lambda \) is a cut-off scale of the divergent integral. We apply the same renormalization condition \(^6\) as shown in Eq. (32) and obtain the renormalized coupling \( \lambda \):
\[ \frac{1}{\lambda_0} = \frac{\mu^2}{\lambda} + \frac{\text{tr} \Gamma(1-D/2)}{(4\pi)^{D/2}} \left[ \Lambda^2 + 2\mu^2 - 3\mu^2 \ln \left( \frac{\Lambda}{\mu} \right) \right]. \] (41)

Substituting Eq. (41) into Eq. (40) we obtain the renormalized effective potential for \( D = 4 \)
\[ \frac{V_{0}^{D=4}(\sigma)}{\mu^2} = \frac{1}{2\lambda} \left( \frac{\sigma}{\mu} \right)^2 - \frac{1}{6} \left[ \ln \left( \frac{\Lambda}{\mu} \right)^2 - \frac{2}{3} \left( \frac{\sigma}{\mu} \right)^2 \right] - \left\{ \frac{1}{2} - \ln \left( \frac{\sigma}{\Lambda} \right) \right\} \left( \frac{\sigma}{\mu} \right)^4. \] (42)

We find that there is a correspondence \([16]\) between Eq. (37) and Eq. (42) if we make a replacement
\[ C_{\text{div}} \leftrightarrow \ln \left( \frac{\Lambda}{\mu} \right)^2. \] (43)

This correspondence rule is to be modified in curved spacetime (see §3.2).

### 2.4 Phase structure in Minkowski spacetime

In terms of the effective potential \( V_0(\sigma) \) we can argue the phase structure of the four-fermion model. Using Eq. (34) the effective potential is calculated numerically. In Fig. 2 we present the typical behavior of the effective potential evaluated at \( D = 2.5 \). In drawing Fig. 2 we introduce, for convenience, the new variables \( m_0 \) which will be defined by Eq. (47) and \( m_0' \) defined by
\[ m_0' = \mu \left[ -\frac{(4\pi)^{D/2}}{\text{tr} \Gamma(1-D/2)} \frac{1}{\lambda} - D + 1 \right]^{1/(D-2)}. \] (44)

As is shown in Fig. 2 the shape of the effective potential is of a single and a double well for sufficiently small and large coupling constant \( \lambda \) respectively. Hence we expect that the effective potential changes its shape at a critical value of the coupling constant \( \lambda_c \). We can see in Fig. 2 that the ground state

---

\(^6\)Since the four-fermion model is not renormalizable in four dimensions, we do not apply the renormalization in four dimensions. Note here that the direct comparison of the dimensional regularization with the cut-off regularization is possible only after renormalizing the coupling constant \( \lambda \) under the common renormalization condition.
is invariant under the discrete chiral transformation for a sufficiently small coupling constant. On the other hand for \( \lambda > \lambda_{cr} \) the minimum of the effective potential is located at non-vanishing \( \sigma \). In the latter case the auxiliary field \( \sigma \) develops the non-vanishing vacuum expectation value and the discrete chiral symmetry is broken down dynamically. As the result the fermion acquires the dynamical mass.

Next we calculate the dynamically generated fermion mass for \( \lambda > \lambda_{cr} \). It is equal to the vacuum expectation value of \( \sigma \) which is obtained by observing the minimum of the effective potential. The necessary condition for the minimum is given by the gap equation:

\[
\left. \frac{\partial V_0 (\sigma)}{\partial \sigma} \right|_{\sigma=m_0} = m_0 f(m_0, \lambda) = 0,
\]

where \( f(m_0, \lambda) \) is given by

\[
f(m_0, \lambda) = \frac{1}{\lambda} \mu^{D-2} + \frac{\text{tr} 1}{(4\pi)^{D/2}} (D-1) \Gamma \left( 1 - \frac{D}{2} \right) \mu^{D-2} - \frac{\text{tr} 1}{(4\pi)^{D/2}} \Gamma \left( 1 - \frac{D}{2} \right) m_0^{D-2}.
\]

The dynamical fermion mass is obtained by the non-trivial solution of the gap equation (45). Differentiating the effective potential (34) and solving the equation \( f(m_0, \lambda) = 0 \) for \( m_0 \), we find the dynamical fermion mass:

\[
m_0 = \mu \left[ \frac{(4\pi)^{D/2}}{\text{tr} 1 \Gamma(1-D/2)} \frac{1}{\lambda} + \frac{1}{D-1} \right]^{1/(D-2)}.
\]

In Fig. 3 we plot the dynamical fermion mass \( m_0 \) as a function of the coupling constant \( \lambda \) for \( D = 2, 2.5, 3, 3.5 \). The dynamical fermion mass decreases as \( 1/\lambda \) increases, so that the dynamical fermion mass disappears at a critical value of \( \lambda \). Since no mass gap is observed at the critical point in Fig. 3, the phase transition at the critical coupling \( \lambda_{cr} \) is of the second order. In two dimensions the dynamical
fermion mass decreases exponentially as $1/\lambda$ increases. Thus the critical coupling constant $\lambda_{cr} = 0$ for $D = 2$.

For second order phase transition the critical coupling constant $\lambda_{cr}$ is obtained by taking the limit $m_0 \to 0$:

$$\lim_{m_0 \to 0} f(m_0, \lambda_{cr}) = 0,$$

where $f(m_0, \lambda_{cr})$ is defined in Eq. (45). Solving Eq. (48) we easily find the critical coupling constant

$$\lambda_{cr} = \frac{(4\pi)^{D/2}}{\tr 1} \left[ (1 - D)\Gamma \left( 1 - \frac{D}{2} \right) \right]^{-1}.$$  

(49)

For $\lambda > \lambda_{cr}$ the dynamical fermion mass is generated and the discrete chiral symmetry is broken down. In Fig. 4 we plot the critical coupling constant $\lambda_{cr}$ as a function of the spacetime dimension $D$.

For some specific values of $D$ the solutions $m_0$ and $\lambda_{cr}$ simplify:

$$m_0 = \mu e^{1 - 2\pi/(\tr 1 \lambda)}, \quad \lambda_{cr} = 0; \quad D = 2,$$

$$m_0 = \mu \left( 2 - \frac{4\pi}{\tr 1 \lambda} \right), \quad \lambda_{cr} = \frac{2\pi}{\tr 1}; \quad D = 3.$$  

(50)

Taking the four-dimensional limit, $D \to 4$, we find

$$\left[ C_{\text{div}} - \ln \left( \frac{m_0}{\mu} \right)^2 \right] \left( \frac{m_0}{\mu} \right)^2 = \frac{(4\pi)^2}{\tr 1 \lambda} + 3 \left( C_{\text{div}} - \frac{2}{3} \right),$$

(51)

$$\frac{1}{\lambda_{cr}} = \frac{3\tr 1}{(4\pi)^2} \left( C_{\text{div}} - \frac{2}{3} \right),$$

(52)

where the divergent part $C_{\text{div}}$ is defined in Eq. (38).
2.5 Renormalization group analysis in Minkowski spacetime

We analyze the \( \beta \)-function for the four-fermion coupling \( \lambda \). It has been often pointed out that the fermions acquire the dynamical mass for a larger coupling than the ultraviolet stable fixed point.\[9, 17\]

The renormalization group \( \beta \)-function for the four-fermion coupling is defined by

\[
\beta(\lambda) = \mu \frac{\partial \lambda}{\partial \mu} \bigg|_{\lambda_0},
\]

Taking into account Eq. (33) with Eq. (49) we obtain

\[
\beta(\lambda) = \frac{D - 2}{\lambda_{cr}} \lambda(\lambda_{cr} - \lambda),
\]

in the leading order of the \( 1/N \) expansion.

In Fig. 5 we show the behavior of the \( \beta \)-functions. As is obviously observed in Fig. 5 the theory is asymptotically free for \( D = 2 \) and the theory for \( 2 < D < 4 \) has a nontrivial ultraviolet stable fixed point at

\[
\lambda = \lambda_{cr}.
\]

It simplifies for some special values of \( D \):

\[
\begin{align*}
\beta(\lambda) &= -\frac{\text{tr} 1}{2\pi} \lambda^2; \quad D = 2, \\
\beta(\lambda) &= -\lambda \left( \frac{\text{tr} 1}{2\pi} \lambda - 1 \right); \quad D = 3.
\end{align*}
\]

Taking the four-dimensional limit, \( D \to 4 \), we find

\[
\beta(\lambda) = -2\lambda \left[ \frac{3\text{tr} 1}{(4\pi)^2} \left( C_{\text{div}} - \frac{5}{3} \right) \lambda - 1 \right]; \quad D \to 4,
\]
where $C_{\text{div}}$ is defined in Eq. (38). For a larger coupling than the ultraviolet stable fixed point a tachyon pole appears in the $\sigma$ propagator. Then the classical vacuum state becomes unstable. The generation of the dynamical fermion mass helps recovering the stability of the vacuum.

The basic properties of the four-fermion models have been briefly reviewed. We calculated the effective potential and discussed the phase structure of the four-fermion model (Gross-Neveu type model) in Minkowski spacetime. In the leading order of the $1/N$ expansion the effective potential is given by the integration of the spinor two-point function $S(x,x;s)$ over mass variable $s$. Evaluating the effective potential in the leading order of the $1/N$ expansion it was shown that the discrete chiral symmetry is broken down dynamically for a sufficiently large coupling constant $\lambda > \lambda_{\text{cr}}$. Dynamically generated fermion mass $m_0$ and the critical point $\lambda_{\text{cr}}$ was obtained analytically by solving the gap equation (45). Since no mass gap appears at the critical point, the phase transition from the broken phase to the symmetric phase is of the second order. Applying the renormalization group analysis it was shown that the critical coupling constant $\lambda_{\text{cr}}$ corresponds to the ultraviolet stable fixed point.

In the present section we basically used the dimensional regularization method to investigate the phase structure for $2 \leq D < 4$ and considered the theory for $D = 4 - \epsilon$ with $\epsilon$ sufficiently small positive as a regularized version of the one in four dimensions. As was shown the effective potential can be calculated also by cut-off regularization. In this case the same result is obtained with recourse to the relation between $\epsilon$ and $\Lambda$ (43).

In the following sections we will apply the similar method to the case of the curved spacetime.

### 3 Four-fermion models in weakly curved spacetime

At an early stage of the universe it is generally assumed that broken symmetries are restored and the physical phenomena are described by a more fundamental theory with a higher symmetry. The spontaneous symmetry breaking may be induced under the influence of the strong curvature, finite volume, non-trivial topology and so on.

Here we discuss the curvature induced phase transition in four-fermion models. For this purpose the
effective potential\(^7\) of the four-fermion models need to be calculated in a general curved spacetime. In the present section we consider the theory in weakly curved spacetime. We shall confine ourselves to the case that the spacetime is curved very moderately and shall neglect terms involving derivatives of the metric higher than second order.

There are many studies of the phase transition in weakly curved spacetime. Spontaneous symmetry breaking of the massless scalar theory with \(\lambda\phi^4\) interaction has been studied in four dimensions (see for example Refs. [18, 19] and the references there in). In the scalar theory the continuous \(U(1)\) symmetry is restored for a sufficiently large positive curvature. The pioneering works in the four-fermion model have been performed by Itoyama,[20] and Buchbinder and Kirillova[21] in two dimensions. In Refs. [22]∼[24] four-fermion models have been studied in three, four and arbitrary dimensions (3 \(\leq D < 4\)) respectively. In these literatures it is found that the possibility of the curvature-induced phase transition in four-fermion models.

In this section we mainly follow Refs. [22] and [24] and discuss the phase structure of the Gross-Neveu type model in weakly curved spacetime by using the similar method explained in the previous section.

3.1 Riemann normal coordinate expansion of \(S(x, y; s)\)

As is shown in the previous section the effective potential of the Gross-Neveu type model is described by the spinor two-point function \(S(x, x; s)\) in the leading order of the \(1/N\) expansion. Thus we start with the analysis of the two-point function \(S(x, x; s)\) in weakly curved spacetime.

For this purpose it is convenient to introduce the Riemann normal coordinate system which is a coordinate system with the affine connection \(\Gamma_{\mu\nu}^{\alpha}\) which vanishes at least locally (i.e., locally inertial frame). At the origin \(x_0\) of the coordinates the metric tensor \(g_{\mu\nu}\) and the affine connection \(\Gamma_{\mu\nu}^{\alpha}\) have the property

\[
g_{\mu\nu}(x_0) = \eta_{\mu\nu}, \qquad \Gamma_{\mu\nu}^{\alpha}(x_0) = 0, \tag{58}\]

where \(\eta_{\mu\nu}\) is \(\text{diag}(1, -1, -1, -1)\). Near the origin \(x_0\) the metric tensor is expanded to be [25]

\[
g_{\mu\nu}(y) = \eta_{\mu\nu} + \frac{1}{3} R_0^{\mu\nu\alpha\beta}(y - x_0)^{\alpha}(y - x_0)^{\beta} + O(R_{\mu\nu}, R^2), \]

\[
g(y) = -\frac{1}{3} R_0^{\alpha\beta}(y - x_0)^{\alpha}(y - x_0)^{\beta} + O(R_{\mu\nu}, R^2), \tag{59}\]

where index 0 for \(R_0^{\mu\nu\alpha\beta}\) and \(R_0^{\alpha\beta}\) designates the tensor at the origin \(x_0\).

By the use of the Riemann normal coordinate expansion (59) we calculate the spinor two-point function \(S(x, x; s)\) in the approximation of slowly varying curvature where we neglect any terms involving derivatives of the metric higher than second order. According to the method developed by Parker and Toms[26] the two-point function \(S(x, x; s)\) is expanded asymptotically around \(R^0 = 0\). We introduce the bispinor function \(G(x_0, y; s)\) defined by

\[
(i\gamma_{\mu}\nabla^{\mu} + s)G(x_0, y; s) = S(x_0, y; s). \tag{60}\]

It satisfies the following equation:

\[
\left( -\nabla^{\mu}\nabla_{\mu} - \frac{R}{4} - s^2 \right) G(x_0, y; s) = \frac{1}{\sqrt{-g}} \delta^{\mu}(x_0, y). \tag{61}\]

\(^7\)Since the assumption \(\sigma_c(x) = \sigma\) (constant) in (22) is not always acceptable in general curved spacetimes, it is necessary to evaluate the effective action instead of the effective potential to study the phase structure. Here we restrict ourselves to the static homogeneous spacetime where the assumption is acceptable.
Using the Riemann normal coordinate expansion (59) we expand the first term on the left-hand side of Eq. (61) and find
\[ \sqrt{-g} \nabla \nabla G(x_0, y; s) = \left[ \eta^{\mu \nu} \partial_\mu \partial_\nu + \frac{1}{6} R^0_{\alpha \beta}(y-x_0)^\alpha (y-x_0)^\beta \eta^{\mu \nu} \partial_\mu \partial_\nu ight. \\
- \frac{1}{3} R^{0 \mu}_{\alpha \beta}(y-x_0)^\alpha (y-x_0)^\beta \partial_\mu \partial_\nu \\
- \frac{2}{3} R^{0 \mu}_{\alpha \beta}(y-x_0)^\alpha \partial_\mu \\
+ \left. \frac{1}{4} R^{0 \mu}_{\alpha \beta \delta \epsilon \gamma \sigma \mu \nu}(y-x_0)^\alpha \partial_\mu + \cdots \right] G(x_0, y; s) , \quad (62) \]
where
\[ \sigma^{ab} = \frac{1}{2} [\gamma^a, \gamma^b], \quad (63) \]
and Latin indices \( a \) and \( b \) are vierbein indices. Here we keep only terms independent of the curvature \( R \) and terms linear in \( R \). Inserting the Eq. (62) into Eq. (61) and performing the following Fourier transformation,
\[ G(x_0, y; s) = \int \frac{d^4p}{(2\pi)^4} e^{-ip(x_0-y)} \tilde{G}(p, y; s) , \quad (64) \]
we find that Eq. (61) reduces to
\[ \left[ \eta^{\mu \nu} p_\mu p_\nu - \frac{1}{4} R^0 - s^2 - \frac{2}{3} R^{0 \mu}_{\alpha \beta} p_\mu \frac{\partial}{\partial p_\alpha} \frac{1}{6} R^{0 \alpha \beta \delta \epsilon \gamma \sigma \mu \nu} p_\nu \frac{\partial}{\partial p_\beta} \right. \\
+ \left. \frac{1}{4} R^{0 \mu}_{\alpha \beta \delta \epsilon \gamma \sigma \mu \nu} p_\mu \frac{\partial}{\partial p_\alpha} \right] \tilde{G}(p, y; s) = 1 . \quad (65) \]
From Eq. (65) \( \tilde{G}(p, y) \) is found to be
\[ \tilde{G}(p, y; s) = \frac{1}{p^2 - s^2} - \frac{1}{12} \frac{R^0}{(p^2 - s^2)^2} + \frac{2}{3} \frac{R^{0 \mu \nu} p_\mu p_\nu}{(p^2 - s^2)^3} + O(R^{0 \mu}_{\alpha \beta}, (R^0)^2) . \quad (66) \]
Inserting Eqs. (64) and (66) into Eq. (60) the two-point function \( S(x_0, y; s) \) in the weak curvature expansion is obtained:[24]
\[ S(x_0, y; s) = (i\gamma_\mu \nabla_\mu + s) \int \frac{d^4p}{(2\pi)^4} e^{-ip(x_0-y)} \tilde{G}(p, y) \\
= \int \frac{d^4p}{(2\pi)^4} e^{-ip(x_0-y)} \left[ \gamma^\mu p_\mu s + s \frac{1}{p^2 - s^2} - \frac{1}{12} R^0 \gamma^\mu p_\mu s + s \right. \\
+ \left. \frac{2}{3} R^{0 \mu \nu} p_\mu p_\nu \gamma^\mu p_\mu s + s \right. \\
+ \left. \frac{1}{4} \gamma^a \sigma^{cd \epsilon \sigma \mu \nu} R^{cd \epsilon \sigma \mu \nu} \frac{1}{(p^2 - s^2)^2} \right] \]
\[ + O(R^{0 \mu}_{\alpha \beta}, (R^0)^2) . \quad (67) \]
It may be easily checked that in the limit \( R^0 \to 0 \) the spinor two-point function in Minkowski space (30) is reproduced.
3.2 Effective potential in weakly curved spacetime

We would like to discuss the curvature induced phase transition in weakly curved spacetime. For this purpose we evaluate the effective potential. In the Gross-Neveu type model the effective potential is given by Eq. (25) in the leading order of the $1/N$ expansion. Substituting Eq. (67) to Eq. (25) we obtain the effective potential in weakly curved spacetime up to terms linear in $R$

$$V(\sigma) = \frac{1}{2\lambda_0} \sigma^2 - i \text{tr} 1 \int_{0}^{s} ds \frac{d^D p}{(2\pi)^D} \left[ \frac{s}{p^2 - s^2} - \frac{1}{12} R \frac{s}{(p^2 - s^2)^2} \right] + \frac{2}{3} R^{\mu\nu} \frac{s}{(p^2 - s^2)^{3/2}}.$$

(68)

Here we neglect the index 0 of $R^0$ for simplicity. In order to regularize the divergent integral in the second term of the right-hand side of Eq. (68) we apply the dimensional regularization. Integrating over $p$ and $s$ in arbitrary spacetime dimensions we obtain the effective potential (68),

$$V(\sigma) = V_0(\sigma) + V_\mu(\sigma) + O(R_{\mu\nu}, R^2),$$

(69)

where $V_0(\sigma)$ is the effective potential at $R = 0$ given in Eq. (31) and $V_\mu(\sigma)$ is the effective potential linear in $R$,\[22, 24\]

$$V_\mu(\sigma) = \frac{\text{tr} 1}{(4\pi)^{D/2}} \frac{R}{24} \Gamma \left( 1 - \frac{D}{2} \right) \sigma^{D-2}.$$

(70)

As is mentioned in §2.3 the effective potential $V_0(\sigma)$ is divergent in two and four dimensions. We apply the same renormalization condition as the one in Eq. (32) and obtain the renormalized effective potential as in the $R = 0$ case. Replacing the bare coupling constant $\lambda_0$ with the renormalized one $\lambda$ in Eq. (33) we obtain the renormalized effective potential,

$$V(\sigma) = \frac{1}{2\lambda} \sigma^2 \mu^{D-2} + \frac{\text{tr} 1}{2(4\pi)^{D/2}} (D - 1) \Gamma \left( 1 - \frac{D}{2} \right) \sigma^2 \mu^{D-2}$$

$$- \frac{\text{tr} 1}{(4\pi)^{D/2}} \frac{R}{24} \Gamma \left( 1 - \frac{D}{2} \right) \sigma^{D-2} - \frac{\text{tr} 1}{(4\pi)^{D/2}} \frac{R}{24} \Gamma \left( 1 - \frac{D}{2} \right) \sigma^{D-2}.\quad (71)$$

Before the discussion of the phase structure we consider the two-, three- and four-dimensional limits of the effective potential (71). Taking the two dimensional limit, $D \to 2$, we get

$$\frac{V^{D=2}(\sigma)}{\mu^D} = \frac{1}{2\lambda} \left( \frac{\sigma}{\mu} \right)^2 + \frac{\text{tr} 1}{8\pi} \left[ -3 + \ln \left( \frac{\sigma}{\mu} \right)^2 \right] \left( \frac{\sigma}{\mu} \right)^2$$

$$+ \frac{\text{tr} 1}{96\pi \mu^2} \frac{R}{2 - D} - \gamma + \ln 4\pi - \ln \left( \frac{\sigma}{\mu} \right)^2.$$

(72)

It is different from the expression obtained in Ref. [21].\[8\] The effective potential $V_\mu(\sigma)$ (i.e. the third term on the right-hand side in the Eq. (72)) is also divergent in two dimensions. For $D = 2$ the

\[8\] Differentiating Eq. (72) we find

$$\frac{\partial V^{D=2}}{\partial \sigma} = \frac{1}{\lambda} \left( \frac{\sigma}{\mu} \right)^2 + \frac{\text{tr} 1}{4\pi} \left[ -2 + \ln \left( \frac{\sigma}{\mu} \right)^2 + \frac{R}{12\sigma^2} \right].$$

(73)

On the other hand Eq. (4.3) in Ref. [21] reads

$$\frac{\partial V^{D=2}}{\partial \sigma} = \frac{1}{\lambda} \left( \frac{\sigma}{\mu} \right)^2 + \frac{\text{tr} 1}{4\pi} \left[ -2 + \ln \left( \frac{\sigma}{\mu} \right)^2 + \frac{R}{4\sigma^2} \right],$$

(74)

in the weak curvature limit where we adopt the notation in the present paper. The common renormalization condition is adopted in the above two cases. As is mentioned in §2 the four-fermion theory is renormalizable. The theory is independent of the regularization method. Thus the above two results should agree with each other. The solution of the equation for the evolution operator in Ref. [21] seems not to be the general solution but a special solution.
divergence comes from the lower momentum region of the $p$-integral in Eq. (68). To show it we cut off the higher and lower momentum regions of the divergent integral in Eq. (68) and find
\[
V^{D=2}(\sigma) = \frac{1}{2\lambda_0}\sigma^2 + \frac{\text{tr} L}{8\pi} \left[ 1 + \ln \left( \frac{\Lambda_{UV}}{\sigma} \right)^2 \right] + \frac{1}{2} \ln \left( \frac{\Lambda_{IR}}{\sigma} \right)^2 + \frac{\Lambda^2_{UV}}{\Lambda^2_{UV} + \sigma^2} + \frac{\Lambda^2_{IR}}{\Lambda^2_{IR} + \sigma^2} + O\left( \frac{\sigma^2 \Lambda^2_{UV}}{\sigma^2} \right)
\]
where $\Lambda_{UV}$ is the ultraviolet cut-off scale and $\Lambda_{IR}$ is the infrared cut-off scale of the divergent integral. The ultraviolet divergence is cancelled out after renormalization and the terms independent of $R$ reduce to Eq. (35). As is seen in Eq. (75) the infrared divergence appears in the terms linear in $R$ for $D = 2$.

Because of the infrared divergence the symmetry restoration always occurs for any positive values of the spacetime curvature $R$ whereas only a broken phase is realized for negative $R$.

Taking the three-dimensional limit $D \to 3$ we find [23]
\[
\frac{V^{D=3}(\sigma)}{\mu^3} = \frac{1}{2\lambda} \left( \frac{\sigma}{\mu} \right)^2 - \frac{\text{tr} L}{4\pi} \left[ \left( \frac{\sigma}{\mu} \right)^2 - \frac{1}{3} \left( \frac{\sigma}{\mu} \right)^3 \right] + \frac{\text{tr} L \mu}{96\pi \mu^2} R \left( \frac{\sigma}{\mu} \right)^2 .
\] (76)
If we take the four-dimensional limit $D \to 4$, the effective potential (71) reduces to
\[
\frac{V^{D=4}(\sigma)}{\mu^4} = \frac{1}{2\lambda} \left( \frac{\sigma}{\mu} \right)^2 - \frac{\text{tr} L}{4(4\pi)^2} \left\{ 6 \left( C_{\text{div}} - \frac{2}{3} \right) \left( \frac{\sigma}{\mu} \right)^2 + C_{\text{div}} + \frac{1}{2} \ln \left( \frac{\sigma}{\mu} \right)^2 \right\} + \frac{\text{tr} L}{4(4\pi)^2 \mu^2} \left( C_{\text{div}} - \ln \left( \frac{\sigma}{\mu} \right)^2 \right) \left( \frac{\sigma}{\mu} \right)^2 ,
\] (77)
where $C_{\text{div}}$ is defined in Eq. (38).

To see the relationship between $C_{\text{div}}$ and a cut-off parameter we calculate the effective potential for $D = 4$ by using the cut-off regularization. Performing the integration in Eq. (68) with the cut-off $\Lambda$ in the higher momentum region one gets [22]
\[
V^{D=4}(\sigma) = V_0^{D=4}(\sigma) + \frac{\text{tr} L}{(4\pi)^2} \left[ \ln \left( \frac{\Lambda}{\sigma} \right)^2 - 1 \right] \sigma^2 + O\left( \frac{\sigma^2 \Lambda^2}{\sigma^2} \right)
\] (78)
where $V_0(\sigma)$ is given by Eq. (40). Applying the renormalization condition given by Eq. (32) $V_0(\sigma)$ reduces to (42). After the renormalization we find that there is a correspondence between two results (77) and (78), if we make a replacement [24]
\[
C_{\text{div}} + \frac{R}{6} \frac{1}{6\mu^2 - \sigma^2} \leftrightarrow \ln \frac{\Lambda^2}{\mu^2} .
\] (79)
We recognize that the previous correspondence rule (43) is now modified by the term including curvature in Eq. (79).

Now we have succeeded to expand the effective potential asymptotically in terms of the spacetime curvature $R$ by using the Riemann normal coordinate. Starting from the effective potential (71) we investigate the phase structure of the theory in weakly curved spacetime in the following subsection.
3.3 Curvature induced phase transition

The vacuum expectation value is determined by observing the minimum of the effective potential. In weakly curved spacetime the effective potential is modified as is mentioned above. Thus the vacuum expectation value of $\sigma$ will be changed by the curvature effect. We expect that the phase transition takes place by varying the spacetime curvature $R$.

3.3.1 Phase structure for $\lambda > \lambda_{cr}$

In Minkowski spacetime, as is shown in §2.3, the system is in broken phase for $\lambda > \lambda_{cr}$. Here in the present subsection we fix the coupling constant $\lambda$ larger than the critical value $\lambda_{cr}$ and see whether the chiral symmetry is restored by the curvature effect. To study the phase structure in curved spacetime we evaluate the effective potential (71) numerically with the varying curvature. In Fig. 6 we present the typical behavior of the effective potential (71) for several values of the curvature in the case $D = 2.5$ and $D = 3.5$. Here we adopted the formula $\text{tr}1 = 2^{D/2}$. We find that the chiral symmetry is restored as $R$ is increased with $\lambda$ fixed. As can be seen in Fig. 6 the phase transition induced by curvature effects is of the first order. On the other hand no phase transition takes place and the vacuum expectation value of the auxiliary field simply increases for the negative curvature as its absolute value increases. We observe the similar behavior of the effective potential in the spacetime dimensions, $2 < D < 4$.

In the leading order of the $1/N$ expansion the dynamical fermion mass is equal to the vacuum expectation value of the auxiliary field $\langle \sigma \rangle$. The dynamical fermion mass is given through the minimum of the effective potential. The necessary condition for the minimum is given by the gap equation:

$$\frac{\partial V(\sigma)}{\partial \sigma} \bigg|_{\sigma = m} = 0.$$  

(80)

The non-trivial solution of the gap equation corresponds to the dynamical mass of the fermion. Inserting the Eqs. (71) and (47) into Eq. (80) the non-trivial solution of the gap equation is expressed as

$$m_0^{D-2} - m^{D-2} + \frac{R}{12} \left(1 - \frac{D}{2}\right) m^{D-4} = 0.$$  

(81)

In Fig. 7 we plot the non-trivial solution of the gap equation as a function of the spacetime dimension $D$.

Since the phase transition induced by curvature effects is of the first-order, two independent solutions appear for a small positive curvature. The larger solution corresponds to the local minimum and the smaller solution represents the first extremum of the effective potential.

At the critical point the two local minima of the effective potential acquire the same value which is zero. Thus the critical point is obtained by the solution of Eq. (81) supplemented by the condition

$$V(m) = V(0) = 0.$$  

(82)

We solve Eqs. (81) and (82) and obtain the critical curvature,[24]

$$R_{cr} = 6(D - 2) \left(\frac{D(4 - D)}{4}\right)^{(4-D)/(D-2)} m_0^2,$$  

(83)

and the mass gap at the critical point,[24]

$$m_{cr} = \left(\frac{D(4 - D)}{4}\right)^{1/(D-2)} m_0.$$  

(84)

If the spacetime curvature $R$ is less than the critical value $R_{cr}$, the vacuum expectation value of the auxiliary field $\langle \sigma \rangle$ is given by the larger solution of Eq. (81). In the case that the spacetime curvature
Figure 6: Behavior of the effective potential is shown at $D = 2.5$ and $D = 3.5$ for fixed $\lambda > \lambda_{cr}$ with the varying curvature where $R_{cr} = 6(D - 2)(D(4 - D)/4)^{(4 - D)/(D - 2)}m_0^2 > 0$. 
Figure 7: Solutions of the gap equation for $D = 2, 2.5, 3, 3.5, 4.$

Figure 8: Dynamical fermion mass as a function of the spacetime curvature.
Figure 9: Critical curvature as a function of dimension $D$ in weakly curved spacetime.

$R$ is larger than the critical value $R_{cr}$, the chiral symmetry is restored and the vacuum expectation value of the auxiliary field $\langle \sigma \rangle$ disappears. The behavior of the dynamical fermion mass as a function of $R$ is given in Fig. 8. We observe that the mass gap appears at $R = R_{cr}$ and disappears in the four-dimensional limit. For some special values of $D$ Eq. (83) simplifies:

$$R_{cr} = 0; \quad D = 2,$$
$$R_{cr} = \frac{9}{2} m_0^2; \quad D = 3,$$
$$R_{cr} = 12m_0^2; \quad D = 4.$$  \hfill (85)

In Fig. 9 we show the critical curvature $R_{cr}$ as a function of the spacetime dimensions $D$. In two dimensions the weak curvature expansion shows that the broken chiral symmetry is restored by the effect of an infrared divergence for any positive curvature. The critical curvature $R_{cr}^{D=2}$ is thus zero.

### 3.3.2 Phase structure for $\lambda \leq \lambda_{cr}$

For $\lambda \leq \lambda_{cr}$ the system is in symmetric phase in Minkowski spacetime. In the present subsection we investigate whether the chiral symmetry is broken down by the curvature effect for $\lambda \leq \lambda_{cr}$. We introduce, for convenience, the scale $m_0'$ defined in Eq. (44) and calculate the effective potential numerically. In Fig. 10 the behavior of the effective potential (71) is presented with the varying curvature at $D = 2.5$ and $D = 3.5$. We observe the second order phase transition as the curvature decreases. The chiral symmetry is always broken down for the negative curvature even if the coupling constant $\lambda$ is very small. Thus the critical curvature is given by $R_{cr} = 0$ in the whole range of the spacetime dimensions $D$ considered here, $2 \leq D < 4$.

In the case with $\lambda \leq \lambda_{cr}$ Eq. (81) is modified to be

$$-m_0'^{-2} - m^{-2} + \frac{R}{12} \left(1 - \frac{D}{2}\right) m^{D-4} = 0.$$  \hfill (86)
Figure 10: The behavior of the effective potential is shown at $D = 2.5$ and $D = 3.5$ for fixed $\lambda \leq \lambda_{cr}$ with the varying curvature where $K = 6(D - 2)(D(4 - D)/4)^{(4 - n)/(n - 2)}m'_0 > 0$. 
The dynamical mass of the fermion is given by the solution of the Eq. (86). In Fig. 11 we present the dynamical mass of the fermion as a function of the spacetime dimensions $D$. As is expected for the second order phase transition, the dynamical mass of the fermion smoothly disappears at $R = R_{\text{cr}} = 0$.

### 3.3.3 Phase structure in four dimensions

We are interested in applying the results to the critical phenomena at the early stage of the universe. Thus we focus on the study of the phase structure in four dimensions here.

In four dimensions the four-fermion models are not renormalizable. In Minkowski spacetime the correspondence (43) does not depend on $\sigma$. Thus both the dimensional and cut-off regularization describe the equivalent theory. The differences are removed by the finite renormalization. Therefore it is not necessary to investigate the theories defined by the different regularizations in Minkowski spacetime.

In curved spacetime the correspondence (79) depends on $\sigma$ and the curvature $R$. Thus above results in weakly curved spacetime may depend on the regularization methods. We regard the theory for $D = 4 - \epsilon$ with $\epsilon$ sufficiently small positive as a regularization of the one in four dimensions. In the previous subsections we found that the broken chiral symmetry was restored for a sufficiently large positive curvature $R > R_{\text{cr}}$ and the phase transition induced by curvature effects was of the first order for a positive finite $\epsilon$.

To see whether the phase structure depends on the regularization method we define the theory by the cut-off regularization.[22] In this case the effective potential is given by Eqs. (40) and (78) and the gap equation defined by Eq. (80) reads\(^9\)

$$
\frac{1}{\lambda_0} - \frac{\text{tr} \mathbf{1}}{(4\pi)^2} \left[ \Lambda^2 - m^2 \ln \left( \frac{\Lambda^2}{m^2} \right) \right] - \frac{\text{tr} \mathbf{1}}{192\pi^2} R \left[ - \ln \left( \frac{\Lambda^2}{\sigma_0^2} \right) + 2 \right] = 0.
$$

(87)

The dynamical fermion mass is obtained by analyzing the gap equation (87). In Fig. 12 the dynamical mass of the fermion is plotted as a function of the curvature $R$ for fixed $\lambda_0$. The coupling constant

\[^9\text{In the present subsection we do not adopt the renormalization procedure. It is not necessary in four dimensions because the theory is not renormalizable.}\]
Figure 12: The behavior of the dynamical fermion mass where $\lambda_0 = 1.25\lambda_{cr}$ and $R_{cr} = 0.656\Lambda^2$.

Figure 13: The phase diagram in $\lambda_0$-$R$ plane for the four-fermion model defined by the cut-off regularization.
is kept in the range \( \lambda_0 > \lambda_{cr} = 4\pi^2/\Lambda^2 \). The behavior of the dynamical mass as shown in Fig. 12 is characteristic for the relatively small coupling constant, \( \lambda_{cr} < \lambda_0 \leq 2\lambda_{cr} \). For larger coupling, \( \lambda_0 \geq 2\lambda_{cr} \), the behavior near the critical point \( R = R_{cr} \) is quite different from the one in Fig. 12: the curve representing the dynamical mass is bent upward near \( R = R_{cr} \). At any rate there is observed the gap in the dynamical mass at the critical point \( R = R_{cr} \) reflecting the nature of the first order phase transition. By the direct numerical analysis we find that there is no gap at \( R = R_{cr} = 0 \) if \( \lambda_0 \leq \lambda_{cr} \) and so the phase transition is of the second order.

It is possible to obtain the critical value of the curvature \( R \) and the coupling constant \( \lambda_0 \) by observing the behavior of the effective potential. The critical values \( R_{cr} \) and \( \lambda_{cr} \) constitute a critical curve in the \( R - \lambda_0 \) plane as shown in Fig. 13. It is found from Fig. 13 that for large positive \( R \) the chiral symmetry is restored even if \( \lambda \) is kept in the region of the broken phase for \( R = 0 \). On the other hand the chiral symmetry is always broken for negative \( R \) irrespective of the value of \( \lambda \). Thus the similar phase structure was found in the theory defined by the cut-off regularization.

To summarize this section we have employed the weak curvature expansion to evaluate the effective potential in the leading order of the \( 1/N \) expansion. We assumed that the spacetime is weakly curved and have investigated the phase structure.

Starting from the theory with broken chiral symmetry for vanishing \( R \) we calculated the effective potential for finite \( R \) in arbitrary dimensions \( 2 \leq D < 4 \). We found that the chiral symmetry is restored for a large positive curvature. The critical curvature dividing the symmetric phase and broken phase is obtained analytically. Starting from the theory with chiral symmetry for vanishing \( R \) it was found that the symmetry is broken down in the spacetime with the negative curvature. Therefore only a broken phase occurs for the negative \( R \).[22, 24]

4 Exact solutions in special spacetimes

In the previous section the curvature induced phase transition was discussed by using the weak curvature approximation. It is found that the broken chiral symmetry is restored for a large positive curvature. But the large positive curvature may spoil the validity of the weak curvature approximation. To study the phase transition in the large positive curvature in more confident manner we need to avoid any approximation in dealing with the spacetime curvature. In some specific spacetimes the effective potential is calculable rigorously in the leading order of the \( 1/N \) expansion without using weak curvature approximation because of the symmetry of the spacetime.

In the present section we try to find the effective potential of the four-fermion model without making any approximation in the spacetime curvature. For this purpose we restrict ourselves to specific spacetimes, i.e., maximally symmetric spacetime \( (S^D, H^D) \) [27, 28, 29, 30] and Einstein universe \( (R \otimes S^{D-1}) \) [31, 32, 33, 34] and calculate the effective potential in the leading order of the \( 1/N \) expansion. The phase structure of the four-fermion model is obtained by using the same method developed in §2 and §3. To justify the weak curvature approximation we compare the result with the one obtained in §3. It is shown that the weak curvature approximation gives the exact result [24, 28, 34] in the limit \( D \to 4 \) or \( \Lambda \to \infty \). We follow here Refs. [27, 34] and [35].

4.1 Spinor two-point function in \( S^D \) and \( H^D \)

As we have seen in §2 the effective potential for the composite field in the leading order of the \( 1/N \) expansion for the Gross-Neveu type model is described by the two-point function \( S(x, x; s) \) of the massive free fermion in curved spacetime. Thus we start with the analysis of the two-point function \( S(x, y; s) \) in the maximally symmetric spacetime.

In the maximally symmetric spacetime \( S^D \) and \( H^D \) \( (2 \leq D < 4) \) the exact expression of the two-point function is known without making any approximation with respect to the spacetime curvature.\[36, 37, \]

\[10\] Large positive curvature may spoil the validity of the weak curvature expansion. This point will be discussed in §4.
We then restrict ourselves to the maximally symmetric spacetime and closely follow the method developed by Camporesi.\cite{39} Here we consider the manifolds $S^D$ and $H^D$. The manifold $S^D$ is defined by the metric

$$ds^2 = a^2(d\theta^2 + \sin^2 \theta d\Omega_{D-1}),$$

(88)

where $d\Omega_{D-1}$ is the metric on a unit sphere $S^{D-1}$ while the manifold $H^D$ is defined by

$$ds^2 = a^2(d\theta^2 + \sinh^2 \theta d\Omega_{D-1}).$$

(89)

The manifold $S^D$ and $H^D$ are constant curvature spacetimes with positive and negative curvature

$$R = \pm D(D - 1)a^{-2},$$

(90)

respectively ($2 \leq D < 4$).

The spinor two-point function $S(x, y; s)$ is defined by the Dirac equation

$$(\nabla + s)S(x, y; s) = -\frac{1}{\sqrt{g}}\delta^D(x, y),$$

(91)

where $\delta^D(x, y)$ is the Dirac delta function in the maximally symmetric spacetime. We introduce the bispinor function $G$ defined by

$$(\nabla - s)G(x, y; s) = S(x, y; s).$$

(92)

According to Eq. (91) $G(x, y; s)$ satisfies the following equation,

$$(\nabla_\nabla - s^2)G(x, y; s) = -\frac{1}{\sqrt{g}}\delta^D(x, y).$$

(93)

On $S^D$ and $H^D$ we rewrite Eq. (93) in the following form

$$\left(\Box_D - \frac{R}{4} - s^2\right)G(x, y; s) = -\frac{1}{\sqrt{g}}\delta^D(x, y),$$

(94)

where $\Box_D$ is the Laplacian on the maximally symmetric spacetime.

The general form of the Green function $G(x, y; s)$ is written as\cite{39}

$$G(x, y; s) = U(x, y)g_D(l),$$

(95)

where $U$ is a matrix in the spinor indices, $g_D$ is a scalar function depending only on $l$, $l = a\theta$ which is the geodesic distance between $x$ and $y$ on the maximally symmetric spacetime, $n_i$ is a unit vector tangent to the geodesic $n_i = \nabla_i l$. Inserting Eq. (95) into Eq. (94) we get

$$\left[U\Box_D g_D + 2(\nabla_j U)\nabla^j g_D + (\Box_D U)g_D - \left(\frac{R}{4} + s^2\right)Ug_D\right] = 0,$$

(96)

where we restrict ourselves to the region $l \neq 0$. To evaluate Eq. (96) we have to calculate the covariant derivative of $U$ and $n_i$. $U$ is the operator which makes parallel transport of the spinor at point $x$ along the geodesic to point $y$. Thus the operator $U$ must satisfy the following parallel transport equations:\cite{37}

$$\begin{cases}
  n^i \nabla_i U &= 0, \\
  U(x, x) &= 1.
\end{cases}$$

(97)

To evaluate the second derivative of $U$ we set

$$\nabla_i U \equiv V_i U.$$  

(98)
From the integrability condition\cite{36} on $V_i$,
\[ \nabla_i V_j - \nabla_j V_i - [V_i, V_j] = \frac{R}{D(D-1)} \sigma_{ij}, \]  
(99)
and the parallel transport equation (97) we easily find that
\[ V_i = -\frac{1}{a} \tan \left( \frac{l}{2a} \right) \sigma_{ij} n^j; \text{ on } S^D, \]
(100)
\[ V_i = \frac{1}{a} \tanh \left( \frac{l}{2a} \right) \sigma_{ij} n^j; \text{ on } H^D, \]
(101)
where $\sigma_{ij}$ are the antisymmetric tensors constructed by the Dirac gamma matrices, $\sigma_{ij} = \frac{i}{4}[\gamma_i, \gamma_j]$. To find $V_i$ we have used the fact that the maximally symmetric bitensors are represented as a sum of products of $n_i$ and $g_{ij}$ with coefficients which are functions only of $l$.\cite{38} After some calculations we get the Laplacian acting on $U$
\[ \Box_a U = -\frac{D-1}{4a^2} \tan^2 \left( \frac{l}{2a} \right) U, \quad \text{on } S^D, \]
(102)
\[ \Box_a U = -\frac{D-1}{4a^2} \tanh^2 \left( \frac{l}{2a} \right) U, \quad \text{on } H^D. \]
The derivative of $n_i$ is also the maximally symmetric bitensor \cite{38} and found to be
\[ \nabla_i n_j = \frac{1}{a} \cot \left( \frac{l}{a} \right) (g_{ij} - n_i n_j), \quad \text{on } S^D, \]
(103)
\[ \nabla_i n_j = \frac{1}{a} \coth \left( \frac{l}{a} \right) (g_{ij} - n_i n_j), \quad \text{on } H^D. \]
Therefore Eq. (96) reads
\[ \left( \partial_a^2 + \frac{D-1}{a} \cot \left( \frac{l}{a} \right) \partial_a - \frac{D-1}{4a^2} \tan^2 \left( \frac{l}{2a} \right) - \frac{R}{4} - s^2 \right) g_{ij} = 0, \quad \text{on } S^D, \]
(104)
\[ \left( \partial_a^2 + \frac{D-1}{a} \coth \left( \frac{l}{a} \right) \partial_a - \frac{D-1}{4a^2} \tanh^2 \left( \frac{l}{2a} \right) - \frac{R}{4} - s^2 \right) g_{ij} = 0, \quad \text{on } H^D. \]
(105)
We define the functions $h_{SD}(l)$ and $h_{HD}(l)$ by $g_{ij}(l) = \cos (l/2a) h_{SD}(l)$ and $g_{ij}(l) = \cosh (l/2a) h_{HD}(l)$ respectively and make a change of variable by $z = \cos^2 (l/2a)$ in Eq. (104) and $z' = \cosh^2 (l/2a)$ in Eq. (105). We then find that Eqs. (104) and (105) are rewritten in the forms of hypergeometric differential equations:
\[ \left( z(1-z) \partial_z^2 + \left( \frac{D+2}{2} - (D+1)z \right) \partial_z - \frac{D^2}{4} - s^2 a^2 \right) h_{SD}(z) = 0, \]
(106)
\[ \left( z'(1-z') \partial_{z'}^2 + \left( \frac{D+2}{2} - (D+1)z' \right) \partial_{z'} - \frac{D^2}{4} + s^2 a^2 \right) h_{HD}(z') = 0. \]
(107)
Noting that the Green functions are regular at the point $l = a \pi$ and fall off for $l \to \infty$ we write the solutions of Eqs. (106) and (107) by the hypergeometric function,
\[ h_{SD}(z) = c_{SD} F \left( \frac{D}{2} + isa, \frac{D}{2} - isa, \frac{D+2}{2}; z \right), \]
(108)
\[ h_{HD}(z') = c_{HD} (-z')^{-D/2-sa} F \left( \frac{D}{2} + sa, sa, 2sa + 1; \frac{1}{z'} \right). \]
(109)
As we remained in the region where \( l \neq 0 \) the normalization constants \( c_{SD} \) and \( c_{HD} \) are yet undetermined. To obtain \( c_{SD} \) and \( c_{HD} \) we consider the singularity of \( G(x, y; s) \) in the limit \( l \to 0 \),

\[
G \to c_{SD} \frac{\Gamma \left( \frac{D + 2}{2} \right) \Gamma \left( \frac{D - 2}{2} \right) \left( \frac{l}{2a} \right)^{2-D}}{\Gamma \left( \frac{D}{2} + isa \right) \Gamma \left( \frac{D}{2} - isa \right)},
\]

\[
G \to c_{HD} (-1)^{-D/2 - sa} \frac{\Gamma \left( (2sa + 1) \right) \Gamma \left( \frac{D - 2}{2} \right) \left( \frac{l}{2a} \right)^{2-D}}{\Gamma \left( \frac{D}{2} + sa \right) \Gamma \left( sa \right)},
\]

and compare them with the singularity of the Green function in the flat spacetime. This procedure is justified because the singularity on a curved spacetime background has the same structure as that in the flat spacetime. For \( l \sim 0 \) the Green function in the flat spacetime behaves as

\[
G^{\text{flat}}(l) \sim \frac{1}{4\pi^{D/2}} \Gamma \left( \frac{D - 2}{2} \right) l^{2-D}.
\]

Comparing Eq. (110) with Eq. (111) the over-all factors \( c_{SD} \) and \( c_{HD} \) are obtained:

\[
c_{SD} = \frac{a^{2-D}}{(4\pi)^{D/2}} \frac{\Gamma \left( \frac{D}{2} + isa \right) \Gamma \left( \frac{D}{2} - isa \right)}{\Gamma \left( \frac{D + 2}{2} \right)},
\]

\[
c_{HD} = (-1)^{-D/2 + sa} \frac{a^{2-D}}{(4\pi)^{D/2}} \frac{\Gamma \left( \frac{D}{2} + sa \right) \Gamma \left( sa \right)}{\Gamma \left( 2sa + 1 \right)}.
\]

Inserting the Eqs. (108), (109) and (112) into Eq. (95) we find on \( S^D \)

\[
G(x, y; s) = U(x, y) \frac{a^{2-D}}{(4\pi)^{D/2}} \frac{\Gamma \left( \frac{D}{2} + isa \right) \Gamma \left( \frac{D}{2} - isa \right)}{\Gamma \left( \frac{D + 2}{2} \right)} \times \cos \left( \frac{l}{2a} \right) \left[ F \left( \frac{D}{2} + isa, D, D - 2, \left( \frac{l}{2a} \right) \right) \right],
\]

and on \( H^D \)

\[
G(x, y; s) = U(x, y) \frac{a^{2-D}}{(4\pi)^{D/2}} \frac{\Gamma \left( \frac{D}{2} + sa \right) \Gamma \left( sa \right)}{\Gamma \left( 2sa + 1 \right)} \times \left[ \cosh \left( \frac{l}{2a} \right) \right]^{1 - D - 2sa} \left[ F \left( \frac{D}{2} + sa, sa, 2sa + 1, \cosh^{-2} \left( \frac{l}{2a} \right) \right) \right].
\]

Thus the Green functions \( G(x, y; s) \) on the maximally symmetric spacetime are obtained.

The spinor two-point function \( S(x, y; s) \) is derived from the Green function \( G(x, y; s) \). From the Eq. (92) we get

\[
S = (\gamma^i \nabla_i - s) U g_D,
\]

\[
S = \begin{cases} 
\gamma_n U \left( \left( \frac{D - 1}{2a} \tan \left( \frac{l}{2a} \right) \right) g_D - s U g_D \right), & \text{on } S^D, \\
\gamma_n U \left( \left( \frac{D - 1}{2a} \tanh \left( \frac{l}{2a} \right) \right) g_D - s U g_D \right), & \text{on } H^D.
\end{cases}
\]

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Substituting Eqs. (113) and (114) in Eq. (115) the spinor two-point function $S(x, y; s)$ is obtained \[39\]

\[
S(x, y; s) = -a^{2-d} \frac{\Gamma \left( \frac{D}{2} + isa \right) \Gamma \left( \frac{D}{2} - isa \right)}{(4\pi)^{d/2} \Gamma \left( \frac{D+2}{2} \right)}
\times \left[ sU(x, y) \cos \left( \frac{l}{2a} \right) F \left( \frac{D}{2} + isa, \frac{D}{2} - isa, \frac{D + 2}{2}; \cos^2 \left( \frac{l}{2a} \right) \right) \right.
\]
\[\left. + \gamma n^i U(x, y) \frac{D}{2a} \sin \left( \frac{l}{2a} \right) \right] 
\times F \left( \frac{D}{2} + isa, \frac{D}{2} - isa, \frac{D}{2}; \cos^2 \left( \frac{l}{2a} \right) \right),
\]

on $S^D$ and

\[
S(x, y; s) = -a^{1-d} \frac{\Gamma \left( \frac{D}{2} + sa \right) \Gamma (sa + 1)}{(4\pi)^{d/2} \Gamma (2sa + 1)} \left[ \cosh \left( \frac{l}{2a} \right) \right]^{1-D-2sa}
\times \left[ \left[ U(x, y) \cosh \left( \frac{l}{2a} \right) F \left( \frac{D}{2} + sa, sa + 1; \cosh^{-2} \left( \frac{l}{2a} \right) \right) \right]
\left. + \gamma n^i U(x, y) \sinh \left( \frac{l}{2a} \right) \right] 
\times F \left( \frac{D}{2} + sa, sa + 1, 2sa + 1; \cosh^{-2} \left( \frac{l}{2a} \right) \right),
\]

on $H^D$. According to the anticommutation relation of spinor fields the two-point function (116) satisfies the antiperiodic boundary condition $S(l) = -S(l + 2\pi na)$ where $n$ is an arbitrary integer.

We succeeded to calculate the spinor two-point functions without making any approximation in the spacetime curvature. Using these functions we evaluate the exact expression of the effective potential in curved spacetime in the following subsections.

4.2 de Sitter background ($S^D$)

First we consider the four-fermion model in de Sitter space \[27, 28, 29\]. The $D$-dimensional de Sitter space is represented as a hyperboloid,

\[
a^2 = \xi_0^2 - \xi_1^2 - \cdots - \xi_D^2,
\]

embedded in the $(D + 1)$-dimensional Minkowski space. It is one of the maximally symmetric spacetime. The de Sitter space is a constant curvature spacetime with curvature

\[
R = D(D - 1)a^{-2}.
\]

We consider the manifold $S^D$ as a Euclidean analog of the $D$-dimensional de Sitter space. As is mentioned in the previous subsection the spinor two-point function $S(x, y; s)$ is known on the manifold $S^D$. $\text{tr}S(x, x; s)$ is required in evaluating the effective potential:

\[
\text{tr}S(x, x; s) = -\frac{\text{tr}1s^2-a^{2-D}}{(4\pi)^{d/2} \Gamma \left( \frac{D}{2} \right)} \left[ 1 - \frac{D}{2} \right] \frac{\Gamma \left( \frac{D}{2} + isa \right) \Gamma \left( \frac{D}{2} - isa \right)}{\Gamma (1 + isa) \Gamma (1 - isa)}.
\]
Performing the Wick rotation and inserting Eq. (120) into Eq. (25) our final expression of the effective potential is obtained.[27]

\[
V(\sigma) = \frac{1}{2\lambda_0} \sigma^2 - \text{tr} 1 \chi^{a^2 - D} (4\pi)^{D/2} \chi \left( 1 - \frac{D}{2} \right) \int_0^\sigma ds \frac{\Gamma \left( \frac{D}{2} + isa \right) \Gamma \left( \frac{D}{2} - isa \right)}{\Gamma (1 + isa) \Gamma (1 - isa)}. \tag{121}
\]

Equation (121) is an exact expression of the effective potential for the model of four-fermion interactions in de Sitter space in the leading order of the $1/N$ expansion. Substituting the renormalized coupling constant $\lambda$ defined by Eq. (33) in the Eq. (121) we find the renormalized expression of the effective potential $V(\sigma)$ in de Sitter space,

\[
V(\sigma) = \frac{1}{2\lambda} \sigma^2 \mu^{-2} + \frac{\text{tr} 1}{(4\pi)^{D/2} (D - 1)} \Gamma \left( 1 - \frac{D}{2} \right) \sigma^2 \mu^{-2} - \frac{\text{tr} 1}{(4\pi)^{D/2} \chi} \left( 1 - \frac{D}{2} \right) \int_0^\sigma ds \frac{\Gamma \left( \frac{D}{2} + isa \right) \Gamma \left( \frac{D}{2} - isa \right)}{\Gamma (1 + isa) \Gamma (1 - isa)}. \tag{122}
\]

Note that Eq. (122) reduces to

\[
\frac{V^{D=2}(\sigma)}{\mu^2} = \left[ \frac{1}{2\lambda} - \frac{\text{tr} 1}{4\pi} (\ln(\mu a^2) + 2) \right] \left( \frac{\sigma}{\mu} \right)^2 + \frac{\text{tr} 1}{4\pi \mu^2} \int_0^\sigma ds \left[ \psi(1 + isa) + \psi(1 - isa) \right], \tag{123}
\]

in two dimensions. Expanding Eq. (122) asymptotically around $1/a = 0$ (weak curvature expansion) the effective potential (34) is reproduced. In Fig. 14 the behavior of the effective potential given by Eq. (122) is illustrated in the case $D = 2.5$ for several typical values of the curvature. It is observed in Fig. 14 that, if $\lambda \leq \lambda_{cr}$, the theory is always in the symmetric phase as the curvature changes while, if $\lambda > \lambda_{cr}$, the symmetry restoration takes place as the curvature exceeds its critical value. This observation remains true if the spacetime dimension is arbitrarily changed.

To discuss the dynamical mass of the fermion we study the minimum of the effective potential more precisely. A necessary condition for the minimum is given by the gap equation (80). Inserting Eq. (122) into Eq. (80) we find [27]

\[
\frac{1}{\lambda} + \text{tr} 1 \frac{D - 1}{(4\pi)^{D/2}} \chi \left( 1 - \frac{D}{2} \right) - \frac{\text{tr} 1}{(4\pi)^{D/2} \chi} \left( 1 - \frac{D}{2} \right) \Gamma \left( \frac{D}{2} + ima \right) \Gamma \left( \frac{D}{2} - ima \right) = 0. \tag{124}
\]

If the coupling constant $\lambda$ is no less than a critical value $\lambda_{cr}$, the gap equation allows a nontrivial solution. In Fig. 15 we present the solution of the gap equation (124). The dynamical fermion mass smoothly disappears as the curvature increases.

Taking the two-dimensional limit Eq. (124) reduces to

\[
\frac{1}{\lambda} - \frac{\text{tr} 1}{4\pi} \left[ \ln(\mu a^2) - \psi(1 + ima) - \psi(1 - ima) \right] = 0. \tag{125}
\]

For three dimensions it reads [29]

\[
\frac{1}{\lambda} - \frac{2\text{tr} 1}{4\pi} + \frac{\text{tr} 1}{4\pi \mu} \Gamma \left( \frac{3}{2} + ima \right) \Gamma \left( \frac{3}{2} - ima \right) = 0, \tag{126}
\]

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Figure 14: Behavior of the effective potential is shown at $D = 2.5$ for fixed $\lambda$ with the varying curvature where $K = 8.23 m_0^2$ and $R_{ct} = 8.23 m_0^2$. 

Figure 15: Behavior of the dynamical fermion mass in de Sitter space as a function of the curvature $R$ at $D = 2, 2.5, 3.0, 3.5, 4$ where $m_0$ is the dynamical fermion mass in flat spacetime.

while it reduces to

$$\frac{1}{\lambda} - 3 \text{tr} \mathbf{1} \left( C_{\text{div}} - \frac{2}{3} \right) = \frac{2 \text{tr} \mathbf{a}}{(4\pi)^2 \mu^2} \text{tr} \mathbf{1} + \frac{a^2 + m^2}{(4\pi)^2 \mu^2} \left[ C_{\text{div}} + \ln(a^2 \mu^2) - \psi(1 + ima) - \psi(1 - ima) \right] = 0,$$

in the limit of $D \to 4$, where the divergent part is expressed by $C_{\text{div}}$ in Eq. (38). Taking the four-dimensional limit, $D \to 4$, the divergent parts of the gap equation (127) is exactly equal to that given by Eq. (81). Hence the same critical behavior is observed for the dynamically generated fermion mass in Figs. 8 and 13 at $D = 4$.

As is seen in Figs. 12 and 13 the phase transition is of the second order. Accordingly by setting $m = 0$ in the gap equation (124) we may derive the equation which determines the critical radius $a_{cr}$,

$$\frac{1}{\lambda} + \text{tr} \mathbf{1} \left( D - \frac{1}{2} \right) - \text{tr} \mathbf{1} \left( \frac{a_{cr} \mu}{(4\pi)^{\nu/2}} \Gamma^2 \left( \frac{D}{2} \right) \Gamma \left( 1 - \frac{D}{2} \right) \right) = 0.$$

Substituting Eqs. (47) and (49) in Eq. (128) the critical radius $a_{cr}$ is found to be [27]

$$a_{cr} = \frac{1}{m_0} \left[ \Gamma \left( \frac{D}{2} \right) \right]^{2/(\nu - 2)}.$$

In Fig. 18(a) the critical radius $a_{cr}$ is plotted as a function of the spacetime dimension $D$. By using the critical radius (129) we find the critical curvature:

$$R_{cr} = D(D - 1)a_{cr}^{-2}.$$
For some special values of $D$ Eq. (130) simplifies to:

\[ R_{cr} = \begin{cases} 2e^{2\gamma}m_0^2 & ; D = 2, \\ \frac{96}{\pi^2}m_0^2 & ; D = 3, \\ 12m_0^2 & ; D = 4, \end{cases} \]  

(131)

where $\gamma$ is the Euler constant. In Fig. 18(b) the critical curvature $R_{cr}$ is presented as a function of the spacetime dimension $D$.

4.3 Anti-de Sitter background ($H^D$)

Next we consider the anti-de Sitter space. It is also the maximally symmetric spacetime with negative curvature

\[ R = -D(D-1)a^{-2}. \]  

(132)

We consider the manifold $H^D$ as a Euclidean analog of the $D$-dimensional anti-de Sitter space.

As is discussed in §4.1 the spinor two-point function $S(x,x;s)$ is known in $H^D$ and $\text{tr}S(x,x;s)$ is given by

\[ \text{tr}S(x,x;s) = -\frac{\text{tr}1a^{1-D}}{(4\pi)^{D/2}} \frac{\Gamma \left( \frac{D}{2} + sa \right) \Gamma \left( 1 - \frac{D}{2} \right)}{\Gamma \left( 1 - \frac{D}{2} + sa \right)}. \]  

(133)

Performing the Wick rotation and inserting Eq. (133) into Eq. (25) the effective potential of the four-fermion model reads

\[ V(\sigma) = \frac{1}{2\lambda_0} \sigma^2 - \frac{\text{tr}1a^{1-D}}{(4\pi)^{D/2}} \Gamma \left( 1 - \frac{D}{2} \right) \int_0^\sigma ds \frac{\Gamma \left( \frac{D}{2} + sa \right)}{\Gamma \left( 1 - \frac{D}{2} + sa \right)}. \]  

(134)

Thus we obtain the exact effective potential (134) in anti-de Sitter space in the leading order of the $1/N$ expansion. Applying the same renormalization condition (32) in Eq. (134) we find the renormalized expression of the effective potential $V(\sigma)$ in anti-de Sitter space,

\[ V(\sigma) = \frac{1}{2\lambda} \sigma^2 \mu^{D-2} + \frac{\text{tr}1}{2(4\pi)^{D/2}} (D-1) \Gamma \left( 1 - \frac{D}{2} \right) \sigma^2 \mu^{D-2} \]

\[-\frac{\text{tr}1a^{1-D}}{(4\pi)^{D/2}} \Gamma \left( 1 - \frac{D}{2} \right) \int_0^\sigma ds \frac{\Gamma \left( \frac{D}{2} + sa \right)}{\Gamma \left( 1 - \frac{D}{2} + sa \right)}. \]  

(135)

Expanding Eq. (135) asymptotically about $1/a = 0$ (weak curvature expansion) the effective potential (34) is reproduced. To study the phase structure in anti-de Sitter space we evaluate the effective potential (135).

In order to illustrate phase structure of four-fermion model in anti-de Sitter space we will consider two-dimensional case only below. That will be enough for our purposes because as we will see negative curvature always supports chiral symmetry breaking. Hence, there is no need to make the exposition of cases for all $D$ as it was above. Note that the fact that negative curvature supports chiral symmetry breaking and only broken phase survives has been already mentioned few times when we have discussed weak curvature expansion.

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In two dimensions the effective potential (135) reduces to [29]

\[
\frac{V^{D=2}(\sigma)}{\mu^2} = \left[ \frac{1}{2\lambda} + \frac{\text{tr}1}{4\pi} \left( \ln \frac{1}{a\mu} - 1 \right) \right] \left(\frac{\sigma}{\mu}\right)^2
\]

\[+ \frac{\text{tr}1}{4\pi\mu^2} \int_0^\sigma ds \left[ \psi(1+sa) + \psi(sa) \right]. \tag{136}\]

Evaluating the effective potential (136) numerically we easily see that only the broken phase is realized in two dimensions. In Fig. 16 the behavior of the effective potential given by Eq. (136) is illustrated.

Differentiating (136) with respect to \( \sigma \) we get

\[
\frac{\partial V^{D=2}}{\partial \sigma} \bigg|_{\sigma=0} = \sigma \left\{ \frac{1}{\lambda} - \frac{\text{tr}1}{4\pi} \left[ \ln(a^2\mu^2) - \psi(1+\sigma) - \psi(\sigma) \right] \right\}. \tag{137}\]

A careful study of \( \partial V/\partial \sigma \) shows that due to the presence of the last term in (137), \( \sigma = 0 \) is never stationary for any value of \( \lambda \) and finite \( a \). Owing to the fact that

\[
\frac{\partial V^{D=2}}{\partial \sigma} \bigg|_{\sigma=0} = -\frac{\text{tr}1}{4\pi a} < 0, \tag{138}\]

the chiral symmetry is always broken in two-dimensional anti-de Sitter space.[29]

In arbitrary dimensions (\( 2 \leq D < 4 \)) we easily find

\[
\frac{\partial V}{\partial \sigma} \bigg|_{\sigma=0} = -\frac{\text{tr}1}{(4\pi)^{D/2}} a^{1-D} \Gamma\left(\frac{D}{2}\right) < 0. \tag{139}\]

Thus the chiral symmetry is always broken down in anti-de Sitter space (i.e., a negative curvature spacetime) in arbitrary dimensions \( 2 \leq D < 4 \) as is mentioned in §3.

4.4 Einstein universe background \((R \otimes S^{D-1})\)

Another universe we can calculate the effective potential without making any approximation in the spacetime curvature is the static Einstein universe.[34] The \( D \)-dimensional Einstein universe is represented
by the metric

$$ds^2 = dr^2 - a^2(d\theta^2 + \sin^2 \theta d\Omega_{D-1}).$$  \hspace{1cm} (140)

It is a constant curvature spacetime with curvature

$$R = (D - 1)(D - 2) \frac{1}{a^2}.$$  \hspace{1cm} (141)

Here we consider the manifold $R \otimes S^{D-1}$ as a Euclidean analog of the $D$-dimensional Einstein universe. In two spacetime dimensions a Euclidean analog of the Einstein universe is represented by a cylinder with radius $a$.

According to a similar analysis developed in §4.1 we are able to solve the Dirac equation in Einstein universe.[35] On $R \otimes S^{D-1}$ we rewrite Eq. (93) in the following form

$$\left( \partial_0^2 + \Box_{D-1} - \frac{R}{4} - s^2 \right) G(x, y; s) = -\frac{1}{\sqrt{g}} \delta^D(x, y),$$  \hspace{1cm} (142)

where $\Box_{D-1}$ is the Laplacian on $S^{D-1}$. Performing the Fourier transformation

$$G(x, y; s) = \int \frac{d\omega}{2\pi} e^{-i\omega(y-x)} \tilde{G}(\omega, y-x),$$  \hspace{1cm} (143)

we rewrite Eq. (142) in the form

$$\left( \Box_{D-1} - \frac{R}{4} - (s^2 + \omega^2) \right) \tilde{G}(\omega, y-x) = -\frac{1}{\sqrt{g}} \delta^{D-1}(y).$$  \hspace{1cm} (144)

Equation (144) is of the same form as the one for the spinor Green function with mass $\sqrt{s^2 + \omega^2}$ on $S^{D-1}$. Thus the explicit expression for the two-point function in $R \otimes S^{D-1}$ is given by

$$\text{tr}S(x, x; s) = -i \frac{\text{tr}1a^{3-D}}{(4\pi)^{(D-1)/2}} \Gamma \left( \frac{3-D}{2} \right) \times \int \frac{d\omega}{2\pi} \frac{\Gamma \left( \frac{D-1}{2} + i\beta \right) \Gamma \left( \frac{D-1}{2} - i\alpha \right)}{\Gamma(1 + i\alpha) \Gamma(1 - i\alpha)},$$  \hspace{1cm} (145)

where the parameter $\alpha$ is defined by

$$\alpha = a\sqrt{s^2 + \omega^2}.$$  \hspace{1cm} (146)

Inserting the two-point function (145) into Eq. (25) we obtain the effective potential for the four-fermion model in Einstein universe in the leading order of the $1/N$ expansion.[34]

$$V(\sigma) = \frac{1}{2\lambda_0} \sigma^2 - \frac{\text{tr}1a^{3-D}}{(4\pi)^{(D-1)/2}} \Gamma \left( \frac{3-D}{2} \right) \times \int_0^\sigma ds \int \frac{d\omega}{2\pi} \frac{\Gamma \left( \frac{D-1}{2} + i\alpha \right) \Gamma \left( \frac{D-1}{2} - i\alpha \right)}{\Gamma(1 + i\alpha) \Gamma(1 - i\alpha)}.$$  \hspace{1cm} (147)

We apply the renormalization condition (32) and obtain the renormalized effective potential by replacing the coupling constant $\lambda_0$ with the renormalized one $\lambda$ defined by Eq. (33)

$$V(\sigma) = \frac{1}{2\lambda} \sigma^2 \mu^{D-2} - \frac{\text{tr}1a^{3-D}}{(4\pi)^{(D-1)/2}} \left( D - 1 \right) \Gamma \left( 1 - \frac{D}{2} \right) \sigma^2 \mu^{D-2}$$

$$- \frac{\text{tr}1a^{3-D}}{(4\pi)^{(D-1)/2}} \Gamma \left( \frac{3-D}{2} \right) \int_0^\sigma ds \int \frac{d\omega}{2\pi} \frac{\Gamma \left( \frac{D-1}{2} + i\alpha \right) \Gamma \left( \frac{D-1}{2} - i\alpha \right)}{\Gamma(1 + i\alpha) \Gamma(1 - i\alpha)}.$$  \hspace{1cm} (148)
Expanding Eq. (148) asymptotically about $1/a = 0$ (weak curvature expansion) the effective potential (34) is reproduced again. For convenience in a numerical calculation we rewrite Eq. (148) as

$$V(\sigma) = \frac{1}{2\lambda} \sigma^2 \mu^{D-2}$$

$$+ \frac{\text{tr} 1}{(4\pi)^{(D-1)/2}} \Gamma \left( \frac{3-D}{2} \right) \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \left[ \omega^2 + (D-2)\mu^2 \right] \left( \omega^2 + \mu^2 \right)^{(D-5)/2}$$

$$- \frac{\text{tr} a^{3-D}}{(4\pi)^{(D-1)/2}} \int_0^a ds \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \Gamma \left( \frac{D-1}{2} + i\alpha \right) \Gamma \left( \frac{D-1}{2} - i\alpha \right) \frac{\Gamma \left( 1 + i\alpha \right) \Gamma \left( 1 - i\alpha \right)}{\Gamma(1+\alpha) \Gamma(1-i\alpha)}. \quad (149)$$

By the use of the expression of the effective potential (149) we may study the behavior of the effective potential as a function of $\sigma$ through the numerical integrations. In the numerical integration in $\omega$ we introduced a suitable upper and lower bound and performed the numerical integration between these bounds. By assuming the sufficiently large absolute value of these bounds and checking the stability of the integral under the change of the bounds we obtained the numerical value for $V(\sigma)$ for each value of $\sigma$ with $\lambda$ and $a$ kept fixed.

A behavior of the effective potential is similar to that in de Sitter space illustrated in Fig. 14. If $\lambda \leq \lambda_{cr}$, the theory is always in the symmetric phase as the curvature changes while, if $\lambda > \lambda_{cr}$, the symmetry restoration takes place as the curvature exceeds its critical value. Only the second order phase transition occurs as the curvature exceeds its critical value. [34]

The dynamical fermion mass is given by the gap equation (80):

$$\frac{1}{\lambda} \mu^{D-2} + \frac{\text{tr} 1}{(4\pi)^{(D-1)/2}} \Gamma \left( \frac{3-D}{2} \right) \int \frac{d\omega}{2\pi} \left[ \omega^2 + (D-2)\mu^2 \right] \left( \omega^2 + \mu^2 \right)^{(D-5)/2}$$

$$- \frac{\text{tr} a^{3-D}}{(4\pi)^{(D-1)/2}} \int_0^a ds \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \Gamma \left( \frac{D-1}{2} + i\alpha \right) \Gamma \left( \frac{D-1}{2} - i\alpha \right) \frac{\Gamma \left( 1 + i\alpha \right) \Gamma \left( 1 - i\alpha \right)}{\Gamma(1+\alpha) \Gamma(1-i\alpha)} = 0. \quad (150)$$

Figure 17 represents the behavior of the dynamical fermion mass which is obtained by solving Eq. (150) numerically for $D = 2.0, 2.5, 3.0$ and $3.5$. In two dimensions Einstein universe becomes a flat spacetime, $R = 0$. To see the mass dependence of the radius $a$ for $D = 2$ we draw the dynamical fermion mass as a function of $(am_0)^{-2}$ instead of the curvature $R$.

The critical radius is given by the massless limit of the gap equation. Taking the massless limit in Eq. (150), we can perform the integration over $\omega$ analytically and find

$$\frac{1}{\lambda} \mu^{D-2} + \frac{\text{tr} 1}{(4\pi)^{(D-1)/2}} \Gamma \left( \frac{3-D}{2} \right) \int \frac{d\omega}{2\pi} \left[ \omega^2 + (D-2)\mu^2 \right] \left( \omega^2 + \mu^2 \right)^{(D-5)/2}$$

$$- \frac{\text{tr} a^{3-D}}{(4\pi)^{(D-1)/2}} \int_0^a ds \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \Gamma \left( \frac{D-1}{2} + i\alpha \right) \Gamma \left( \frac{D-1}{2} - i\alpha \right) \frac{\Gamma \left( 1 + i\alpha \right) \Gamma \left( 1 - i\alpha \right)}{\Gamma(1+\alpha) \Gamma(1-i\alpha)} = 0. \quad (151)$$

Taking into account Eq. (47) with Eq. (49) we rewrite Eq. (151) in the following form [34]

$$a_{cr} = \frac{1}{m_0} \left[ \frac{1}{\sqrt{\pi}} \Gamma \left( \frac{D-1}{2} \right) \Gamma \left( \frac{D}{2} \right) \right]^{1/(D-2)}. \quad (152)$$

Solving Eq. (152) the critical radius is obtained in an arbitrary dimension. In Fig. 18(a) we show the dependence of the critical radius to dimension $D$. It should be noted that Eq. (152) reduces to

$$a_{cr} = \frac{e^{-\gamma}}{2m_0}, \quad (153)$$

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Figure 17: Behavior of the dynamical fermion mass in Einstein universe as a function of $(am_0)^{-2}$ at $D = 2.0, 2.5, 3.0, 3.5$ where $m_0$ is the dynamical fermion mass in flat spacetime.

in two dimensions which reproduces the result obtained in Refs. [40] and [41]. Using the relationship (141), the critical curvature is obtained from the critical radius. For some special values of $D$ the critical curvature $R_{cr}$ simplifies to

$$ R_{cr} = \begin{cases} 0 & ; D = 2, \\ 8m_0^2 & ; D = 3, \\ 12m_0^2 & ; D = 4. \end{cases} $$

The critical curvature $R_{cr}$ is plotted in Fig. 18(b) as a function of the spacetime dimension $D$. It is clearly seen in Fig. 18(b) that three lines of the critical curvature reach the same value $12m_0^2$ at $D \to 4$. For $D = 2$ the critical curvature obtained by weakly curvature approximation is exactly equal to that in Einstein universe. As is explained in the last section the chiral symmetry is restored by the effect of an infra-red divergence for any finite curvature in two-dimensional weakly curved spacetime. The critical curvature is thus zero. In two-dimensional Einstein universe it is obtained that the critical curvature $R_{cr} = 0$. The situation is, however, different from that in weakly curved spacetime. By definition two-dimensional Einstein universe is a flat spacetime, $R = 0$. The symmetry restoration is induced by finite size effects of the compact and closed space. The spacetime curvature $R$ is not suitable to represent the phase structure in two-dimensional Einstein universe.

Thus we have considered the Gross-Neveu type model as one of the prototype models of the dynamical symmetry breaking and investigated the phase transition induced by the curvature effect without making any approximation in the spacetime curvature.

In de Sitter, anti-de Sitter space and Einstein universe the two-point functions have been solved exactly. Then the exact expression of the effective potential is obtained in such spacetimes. We calculated the renormalized effective potential for finite $R$ in the leading order of the $1/N$ expansion in arbitrary dimensions $2 \leq D < 4$. It is found that the broken chiral symmetry is restored at a certain critical curvature in de Sitter space and Einstein universe (i.e., positive curvature spacetimes). The phase transition from the broken phase to the symmetric phase is of second order. In anti-de Sitter space (i.e., negative curvature spacetime) only the broken phase is realized irrespective of the value of $\lambda$. 

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Figure 18: Phase diagrams in de Sitter space and Einstein universe. The full and the dashed lines represent the exact solution in de Sitter space and Einstein universe respectively. The dotted line represents the critical curvature obtained by the weak curvature approximation.
For a negative curvature spacetime the weak curvature approximation gives the exact phase structure, $R_{cr} = 0$, in the whole range of $D$ considered here: $2 \leq D < 4$. The critical curvature $R_{cr}$ obtained by weak curvature approximation is exactly equal to the one obtained in de Sitter space and Einstein universe at $D = 4$ shown in Fig. 18(b). In four dimensions ultraviolet divergences appear in terms independent of the curvature $R$ and terms linear in $R$ only. The higher order terms in $R$ of the effective potential are ultraviolet finite. Expanding the exact results in de Sitter space and Einstein universe asymptotically about $R = 0$, we obtain the $R^2$ term of the effective potential in a compact spacetime with a constant curvature.

$$V(\sigma) = V_0(\sigma) + V_\mu(\sigma) + V_{\alpha}(\sigma) + O(R^3),$$

where $V_0(\sigma)$ and $V_\mu(\sigma)$ are given by Eqs. (31) and (70) respectively, the $R^2$ term $V_{\alpha}(\sigma)$ reads [24]

$$V_{\alpha}(\sigma) = -\frac{\text{tr}_1}{(4\pi)^2} \Gamma \left(1 - \frac{D}{2}\right) \frac{R^2 (D-2)(D-3)(2+5D)}{5760 D(D-1)} \sigma^{D-4}. \quad (156)$$

At the four-dimensional limit the $R^2$ term (156) reduces to

$$\frac{V_{\alpha}^{D=4}(\sigma)}{\mu^D} = \frac{\text{tr}_1}{(4\pi)^2} \frac{11R^2}{17280} \left(C_{\text{div}} = \frac{173}{66} - \ln \left(\frac{C}{\mu}\right)^2 \right). \quad (157)$$

The divergent parts in Eq. (157) appear from the mass singularity at $\sigma \to 0$ and the normalization condition $V(0) = 0$. Only an infrared divergence appears in the $R^2$ term (157). The infrared divergence does not appear in the de Sitter space and Einstein universe since the space components are compact and closed (i.e. there is an infrared cut-off $\Lambda_{IR} = 1/(2a)$). The $R^2$ term has no contribution to the symmetry restoration in a compact space (de Sitter space and Einstein universe). Thus the critical curvature is determined by the terms involving ultraviolet divergences in four dimensions and the weak curvature approximation gives the exact result. Therefore the weak curvature approximation seems to be useful near four dimensions.[24] If we use the cut-off regularization in four dimensions we also see the similar situation. In the case the weak curvature approximation also gives the exact result at the limit $\Lambda \to \infty$.[28]

In the other dimensions the critical curvature (83) is unable to compare with the results obtained in de Sitter space and Einstein universe directly, because the global topology of the spacetime may play a crucial role for the symmetry restoration. In two-dimensional Einstein universe $R \otimes S$ not the curvature but the size of the space coordinate $a$ determines the minimum of the effective potential. (See §6.) The critical radius $a_{cr}$ in Einstein universe is half as many as that in de Sitter space. It seems that the number of the compactified directions determine the critical radius. Thus the finite size effect may play an essential role in two dimensions.11

5 Extensions of four-fermion model

5.1 Gauged NJL model in curved spacetime

Gauged NJL model maybe considered as one of the most realistic four-fermion models.[42] From one side, that is one of the best effective theories to describe QCD. From another side, gauged NJL model may well describe SM (standard model) where composite bound states play the role of an elementary Higgs fields in the process of dynamical symmetry breaking.

11For $D = 2$ the chiral symmetry may be restored at any value of $a$ in the case of finite $N$ through the creation of an instanton-antinstanton or kink-antikink condensation as is discussed at finite temperature.[41] In our method we cannot deal with the influence of time or space dependent configurations. Instantons and kinks are, however, suppressed at the large $N$ limit and the phase transition can take place in two dimensions.
For a long time the only way to study gauged NJL model has been related with Schwinger-Dyson
equation. However in Ref. [44] (following the renormalization group (RG) approach of Ref. [45])
it has been shown that some class of gauge Higgs-Yukawa models in leading order of a modified $1/N_c$
expansion leads to a well-defined, non-trivial theory. This theory is equivalent to the gauged NJL
model when the ultraviolet cut-off goes to infinity (using corresponding compositeness conditions). Then, it
appears the possibility to find the effective potential and to discuss the phase structure of gauged NJL
model without use of Schwinger-Dyson equations. It has been done in flat space in Ref. [44] and in
curved space in Ref. [46]. Below, we will discuss the phase structure of gauged NJL model following
closely Ref. [46].

5.1.1 Renormalization group equations for the class of gauge Higgs-Yukawa models

In this subsection we review the class of gauge Higgs-Yukawa models in flat spacetime we are going to
extend and the approximations we are using in such an analysis. Thereby we closely follow Ref. [44]
where, using the RG approach, this model has been considered in detail.

The Lagrangian of the model in flat spacetime is given by (we use the notations of Ref. [44])

\[ L_m = -\frac{1}{4} G_{\mu\nu}^a G^{a\mu\nu} + \frac{1}{2} (\partial_\mu \sigma)^2 - \frac{1}{2} m^2 \sigma^2 - \frac{\lambda}{4} \sigma^4 + \sum_{i=1}^{N_f} \bar{\psi}_i i \gamma^\mu D_\mu \psi_i - \sum_{i=1}^{N_f} g y \sigma \bar{\psi}_i \psi_i. \]  

(158)

Here the gauge group $SU(N_c)$ is chosen, with $N_f$ fermions $\psi_i (i = 1, 2, ..., N_f)$ belonging to the repre-
sentation $R$ of $SU(N_c)$, $\sigma$ is the scalar field.

Let us now describe the $1/N$ approximation for the perturbative study of this theory at high energies
through the RG equations:

a) The gauge coupling constant is assumed to be small:

\[ \frac{g^2 N_c}{4\pi} \ll 1, \]  

(159)

and the RG equations are considered only in the first non-trivial order on $g^2$.

b) The number of fermions should be large enough:

\[ N_f \sim N_c. \]  

(160)

On the same time it is supposed that only $n_f$ fermions ($n_f \ll N_f$) have large Yukawa couplings, whereas
the remaining fermions have vanishing Yukawa couplings.

c) The approach is assumed to be perturbative also in $1/N_c$, so only the leading order of the $1/N_c$
expansion survives; this means that scalar loop contributions should be negligible

\[ \left| \frac{\lambda}{g^2} \right| \leq N_c. \]  

(161)

Note that above approximation maybe reasonable for the minimal SM.

Within the above approximation the RG equations for the coupling constants in (5·1) are\textsuperscript{13}

\[ \frac{dg(t)}{dt} = - \frac{b}{(4\pi)^2} g^3(t), \]

\[ \textsuperscript{13} \text{It is not known how to formulate consistent Schwinger Dyson equation even in ladder approximation in general curved spacetime.} \]  

\[ \textsuperscript{14} \text{For general discussion of one-loop RG equations in gauge Higgs-Yukawa models, see Refs. [48] and [49].} \]
The Lagrangian $L$ of the theory should be:

\[ \frac{dy(t)}{dt} = \frac{y(t)}{4\pi^2} \left[ a \, y^2(t) - c \, g^2(t) \right], \]

\[ \frac{d\lambda(t)}{dt} = \frac{u \, y^2(t)}{4\pi^2} \left[ \lambda(t) - y^2(t) \right], \]

where $b = (11N_c - 4T(R)N_f)/3$, $c = 6 \, C_2(R)$, $a = u/4 = 2 \, n_f N_c$. For the fundamental representation we have $T(R) = 1/2$, $C_2(R) = (N_c^2 - 1)/(2N_c)$. Here $t = \ln(\mu/\mu_0)$ and $\mu_0$ is the reference scale to discuss low energy physics.

The RG equation for $g^2$ (respectively $\alpha = g^2/4\pi$) is solved by

\[ \eta(t) \equiv \frac{g^2(t)}{g^2_0} \equiv \frac{\alpha(t)}{\alpha_0} = \left(1 + \frac{b \, \alpha_0}{2\pi \, t}\right)^{-1}. \]

To solve the RG equations for the Yukawa and the scalar couplings it is convenient to introduce the following RG invariants:

\[ h(t) \equiv -\eta^{-1+c/b}(t) \left[ 1 - \frac{c - b}{a} \, \frac{g^2(t)}{y^2(t)} \right], \]

\[ k(t) \equiv -\eta^{-1+2c/b}(t) \left[ 1 - \frac{2c - b}{2a} \, \frac{\lambda(t)}{y^2(t)} \, \frac{g^2(t)}{y^2(t)} \right]. \]

With their help we obtain

\[ g^2(t) = \frac{c - b}{a} \, g^2(0) \left[ 1 + h_0 \eta^{1-c/b}(t) \right]^{-1}, \]

\[ \lambda(t) = \frac{2a}{2c - b} \, \frac{y^4(t)}{g^2(t)} \left[ 1 + k_0 \eta^{1-2c/b}(t) \right]. \]

Of course, these solutions are explicitly known only when the values of the RG invariants $h$ and $k$ are given. Note that in (5-8) and below we use notations $h_0$ and $k_0$ for $h$ and $k$ in order to show that they are constants (RG invariants).

As it has been shown in detail in Ref. [44] for the solutions (5-6) $\sim$ (5-8) to be consistent with the above approximation (5-2) $\sim$ (5-4), and in order for the theory to be non-trivial one, we should have

\[ c > b \quad \text{and} \quad 0 \leq h_0 < \infty. \]

That makes the Yukawa coupling to be asymptotically free and the theory to be non-trivial. Further analysis shows that the scalar coupling constant is nontrivial in the above approximation only if

\[ k_0 = 0. \]

Below we will discuss the solutions (5-8) with conditions (5-9) and (5-10) only.

Let us extend now this model to curved spacetime. It is by now well-known (see Ref. [7] for an introduction) that in order to be multiplicatively renormalizable in curved spacetime, the Lagrangian of the theory should be:

\[ L = L_m + L_{\text{ext}} - \frac{1}{2} \xi R a^2, \]

where $L_m$ is given by (5-1) with a change of flat (partial) derivatives to the corresponding covariant derivatives: $\partial_\mu \rightarrow \nabla_\mu$, $\xi$ is the non-minimal scalar-gravitational coupling constant, and

\[ L_{\text{ext}} = a_1 R^2 + a_2 C_{\mu\nu\alpha\beta}^2 + a_3 G + a_4 R^\Lambda + \Lambda - \frac{1}{\kappa} R. \]

The Lagrangian $L_{\text{ext}}$ of the external gravitational field is necessarily introduced in order to have the theory to be multiplicatively renormalizable one; $a_1, a_2, a_3, a_4, \kappa$ are gravitational coupling constants.
Since the RG equations for those coupling constants which are present in flat space do not change in curved spacetime,[7] all the discussions of RG equations for $g, y$ and $\lambda$ (as well as the approximations introduced above) are also valid here. In addition, the effective coupling constants corresponding to $\xi, a_1, \ldots, a_4, G$ appear. However, for our purposes, we only need the RG equation for the coupling constant $\xi$ which maybe written as following (within the above described approximation),

$$\frac{d\xi(t)}{dt} = \frac{1}{(4\pi)^2} 2a y^2(t) \left(\xi(t) - \frac{1}{6}\right).$$

(170)

Taking into account that

$$\frac{d}{dt} f(t) \equiv \frac{d}{dt} \left[\eta \frac{y^{c/b}}{y^2(t)} \left(\xi(t) - \frac{1}{6}\right)\right] = 0,$$

(171)

we may use this RG invariant $f(t)$ to find the solution of (5.13):

$$\xi(t) = \frac{1}{6} + y^2(t) \eta^{c/b} f_0,$$

(172)

where $f_0 \neq 0$ is the value of the RG invariant. Note that in the UV-limit ($t \to \infty$) we have:

$$y^2(t) \sim \frac{c-b}{a} \frac{g_0^2}{h_0} \eta^{c/b} \rightarrow +0, \quad (h_0 \geq 0)$$

and

$$\xi(t) \sim \frac{1}{6} + \frac{c-b}{a} \frac{g_0^2}{h_0} f_0.$$

(173)

For $f_0 \approx 0$ the scalar-gravitational coupling constant $\xi(t) \rightarrow 1/6$, i.e., asymptotic conformal invariance is realized.[50] Hence, the RG invariant $f_0$ characterizes the deviation from conformal invariance ($\xi = 1/6$). Different types of UV-behavior of $\xi(t)$ for different GUT’s have been listed in Ref. [7]; the most typical ones are:

a) $\xi(t) \rightarrow \frac{1}{6}$,

b) $|\xi(t)| \rightarrow \infty$,

c) $\xi(t) \rightarrow \xi$.

(174)

As we can see, unlike the analysis in the case of flat spacetime, there will not appear any restrictions to the sign and the value of $f_0$ from the study of the UV-asymptotics of $\xi(t)$.

Let us discuss now the one-loop effective potential for the theory (5.1) in the approach where we keep only terms with the accuracy up to linear curvature,[19, 51] i.e., $\sigma^2 \gg |R|$. This approach is actually equivalent to the one described in §3. Within above approximation we get

$$V = \frac{1}{2} m^2 \sigma^2 + \frac{\lambda}{4} \sigma^4 + \frac{1}{2} \xi R \sigma^2 - \frac{a \mu_F^2}{2(4\pi)^2} \left[\ln \frac{\mu_F^2}{\mu^2} - \frac{3}{2}\right]$$

$$- \frac{a R \mu_F^2}{12(4\pi)^2} \ln \frac{\mu_F^2}{\mu^2} - 1,$$

(175)

where $\mu_F \equiv y\sigma$.

Using such a form for the potential $V$ one may investigate the curvature-induced phase transitions which have been discussed in detail in Refs. [19, 7] and [51] for different GUT’s.

5.1.2 Gauged NJL model in curved spacetime

Let us discuss now the gauged NJL model in curved spacetime.
The assumption that not elementary ones but composite bound states are mainly relevant in quantum cosmology motivated the study of the chiral symmetry breaking under the influence of the external gravitational field (see previous sections.) Such an investigation has been done via explicit calculation of the effective potential for the composite field \( \langle \bar{\psi} \psi \rangle \). Our purpose here will be to discuss the same question on RG language for the gauged NJL model using its equivalence with the gauged Higgs-Yukawa model (5.1) (the explicit loop calculations in such a model are extremely hard to do).

We start from the gauged NJL model with four-fermion coupling constant \( G \) in curved spacetime

\[
L = -\frac{1}{4} G_{\mu\nu}^2 + \sum_{i=1}^{N_f} \bar{\psi}_i i\gamma^\mu D_\mu \psi_i + G \sum_{i=1}^{N_f} (\bar{\psi}_i \psi_i)^2 .
\]

As usually one introduces an auxiliary field \( \sigma \) to replace the NJL model by the equivalent Higgs-Yukawa model. As is shown in Ref. [45], it is possible within the RG approach to use a set of boundary conditions for the effective couplings of the gauge Higgs-Yukawa model at \( t_\Lambda = \ln(\Lambda/\mu_0) \) (where \( \Lambda \) is the UV-cut off of the gauged NJL model) in order to identify the gauged NJL model with the gauge Higgs-Yukawa model. Taking into account Eqs. (5.8) the explicit expressions for these compositeness conditions (which are specified in Ref. [45]) are given as (see also Eqs. (4.7) and (4.8) of Ref. [44]):

\[
y^2(t) = \frac{c-b}{a} g^2(t) \left[ 1 - \left( \frac{\alpha(t)}{\alpha(t_\Lambda)} \right)^{1-c/b} \right]^{-1} \equiv y^2_\Lambda(t),
\]

\[
\lambda(t)/y^2(t) = \frac{2a}{2c-b} \frac{1}{g^2(t)} \left[ 1 - \left( \frac{\alpha(t)}{\alpha(t_\Lambda)} \right)^{1-2c/b} \right] \equiv \lambda_\Lambda(t)/y^2_\Lambda(t) ,
\]

where \( t < t_\Lambda \). Hence, the system (5.1) with cut-off \( \Lambda \) is equivalent to the system (5.19) with the same cut-off when the running coupling constants are given by (5.20). In this sense the gauged NJL model may be called a renormalizable one. In addition, as it has been shown in Ref. [44], one should have also a compositeness condition for the mass, whose most convenient form will be (for \( b \rightarrow +0 \) [44]):

\[
m^2(t) = \frac{2a}{(4\pi)^2} (\frac{\Lambda^2}{\mu^2})^2 \frac{1}{g^2_\Lambda(t)} \mu^2 \left[ \frac{1}{g_4(\Lambda)} - \frac{1}{w} \right] ,
\]

where \( G \equiv ((4\pi)^2/a)g_4(\Lambda)/\Lambda^2 \), \( g_4(\Lambda) \) is a dimensionless constant, and \( w \equiv 1 - \alpha/(2\alpha_C) \), \( \alpha \equiv \alpha_0 \) and where \( y^2_\Lambda(t) \) is given by the first of Eqs. (5.20) at \( b \rightarrow +0 \).

The above compositeness conditions define the gauged NJL model as the gauge Higgs-Yukawa model in flat spacetime. Now, since we are working in curved spacetime (in linear curvature approximation), one has to add the compositeness condition for \( \xi(t_\Lambda) \). Making the calculations in the same way as in Ref. [52] one obtains

\[
\xi(t_\Lambda) = \frac{1}{6} .
\]

Obviously, the compositeness condition for \( \xi(t) \) is again the same one as in the non-gauged NJL model [52] (for finite corrections see also Ref. [53]). Hence, the conformal invariance due to the renormalization group [50] takes place again. Analyzing the RG equation (5.13) for \( \xi(t) \) we will find that in all cases (i.e., for the general situation, where \( y(t) \) is given by (5.20), for the case with \( b \rightarrow 0 \) and with the corresponding \( y(t) \), and also for the case \( \alpha_0 \rightarrow 0 \), i.e., the non-gauged NJL model) we have

\[
\xi(t) = \frac{1}{6} .
\]
is widely known for the flat space [54, 55] as well as for curved spacetime [19, 51] we will not give any
details of its derivation.

Using the fact that the effective potential satisfies the RG equation, one can explicitly solve this
equation by the method of characteristics, and we find:

\[ V(g, y, \lambda, m^2, \xi, \sigma, \mu) = V(g(t), y(t), \lambda(t), m^2(t), \xi(t), \sigma(t), \mu e^t) , \]

(181)

where the effective coupling constants \( g(t), ..., \sigma(t) \) are defined by the RG equations (5.5), (5.13) (at the
scale \( \mu e^t \)) and corresponding RG equations for \( m^2(t), \sigma(t) \) written in Ref. [44] for the case of the above
gauge-Higgs-Yukawa model (\( t \) is left unspecified for the moment). As boundary condition it is convenient
to use the one-loop effective potential (5.18).

In the case of the gauged NJL model one has to substitute into the effective potential (5.18) the
effective couplings fulfilling compositeness conditions in accordance with Eq. (5.24). In this way, one
obtains the RG improved effective potential in the gauged NJL model. In flat spacetime such calculation
has been already done in Ref. [44] for the case of fixed gauge coupling (\( b \to +0 \)), so we may use for the
running coupling constants (except scalar-gravitational coupling constant) the results which are known
from flat spacetime calculations.

Taking into account that the condition

\[ \mu e^t = \mu F(t) , \]

(182)
serves actually for finding \( t \), we get for the case of fixed gauge coupling

\[ e^t = \left( \frac{\mu F(\mu)}{\mu} \right)^{1/(2-w)} , \]

(183)

where \( \mu F(\mu) = \mu F(t = 0) \). Using (5.20), (5.23) and (5.26) in the case \( b \to 0 \) one gets (for the case of flat
space see expression (6-13) of Ref. [44]):

\[
\frac{(4\pi)^2}{2a} \frac{V}{\mu^4} = \frac{1}{2} \frac{y_2^2(\mu)}{\mu^2} \frac{\Lambda^2}{\mu^2} \left( \frac{1}{g_4(\Lambda)} - \frac{1}{g_4^*} \right) \frac{\sigma^2(\mu)}{\mu^2} + \alpha_c \frac{4}{\alpha} \left[ \frac{y_\lambda(\mu)\sigma(\mu)}{\mu} \right]^{4/(2-w)} - \frac{\mu^2}{\Lambda^2} \frac{\alpha_c}{\alpha} \left( \frac{y_\lambda(\mu)\sigma(\mu)}{\mu} \right)^4 \]

\[ + \frac{3}{8} \left( \frac{y_\lambda(\mu)\sigma(\mu)}{\mu} \right)^{4/(2-w)} + \frac{R}{24\mu^2} \left( \frac{y_\lambda(\mu)\sigma(\mu)}{\mu} \right)^{2/(2-w)} \left( 1 + \frac{2\alpha_c}{\alpha} \right) - \frac{R\alpha_c}{12\alpha} \left( \frac{\mu}{\Lambda} \right)^{\alpha_{\alpha_c}} \frac{y_2^2(\mu)}{\mu^2} , \]

(184)

where \( g_4^* \equiv w \). Thus, we have got the RG-improved effective potential for the gauged NJL model (with
finite cut-off) in curved spacetime.

Taking the limit \( \Lambda \to \infty \) we will get the renormalized effective potential of the gauged NJL model in
curved spacetime:

\[
\frac{(4\pi)^2}{2a} \frac{V}{\mu^4} = \frac{1}{2} \frac{1}{\left( \frac{1}{g_4(\mu)} - \frac{1}{g_4^*} \right)} \frac{y_2^2(\mu)}{\mu^2} \sigma^2(\mu) + \alpha_c \frac{4}{\alpha} \left( 1 + \frac{3\alpha}{2\alpha_c} \right) \left( \frac{y_\lambda(\mu)}{\mu} \right)^{4/(2-w)} \]

\[ + \frac{R}{24\mu^2} \left( 1 + \frac{2\alpha_c}{\alpha} \right) \left( \frac{y_\lambda(\mu)}{\mu} \right)^{2/(2-w)} \]

(185)
where $y_2^2 = (4\pi)^2\alpha/2a\alpha C$ and the four-fermion coupling renormalization has been done; in particular $g_{4R}$ is a finite constant (for discussion of that renormalization see Refs. [44] and [56]).

Note that in flat space ($R = 0$) the potential (5-28) in the same approach has been obtained in Ref. [56] (for non-renormalized potential see also Ref. [57]) using the ladder SD equation. Hence, the RG-improved effective potential may serve as a very useful tool to study the non-perturbative effects on equal footing with the SD equation. It is really surprising that RG improved effective potential gives the same results as ladder SD equation.

Using the finite effective potential (5-28) we are able to discuss the chiral symmetry breaking in the gauged NJL model under consideration. In flat spacetime, the possibility of chiral symmetry breaking is defined by the sign of the first term in (5-28). That gives the value of the critical four-fermion coupling constant $g_4^*(\Lambda) = \alpha C\alpha \approx \frac{2\pi}{3N_c}g_4^2 \gg 1$.

Taking into account that

\[ w = 1 - \frac{3C_2(R)}{2(2\pi)^2} g_4^2 \simeq 1 - \frac{3}{4\pi} \frac{N_c g_4^2}{4\pi} \simeq 1, \]

where

\[ \frac{\alpha C}{\alpha} \approx \frac{2\pi}{3} \frac{4\pi}{N_c g_4^2} \gg 1, \]

and introducing $x = y_\star \sigma(\mu)/\mu$, one can rewrite the quadratic part of the potential (5-28) as follows:

\[ \frac{(4\pi)^2}{2a} \frac{V^{(2)}}{\mu^4} \simeq \left\{ \frac{1}{2} \left( \frac{1}{g_4 R(\mu)} - \frac{1}{g_4 R} \right) + \frac{R}{12\mu^2} \right\} x^2. \tag{187} \]

Relation (5-30) determines the way to estimate the chiral symmetry breaking for the gauged NJL model in curved spacetime.

In particular, in flat spacetime and for (cut-off) dependent effective potential we observe that the chiral symmetry is broken for

\[ \frac{1}{g_4(\Lambda)} - \frac{1}{g_4} < 0 \quad \text{or} \quad \frac{1}{g_4 R(\mu)} - \frac{1}{g_4 R} < 0. \tag{188} \]

Hence, the critical value of the four-fermion coupling constant which divides the chiral symmetric and non-symmetric phase is $g_4^\star = w$.

In curved spacetime, chiral symmetry is always broken if

\[ \left( \frac{1}{g_4 R(\mu)} - \frac{1}{g_4 R} \right) + \frac{\alpha C}{6\alpha} \frac{R}{\mu^2} < 0. \tag{189} \]

When this condition (5-32) is valid one can easily find the curvature-induced dynamical fermion mass. In a similar way one can find the chiral symmetry breaking condition for cut-off dependent effective potential,

\[ \left( \frac{\Lambda^2}{\mu^2} \right)^w \left( \frac{1}{g_4(\Lambda)} - \frac{1}{g_4^\star} \right) + \frac{R}{12\mu^2} \frac{\alpha C}{\alpha} \left[ 1 - \left( \frac{\mu}{\Lambda} \right)^{\alpha/\alpha C} \right] < 0. \tag{190} \]

Thus, we have got the condition for chiral symmetry breaking in terms of the dimensionless curvature $\tilde{R} \equiv R/\mu^2$. From (5-32) we see that the critical coupling constant depends on the curvature. For negative curvature the critical coupling is higher and one has a greater chance to find the system in the phase with broken chiral symmetry. As an example of spaces being in correspondence with (5-33) one can consider the inflationary universe $S^4$ with small curvature.

Note that the critical value of the curvature at which symmetry breaking is absent is defined according to (5-32) by

\[ \frac{R_{\text{cr}}}{\mu^2} = \frac{6\alpha}{\alpha C g_4^2}. \tag{191} \]
At all positive curvatures below $R_{\text{cr}}$ as well as at small negative curvatures the chiral symmetry is broken.

It is quite remarkable that such a simple quantum condition of chiral symmetry breaking in gauged NJL model in curved spacetime is obtained explicitly. Up to now such a simple symmetry breaking tree-level condition in curved spacetime has been known only for the Higgs sector of (5-11):

$$\sigma^2 = -\frac{\xi R + m^2}{\lambda},$$

(192)

where $\xi R + m^2 < 0$.

Thus we discussed the gauged NJL model in curved spacetime using quite standard RG language. The effective potential for composite fermions is found explicitly and its phase structure is discussed. Some analytical results are obtained explicitly (in particularly, the condition of chiral symmetry breaking in the case of fixed gauge coupling). There are different possibilities to extend above methods. First, one can consider other gauged NJL models (with more scalars) as equivalent to gauge-Higgs-Yukawa models typical for GUTs. Second, one can consider other gravitational backgrounds (say, de Sitter space). There is no problem also to generalize this approach for the situation when external gravity is sufficiently strong (see Ref. [58]).

5.2 Higher derivative four-fermion model in curved spacetime

As it has been mentioned several times, four-fermion models are considered now as non-renormalizable effective theories where the presence of an ultraviolet cut-off $\Lambda$ at loop diagrams is a necessary condition. Then there are different possibilities to extend the NJL model. Among such possibilities, a quite interesting one is the introduction in the original Lagrangian of higher derivative terms in the four-fermion interaction [59] or the kinetic term [60]. It may be shown [61] that the physics of such higher derivative four-fermion model is still equivalent to the physics of SM. Note also that the inclusion of higher derivatives in effective theories gives the possibility to take into account the structural effects of the medium and external fields. In the present subsection we discuss the effective potential and phase structure of higher derivative four-fermion model in curved spacetime. We mainly follow Ref. [62].

We start by presenting the model which we set out to study (it was introduced in Ref. [59]). Its Lagrangian in curved spacetime is given by

$$\mathcal{L} = \bar{\psi}i\gamma^\mu(x)\nabla_\mu \psi + \frac{1}{4N_cA^2} \left\{ \lambda_1 (\bar{\psi}\psi)^2 + 3\lambda_2 \left[ \bar{\psi} \left( 1 - \frac{2\nabla^2}{\Lambda^2} \right) \psi \right]^2 \right\},$$

(193)

where $N_c$ is the number of fermionic species, $\Lambda$ a cut-off parameter, and $\lambda_1$ and $\lambda_2$ are coupling constants. We work in the $1/N_c$ expansion scheme.

By introducing some auxiliary fields $\chi_1$ and $\chi_2$ we can give a description of this nonrenormalizable theory by means of the action

$$S = \int d^4x \sqrt{g} \left\{ \bar{\psi}i\gamma^\mu(x)\nabla_\mu \psi - N_cA^2 \left( \frac{\chi_1^2}{\lambda_1} + \frac{\chi_2^2}{\lambda_2} \right) \right\} - \left[ \chi_1 \bar{\psi}\psi + \sqrt{3}\chi_2 \bar{\psi} \left( 1 - \frac{2\nabla^2}{\Lambda^2} \right) \psi \right].$$

(194)

Our purpose is to study the influence of external gravity on the dynamical breaking and restoration patterns of the symmetry possessed by the Lagrangian (193), which are given by the transformations $\psi \rightarrow \gamma_5\psi$ and $\bar{\psi} \rightarrow \bar{\psi}\gamma_5$. If we refer to the action (194), the symmetry is realized by adding the following transformations for the auxiliary fields: $\chi_1 \rightarrow -\chi_1$ and $\chi_2 \rightarrow -\chi_2$.

The effective potential of this model in the $N_c \rightarrow \infty$ limit is given by

$$V = \Lambda^2 \left( \frac{\chi_1^2}{\lambda_1} + \frac{\chi_2^2}{\lambda_2} \right) + V_1$$

(195)
with

\[ V_1 = i \left( \mathcal{V} \text{ol} \right)^{-1} \ln \text{Det} \left( i \mathcal{V} - m + 2\sqrt{3} \chi^2 \mathcal{V}^2 / \Lambda^2 \right), \]

where \( \text{Vol} \) is the volume of the spacetime and \( m = \chi_1 + \sqrt{3} \chi_2 \). Thus, to leading order in the \( 1/N_c \) expansion we have the gap equations, as follows:

\[ \frac{\partial V_1}{\partial m} \bigg|_{\chi^2} = -i \left( \mathcal{V} \text{ol} \right)^{-1} \text{tr} \frac{1}{i \mathcal{V} - m + 2\sqrt{3} \chi^2 \mathcal{V}^2 / \Lambda^2}, \quad (196) \]

\[ \frac{\partial V_1}{\partial \chi^2} \bigg|_{m} = i \left( \mathcal{V} \text{ol} \right)^{-1} \text{tr} \frac{2\sqrt{3} \mathcal{V}^2 / \Lambda^2}{i \mathcal{V} - m + 2\sqrt{3} \chi^2 \mathcal{V}^2 / \Lambda^2}. \quad (197) \]

First of all, we have to calculate the fermionic propagator, which satisfies

\[ \left( i \mathcal{V} - m + 2\sqrt{3} \chi^2 \mathcal{V}^2 / \Lambda^2 \right) \mathcal{G}(x, x\') = \delta(x, x'). \]

Once this propagator is obtained (apart from terms which disappear under the \( \text{tr} \) operation) it will be immediate to produce Eqs. (196) and (197) in explicit form and also the dependence of the effective potential itself on the fields \( \chi_1 \) and \( \chi_2 \). The details of this rather lengthy derivation are written in the Appendix of Ref. [62], where we give details about the calculation of the effective potential up to terms linear in the curvature. The technique is actually equivalent to weak curvature expansion method given in \( \S \). (The difference is in the presence of higher derivatives in propagator equation).

The final outcome is shown below in natural variables, which are obtained by using the following definitions

\[ r = \frac{R}{\Lambda^2}, \quad x_1 = \frac{\chi_1}{\Lambda}, \quad x_2 = \frac{\chi_2}{\Lambda}, \quad a = x_1 + \sqrt{3} x_2, \quad v = \frac{V}{\Lambda^4}. \quad (198) \]

The ‘dimensionless’ effective potential \( v \) may be calculated as [62]

\[ v = \frac{a^2}{l_1} - \frac{a^2}{8\pi^2} + \frac{a^4}{32\pi^2} - \frac{a^2}{96\pi^2} - 2\sqrt{3} \frac{a x_2}{l_1} + \sqrt{3} \frac{a x_2}{4\pi^2} - \sqrt{3} \frac{a^3 x_2}{2\pi^2} + \frac{3 x_2^2}{l_1} \]

\[ + \frac{x_2^2}{2\pi^2} + 9 \frac{a^2 x_2^2}{4\pi^2} + \frac{r x_2^2}{16\pi^2} - 2\sqrt{3} \frac{a x_2^2}{\pi^2} + 9 \frac{x_2^4}{4\pi^2} - \frac{a^4 \ln a^2}{16\pi^2} - \frac{a^2 r \ln a^2}{96\pi^2}, \quad (199) \]

where \( l_1 = \lambda_1 / \Lambda^2, \ l_2 = \lambda_2 / \Lambda^2 \). Note that in deriving this expression we have taken into account only terms up to quartic order on the fields \( a \) and \( x_2 \). The gap equations —which one obtains by differentiating \( v \) with respect to \( x_1 \) and \( x_2 \) and equating the results to zero— are, respectively

\[ x_1 \left( 1 - \frac{8 \pi^2}{l_1} \right) = -2 \times 3^{3/2} x_1^2 x_2 - 18 x_1 x_2^2 - 8 \sqrt{3} x_2^3 - a^3 \ln a^2 \]

\[ - r \left( \frac{a}{6} + \frac{\ln a^2}{12} \right), \quad (200) \]

\[ x_2 \left( 1 - \frac{8 \pi^2}{l_2} \right) = -2 \sqrt{3} x_1^3 - 18 x_1 x_2^2 - 8 \times 3^{3/2} x_1 x_2^2 - 24 x_2^3 - \sqrt{3} a^3 \ln a^2 \]

\[ - r \left( \frac{x_1}{2\sqrt{3}} + \frac{\ln a^2}{4\sqrt{3}} \right). \quad (201) \]

One can study now the influence of gravity on the symmetry breaking pattern of the theory. A simple inspection reveals that a positive curvature tends to protect the symmetry of the vacuum, and also a negative one may trigger the breaking of the symmetry.

In flat spacetime the symmetry is broken whenever either \( \lambda_1 \) or \( \lambda_2 \) are greater than \( 8 \pi^2 / \Lambda^2 \), that is why we shall take in the following \( k_1 \) and \( k_2 \) to be \( l_1 - 8 \pi^2 \) and \( l_2 - 8 \pi^2 \) respectively. We may illustrate
these remarks by studying the evolution of the minimum, given by two coordinates (which we choose to be \(a\) and \(x_2\)) in several circumstances. Before including gravity, it is worth discussing in more detail the situation in flat space. In fact, several cases may be analyzed (see Ref. [59]). To introduce this study it is convenient to define

\[
\pm \mu_i^2 \equiv \Lambda^2 \left(1 - \frac{8\pi^2}{\lambda_i^3}\right),
\]

where \(\lambda_i^+\) means \(\lambda_i > 8\pi^2\) (for the + sign) or \(\lambda_i < 8\pi^2\) (for the − sign). One sticks here to the case when \(\mu_i < \Lambda\). We only repeat the two situations given when \(\mu_2^3 > 3\mu_1^2\) and \((\lambda_1^+, \lambda_2^-)\), or \(\mu_2^3 < 3\mu_1^2\) and \((\lambda_1^-, \lambda_2^+)\). In both cases the results may be summarized by

\[
\begin{align*}
\chi_1^2 & = \frac{\mu_1^2}{\ln(\Lambda^2/m^2)} \left|1 - \frac{3\mu_1^2}{\mu_2^3} \right| \left[1 + O \left(\frac{1}{\ln(\Lambda^2/m^2)}\right)\right], \\
\chi_2 & = -\sqrt{3}\chi_1 \left(\frac{\mu_1}{\mu_2}\right)^2 \left[1 + O \left(\frac{1}{\ln(\Lambda^2/m^2)}\right)\right], \\
m^2 & = \frac{\mu_1^2 \mu_2^3}{|\mu_2^3 - 3\mu_1^2| \ln(\Lambda^2/m^2)}.
\end{align*}
\]

As a first example, consider the case in which the coupling constants are such that the symmetry is not broken in flat space, that is to say, \(\lambda_i < 8\pi^2\). One can see here the situation is modified as we move from negative to positive curvature by the numerical analysis.[62] We observe that there is a negative value of the curvature above which the symmetry is restored (phase transition.)

Note that usual four-fermion model was studied under the influence of a gravitational field in different situations (see §3). The conclusion of this section was always that if the coupling constant is greater than the critical value in flat space, there is a first order phase transition at some positive value of the curvature. In the present case there is no actual contradiction. The point is that, till now, we have concentrated ourselves in cases where the coupling constants are around \(8\pi^2\). We observe that if we explore regions in the space of parameters where \(\lambda_2\) is much smaller, and \(\lambda_1\) is greater than \(8\pi^2\)—which would correspond to a limit where our theory approaches the Gross-Neveu model—then there is a first order phase transition for some positive value of the curvature. Similarly, one can study numerically phase structure of the theory for other values of parameters and curvature.

There is an interesting question about this model—namely whether it would be possible for some generalization of it to be represented in renormalizable form. For \(\lambda_2 = 0\) it is possible to transform theory to Yukawa-type model which describes near the critical point the physics of chiral symmetry breaking. Note that the higher derivative term in (193) acts against such generalization at \(\lambda_2 \neq 0\).

Finally, note that in the same way, one can study other higher derivative four-fermion models in curved spacetime.

### 5.3 Supersymmetric NJL model in curved spacetime

Next we consider the supersymmetric NJL model non-minimally interacting with external supergravity. It is known that in flat spacetime [63] the fermion and the boson loop effects in this model are canceled out. Thus the supersymmetric NJL model does not show the dynamical chiral symmetry breaking in flat spacetime. Hence, could we expect that external (super) gravitational field could be the source of phase transitions in SUSY NJL model?

Below, we will discuss the curvature induced chiral symmetry breaking in supersymmetric NJL model following closely Ref. [64].

We will start with the action of SUSY NJL model in an external supergravitational background. This model can be considered as local supersymmetry generalization of the model given in Ref. [63]:

\[
S = \int d^8z E^{-1} \left[\bar{Q}Q + \bar{Q}^cQ^c + \lambda_0(\bar{Q}^cQ)(\bar{Q}^cQ) + \bar{\xi}_1(Q\bar{Q}) + \bar{\xi}_2(Q^c\bar{Q}^c)\right],
\]

\(51\)
where chiral superfields $Q^a$, $Q^c$ carry the color index $\alpha = 1, \cdots, N$ and belong to the representations of $SU(N)$, $E = \text{Ber}E^a_A$, $E_A^c$ is supertetrad, $\xi_1$ and $\xi_2$ are non-minimal coupling constants of SUSY NJL model with external supergravity. We follow the notation of book [65]. Note also that there exists more general form of non-minimal interaction above SUSY theory with supergravity but we consider only a simplest variant.

After the standard introduction of auxiliary superfields in a similar way which is explained in §2 and rewriting the action (203) in component fields we will limit ourselves to purely gravitational background. In the leading order of the 1/N-expansion the spinor auxiliary fields are dropped away since they may contribute only to next-to-leading order terms in the 1/N-expansion. Then the action to start with takes the form:

$$S = \int d^4x \sqrt{-g} \left[ -\bar{\psi} i\gamma^\mu \nabla_\mu - \rho^2 + \xi_1 R \right] \psi - \phi \xi_1 R \phi - \phi^c \xi_2 R \phi^c + \bar{\psi} (i\gamma^\mu \nabla_\mu - \rho) \psi - \frac{1}{\lambda_0} \rho^2 \right],$$

(204)

where $\rho^2 = \sigma^2 + \pi^2$ is an auxiliary scalar as in the original NJL model, $\psi$ is $N$ component Dirac spinor, $\xi_1 = (1 + \xi_1)/6$, $\xi_2 = (1 + \xi_2)/6$. The minimal interaction with external supergravity corresponds to $\xi_1 = \xi_2 = 1/6$. The start point action has the chiral symmetry. If the auxiliary field $\rho$ develops the non-vanishing vacuum expectation value, $\langle \rho \rangle = m \neq 0$, the fermion $\psi$ and the scalar $\phi$ acquire the dynamical mass $m$ and the chiral symmetry is eventually broken.$^{14}$

To find the phase structure of the model given by action (204) we introduce an effective potential. To evaluate the effective potential ones start with generating functional of Green functions.

$$Z = \int \mathcal{D}\psi \mathcal{D}\bar{\psi} \mathcal{D}\rho \, e^{iS}$$

$$= \int \mathcal{D}\rho \frac{\text{Det}(i\gamma^\mu \nabla_\mu - \rho)}{\text{Det}(\nabla^\mu \nabla_\mu + \rho^2 + \xi_1 R)(\nabla^\mu \nabla_\mu + \rho^2 + \xi_2 R)} \exp i \int d^4x \sqrt{-g} \left( -\frac{1}{\lambda_0} \rho^2 \right)$$

$$= \int \mathcal{D}\rho \exp \left[ \int d^4x \sqrt{-g} \left( -\frac{1}{\lambda_0} \rho^2 \right) - i \text{ln Det}(i\gamma^\mu \nabla_\mu - \rho) + i \text{ln Det}(\nabla^\mu \nabla_\mu + \rho^2 + \xi_1 R) + i \text{ln Det}(\nabla^\mu \nabla_\mu + \rho^2 + \xi_2 R) \right].$$

(205)

An internal line of the $\rho$-propagator has no contribution in the leading order of the 1/N-expansion. Assuming that the $\rho$ is slowly varying field and applying the 1/N-expansion method the effective potential for $\rho$ is found to be

$$V(\rho) = \frac{1}{\lambda_0} \rho^2 - i \text{tr} \int_0^\rho ds \, S(x, x; s) - 2i \int_0^\rho ds [G_1(x, x; s) + G_2(x, x; s)] + O \left( \frac{1}{N} \right),$$

(206)

where $S(x, x; s)$ and $G_i(x, x; s)$ are the spinor and scalar two-point functions respectively. It should be noted that the effective potential (206) is normalized as $V(0) = 0$.

We will evaluate the effective potential taking into account the terms up to linear curvature and using local momentum representation of propagators (see §3 above). Hence the effective potential reads

$$V(\rho) = \frac{1}{\lambda_0} \rho^2 - 4i \int_0^\rho dsd \int \frac{dp}{(2\pi)^4} \left[ \frac{1}{p^2 - s^2} - \frac{1}{12} R \frac{1}{(p^2 - s^2)^2} + \frac{2}{3} R^{\mu\nu} p_\mu p_\nu \frac{1}{(p^2 - s^2)^3} \right].$$

$^{14}$We started with the locally supersymmetric action (203) in order to justify the consideration of the Lagrangian (204) and clarify its origin. However, after restriction to pure gravitational background the theory with Lagrangian (204) is not supersymmetric since the gravitino field (fermionic superpartner for gravitational field) is absent. However, taking into account that we started from supersymmetric theory (in flat spacetime) we continue to call our model as SUSY NJL model.
$$+ 2i \int_0^\rho \text{d}s \int \frac{\text{d}^4p}{(2\pi)^4} \left[ \frac{2}{p^2 - s^2} - \left( \frac{2}{3} - \xi_1 - \xi_2 \right) R \frac{1}{(p^2 - s^2)^2} \right. $$

$$= \left. \frac{4}{3} R^{\mu\nu} p_\mu p_\nu \frac{1}{(p^2 - s^2)^3} \right]$$

$$= \frac{1}{\Lambda^0} \rho^2 + 2i R \left( \frac{1}{6} - \frac{2}{3} + \xi_1 + \xi_2 \right) \int_0^\rho \text{d}s \int \frac{\text{d}^4p}{(2\pi)^4} \frac{1}{(p^2 - s^2)^2} + O \left( \frac{1}{N} \right). \hspace{1cm} \text{(207)}$$

Thus for $\xi_1 + \xi_2 = 1/2$ the fermion loop contribution is cancelled with the boson loop contribution. To evaluate the integration in Eq. (207) we perform the Wick rotation $p^0 \to ip^0$

$$I = iR \int_0^\rho \text{d}s \int \frac{\text{d}^4p}{(2\pi)^4} \frac{1}{(p^2 - s^2)^2}$$

$$\to - R \int_0^\rho \text{d}s \int \frac{\text{d}^4p}{(2\pi)^4} \frac{1}{(p^2 + s^2)^2}. \hspace{1cm} \text{(208)}$$

Applying the Schwinger proper time method \[12\] $I$ is rewritten as

$$I = - R \int_0^\rho \text{d}s \int \frac{\text{d}^4p}{(2\pi)^4} \int_0^\infty \text{d}t \ e^{-t(p^2 + s^2)}$$

$$= R \int_0^\infty \text{d}t \left( e^{-t\rho^2} - 1 \right). \hspace{1cm} \text{(209)}$$

Since the integration over $t$ is divergent around $t \sim 0$, we introduce the proper time cut-off $\Lambda$ and find

$$I \to - R \int_1^{\Lambda^2} \frac{1}{(4\pi)^2} \text{d}t \left( e^{-t\rho^2} - 1 \right)$$

$$= - R \int_0^\infty \frac{1}{(4\pi)^2} \left[ \rho^2 \text{Ei} \left( \frac{-p^2}{\Lambda^2} \right) + \Lambda^2 (e^{-\rho^2/\Lambda^2} - 1) \right]. \hspace{1cm} \text{(210)}$$

where $\text{Ei}(-x)$ is the exponential-integral function which is defined by

$$\text{Ei}(-x) = - \int_x^\infty \frac{e^{-t}}{t} \ dt = \ln x + \gamma + \sum_{n=1}^{\infty} \frac{(-x)^n}{n \cdot n!} < 0 ; (x > 0). \hspace{1cm} \text{(211)}$$

Hence the effective potential in the leading order of the $1/N$-expansion for the supersymmetric NJL model in curved spacetime reads

$$V(\rho) = \frac{1}{\Lambda^0} \rho^2 - R \int \frac{\text{d}s}{(4\pi)^2} f(\xi_1, \xi_2) \left[ \rho^2 \text{Ei} \left( \frac{-p^2}{\Lambda^2} \right) + \Lambda^2 (e^{-\rho^2/\Lambda^2} - 1) \right], \hspace{1cm} \text{(212)}$$

where $f(\xi_1, \xi_2)$ is

$$f(\xi_1, \xi_2) = \frac{1}{2} - \xi_1 - \xi_2. \hspace{1cm} \text{(213)}$$

The ground state of the system corresponds to the minimum of the effective potential. To find the ground state we evaluate the effective potential with varying the $f(\xi_1, \xi_2)R\lambda_0$. \cite{64}

In flat spacetime $R = 0$ the effective potential (212) takes a very simple form of quadratic function. It is evident that the minimum of the effective potential will be only at $\rho = 0$ for arbitrary values of the nonminimal coupling parameters.

For $f(\xi_1, \xi_2)R\lambda_0 > 0$ taking into account the property (211) of the exponential-integral function we see that the effective potential (212) is non-negative and takes minimum at $\rho = 0$.

To understand the phase structure of the SUSY NJL model for a negative $f(\xi_1, \xi_2)R\lambda_0$ we will analyse the effective potential more precisely. The dynamically generated mass $m$ of fermion $\psi$ and scalar $\phi$ is
given by the value of $\rho$ at the minimum of the effective potential. Stationary condition for the effective potential (212) is given by

$$\frac{\partial V(\rho)}{\partial \rho} = \rho \left[ \frac{2}{\lambda_0} - \frac{2R}{(4\pi)^2} f(\xi_1, \xi_2) \text{Ei} \left( -\frac{\rho^2}{\Lambda^2} \right) \right] = 0. \quad (214)$$

Thus the dynamical mass $m = \langle \rho \rangle$ satisfies

$$\frac{16\pi^2}{R\lambda_0} = f(\xi_1, \xi_2) \text{Ei} \left( -\frac{m^2}{\Lambda^2} \right) < 0. \quad (215)$$

This equation has a solution only for a negative $f(\xi_1, \xi_2)R\lambda_0$. In Fig. 19 the dynamically generated mass is plotted as a function of $f(\xi_1, \xi_2)R\lambda_0$. The dynamical mass $m = \langle \rho \rangle$ which corresponds to the order parameter smoothly disappears as the $f(\xi_1, \xi_2)R\lambda_0$ increases.

The critical curvature $R_{\text{cr}}$ which divides the symmetric phase and the broken phase is given by the curvature where the order parameter $m$ disappears. If we take the limit $m/\Lambda \to 0$ in Eq. (215), we find

$$R\lambda_0 = \frac{16\pi^2}{f(\xi_1, \xi_2)} \left[ \ln \frac{m^2}{\Lambda^2} + \gamma + O \left( \frac{m^2}{\Lambda^2} \right) \right]^{-1} \to 0 \quad (f(\xi_1, \xi_2) \neq 0). \quad (216)$$

Due to the stability of the ground state the coupling $\lambda_0$ must be real and positive. Hence the critical curvature is given by $R_{\text{cr}} = 0$ and the chiral symmetry is broken down for an arbitrary negative $f(\xi_1, \xi_2)R$ and an arbitrary positive $\lambda_0$.

For $m/\Lambda \ll 1$ the dynamical mass $m$ is obtained by

$$m^2 = \Lambda^2 \left[ \exp \left( \frac{16\pi^2}{f(\xi_1, \xi_2)R\lambda_0} - \gamma \right) + O \left( \frac{m^2}{\Lambda^2} \right) \right]. \quad (217)$$

The dynamical mass is suppressed exponentially for a small negative $f(\xi_1, \xi_2)R\lambda_0$. 

---

**Figure 19**: Behaviour of the dynamically generated mass $m$ with varying the $f(\xi_1, \xi_2)R\lambda_0$. 

---
Thus it is found that the chiral symmetry is broken down for an arbitrary negative \( f(\xi_1, \xi_2) R \) and an arbitrary positive \( \lambda_0 \) in SUSY NJL model within our approximation. For \( \xi_1 + \xi_2 < 1/2 \) the broken phase is realised in a spacetime with an arbitrary negative curvature and the symmetric phase is realized for an arbitrary positive curvature. On the other hand there is only the broken phase in positive curvature spacetime and the symmetric phase at a negative curvature for \( \xi_1 + \xi_2 > 1/2 \). In both cases the critical curvature is given by \( R_{cr} = 0 \). The dynamically generated fermion mass exponentially disappears at the critical point. If the dynamically generated fermion mass is extremely small, the large \( R^2/m^2 \) compared with \( R \) may spoil the validity of the linear curvature approximation. The terms of the order \( O(R^2/m^2) \) may change the exponential behavior of the dynamical mass near the critical point.

Note finally that there are few possibilities to generalize the results of the present consideration. First of all, it could be interesting to study phase structure of SUSY NJL model on non-trivial supergravitational background (with non-zero gravitino). Second, it could be interesting to investigate the gauged SUSY NJL model where compositeness conditions a la Bardeen-Hill-Lindner [45] maybe implemented. That could lead to the formulation of compositeness condition for \( \xi_1 \) and \( \xi_2 \) which presumably should be given by asymptotically (super) conformal invariant values. [7, 50]

6 Thermal and curvature induced phase transition

At the early stage of the universe the combined effects of the temperature and curvature to the DSB may play an important role. In the present sections we will apply the similar analysis as in \( \S 2 \) at finite temperature and curvature. To show the combined effects of the temperature and curvature to the DSB, we fix the coupling constant \( \lambda \) above the critical one and see whether the broken chiral symmetry is restored in an environment of the high temperature and/or large curvature. To introduce the temperature in the theory it is supposed that the system is in equilibrium. This assumption is not accepted in a general curved spacetime. In the spacetime which has no time evolution the equilibrium state can be defined. We then restrict ourselves in the positive curvature spacetime \( R \otimes S^{D-1} \) and the negative curvature spacetime \( R \otimes H^{D-1} \). To find the ground state at finite temperature and curvature we calculate the effective potential and analyze its stationary condition by the gap equation in the leading order of the \( 1/N \) expansion.

In this section we mainly follow Ref. [67] and discuss the thermal and curvature effects to the dynamical symmetry breaking.

\(^{15}\)In Ref. [66] the combined effects of the temperature and curvature to the DSB is discussed in the positive weak curvature limit. In the paper the variable \( k \) is introduced by using the trace formula:

\[
1 = \frac{1}{(4\pi)^D/2} \int d^Dk e^{-sk^2}. \tag{218}
\]

Inserting Eq. (218) into Eq. (25) the effective potential is rewritten as

\[
V(\sigma) = \frac{1}{2\lambda_0} \sigma^2 - i \text{tr} \int_0^\infty ds \frac{1}{(4\pi s)^{D/2}} \int d^Dk \frac{k}{(2\pi)^D} e^{-sk^2} S(x, x; s). \tag{219}
\]

The temperature \( T \) and chemical potential \( \mu \) are introduced by the replacement

\[
k^0 \rightarrow \sqrt{(\omega_n - i\mu)^2}, \quad \int \frac{dk^0}{2\pi} \rightarrow k_B T \sum_n , \tag{220}
\]

where \( k_B \) is the Boltzmann constant and \( \omega_n = (2n + 1)\pi k_B T \). The variable \( k \) in Eq. (219) does not always correspond to the momentum of the fermion field in a general curved spacetime. Thus above method is useful only in flat spacetime or special spacetime where the variable \( k \) is regarded as the momentum of the fermion field. So we do not use the trace formula to introduce temperature here.
6.1 Effective potential at finite temperature

It is known that the broken chiral symmetry is restored for a sufficiently high temperature through the second order phase transition in flat spacetime.\cite{16, 40} Here we introduce the effect of the finite temperature to the four-fermion models and give the explicit expression of the effective potential at finite temperature.

As we have seen in §2, the effective potential is expressed by the two-point function \( S(x, x; s) \) of a massive free fermion. The two-point function at finite temperature is defined by

\[
S_T(x, x; s) = \sum_{\alpha} e^{-\beta E_\alpha} \frac{\langle \alpha | T(\psi(x)\bar{\psi}(x)) | \alpha \rangle}{\sum_{\alpha} e^{-\beta E_\alpha}}, \tag{221}
\]

where \( E_\alpha \) is the energy in the state specified by quantum number \( \alpha \) respectively, \( \beta = 1/k_B T \) with \( k_B \) the Boltzmann constant and \( T \) the temperature.

Following the standard procedure of the dealing with Matsubara Green function, the two-point function at finite temperature is obtained from the one at \( T = 0 \) by the Wick rotation and the replacements\cite{68}

\[
\begin{align*}
\int_{-\infty}^{\infty} & \frac{dk^0}{2\pi i} \rightarrow \frac{1}{\beta} \sum_{n=\infty}, \\
k^0 & \rightarrow i\omega_n \equiv \frac{2n + 1}{\beta} \pi, \\
\gamma^0 & \rightarrow i\gamma^0.
\end{align*} \tag{222}
\]

In Minkowski space the effective potential at finite temperature in the leading order of the \( 1/N \) expansion reads\cite{40}

\[
V_T(\sigma) = \frac{1}{2\lambda_0} \sigma^2 + \text{tr} \int_0^\sigma ds \frac{1}{\beta} \sum_{n=-\infty}^\infty \int \frac{d^{D-1}k}{(2\pi)^{D-1}} \frac{1}{k + s}. \tag{223}
\]

If we perform the integration over \( k \), we obtain

\[
V_T(\sigma) = \frac{1}{2\lambda_0} \sigma^2 - \int_0^\sigma ds \frac{\text{tr} 1}{(4\pi)^{(D-1)/2}} \Gamma\left(\frac{3-D}{2}\right) \frac{1}{\beta} \sum_{n=-\infty}^\infty s (s^2 + \omega_n^2)^{(D-3)/2}. \tag{224}
\]

Performing a summation and integrating over angle variables and \( s \) in Eq. (223), we get

\[
V_T(\sigma) = \frac{1}{2\lambda_0} \sigma^2 - \frac{\text{tr} 1}{(4\pi)^D} \frac{1}{D} \Gamma\left(1 - \frac{D}{2}\right) \sigma^D \\
- \frac{\text{tr} 1}{(4\pi)^{(D-1)/2}} \Gamma\left(\frac{D-1}{2}\right) \frac{1}{\beta} \int_0^\infty dt t^{(D-3)/2} \ln \frac{1 + e^{-\beta \sqrt{t + \sigma^2}}}{1 + e^{-\beta \sqrt{t}}}. \tag{225}
\]

Comparing Eq. (224) with Eq. (225) we find the following relation.

\[
\frac{\text{tr} 1}{(4\pi)^D} \frac{1}{D} \Gamma\left(1 - \frac{D}{2}\right) \sigma^D + \frac{\text{tr} 1}{(4\pi)^{(D-1)/2}} \Gamma\left(\frac{D-1}{2}\right) \frac{1}{\beta} \int_0^\infty dt t^{(D-3)/2} \ln \frac{1 + e^{-\beta \sqrt{t + \sigma^2}}}{1 + e^{-\beta \sqrt{t}}}
= \int_0^\sigma ds \frac{\text{tr} 1}{(4\pi)^{(D-1)/2}} \Gamma\left(\frac{3-D}{2}\right) \frac{1}{\beta} \sum_{n=-\infty}^\infty s (s^2 + \omega_n^2)^{(D-3)/2}. \tag{226}
\]

This relation will be used for numerical calculation of the effective potential at finite temperature in curved spacetime.
Starting from the theory with the broken chiral symmetry at vanishing $T$ and evaluating the effective potential at finite temperature, it is known that the broken chiral symmetry is restored at a critical temperature $T_{cr}$ through the second order phase transition. The critical temperature is given by \[ \frac{k_B T_{cr}}{m_0} = \frac{1}{2\pi} \left[ \frac{2\Gamma \left( \frac{3-D}{2} \right)}{\sqrt{\pi} \Gamma \left( \frac{2-D}{2} \right)} \right]^{1/(2-D)} (3^D - 1) \zeta(3-D). \] (227)

Below, we investigate the combined effects of the temperature and curvature on $R \otimes S^{D-1}$ and $R \otimes H^{D-1}$.

### 6.2 Positive curvature space ($R \otimes S^{D-1}$)

In the positive curvature space, $R \otimes S^{D-1}$, the effective potential is given by (4.60). According to the definition of the two-point function at finite temperature (221), we obtain the effective potential in the space at finite temperature by the replacements (222)

\[
\int_{0}^{\sigma} ds \frac{\rho}{(4\pi)^{(D-1)/2}} \frac{a^{3-D}}{\beta} 
\times \sum_{n=-\infty}^{\infty} \frac{\Gamma \left( \frac{D-1}{2} + i\alpha_n \right) \Gamma \left( \frac{D-1}{2} - i\alpha_n \right)}{\Gamma \left( 1 + i\alpha_n \right) \Gamma \left( 1 - i\alpha_n \right)} \Gamma \left( \frac{3-D}{2} \right),
\]

where $\alpha_n$ is defined by

\[
\alpha_n \equiv a \sqrt{s^2 + \omega_n^2}. \tag{229}
\]

Evaluating the effective potential (228) we will find the phase structure of the model at finite temperature in positive curvature space.

For numerical calculations we need the finite expression of the effective potential in summation and integration. Inserting Eq. (2.33) and Eq. (226) into Eq.(228) the renormalized effective potential reads

\[
V_{TR}^R(\sigma) = \frac{1}{2 \lambda_R} \sigma^2 - \int_{0}^{\sigma} ds \frac{\rho}{(4\pi)^{(D-1)/2}} \frac{a^{3-D}}{\beta} \times \sum_{n=-\infty}^{\infty} \frac{\Gamma \left( \frac{D-1}{2} + i\alpha_n \right) \Gamma \left( \frac{D-1}{2} - i\alpha_n \right)}{\Gamma \left( 1 + i\alpha_n \right) \Gamma \left( 1 - i\alpha_n \right)} \Gamma \left( \frac{3-D}{2} \right),
\]

where $\alpha_n$ is defined by

\[
\alpha_n \equiv a \sqrt{s^2 + \omega^2}. \tag{230}
\]

In this representation of the effective potential the divergence is canceled out in the summation.

The phase structure of the theory is obtained by observing the minimum of the effective potential. The necessary condition for the minimum of the effective potential is given by the gap equation:

\[
\frac{\partial V_{TR}^R(\sigma)}{\partial \sigma} \bigg|_{\sigma=m} = 0. \tag{231}
\]

If the gap equation has a non-trivial solution which corresponds to the minimum of the effective potential, the chiral symmetry is broken down and the dynamical fermion mass is generated. The non-trivial
solution of the gap equation is given by

\[
\left(\frac{1}{\lambda_R} - \frac{1}{\lambda_{cr}}\right)\mu^{D-2} - \frac{\text{tr} 1}{(4\pi)^{D/2}} \Gamma\left(1 - \frac{D}{2}\right) m^{D-2} + \frac{\text{tr} 1}{(4\pi)^{(D-1)/2}} \Gamma\left(\frac{D-1}{2}\right) \int_0^\infty dt t^{(D-3)/2} \frac{1}{\sqrt{t + m^2}} e^{-\beta\sqrt{t + m^2}} + \frac{\text{tr} 1}{(4\pi)^{(D-1)/2}} \Gamma\left(\frac{3 - D}{2}\right) \sum_{n=-\infty}^{\infty} \left[ (m^2 + \omega_n^2)^{(D-3)/2} - a^{3-D} \frac{\Gamma\left(\frac{D-1}{2} + i\alpha_n\right) \Gamma\left(\frac{D-1}{2} - i\alpha_n\right)}{\Gamma\left(1 + i\alpha_n\right) \Gamma\left(1 - i\alpha_n\right)} \right] = 0,
\]

where \(\alpha_n = a\sqrt{m^2 + \omega_n^2}\), \(\lambda_{cr}\) is defined in Eq. (49) and \(m\) corresponds to the dynamically generated fermion mass.

In Ref. [67] the typical behaviors of the dynamical fermion mass \(m\) was evaluated numerically as a function of temperature \(T\) or curvature \(R\). It was found that the broken chiral symmetry is restored for a sufficiently high temperature and large curvature and only the second order phase transition is realized with varying temperature and/or curvature for \(2 \leq D < 4\).

Since the dynamical fermion mass smoothly disappears at the critical point for second order phase transition, the critical line on \(T-R\) plane is given by the massless limit of Eq. (232). To find the equation for the critical line in an analytic form we take the limit \(m \to 0\) in Eq. (232) and find

\[
\frac{\text{tr} 1}{(4\pi)^{D/2}} \Gamma\left(1 - \frac{D}{2}\right) m_0^{D-2} - \frac{\text{tr} 1}{(4\pi)^{(D-1)/2}} \frac{2}{\beta_{cr}} \Gamma\left(\frac{3 - D}{2}\right) \left(\frac{2\pi}{\beta_{cr}}\right)^{D-3} \zeta(3-D, 1/2) + \frac{\text{tr} 1}{(4\pi)^{(D-1)/2}} \Gamma\left(\frac{3 - D}{2}\right) \sum_{n=-\infty}^{\infty} \left[ \omega_{\text{cr} n}^{D-3} - a^{3-D} \frac{\Gamma\left(\frac{D-1}{2} + i\alpha_{\text{cr} n}\right) \Gamma\left(\frac{D-1}{2} - i\alpha_{\text{cr} n}\right)}{\Gamma\left(1 + i\alpha_{\text{cr} n}\right) \Gamma\left(1 - i\alpha_{\text{cr} n}\right)} \right] = 0,
\]

where \(\zeta(z, a)\) is the generalized zeta function, \(\omega_{\text{cr} n} = (2n + 1)i\beta_{cr}\), \(\alpha_{\text{cr} n} = a_{\text{cr}} |\omega_{\text{cr} n}|\) and \(m_0\) is the dynamical fermion mass in Minkowski spacetime at \(T = 0\). The critical lines are shown in Fig. 20.

The two-dimensional spacetime \(R \otimes S^1\) is a flat compact spacetime, \(R = 0\). Thus the symmetry restoration which is caused by decreasing \(a\) is induced by the finite size effect of the compact space. In four dimensions the effective potential at \(T = 0\) is divergent. However the thermal effect gives only a finite correction to the effective potential. Thus the critical temperature \(T_{cr}\) at \(R = 0\) is divergent at the four-dimensional limit. Near four dimensions the curvature effect seems to give the main contribution to the phase transition. But it results from the non-renormalizability of the four-fermion model. In a renormalizable theory the situation must be changed.

### 6.3 Negative curvature space (\(R \otimes H^{D-1}\))

In a negative curvature spacetime the chiral symmetry is always broken down irrespective of \(\lambda\) at \(T = 0\). Can the thermal effect restore the symmetry in the negative curvature spacetime? Here we consider the model at finite temperature in the negative curvature spacetime, \(R \otimes H^{D-1}\), for both \(\lambda > \lambda_{cr}\) and \(\lambda \leq \lambda_{cr}\). The manifold \(R \otimes H^{D-1}\) is defined by the metric

\[
ds^2 = dt^2 + a^2(d\theta^2 + \sinh^2 \theta \ d\Omega_{D-2}).
\]

It is a constant curvature spacetime with negative curvature

\[
R = -(D - 1)(D - 2)a^{-2}.
\]

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Figure 20: The phase diagram at $D = 2.0, 2.5, 3.0, 3.5$.

respectively ($2 \leq D < 4$).

According to the similar method used in the previous subsection the effective potential at finite temperature in $R \otimes H^{D-1}$ reads

$$V^{TR}(\sigma) = \frac{1}{2\lambda_0} \sigma^2 - \int_0^\sigma ds \frac{\text{tr} \mathbf{1}}{(4\pi)^{(D-1)/2}} \frac{s^D - D}{D} \frac{\Gamma \left( \frac{D-1}{2} + \alpha_n \right) \Gamma \left( \frac{3-D}{2} \right)}{\Gamma \left( \frac{1}{2} \right)} \sum_{n=-\infty}^{\infty} \frac{\Gamma \left( \frac{D-1}{2} + \alpha_n \right)}{\alpha_n \Gamma \left( \frac{1}{2} \right)} \frac{\Gamma \left( \frac{3-D}{2} \right)}{\alpha_n \Gamma \left( \frac{3-D-1}{2} \right)}.$$

(235)

$R \otimes H^1$ is equivalent to the two-dimensional Minkowski space $R^2$. At the two-dimensional limit of Eq. (235) the effective potential in two-dimensional Minkowski space is reproduced. Because of the convenience for numerical calculations we rewrite the effective potential (235) in the same form described in the previous subsection. Inserting Eq. (233) and Eq. (226) into Eq. (235) we get

$$V^{TR}_R(\sigma) = \frac{1}{2} \left[ \frac{1}{\lambda_R} + \frac{\text{tr} \mathbf{1}}{(4\pi)^{D/2}} \frac{1}{\Gamma \left( \frac{D}{2} \right)} \right] \mu^{D-2} \sigma^2 - \frac{\text{tr} \mathbf{1}}{(4\pi)^{D/2}} \frac{1}{\Gamma \left( \frac{D}{2} \right)} \int_0^\sigma \frac{dt}{\Gamma \left( \frac{(D-3)/2}{2} \right)} \ln \frac{1 + e^{-\beta \sqrt{1+\sigma^2}}}{1 + e^{-\beta \sqrt{1+\sigma^2}}}$$

$$+ \int_0^\sigma ds \frac{\text{tr} \mathbf{1}}{(4\pi)^{(D-1)/2}} \frac{1}{\Gamma \left( \frac{3-D}{2} \right)}$$

$$\sum_{n=-\infty}^{\infty} \frac{s \left( (s^2 + \omega_n^2)^{(D-3)/2} - a^{3-D} \frac{\Gamma \left( \frac{D+1}{2} + \alpha_n \right)}{\alpha_n \Gamma \left( \frac{1}{2} \right)} \Gamma \left( \frac{D+1}{2} + \alpha_n \right) \right)}{\alpha_n \Gamma \left( \frac{1}{2} \right)}.$$

(236)

To find the minimum of the effective potential (236) we analyze the non-trivial solution of the gap equation. Substituting Eq. (236) to Eq. (231) the gap equation reads

$$\left( \frac{1}{\lambda_R} - \frac{1}{\lambda_{cr}} \right) \mu^{D-2} = \frac{\text{tr} \mathbf{1}}{(4\pi)^{D/2}} \frac{1}{\Gamma \left( \frac{D}{2} \right)} m^{D-2}$$

$$+ \frac{\text{tr} \mathbf{1}}{(4\pi)^{D/2}} \frac{1}{\Gamma \left( \frac{D}{2} \right)} \int_0^\infty \frac{dt}{\Gamma \left( \frac{(D-3)/2}{2} \right)} \frac{1}{\sqrt{t + m^2}} e^{-\beta \sqrt{t + m^2}}$$

$$- \frac{\text{tr} \mathbf{1}}{(4\pi)^{D/2}} \frac{1}{\Gamma \left( \frac{D}{2} \right)} \int_0^\infty \frac{dt}{\Gamma \left( \frac{(D-3)/2}{2} \right)} \frac{1}{\sqrt{t + m^2}} e^{-\beta \sqrt{t + m^2}}$$

(237)

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Figure 21: The phase diagram at $D=2.5, 3.0, 3.5$ for $\lambda > \lambda_{cr}$.

\[ E = \frac{\text{tr} 1}{(4\pi)^{(D-1)/2}} \frac{1}{\beta} \Gamma \left( \frac{3-D}{2} \right) \times \sum_{n=-\infty}^{\infty} \left[ (m^2 + \omega_n^2)^{(D-3)/2} - a^{-D} \frac{\Gamma(D-1)}{\Gamma \left( \frac{3-D}{2} + \alpha_n \right)} \right] = 0. \]

Evaluating the gap equation (237) the dynamical fermion mass $m$ is obtained.\cite{67} The dynamical fermion mass smoothly disappears as the temperature increases with the curvature fixed for both $\lambda > \lambda_{cr}$ and $\lambda \leq \lambda_{cr}$. Then the broken chiral symmetry is restored for a sufficiently high temperature. On the other hand the dynamical fermion mass becomes heavier as the curvature $|R|$ decreases with the temperature fixed. Calculating Eq. (227) in three dimensions the critical temperature for $a \to \infty$ is given by

\[ k_B T_{cr} = \frac{1}{2 \ln 2}. \] \hfill (238)

The curvature effects enhance the symmetry breaking on $R \otimes H^2$. Hence there is only the broken phase for $k_B T < 1/(2 \ln 2)$ in the model $\lambda > \lambda_{cr}$. For $k_B T \geq 1/(2 \ln 2)$ or $\lambda \leq \lambda_{cr}$ the dynamical fermion mass is smoothly generated as the curvature $|R|$ increases and the chiral symmetry is broken down by the curvature effect. After the same analysis done in arbitrary dimensions $2 < D < 4$ we find the same behavior for the dynamical fermion mass. The thermal effect restores the broken chiral symmetry while the negative curvature effect breaks the chiral symmetry. Only the second order phase transition occurs with varying temperature and curvature in $2 < D < 4$. In two dimensions Eq. (237) has the same behavior in Minkowski space.

For the second order phase transition the critical point is obtained by the massless limit of the gap equation. Taking the massless limit $m \to 0$ in Eq. (237) we find the equation that gives the relation between critical temperature $\beta_{cr}$ and critical radius $a_{cr}$:

\[ \frac{\text{tr} 1}{(4\pi)^{D/2}} \frac{1}{\Gamma \left( 1 - \frac{D}{2} \right)} m_0^{D-2} \]
$K_B T_{cr} / m_0$

Figure 22: The phase diagram at $D = 2.5, 3.0, 3.5$ for $\lambda \leq \lambda_{cr}$.

\[
- \frac{\text{tr} 1}{(4\pi)^{(D-1)/2}} \frac{2}{\beta_{cr}} \Gamma \left( \frac{3 - D}{2} \right) \left( \frac{2\pi}{\beta_{cr}} \right)^{D-3} \zeta(3 - D, 1/2) \\
+ \frac{\text{tr} 1}{(4\pi)^{(D-1)/2}} \frac{1}{\beta_{cr}} \Gamma \left( \frac{3 - D}{2} \right) \\
\times \sum_{n=-\infty}^{\infty} \left[ \omega_{cr_n} (D-3) - a_{cr_n}^3 \frac{\Gamma \left( \frac{D-1}{2} + \alpha_{cr_n} \right)}{\alpha_{cr_n} \Gamma \left( \frac{3-D}{2} + \alpha_{cr_n} \right)} \right] = 0.
\]

Evaluating Eq. (239) numerically we draw the phase diagram of the four-fermion model with varying temperature and/or curvature on $R \otimes H^{D-1}$ at $D = 2.5, 3.0, 3.5$ in Figs. 21 and 22. In those figures the normalization scale $m_0$ is taken to the value defined in Eq. (47) for $\lambda > \lambda_{cr}$ and

\[
m_0 = \mu \left[ - \frac{(4\pi)^{D/2}}{\text{tr} \Gamma(1 - D/2) \lambda_R} \right]^{1/(D-2)}
\]

for $\lambda \leq \lambda_{cr}$. At a scale $T \sim a^{-1} \sim m_0$ the thermal effect gives the main contribution to the phase structure for $D \leq 3$ and the symmetric phase is realized. At four-dimensional limit $T_{cr}$ is divergent for $\lambda > \lambda_{cr}$. It comes from the divergence which appears at the four-dimensional limit. Thus it may be a result caused mainly by the non-renormalizability of the theory.

Thus we have investigated the phase structure of the four-fermion model at finite temperature and curvature in arbitrary dimensions (2 < D < 4). In positive curvature space the curvature effect restores the broken chiral symmetry. On the contrary the curvature effect enhances the chiral symmetry breaking in negative curvature space. At finite temperature the lower limit of the momentum for the fermion field ($k^0 < \pi / \beta$) appears from the antiperiodicity and then the long range effect is suppressed. Thus the thermal effect restore the broken symmetry even in a negative curvature spacetime.

In two spacetime dimensions it is impossible to introduce the combined effect of the temperature and curvature within our method because of the assumption of the equilibrium. This assumption is not fulfilled in inflationary expanding universe, but we may discuss the phase transition at early universe using our results.
7 Finite size and topological effects

In the previous sections we discussed four-fermion models in curved spacetime. The main motivation to do such a study was the cosmological applications. However, one may expect that the very early universe had a non-trivial topology (the introduction of the temperature in the hot universe maybe classified as the similar type of effects). This renders interesting to investigate the phenomenon of dynamical chiral symmetry breaking in curved spacetime with non-trivial topology. This section will be devoted to such a study on the background $\mathcal{M}^{D-1} \otimes S^1$ where $S^1$ is the one-dimensional sphere and $\mathcal{M}^{D-1}$ is a $(D - 1)$-dimensional arbitrary curved manifold of trivial topology (see Ref. [69]).

7.1 Effective potential in $\mathcal{M}^{D-1} \otimes S^1$

We start again from the action of four-fermion models in curved spacetime defined in (1) or (26). Making the explicit calculations in the same way as in §2 we easily get the effective potential in the leading order of $1/N$ expansion:

$$V(\sigma) = \frac{1}{2\lambda_0} \sigma^2 - \frac{i}{2} \text{tr} \int_0^\sigma dsS(x,x;s) + O\left(\frac{1}{N}\right).$$

Using the known expression for the propagator of a free Dirac field in a weakly-varying gravitational background (see expression (67)), we will then write the effective potential with accuracy up to linear curvature terms. In the spacetime $\mathcal{M}^{D-1} \otimes S^1$ there is no constraint for the boundary condition [70, 71] along the compactified direction. Some of the independent spin structures are allowed in our universe. If we take the antiperiodic boundary condition along the compactified direction, the field theory in $\mathcal{M}^{D-1} \otimes S^1$ is equivalent to the finite temperature field theory (see §6). Here we consider the fermion fields with periodic and antiperiodic boundary conditions independently. Thus, the effective potential is found to be

$$V(\sigma) = \frac{1}{2\lambda_0} \sigma^2 - \frac{i}{2} \text{tr} \int_0^\sigma dsL \sum_{n=-\infty}^{\infty} \int \frac{d^{D-1}k}{(2\pi)^{D-1}} 
\times \left[ \frac{1}{k^2 - s^2} - \frac{1}{12} R \left( \frac{1}{(k^2 - s^2)^2} \right) + \frac{2}{3} R_{\mu\nu} k^\mu k^\nu \frac{1}{(k^2 - s^2)^3} \right],$$

(241)

where one should integrate over $k_0, \cdots, k_{D-2}$ and sum over the coordinate $k_{D-1}$, which is given by: $k_{D-1} = (2n + \delta_{p,1}) \pi/L$.

Evaluating the effective potential (241) we study the phase structure of the four-fermion models in weakly curved spacetime with nontrivial topology below.

7.2 Flat spacetime with nontrivial topology ($R^{D-1} \otimes S^1$)

Now it is not so clear if the symmetry restoration is mainly caused by the curvature effect or finite size effect in an arbitrary dimension. In flat spacetime with nontrivial topology we clearly see only the finite size effect. In Refs. [72, 73, 74] the dynamical symmetry breaking has been investigated in flat spacetime with nontrivial topology. In the present subsection we calculate the effective potential (241) in flat spacetime $R^{D-1} \otimes S^1$ to see only a finite size effect.

In $R^{D-1} \otimes S^1$ the effective potential (241) reduces to

$$V(\sigma) = \frac{1}{2\lambda_0} \sigma^2 - \frac{i}{2} \text{tr} \int_0^\sigma dsL \sum_{n=-\infty}^{\infty} \int \frac{d^{D-1}k}{(2\pi)^{D-1}} \frac{1}{k^2 - s^2}. $$

(242)

If we perform the integration in Eq. (242) we find that [16]

$$V(\sigma) = \frac{\sigma^2}{2\lambda_0} + \frac{\text{tr} 1}{2(4\pi)^{(D-1)/2}} \Gamma\left(\frac{1 - D}{2}\right).$$

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\[ \times \frac{1}{L} \sum_{n=-\infty}^{\infty} \left[ (k_{D-1}^2 + \sigma^2)^{(D-1)/2} - k_{D-1}^2 \right]. \]  

(243)

Thus the effective potential is modified by finite size effect which is interpreted as a generalized version of the Casimir effect.[75] Analyzing Eq. (243) it is found that the divergences in the effective potential are of the same form in Minkowski spacetime. Thus we need not renormalize the parts which include the finite size effects contributions. We apply the same renormalization condition as that shown in Eq. (32) and obtain the renormalized effective potential as in Minkowski spacetime. The renormalized effective potential is given by replacing \( \lambda_0 \) in Eq. (243) with the renormalized one \( \lambda \) in Eq. (33).

If we apply the expression (243) in the gap equation (80), we find that

\[ \frac{1}{\lambda} - \frac{1}{\lambda_{cr}} - \frac{\text{tr} 1}{(4\pi)^{(D-1)/2}} \Gamma \left( \frac{3 - D}{2} \right) \frac{1}{L \mu \sigma} \sum_{n=-\infty}^{\infty} \left( \frac{k_{D-1}^2 + m^2}{\mu^2} \right)^{(D-3)/2} = 0. \]  

(244)

In Eq. (244) we neglected the trivial solution \( m = 0 \).

In the case of antiperiodic boundary condition it is well-known that the field theory in \( R^{D-1} \otimes S^1 \) is equivalent to the finite temperature field theory. The effective potential (244) is in agreement with that of the finite temperature four-fermion theory (see for example Ref. [16] and references there in) with recourse to the relation between the size \( L \) of the compactified direction and the temperature \( T \)

\[ L \leftrightarrow \frac{1}{k_B T} \]  

(245)

with \( k_B \) the Boltzmann constant. Evaluating the effective potential (243) and the gap equation (244) one can study numerically phase structure of the theory and find that the second order phase transition takes place and the broken chiral symmetry is restored for sufficiently small \( L < L_{cr} \) at the large \( N \) limit. It agrees with the known result at finite temperature. Analyzing the gap equation (244) the critical size is given by

\[ L_{cr} m_0 = 2\pi \left[ \frac{2\Gamma((3 - D)/2)}{\sqrt{\pi} \Gamma(1 - D/2)} (2^{3-D} - 1) \zeta(3-D) \right]^{1/(D-2)}, \]  

(246)

where \( m_0 \) is the dynamical fermion mass in Minkowski spacetime. It is equal to the well-known formula of the critical temperature for the four-fermion model.[16] In Fig. 23 the critical size is plotted as a function of dimension \( D \) for \( \lambda > \lambda_{cr} \) by the use of Eq. (246).

On the other hand finite size \( L \) has an opposite effect for the fermion field with periodic boundary condition. In this case the first derivative of the effective potential (244) has a negative value for \( D = 2 \) at the limit \( \sigma \to 0 \)

\[ \frac{dV}{d\sigma} \bigg|_{\sigma \to 0} \to -\frac{\text{tr} 1}{2} \frac{1}{L} < 0. \]  

(247)

For \( 2 < D \leq 3 \) the first derivative of \( V \) vanishes but the second derivative of \( V \) has a negative value at the limit \( \sigma \to 0 \)

\[ \frac{1}{\mu^{D-2}} \frac{\partial^2 V}{\partial \sigma^2} \bigg|_{\sigma \to 0} \to -\frac{\text{tr} 1}{(4\pi)^{(D-1)/2}} \Gamma \left( \frac{3 - D}{2} \right) \frac{D - 2}{L \mu} \left( \frac{\mu^2}{\sigma^2} \right)^{(D-3)/2} < 0. \]  

(248)

Thus the symmetric phase cannot be realized for \( 2 \leq D \leq 3 \) even if the coupling constant \( \lambda \) is sufficiently small.

For \( 3 < D < 4 \) the first derivative of \( V \) also vanishes at the limit \( \sigma \to 0 \) but the sign of the second derivative of \( V \) depends on the coupling constant \( \lambda \). Evaluating the effective potential (243) numerically it is observed that the chiral symmetry is broken down for a sufficiently small length \( L \) in the case \( \lambda < \lambda_{cr} \), while only the broken phase is realized for \( \lambda > \lambda_{cr} \).

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Figure 23: Critical size $L_{cr}$ as the function of $D$ for $\lambda > \lambda_{cr}$ in the case of the anti-periodic boundary condition.

Figure 24: Critical size $L_{cr}$ as the function of $D$ for $\lambda \leq \lambda_{cr}$ in the case of the periodic boundary condition.
Analyzing the gap equation (244) the critical value of \( L \) is given by

\[
L_{\text{cr}} m_0^2 = 2\pi \left[ -\frac{2\Gamma((3-D)/2)}{\sqrt{\pi}\Gamma((3-D)/2)} \xi((3-D)) \right]^{1/(D-2)},
\]

where \( m_0^2 \) is defined in Eq. (44). In Fig. 24 we present the critical size \( L_{\text{cr}} \) as a function of dimension \( D \) by the use of Eq. (249). In drawing Fig. 24 we fix the coupling constant \( \lambda \leq \lambda_{\text{cr}} \). Since the finite size effect gives a finite correction to the effective potential even at the four-dimensional limit, the finite size effects disappear at \( D \to 4 \) as is shown in Figs. 23 and 24.

Thus the boundary condition of the fermion field has large contributions to the phase structure. If the fermion field possesses the antiperiodic boundary condition the finite size effect raise the vacuum energy \( V(\langle \sigma \rangle) \). Hence the broken chiral symmetry is restored for a sufficiently small \( L \). The fermion field which possesses the periodic boundary condition has an opposite finite size effect. The vacuum energy \( V(\langle \sigma \rangle) \) reduces by the finite size effect. Thus the chiral symmetry is broken down for a small \( L \).

We can interpret above results in the following way. Under the antiperiodic boundary condition minimum momentum allowed for the fermion field becomes \( p = \pi/L \). Then the infrared cut-off induced in a finite \( L \) spacetime. Since the lower momentum modes have an essential role to break the chiral symmetry, the symmetry restoration occurs for a sufficiently small \( L \). On the other hand the vanishing momentum is allowed under the periodic boundary condition. In \( R^{D-1} \otimes S^1 \) a fermion field \( \psi(x) \) is able to interact with \( \psi(x+nL) \) where \( n \) is an arbitrary integer. Summing up all the correlations, \( \langle \psi(x)\psi(x) \rangle \), \( \langle \psi(x)\psi(x+L) \rangle \), \( \langle \psi(x)\psi(x+2L) \rangle \), \( \ldots \), the vacuum expectation value of the composite operator increases as \( L \) decreases. It is understood as a dimensional reduction. Compactifying one direction to the size \( L \) in \( D \)-dimensional space, it looks \( (D-1) \)-dimensional space for particles with Compton wavelength much larger than the size \( L \). In the lower dimensional space the influence from the lower momentum fermion exceeds. Then the finite size effect breaks the chiral symmetry under the periodic boundary condition.

### 7.3 Curved spacetime with nontrivial topology \((\mathcal{M}^3 \otimes S^1)\)

Now we are interested in the combined effect of the finite size and curvature. The fermion field with an antiperiodic boundary condition has been already discussed on compact spaces \( S^D \) and \( R \otimes S^{D-1} \) in §4. Thus we consider the universe where both the independent spin structures exist below.

For this purpose we evaluate the effective potential (241) in weakly curved spacetime. Since we are interested in the theory in four dimensions, we restrict ourselves in four spacetime dimensions and calculate the effective potential by using the cut-off regularization. Notice also that a summation over the two inequivalent spin structures which are admitted by the spacetime \( \mathcal{M}^3 \times S^1 \) will be performed. Of course, one can consider also the corrections corresponding to periodic and antiperiodic (non-zero temperature) boundary conditions independently.

Integration over \( s \) in Eq. (241) is immediate. To perform the momentum integration, one first makes the Wick rotation \( (k^0 = ik^4) \) and puts then a cut-off to regularize the resulting expressions. In our case, we simply restrict

\[
(k^4)^2 + (k^1)^2 + (k^2)^2 \leq \Lambda^2,
\]

so that our cut-off is different, when compared with the cut-off for the case of trivial topology discussed in §3.2.

From now on, we shall call \( V_1 \) the contribution to the effective potential which comes from the logarithm of the determinant of the operator which appears in Eq. (241). After carrying out the integrations over \( s \) and the momenta, we are led to the following expression for the contribution to \( V_1 \) coming from the \( p = 0 \) case, namely, purely periodic boundary conditions (this corresponds to the contribution to \( V(\sigma) \) obtained by taking only the \( p = 0 \) term in Eq. (241)). Of course, we want to study \( V_1 \), which is given by the sum over the two values of \( p \), but —as we discuss below— \( V_1 \) may be written
Another remark is in order here: when computing the term

\[ V_i^{p=0} = -\frac{1}{L} \sum_{n=-\infty}^{\infty} \left\{ \frac{1}{3\pi^2} \Lambda^3 \log \left( 1 + \frac{1}{4\pi^2 n^2 / L^2} \right) \right\} \]

+ \frac{2}{3\pi^2} \left[ \sigma^2 \Lambda - \frac{4\pi^2 n^2}{L^2} \right]^{3/2} \arctan \left( \frac{\Lambda}{\sqrt{4\pi^2 n^2 / L^2 + \sigma^2}} \right) \\
+ \frac{R}{3 \pi^2} \left[ \frac{2\pi n}{L} \arctan \left( \frac{\Lambda L}{2\pi n} \right) - \sqrt{4\pi^2 n^2 / L^2 + \sigma^2} \arctan \left( \frac{\Lambda}{\sqrt{4\pi^2 n^2 / L^2 + \sigma^2}} \right) \right] \\
- \frac{R}{2 \pi^2} \left[ \frac{(2\pi n/L)^2 \Lambda}{\Lambda^2 + 4\pi^2 n^2 / L^2} - \frac{(4\pi^2 n^2 / L^2 + \sigma^2) \Lambda}{4\pi^2 n^2 / L^2 + \sigma^2 + \Lambda^2} \right]. \tag{250} \]

To simplify this expression we use standard techniques drawn from complex analysis, such as the expression

\[ \sum_{n=-\infty}^{\infty} f \left( \frac{2\pi n}{L} \right) = \frac{L}{2\pi i} \int_{-\infty}^{\infty} dp \frac{1}{2} [f(p) + f(-p)] \\
+ \frac{L}{2\pi i} \int_{-\infty+\epsilon}^{\infty+\epsilon} dp \frac{f(p) + f(-p)}{\exp (Lp) - 1}. \tag{251} \]

One has just to identify the function \( f \) for each term in Eq. (250) and then perform the integrals in (251). Another remark is in order here: when computing the term

\[ -\frac{2}{L} \sum_{n=1}^{\infty} \left\{ \left( \frac{4\pi^2 n^2}{L^2} + \sigma^2 \right)^{3/2} \arctan \left( \frac{\Lambda}{\sqrt{4\pi^2 n^2 / L^2 + \sigma^2}} \right) \right\} \]

+ \left( \frac{2\pi n}{L} \right)^3 \arctan \left( \frac{\Lambda}{2\pi n} \right),

it is better to rewrite it as

\[ -\lim_{\sigma' \to 0} \frac{1}{\pi^2} \sum_{n=1}^{\infty} \left\{ \sigma^2 \Lambda - \left( \frac{4\pi^2 n^2}{L^2} + \sigma^2 \right)^{3/2} \arctan \left( \frac{\Lambda}{\sqrt{4\pi^2 n^2 / L^2 + \sigma^2}} \right) \right\} - \sigma^2 \Lambda + \left( \frac{4\pi^2 n^2}{L^2} + \sigma^2 \right)^{3/2} \arctan \left( \frac{\Lambda}{\sqrt{4\pi^2 n^2 / L^2 + \sigma^2}} \right), \]

since now the expression within square brackets satisfies the properties which justify the use of Eq. (251).

As we said before, once \( V_i^{p=0} \) has been computed, one may write immediately \( V_i^{p=1} \) and \( V_i \), which is given (by definition) by \( V_i = V_i^{p=0} + V_i^{p=1} \), because

\[ \frac{1}{L} \sum_{n=-\infty}^{\infty} F \left( \frac{2n + 1}{L} \right) = \frac{1}{L} \sum_{n=-\infty}^{\infty} F \left( \frac{n}{L} \right) - \frac{1}{L} \sum_{n=-\infty}^{\infty} F \left( \frac{2n}{L} \right), \tag{252} \]

and, then

\[ \frac{1}{L} \sum_{n=-\infty}^{\infty} \left[ F \left( \frac{2n + 1}{L} \right) + F \left( \frac{2n}{L} \right) \right] = \frac{1}{L} \sum_{n=-\infty}^{\infty} F \left( \frac{n}{L} \right). \tag{253} \]
Now it is apparent that the physics displayed by the model in the case of purely periodic boundary conditions and in the one where we consider both spin structures will be essentially the same, since

\[ V_{1,L} = V_{1,L}^{p=0} + V_{1,L}^{p=1} = 2V_{1,2L}^{p=0}. \]

Thus we see that both cases are related by a trivial rescaling of the length and an overall factor which multiples \( V_1 \).

However, the last remark does not apply to the case of purely antiperiodic boundary conditions, where we only take \( p = 1 \) — which gives the thermodynamics of a system in three-dimensional space with trivial topology. In fact, from the last expression we get

\[ V_{1,L}^{p=1} = V_{1,L} - V_{1,L}^{p=0} = V_{1,L} - \frac{1}{2}V_{1,L/2}. \]

Henceforth, we shall concentrate below only on the analysis of the case in which both spin structures are taken into account, as they appear in Eq. (241).

The first term on the right-hand side of Eq. (251) may be computed without difficulty in all cases. The second term is, in general, rather more involved. In order to simplify this contribution as much as possible, we have taken into account, as they appear in Eq. (250), one should pay careful attention to the determination of the integrand along the contour of integration. After some work, the final result is found to be

\[
\frac{V_1}{\Lambda^3} = -\frac{2}{3\pi^2} \left[ 1 - \sqrt{1+x^2} + \frac{1}{7} \log \left( \frac{\exp (2l\sqrt{1+x^2} - 1)}{\exp (2l) - 1} \right) \right]
+ \frac{1}{6\pi^2} \left[ \sqrt{1+x^2} - 1 - \frac{3}{2}x^2 \sqrt{1+x^2} + \frac{3}{2}x^4 \arcsinh \left( \frac{1}{x} \right) \right]
+ \frac{4}{3\pi^2} \left[ \int_x^{\sqrt{1+x^2}} d\tau \left( \frac{\exp (2l\tau - x^2)}{\exp (2l\tau - 1)} - \int_0^1 \frac{d\tau}{\exp (2l\tau - 1)} \right) \right]
+ \frac{r}{6(2\pi)^2} \left[ 1 - \sqrt{1+x^2} + x^2 \arcsinh \left( \frac{1}{x} \right) \right]
+ \frac{2r}{3(2\pi)^2} \left[ \int_0^1 \frac{d\tau}{\exp (2l\tau - 1)} - \int_x^{\sqrt{1+x^2}} \frac{d\tau}{\exp (2l\tau - 1)} \right]
\]

\[ - \frac{r}{2(3\pi)^2} \left( \frac{1}{\sqrt{1+x^2}} \right)
+ \frac{r}{(3\pi)^2} \left[ \frac{1}{\sqrt{1+x^2}} \exp (2l\sqrt{1+x^2} - 1) - \exp (2l) - 1 \right]. \]

where \( x = \sigma/\Lambda, l = \Lambda L \) and \( r = R/\Lambda^2 \).

The value of the field \( \sigma \) which satisfies the gap equation, \( V' \) = \( \partial V(\sigma) \)/\( \partial \sigma = 0 \), gives a dynamical mass to the fermions. This last equation, when written in terms of the natural variables \( x, l, r \) and \( c (c \equiv \lambda \Lambda^2) \), reads

\[
0 = \frac{V'(x)}{\Lambda^4} = \frac{x^2}{2c} + \frac{5}{6\pi^2} \sqrt{\frac{x}{1+x^2}} - \frac{4}{3\pi^2} \sqrt{\frac{x}{1+x^2}} - \frac{1}{\exp (-2l\sqrt{1+x^2})}
+ \frac{x}{\pi^2} \left[ x^2 \arcsinh \left( \frac{1}{x} \right) - \sqrt{1+x^2} \right]
+ \frac{4}{3\pi^2} \left[ \frac{x}{\sqrt{1+x^2}} \exp (2l\sqrt{1+x^2} - 1) - \frac{1}{\exp (2l) - 1} \right]
\]

\[ + \frac{1}{2\pi^2} \sqrt{\frac{x}{1+x^2}} \]
\[
+ \frac{rx}{3(2\pi)^2} \left[ \arcsinh \left( \frac{1}{x} \right) - \frac{1}{\sqrt{1+x^2}} \right] - \frac{r}{4(3\pi)^2} \frac{x}{(1+x^2)^{3/2}} \\
- \frac{2rx}{3(2\pi)^2} \left[ \frac{1}{\sqrt{1+x^2}} \exp \left( 2\sqrt{1+x^2} \right) - 1 \right] \\
- \frac{r}{2(3\pi)^2} \frac{x}{(1+x^2)^{3/2}} \exp \left( 2\sqrt{1+x^2} \right) - 1 \\
- \int_x^{\sqrt{1+x^2}} d\tau \left[ \frac{1}{\sqrt{\tau^2-x^2}} \exp(2\tau) - 1 \right]
\]

(255)

Let us discuss some limiting case of above expression (255). Assuming \( L \Lambda \ll 1 \), let us expand \( V/\Lambda^4 \) in powers of \( l \). Assuming now that \( l/\sqrt{1+x^2} \ll 1 \), we readily obtain

\[
\frac{LV(x)}{\Lambda^4} = \frac{x^2}{2g} - \frac{1}{3\pi^2} \log \left( 1+x^2 \right) + \frac{2}{3\pi^2} \left( -x^2 + x^3 \arcsin \left( \sqrt{1 + \frac{1}{x^2}} \right) \right) \\
+ \frac{r}{3(2\pi)^2} x \arcsin \left( \frac{1}{\sqrt{1+x^2}} \right) + \frac{r}{2(3\pi)^2} \left( \frac{1}{1+x^2} - 1 \right) \\
+ O \left( l^2 \right),
\]

(256)

where \( g = \lambda_0 \Lambda^2/l \). Using (256) one can study the phase structure of the model.

### 7.3.1 Phase structure in \( \mathcal{M}^3 \otimes S^1 \)

Because we are making expansion on the curvature, i.e., \( r \ll 1 \), it is worthwhile studying first the case \( r = 0 \). Here it can be proven that it is impossible to have a first order phase transition, in fact there is a second order phase transition at \( g_{cr} = \pi^2/2 \). For \( g > g_{cr} \) there is chiral symmetry breaking and the symmetry is restored for \( g < g_{cr} \); in other words, for a given value of \( \lambda \), the symmetry is restored when \( l \) grows beyond a critical point. In the broken phase the order parameter is given by \( x_{br} = \pi \left( 2/\pi^2 - 1/g \right) \). It is straightforward to study the behavior of the value of the effective potential at \( x_{br} \), varying \( g \) or \( l \), in this limit of small compactification length. One finds

\[
V(\sigma_{br}) = -\Lambda^4 \frac{\pi^2}{6l} \left( \frac{2}{\pi^2} - \frac{1}{g} \right)^3,
\]

and for \( L \to 0 \) we find

\[
\lim_{L \to 0} \frac{LV(\sigma_{br})}{L} = -\Lambda^4 \frac{4}{3\pi^2}.
\]

Now we shall study the influence that the presence of curvature has on this behavior. It is immediate to notice that for negative values of the curvature, the symmetry is always broken (the slope of the effective potential is \( r/24\pi \) at the origin.) Moreover, for fixed positive values of \( r \) (which is kept always small), as \( l \) grows there is a first-order phase transition: it has actually lost its continuous character. Now the critical value of the parameter \( g \) is

\[
g_{cr} = \frac{\pi^2}{2} \left( 1 + \frac{2\pi}{3} \sqrt{\frac{r}{8}} \right).
\]

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The approximation is consistent with the small-curvature limit. The difference between the values of the order parameter \( x \) in the broken and disordered phases at the phase transition is given by \( x = r/8 \) (retaining again only the first correction coming from the curvature).

We can now, on the other hand, study the situation when \( g \) is held fixed and the curvature takes on different values.

If \( g < \pi^2/2 \) there is a continuous phase transition at \( r = 0 \) (the symmetry is broken for negative values of \( r \) and is restored for positive curvature). As \( r \) approaches 0 from below, the order parameter tends to zero according to the expression

\[
x = \frac{|r|}{24\pi \Delta},
\]

or \( m_{\text{gen}} = \Lambda^{-1}|R|/24\pi \Delta \), being \( \Delta \equiv 1/g - 2/\pi^2 \). In deriving this result we assume that \(|r| \ll 1\) and that \(|r| \ll \Delta\).

One may also consider the case \( \Delta < 0 \) or, equivalently, \( g > \pi^2/2 \). To be consistent with the requirement of small curvature near the critical point, we have to assume now that \(|\Delta| \ll 1\). In this setting one may expect that, as \( r \) grows, there will be a phase transition from the ordered to the disordered phase at some positive value of the curvature. Retaining again the first terms of the expansions only, we obtain

\[
r_{\text{cr}} = \left(\frac{3\pi}{2}\right)^2 \Delta^2.
\]

We shall here consider again the phase structure in the fixed-curvature case but corresponding now to the opposite limit, when \( L \) is large.

It is easy to see from Eq. (255) that, dropping exponentially vanishing contributions, one may approximate it with

\[
\frac{V(x)}{\Lambda^4} = \frac{x^2}{2c} - \frac{1}{4\pi^2} \left[ 2\left(\sqrt{1+x^2} - 1\right) + x^2\sqrt{1+x^2} - x^4\arcsinh \frac{1}{x} \right],
\]

(257)

where \( c \) is, as before, \( c = \lambda_0 \Lambda^2 \).

From the analysis of gap equation it is trivial to see that the symmetry is broken when

\[
\lambda_0 > \frac{\pi^2}{\Lambda^2}.
\]

Otherwise, the symmetry is respected by the vacuum.

If one takes into account the presence of a background gravitational field, one can check that the effective potential at large \( L \) is becoming

\[
\frac{V(x)}{\Lambda^4} = \frac{x^2}{2c} - \frac{1}{4\pi^2} \left[ 2\left(\sqrt{1+x^2} - 1\right) + x^2\sqrt{1+x^2} - x^4\arcsinh \frac{1}{x} \right]
+ \frac{r}{6(2\pi)^2} \left(1 - \sqrt{1+x^2} + x^2\arcsinh \frac{1}{x}\right)
- \frac{r}{2(3\pi)^2} \left(1 - \frac{1}{\sqrt{1+x^2}}\right).
\]

(258)

To analyze phase structure let us consider \( \lambda < \pi^2/\lambda_0^2 \). Then the order parameter maybe found approximately as

\[
x = \left[ \sinh \left( \frac{12}{|r|}(\pi^2/c - 1) - \frac{5}{3} \right) \right]^{-1},
\]

(remember that this is valid for negative \( r \)). The symmetry is restored for positive values of \( r \). The numerical analysis for this and some other cases maybe found in Ref. [69].
Thus we discussed phase structure of four-fermion model on $M^{D-1} \otimes S^1$. In compact flat spacetime $R^{D-1} \otimes S^1$ it was found that the boundary condition for the fermion field drastically changes the phase structure. In the case of an antiperiodic boundary condition finite size effect restores the broken chiral symmetry, while the finite size effect breaks the chiral symmetry for the fermion field with a periodic boundary condition. In $M^3 \otimes S^1$ we found the possibility of (topology combined with curvature)-induced phase transitions.

8 Chiral symmetry breaking in curved spacetime with magnetic field

The external electromagnetic (EM) fields play an important role as a possible probe of quantum field theory (for a review of quantum field theory in an external EM fields, see Ref. [76]). It is quite well-known fact that an external magnetic field supports chiral symmetry breaking in four-fermion models. (see Refs. [77] ~ [79] and references there in). There was some activity on the study of phase structure of four-fermion models in an external magnetic field (for a review, see for example Ref. [80]).

From another point it maybe possible that strong primordial magnetic fields should be considered on equal footing with the strong curvature in the early universe. Then, different consequences of the combined effect of the magnetic and gravitational fields may occur (for a example, it maybe significant increase in the number of created particles in the early universe filled with the constant EM field [81]). Hence, for cosmological applications it could be interesting to discuss the chiral symmetry breaking phenomenon in four-fermion models under the action of external gravitational and magnetic fields. This section will be devoted to such situation where gravitational field will be taken in linear curvature approach (see §3). We follow Refs. [82] and [83].

8.1 Three-dimensional four-fermion model with magnetic field

We start from three-dimensional four-fermion model. The effective potential maybe easily written as

$$V(\sigma) = \frac{\sigma^2}{2\lambda_0} + i \ln \text{Det} [i\gamma^\mu(x)\nabla_\mu - \sigma],$$

where at first step there is no magnetic field yet. It is convenient to work in terms of the derivative from $V(\sigma)$. Then, using the local momentum representation of the propagator (see §3), we get

$$\frac{\partial V(\sigma)}{\partial \sigma} = \frac{\sigma}{\lambda_0} - i \text{tr} \int \frac{d^3k}{(2\pi)^3} \left[ \frac{\sigma}{k^2 - \sigma^2} - \frac{R}{12} \frac{1}{(k^2 - \sigma^2)^2} \right].$$

Making use of Wick rotation and calculating the trace in (260), we obtain

$$\frac{\partial V(\sigma)}{\partial \sigma} = \frac{\sigma}{\lambda_0} + \frac{2\sigma}{\pi^2} \int_0^\infty k^2 dk \left[ -\frac{1}{k^2 + \sigma^2} - \frac{R}{12} \frac{1}{(k^2 + \sigma^2)^2} + \frac{2}{9} \frac{R}{(k^2 + \sigma^2)^3} \right].$$

In derivation of Eq. (261) we have used the expression for proper-time representation

$$A^{-\nu} = \frac{1}{\Gamma(\nu)} \int_0^\infty ds s^{\nu-1} e^{-sA},$$

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and after that, the ultraviolet proper-time cut-off $\Lambda^2$ has been introduced in the low limit of proper-time integral. This cut-off is different from the cut-off used in §3.

Performing the integration over $s$ and over $\sigma$, we get (compare with Ref. [23] or §3 where other regularization has been used):

$$V(\sigma) = \frac{\sigma^2}{2\lambda_0} + \frac{1}{6\pi^{3/2}} \left\{ \Lambda^2 \exp \left( -\frac{\sigma^2}{\Lambda^2} \right) - 2\sigma^2 \Lambda \exp \left( -\frac{\sigma^2}{\Lambda^2} \right) + 2\sqrt{\pi} \sigma^3 \text{erfc} \left( \frac{\sigma}{\Lambda} \right) \right\} - \frac{R}{4} \left[ \Lambda \exp \left( -\frac{\sigma^2}{\Lambda^2} \right) - \sqrt{\pi} \sigma \text{erfc} \left( \frac{\sigma}{\Lambda} \right) \right],$$

(263)

where $\text{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-t^2} dt$. Hence, we obtained the effective potential in three-dimensional NJL model in curved spacetime using proper-time cut-off.

The renormalized effective potential may be found in the limit $\Lambda \to \infty$ as following:

$$V(\sigma) = \frac{\sigma^2}{2\lambda} + \frac{|\sigma|^3}{3\pi} + \frac{R|\sigma|}{24\pi},$$

(264)

where

$$\frac{1}{\lambda} = \frac{1}{\lambda_0} - \frac{\Lambda}{\pi^{3/2}}.$$

(265)

Let us analyze now the phase structure of the potential (264). In the absence of curvature, $R = 0$, for $\lambda > 0$ the minimum of the potential (264) is given by $\sigma = 0$. Hence, there is no chiral symmetry breaking. For $\lambda < 0$ the chiral symmetry is broken, the dynamically generated fermion mass is given as

$$m_0 \equiv \sigma_{\text{min}} = -\frac{\pi}{\lambda}.$$

(266)

When the curvature is not zero we find the following picture. Let first $\lambda > 0$. Then the ground state (and dynamically generated mass) is defined by

$$m \equiv \sigma_{\text{min}} = -\frac{\pi}{2\lambda} + \frac{1}{2} \sqrt{\frac{\pi^2}{\lambda^2} - \frac{R}{6}}.$$

(267)

One can see that unlike the case of flat space for positive $\lambda$ we have the chiral symmetry breaking. We also see the possibility of curvature-induced phase transitions. The critical curvature is given by $R_{\text{cr}} = 0$ (flat space). For negative curvature, $R < R_{\text{cr}} = 0$, we observe the chiral symmetry breaking, while for positive curvatures symmetry is not broken.

For negative four-fermion coupling constant $\lambda < 0$ we get the following ground state:

$$\sigma_{\text{min}} = -\frac{\pi}{2\lambda} + \frac{1}{2} \sqrt{\frac{\pi^2}{\lambda^2} - \frac{R}{6}}.$$

(268)

The critical curvature is defined by the condition: $R_{\text{cr}} = 9\pi^2/2\lambda^2 = (9/2)m_0^2$. Between $0 < R \leq R_{\text{cr}}$ the chiral symmetry is broken.

Now, let us discuss the situation when four-fermion model is considered in curved spacetime with external magnetic field. That means that spinor covariant derivative also contains the electromagnetic piece. Treating the magnetic field exactly,[84] one can easily find the effective potential for our model.

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16 Three-dimensional four-fermion model is renormalizable in the sense of $1/N$ expansion.[15] Thus the theory is defined independent of the regularization method. If we perform the finite renormalization

$$\frac{1}{\lambda} \to \frac{1}{\lambda} - \frac{2}{\pi},$$

and put tr1 = 4, the effective potential (76) is reproduced.
Considering now four-fermion model in curved spacetime with magnetic field, we again work in linear curvature approximation as above (but making no approximations for an external magnetic field). Moreover, we take into account only leading contribution on curvature which does not depend on magnetic field, i.e., the contribution discussed above. (One can show that for not a very large magnetic field the curvature correction depending explicitly from magnetic field is not essential). Then, using Eq. (264) and the results of the calculation of three-dimensional four-fermion effective potential in an external magnetic field,[77, 79] one can get:

$$V = \frac{\sigma^2}{2\lambda} + \frac{|\sigma|^3}{3\pi} + \frac{R|\sigma|}{24\pi} + \frac{eH}{4\pi^{3/2}} \int_0^\infty ds \frac{e^{-s\sigma^2}}{s^{3/2}} \left[ \coth(eHs) - \frac{1}{eHs} \right],$$

(269)

where $H$ is magnetic field and $\epsilon$ is electric charge.

The renormalized effective potential (269) may be represented as following

$$V = \frac{\sigma^2}{2\lambda} + \frac{R|\sigma|}{24\pi} + \frac{eH|\sigma|}{2\pi} - \frac{(2eH)^{3/2}}{2\pi} \zeta \left( -\frac{1}{2}, \frac{\sigma^2}{2eH} \right),$$

(270)

where properties of generalized zeta-function may be found in Ref. [30].

Working in frames of our approximation and considering also $\epsilon H \rightarrow 0$, we find that for positive $\lambda$

$$\sigma_{\text{min}} \simeq \frac{\lambda}{\pi} \left( \frac{eH}{2} - \frac{R}{24} \right).$$

(271)

Hence, for $R = 0$ chiral symmetry breaking due to magnetic field occurs. Negative curvature increases chiral symmetry breaking. However, positive curvature acts against the chiral symmetry breaking. On the critical line of phase diagram

$$\frac{R}{12} \simeq \epsilon H$$

(272)

the restoration of chiral symmetry breaking occurs. Hence, in this example magnetic field and gravitational field act in the opposite directions with respect to chiral symmetry breaking. Gravity tends to restore the chiral symmetry while magnetic field tends to break it. On the critical line both effects are compensated and there is no chiral symmetry breaking.

Similarly, one can analyze other choices for $\lambda$, $\epsilon H$ (see Ref. [82] for more details).

### 8.2 Four-dimensional NJL model in curved spacetime with magnetic field

Let us start again from the action for the NJL model in curved spacetime:

$$S = \int d^4x \sqrt{-g} \left\{ \bar{\psi}i\gamma^\mu(x)\nabla_\mu \psi + \frac{\lambda_0}{2N} \left[ (\bar{\psi}\psi)^2 + (\bar{\psi}i\gamma_5\psi)^2 \right] \right\}. \quad (8 \cdot 15)$$

Standard calculation gives the effective potential as (where as usually it is enough to put pseudo-scalar $\pi = 0$ as is shown in §2.)

$$V(\sigma) = \frac{\sigma^2}{2\lambda_0} + i \ln \text{Det}[i\gamma^\mu(x)\nabla_\mu - \sigma]. \quad (8 \cdot 16)$$

Then calculation similar to the one in previous subsection gives the effective potential in proper-time cut-off

$$V(\sigma) = \frac{\sigma^2}{2\lambda_0} + \frac{1}{8\pi^2} \int_{1/\Lambda^2}^\infty ds \exp(-s\sigma^2) \left[ \frac{1}{s^3} - \frac{R}{12s^2} \right]$$

$$= \frac{\sigma^2}{2\lambda} + \frac{1}{(4\pi)^2} \left( \Lambda^4 - \Lambda^2 \sigma^2 \right) \exp \left( -\frac{\sigma^2}{\Lambda^2} \right),$$
\[ -\sigma^4 \text{Ei}\left(-\frac{\sigma^2}{\lambda^2}\right) = \frac{R}{6} \left[ \Lambda^2 \exp\left(-\frac{\sigma^2}{\lambda^2}\right) + \sigma^2 \text{Ei}\left(-\frac{\sigma^2}{\lambda^2}\right) \right], \quad (8.17) \]

where \(\text{Ei}(-x)\) is defined in Eq.(5.54).

Expanding Eq. (8.17) and keeping only terms which are not zero at \(\Lambda \to \infty\) we get

\[ V(\sigma) = \frac{\sigma^2}{2\lambda_0} - \frac{1}{(4\pi)^2} \left[ 2\Lambda^2 \sigma^2 + \sigma^4 \left( \ln \frac{\sigma^2}{\Lambda^2} + \frac{\lambda^2}{2} \right) + \frac{R\sigma^2}{6} \left( \ln \frac{\sigma^2}{\Lambda^2} + \gamma - 1 \right) \right] + O\left(\frac{\sigma^2}{\Lambda^2}\right). \quad (8.18) \]

Thus, we have got the effective potential with proper-time cut-off.

Using Eq. (8.18) the gap equation is found as follows

\[ \frac{4\pi^2}{\lambda_0\Lambda^2} - 1 = \frac{\sigma^2}{\Lambda^2} \left( \ln \frac{\sigma^2}{\Lambda^2} + \gamma - 1 \right) + \frac{R}{12\Lambda^2} \left( \ln \frac{\sigma^2}{\Lambda^2} + \gamma \right). \quad (8.19) \]

This gap equation defines the possibility of chiral symmetry breaking in curved spacetime (in linear curvature approximation). If our weakly curved spacetime is filled by the magnetic field (the covariant derivative is now \(\nabla_\mu = \partial_\mu - ieA_\mu\), \(A_\mu = -Bx_\mu\)) one can get the following effective potential

\[ V(\sigma) = \frac{\sigma^2}{2\lambda_0} + i \ln \text{Det} \left[ i\gamma^\mu(x)\nabla_\mu - \sigma \right] \]

\[ = \frac{\sigma^2}{2\lambda_0} + \frac{1}{8\pi^2} \int_{1/\Lambda^2}^\infty ds \exp(-so^2) \left[ |eB| \coth(s|eB|) + \left( -\frac{R}{12} + O(R^2) \right) \right]. \quad (8.20) \]

In the absence of the gravitational field \((R = 0)\), the effective potential corresponds to flat space situation where the magnetic field is treated exactly.\[^{[12]}\] In the absence of the magnetic field \((B = 0)\) we are back to the potential (8.17). In addition, in the linear curvature terms the effect of the magnetic field is not taken into account as in the previous subsection.

Making the calculation of the integrals in Eq. (8.20) up to \(O(1/\Lambda^2)\), and taking the derivative with respect to \(\sigma\) one gets the gap equation, \(\partial V/\partial \sigma = 0\), as follows

\[ \frac{4\pi^2}{\lambda_0\Lambda^2} - 1 = \frac{\sigma^2}{\Lambda^2} \ln \left( \frac{\sigma^2}{\Lambda^2} \right) + \frac{|eB| \ln (\sigma |eB|^{-1/2})}{\Lambda^2} \]

\[ + \frac{2|eB|}{\Lambda^4} \ln \left( \frac{\sigma^2 |eB|^{-1}}{2} \right) - \frac{R}{12\Lambda^2} \left( \ln \frac{\Lambda^2}{\sigma^2} - \gamma \right) + O\left(\frac{1}{\Lambda}\right). \quad (8.21) \]

Using this gap equation one can study the dynamical symmetry breaking in different cases. In some cases it can be given analytically, for example, for values of the coupling constant \(\lambda\) much below the critical value, i.e.,

\[ \lambda_0 \ll \frac{4\pi^2}{\Lambda^2}. \quad (8.22) \]

One can find (supposing that the second term on the r.h.s. of (8.21) is the leading one)

\[ \frac{4\pi^2}{\lambda_0\Lambda^2} - 1 \approx \frac{|eB|}{\Lambda^2} \ln \left( \frac{\sigma |eB|^{-1/2}}{4\pi} \right) - \frac{R}{12\Lambda^2} \ln \frac{\Lambda^2}{\sigma^2}, \quad (8.23) \]

and finally, the dynamically generated fermionic mass is given by

\[ \sigma^2 \approx \left( \frac{|eB|^{-1}}{4\pi} \right)^{-1} \left( \frac{1}{\Lambda^2} \right)^{-1} \exp\left( \frac{4\pi^2}{\lambda_0\Lambda^2} - 1 \right). \quad (8.24) \]
In the absence of the magnetic field it gives the analytic expression for the dynamically generated fermionic mass due to the curvature

$$\sigma^2 \approx \Lambda^2 \exp \left[ \frac{12\Lambda^2}{R} \left( \frac{4\pi^2}{\lambda_0\Lambda^2} - 1 \right) \right].$$

(8.25)

One can see that positive curvature tends to make the first term in (8.24) less, i.e., it acts against the dynamical symmetry breaking. At the same time, the negative curvature always favors the dynamical chiral symmetry breaking in accordance with the explicit calculations in an external gravitational field (see §3).

The numerical analysis shows the similar qualitative behavior (for more details, see Ref. [82]). Note finally that in the same way one consider more complicated situations when both gravitational and magnetic fields are treated exactly (see, for example, Ref. [85] where two dimensional Gross-Neveu model on sphere with magnetic monopole has been considered).

9 Quantum gravity and four-fermion models

In the present section we give few examples where four-fermion models may be relevant to study the dynamics of quantum gravity. In particular we apply the effective potential of \(D = 2\) four-fermion model to investigate \(D = 2\) black hole solution of dilatonic gravity. We discuss conformal factor dynamics in frame of \(1/N\) expansion. The effective potential for composite gravitino on de Sitter background is also investigated.

9.1 Semiclassical approach in the dilatonic gravity with Gross-Neveu model

Motivated by the idea that 2D quantum gravity is easier to study than 4D quantum gravity there was recently much of interest in 2D quantum gravity and its black holes solutions.[86, 87, 88, 89, 90] We consider now one of such solutions in the 2D dilatonic gravity with Gross-Neveu model following Ref. [90].

The action to start with is

$$S = \frac{1}{2\pi} \int \sqrt{-g} d^2x \left[ e^{-2\phi}(R + \Lambda) + \sum_{k=1}^{N} \bar{\psi}_k(i\gamma^\mu \nabla_\mu + \sigma)\psi_k - \frac{N}{2\lambda_0} \sigma^2 \right],$$

(273)

where \(\phi\) is the dilaton field and Gross-Neveu action is the same as Eq. (2.12).

Notice that if we redefine the metric by the relation

$$g_{\mu\nu} = e^{-2\phi} \hat{g}_{\mu\nu},$$

(274)

the gravitational part of the action (273) takes the form

$$S_1 = \frac{1}{2\pi} \int \sqrt{-\hat{g}} d^2x e^{-2\phi} \left( \hat{R} + 4\hat{g}^{\mu\nu} \partial_\mu \phi \partial_\nu \phi + \Lambda e^{-2\phi} \right),$$

(275)

which is similar to the CGHS action [88] (the cosmological constant term is a fixed number in Ref. [88] while it is accompanied by the Liouville type potential in Eq. (275).) However, we should emphasize that the theory with action (273) is different from the theory with action (275) coupled to the Gross-Neveu fermions (with metric tensor \(\hat{g}_{\mu\nu}\) already at the classical level. The difference becomes more serious in the semiclassical approach.

Let us start our discussion of the theory described by Eq. (273) in the semiclassical approach (1/N expansion where dilatonic quantum effects are subleading). We will assume that background spinors are

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absent. In the semiclassical approach we have to add to the action (273) the conformal anomaly term [88, 91]

\[ S_c = \frac{1}{2\pi} \int \sqrt{-g} d^2 x \left[ -\frac{1}{2} (\nabla Z)^2 + \sqrt{\frac{N}{48}} Z R \right], \tag{276} \]

where \( Z \) is the auxiliary scalar field. For the semiclassical argument we need the Gross-Neveu effective potential \( V(\sigma, R) \) in curved spacetime which replaces classical potential \( \sigma^2/2 \) in Eq. (273). The dilatonic equation of motion derived from the action (273)

\[ R + \Lambda = 0, \tag{277} \]

apparently constrains the curvature to be constant. Using the Gross-Neveu effective potential in curved spacetime with constant curvature in the leading order of the \( 1/N \) expansion, our effective action in the semiclassical approach reads

\[ S_{\text{eff}} = \frac{1}{2\pi} \int \sqrt{-g} d^2 x \left[ e^{-2\rho} (R + \Lambda) - V(\sigma, R) \right] + S_c, \tag{278} \]

where \( V(\sigma, R) \) is given by Eq. (3.15) with \( \sigma \)-independent part to be discarded.

Applying the conformal gauge

\[ g_{++} = e^{\rho}, \quad g_{--} = -\frac{1}{2} e^{-2\rho}, \quad g_{+-} = g_{-+} = 0, \tag{279} \]

we derive the following semiclassical equations of motion from the effective action (278) (with constant curvature according to Eq. (277) and without spinors):

\[ T_{++} = e^{-2\rho} \left[ 4 \partial_+ \rho \partial_+ \phi + 4 (\partial_+ \phi)^2 - 2 \partial_+^2 \phi \right] \]
\[ + \frac{1}{2} (\partial_+ Z)^2 + \sqrt{\frac{N}{48}} (\partial_+^2 - 2 \partial_+ \rho \partial_+) Z = 0, \tag{280} \]

\[ T_{--} = e^{-2\rho} \left[ 4 \partial_- \rho \partial_- \phi + 4 (\partial_- \phi)^2 - 2 \partial_-^2 \phi \right] \]
\[ + \frac{1}{2} (\partial_- Z)^2 + \sqrt{\frac{N}{48}} (\partial_-^2 - 2 \partial_- \rho \partial_-) Z = 0, \tag{281} \]

\[ T_{+-} = e^{-2\rho} \left[ 2 \partial_+ \partial_- \phi - 4 \partial_+ \phi \partial_- \phi - \frac{1}{4} \Lambda e^{2\rho} \right] - \sqrt{\frac{N}{48}} \partial_+ \partial_- Z \]
\[ + \frac{1}{4} e^{2\rho} V(\sigma, -\Lambda) + 2 \partial_+ \partial_- \frac{\partial V(\sigma, -\Lambda)}{\partial \Lambda} = 0, \tag{282} \]
\[ \partial_+ \partial_- \rho + \frac{\Lambda}{8} e^{2\rho} = 0, \tag{283} \]
\[ \partial_+ \partial_- Z - \sqrt{\frac{N}{12}} \partial_+ \partial_- \rho = 0 \tag{284} \]

and

\[ \frac{\partial V(\sigma, -\Lambda)}{\partial \sigma} = 0. \tag{285} \]

The general solution of Eq. (283) is well known:

\[ \rho(x) = \frac{1}{2} \ln \left[ \frac{F'_+(x_+)}{F'_-(x_-)} \right], \tag{286} \]
where \( F_\pm \) is an arbitrary holomorphic (anti-holomorphic) function. The conformal gauge in Eq. (286) can be fixed completely by choosing \( F_\pm = x^\pm \).

The solution of Eq. (284) is given by
\[
Z = \sqrt{\frac{N}{12}} \theta + u_+(x^+) + u_-(x^-),
\] (287)
where \( u_\pm \) is an arbitrary holomorphic (anti-holomorphic) function. Using Eqs. (283) and (284) in Eq. (282), we get
\[
\partial_+ \partial_- e^{-2\phi} + \frac{\Lambda}{4} e^{2\phi}(e^{-2\phi} - a) = 0,
\] (288)
where
\[
a = \frac{N}{48} + \frac{V(\sigma_0, -\Lambda)}{\lambda} - \frac{\partial V(\sigma_0, -\Lambda)}{\partial \Lambda},
\] (289)
and \( \sigma_0 \) is a solution of Eq. (285). Note that the quantity \( \Lambda a \) plays the role of the effective cosmological constant. It may be interesting to note that in the weak curvature limit the effective cosmological constant disappears (\( a = 0 \)) if
\[
\Lambda = \frac{2\lambda \mu^2}{N} \exp \left( 2 - \frac{2\pi}{\lambda} \right) \left[ -\left( 1 + \frac{\pi}{3} \right) \pm \sqrt{\left( 1 + \frac{\pi}{3} \right)^2 + 4} \right].
\] (290)

Accordingly we observe that the quantum effects in the Gross-Neveu model provides us with the way to solve the cosmological constant problem within the model under investigation.

The general solution of Eq. (288) is
\[
e^{-2\phi(x^+, x^-)} = v_+(x^+) + \frac{1 - \frac{\Lambda}{8} F_+ F_-^*}{1 + \frac{\Lambda}{8} F_+ F_-} \int^x dy_F^+(y) v_+(y) + \frac{a}{2} + (\pm \pm -).\] (291)
Here the function \( u_\pm \) and \( v_\pm \) are determined by solving the equations for \( T_{++} \) and \( T_{--} \). The equation for \( T_{++} \) with the choice \( F_\pm = x^\pm \) reads
\[
T_{++} = v''_+ + \frac{v'}{x^+} - \frac{v}{x^+ x^2} + \frac{1}{2} u'_+^2 + \sqrt{\frac{N}{48}} u''_+ = 0,
\] (292)
and the same for \( T_{--} \) with the change of index + to −.

We rewrite these two equations in the form
\[
v'' + \frac{v'}{x^+} - \frac{v}{x^+ x^2} = \frac{N}{24} \{ p(x), x \},
\frac{1}{2} u'^2 + \sqrt{\frac{N}{48}} u'' = -\frac{N}{24} \{ p(x), x \},
\] (293)
where we omit indices + and −, \( p(x) \) is an unknown function and \( \{ , \} \) is the Schwarzian derivative. By solving Eq. (293) for \( v \) and \( u \) and substituting the solutions into the equations into Eq. (291), we finally have
\[
e^{-2\phi(x^+, x^-)} = \frac{N}{96} \left[ C^+_3 x^+ + C^+_4 x^+ + x^+ \int^{x^+} \{ p_+(z), z \} dz - \frac{1}{x^+} \int^{x^+} z^2 \{ p_+(z), z \} dz + \frac{1 - \frac{\Lambda}{8} x^+ x^-}{1 + \frac{\Lambda}{8} x^+ x^-} \int^{x^+} dy \left( C^+_3 y + C^+_4 y + y \int^{y} \{ p_+(z), z \} dz \right) - \frac{1}{y} \int^{y} z^2 \{ p_+(z), z \} dz \right] + \frac{a}{2} + (\pm \pm -),\] (294)

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where $C_3^\pm$ and $C_4^\pm$ are arbitrary constants. For the simplest case in which $p_\pm(x^\pm) = x^\pm$, the solution for dilaton field is given by

$$e^{-2\phi(x^+,x^-)} = \frac{N}{48} \left( C_3^+ + \frac{\Lambda}{8} C_4^+ \right) x^+ + \left( C_3^- + \frac{\Lambda}{8} C_4^- \right) x^- + a. \quad (295)$$

This is the main result of this subsection. We found the static solution for our model in an explicit form. In order to compare it with the CGHS-Witten black hole solution,[88] we make the redefinition (274) of the metric. The new spacetime interval squared becomes

$$ds^2 = -e^{-2\phi(x^+,x^-)} e^{-2\rho(x^+,x^-)} dx^+ dx^- = \frac{-dx^+ dx^-}{\left(1 + \frac{\Lambda}{8} x^+ x^-\right) \left(Ax^+ + Bx^- + a + \frac{\Lambda}{8} ax^+ x^-\right)}, \quad (296)$$

where

$$A = \frac{N}{48} \left( C_3^+ + \frac{\Lambda}{8} C_4^+ \right), \quad (297)$$

$$B = \frac{N}{48} \left( C_3^- + \frac{\Lambda}{8} C_4^- \right). \quad (298)$$

This new solution may be regarded as a modified version of the CGHS-Witten black hole.[86, 88] For the simplest case where all arbitrary constants are chosen to vanish ($e^{-2\phi} = a$), we get

$$ds^2 = \frac{-dx^+ dx^-}{a \left(1 + \frac{\Lambda}{8} x^+ x^-\right)^2}. \quad (299)$$

It is interesting to note that for $\Lambda = 0$ in Eq. (273) the semiclassical solution becomes

$$\rho = 0,$$

$$\sigma_0^2 = \mu^2 \exp \left(2 - \frac{2\pi}{\lambda}\right),$$

$$e^{-2\phi} = \frac{1}{4} V_0 x^+ x^- - C_0, \quad (300)$$

where $V_0 = V(\sigma_0, R = 0)$ and $C_0$ is an arbitrary constant. In terms of the new metric (274) we obtain

$$ds^2 = \frac{dx^+ dx^-}{C_0 - \frac{1}{4} V_0 x^+ x^-}. \quad (301)$$

Equation (301) exactly coincides with the black hole solution of Refs. [86] and [88].

Thus we presented the example of 2D black hole solution with constant curvature in dilatonic gravity with Gross-Neveu model. It is interesting to remark that one can use the renormalization group improved effective potential instead of one-loop Gross-Neveu effective potential. Then, one has only change the coupling constant ($\lambda = Ng^2$) in the semiclassical effective action by the running coupling constant

$$g^2(t) = \frac{g^2}{1 - \lambda t/\pi}, \quad (302)$$

where $t$ is the renormalization group parameter. Working in the regime of strong curvature, $t = \ln R/\mu^2$.[92] Hence, asymptotic freedom in Gross-Neveu model is induced by black hole curvature.
Stronger black hole curvature induces stronger asymptotic freedom as in $d = 4$ gauge theories in curved spacetime. In other words, in vicinity of constant curvature black hole, Gross-Neveu model tends to become free theory.

Note also that one can consider other types of dilatonic gravity interacting with Gross-Neveu model and investigate the constant curvature black holes solutions there. However, to study other types of black hole solutions one should calculate the effective potential of Gross-Neveu model in an arbitrary curved spacetime.

9.2 Conformal factor dynamics in $1/N$ expansion

The conformal factor dynamics describes the conformally-flat solution of the gravitational theories at classical level. At the quantum level the dynamics of conformal factor (induced by the conformal anomaly) was suggested as the tool for the description of quantum gravity in the infrared phase. Conformal factor dynamics gives rise to the effective potential for the conformal factor which seems to be quite relevant in quantum gravity. In particular it maybe responsible for infrared phase transitions. It is very interesting to note that composite bound states which are typical for four-fermion theories maybe also the origin of the conformal factor dynamics. We follow Ref. in discussion of this question.

Our starting point is the two-dimensional theory with action

$$S = \int d^2x \sqrt{-g} \left[ \psi (i\gamma^\mu (x) \nabla_\mu - m) \psi + R - \frac{\Lambda}{2} \right],$$

where the massive $N$-component spinor $\psi$ is considered to be a quantum field. The gravitational field, on the other hand, may be either classical or quantum. We also consider the conformal parametrization of the metric

$$g_{\mu\nu} = \rho^2 \eta_{\mu\nu},$$

where $\rho$ is the conformal factor (in general it is $\rho = \rho(x)$) and $\eta_{\mu\nu}$ is the flat fiducial metric. In the QG case, the choice (304) corresponds to the gauge fixing. Substituting (304) into (303) one gets, at the classical level,

$$S = \int d^2x \left[ \chi (i\gamma^\mu \partial_\mu - m\rho) \chi - \frac{\Lambda}{2} \rho^2 \right],$$

where $\chi = \rho^{1/2} \psi$. Rescaling $\rho \rightarrow \Lambda^{1/2} \rho$, we get

$$S = \int d^2x \left[ \chi (i\gamma^\mu \partial_\mu - h\rho) \chi - \frac{1}{2} \rho^2 \right],$$

where $h = m\Lambda^{-1/2}$. As one can see, action (306) has the form that is typical for the Gross-Neveu model ($\rho \sim \chi$). The dynamics of this model are quite well known: asymptotic freedom in the UV limit

$$h^2(t) = \frac{h^2}{1 + h^2 N t / \pi},$$

where $t$ is the RG parameter. However, since (306) describes also the dynamics of the conformal factor, the interpretation of the function $h^2(t)$ is now completely different from the original interpretation. The $h^2(t)$ here is a combination of the fermionic mass in (303) and of the two-dimensional cosmological constant $\Lambda$. Using the anomalous scaling dimension one gets the running composite field

$$\rho(x, t) = \rho(x) \left(1 + h^2 N t / \pi \right)^{-1/2}.$$
\( \Lambda(t) \sim \Lambda (1 + h^2 N t/\pi)^{-1} \). Therefore, there appears to be a screening of the cosmological constant in the 1/N expansion in the UV regime (quantum gravitational corrections are subleading in 1/N expansion).

One can also investigate specific features of the effective potential for the conformal factor, which coincides with the GN effective potential.\(^9\) In particular, the appearance of a minimum, i.e., a non-zero vacuum expectation value (v.e.v.), for the conformal factor

\[
\rho = \rho_0 \exp \left( 1 - \frac{\pi}{h^2 N} \right)
\]  

(309)

is interesting. After having shown the possibility to study the dynamics of the conformal factor in two dimensions as the dynamics of the GN model, we now turn to the four-dimensional theory, which is physically more interesting.

We start from the multiplicatively renormalizable theory\(^7\) with action

\[
S = \int d^4x \sqrt{-g} \left[ \bar{\psi} (i \gamma^\mu \nabla_\mu - m) \psi - \frac{\Lambda}{\kappa^2} - \frac{R}{\kappa^2} + \frac{W}{\Lambda_1} - \frac{UR^2}{3\Lambda_1} \right],
\]  

(310)

where \( \psi \) is an \( N \)-component spinor and \( W \) the square of the Weyl tensor. The gravitational field may be chosen to be classical or quantum, and the theory remains multiplicatively renormalizable in both cases.

We shall work again with the conformal parametrization (304) for the four-dimensional metric. In the case of four-dimensional QG this does not fix the gauge, contrary to what happens in two dimensions, but it can still be considered as a convenient background.\(^94\) Bewriting action (310), we get

\[
S = \int d^4x \left\{ \bar{\chi} (i \gamma^\mu \partial_\mu - m \rho) \chi - \frac{\Lambda}{\kappa^2} \rho^4 - \frac{6}{\kappa^2} (\partial \rho)^2 \right. \\
+ \frac{12U}{\Lambda_1} [\sigma \Box^2 \sigma + 2 (\partial \sigma)^2 \Box \sigma + (\partial \sigma)^2 (\partial \sigma)^2] \right\},
\]  

(311)

where \( \chi = \rho^{3/2} \psi \) and \( \sigma = \ln \rho \). In this way we have got the classical theory for the conformal factor. At the quantum level, the theory (311) may be considered as an effective theory for QG (see also Ref. [93]). If we drop the \( \rho \)-terms with derivatives from action (311), we obtain a model that is reminiscent of the NJL model (where, of course, owing to the absence of the \( M^2 \rho^2 \)-term, it is \( \rho \sim (\chi \chi)^{1/3} \)).

Now we are going to study the theory (311) in the large-\( N \) limit, while concentrating our attention on the RG and low-derivative terms in (311). The higher-derivative terms are actually of lesser importance, moreover, they simply disappear in the subsequent analysis of the effective potential for the conformal factor. First of all, we rescale \( \rho \to \sqrt{12} \rho / \kappa \) and denote \( h = m\kappa / \sqrt{12} \) and \( \lambda = \Lambda \kappa^2 / 6 \). Thus,

\[
S = \int d^4x \left[ \bar{\chi} (i \gamma^\mu \partial_\mu - h \rho) \chi - \frac{\lambda}{24} \rho^4 - \frac{1}{2} (\partial \rho)^2 \right].
\]  

(312)

By integrating over the fermionic field, we get the effective potential for the conformal factor at large \( N \):

\[
V(\rho) = \frac{\lambda \rho^4}{24} + iN \text{Tr} \ln (i \gamma^\mu \partial_\mu - h \rho),
\]  

(313)

where \( \rho \) is constant. Supposing that, as usually, for large \( N \) \( \lambda \) scales as \( \lambda \sim \tilde{\lambda} N \), where \( \tilde{\lambda} \) does not depend on \( N \), and using a finite cut-off \( \mu \) (see also §3) we get (notice that \( N \) has been factored out)

\[
V(\rho) = \frac{\tilde{\lambda} \rho^4}{24} - \frac{1}{(4\pi)^2} \left[ \rho^2 \mu^2 + \mu^4 \ln \left( 1 + \frac{\rho^2}{\mu^2} \right) - \rho^4 \ln \left( 1 + \frac{\mu^2}{\rho^2} \right) \right].
\]  

(314)

The v.e.v. of the conformal factor can be found from (314) as the solution of the equation

\[
\frac{\partial V(\rho)}{\partial \rho} = \frac{\tilde{\lambda}}{6} \rho^2 \left[ \mu^2 - \rho^2 \ln \left( 1 + \frac{\mu^2}{\rho^2} \right) \right] = 0.
\]  

(315)
One can present a sample of numerical values of the solution $\rho^2/\mu^2$ (which lie between 0 and 1 for $0.04644 \leq \tilde{\lambda} \leq 1$). Notice that from the point of view of the original theory, a non-zero v.e.v. for $\rho$ is more acceptable physically, because for $\rho = 0$ the conformal parametrization (304) becomes degenerate.

Now we turn to the study of the renormalization structure of theory (312), which is rather non-trivial. There are a few different ways to renormalize this theory, the actual problem being the fact that $\rho$ is dimensionless.

We may consider a theory (312)—as it stands—which will correspond eventually to some QG phase. After renormalization of $\rho$ (taking into account the negative sign for $(\partial \rho)^2$), we obtain

$$\beta_h = -\frac{4Nh^4}{(4\pi)^2}, \quad \beta_{\lambda} = \frac{3\lambda^2 - 8N\lambda h^2 - 48Nh^4}{(4\pi)^2}.$$  \hspace{1cm} (316)

The theory has now a UV stable fixed point ($t \to +\infty$), where the behavior of $h^2$ and $\lambda$ is

$$h^2(t) \sim \frac{4\pi^2}{Nt}, \quad \lambda(t) \sim -\frac{48\pi^2}{Nt}.$$  \hspace{1cm} (317)

Hence, we now obtain a decrease of the cosmological constant in the UV limit. Notice that in comparing this theory with the standard scalar self-interacting theory, we do not have here physical restrictions on the sign of $\lambda$, and a negative sign is perfectly acceptable. That indicates to the possibility of solution of cosmological constant problem in frames of conformal factor dynamics based on four-fermion like theory.

### 9.3 Effective action in $N=1$ supergravity

In §5.3 we already discussed supersymmetric extension of NJL model. There are also other possibilities to introduce supersymmetry into the theory. In particular, in present subsection we will discuss $N=1$ supergravity on de Sitter background. We calculate the effective action for non-zero gravitino condensate $\sigma \sim \langle \bar{\psi}_a \Gamma^a \psi_b \rangle$. Hence, such a model maybe considered also as kind of four-fermion (four-gravitino model). Note, however, that unlike the above discussion we work in one-loop approximation (not in $1/N$-expansion). We follow Ref. [96] below.

The Lagrangian for the theory of $N=1$ supergravity is

$$\mathcal{L} = -\frac{1}{k^2} R(e, \psi) + \frac{1}{2} \epsilon_{\mu\nu\rho\sigma} \bar{\psi}_\mu \gamma_5 \gamma_\rho \nabla_\sigma \psi_\nu + \frac{1}{3} (A^2_\mu - S^2 - P^2),$$ \hspace{1cm} (318)

where $\Gamma^{\mu\nu} = \frac{1}{2} [\gamma^\mu, \gamma^\nu], R(e, \psi) = R(e) + \frac{1}{4} k^4 (\bar{\psi}_\mu \Gamma^{\mu\nu} \psi_\nu)^2 + \cdots$ (in this expression the terms corresponding to the interaction of the gravitino with the gravitational field and four-fermion terms with insertions containing the $\gamma_5$ matrix are not written in the explicit form), $\nabla_\lambda \equiv \nabla_\lambda (e), A_\mu, S, P$ is the minimal set of auxiliary fields. Let us represent the Lagrangian (318) in the form:[96, 97]

$$\mathcal{L} = -\frac{1}{k^2} R(e, \psi) + \frac{1}{2} \epsilon_{\mu\nu\rho\sigma} \bar{\psi}_\mu \gamma_5 \gamma_\rho \nabla_\sigma \psi_\nu + \sigma^2 - \sqrt{11} k\sigma (\bar{\psi}_\mu \Gamma^{\mu\nu} \psi_\nu) + \frac{1}{3} (A^2_\mu - S^2 - P^2),$$ \hspace{1cm} (319)

where $\sigma$ is the auxiliary scalar field. Let us calculate the effective action in the theory with the Lagrangian (319) in de Sitter spacetime ($R = 4\Lambda$) with constant scalar field $\sigma$. If the effective equations

$$\frac{\delta \Gamma}{\delta \sigma} = 0,$$ \hspace{1cm} (320)

$$\frac{\delta \Gamma}{\delta \Lambda} = 0,$$ \hspace{1cm} (321)

have the real solutions $\sigma \neq 0$, then these solutions indicate dynamical supersymmetry breaking.[97] Equation (321) represents the effective equation for the gravitational field which depends on the parameter $\Lambda$ only.
Let us write $g_{\mu\nu} \rightarrow g_{\mu\nu} + h_{\mu\nu}, \sigma \rightarrow \sigma + \sigma_q$, where $g_{\mu\nu}$ is the de Sitter space metric, $\sigma =$ const, and $h_{\mu\nu}, \sigma_q$ are quantum fields. The background values of other fields of the theory we put equal to zero. One can obtain that the sector of quantum fields $(h_{\mu\nu}, \sigma_q)$ does not interact with the gravitino sector.

At first one can integrate over the field $\sigma_q$ in the functional integral. Then the additional term $\Delta \mathcal{L}_{\sigma_q} = -\sigma^2 h^2/4$ appears in the expansion of the classical gravitational Lagrangian on the quantum field $h_{\mu\nu}$.

In order to calculate the effective action, one needs to add a gauge-fixing term to the Lagrangian. We shall choose the one-parameter gauge:

$$\mathcal{L}_{\text{gauge fix}} = \frac{\gamma}{2k^2} \left( \nabla^\mu h_{\mu\nu} - \frac{1}{2} \nabla_\nu h \right)^2,$$

where $\gamma$ is the gauge parameter, $\gamma = 1$ corresponds to the De Donder gauge, $\gamma \rightarrow \infty$ corresponds to the Landau-DeWitt gauge. Let us write the bilinear part of the Einstein Lagrangian on the $S^4$ background [98] with account of $\Delta \mathcal{L}_{\sigma_q}$ and $\mathcal{L}_{\text{gauge fix}}$ (322).

$$\mathcal{L}_2 = \frac{1}{2k^2} \left[ \frac{1}{2} \partial^i \partial^j \bar{h}^{ij} \left( \frac{8}{3} \Lambda - 2\Lambda_0 \right) \bar{h}^{ij} + 2(\Lambda - \Lambda_0) \xi^i \partial_i (-\Lambda) \xi^j ight. - \frac{3}{16} \partial_i \left( \frac{4}{3} \Lambda \right) \left( \partial_i + 4\Lambda_0 - 4\Lambda + 3\gamma \partial_i + 4\gamma \Lambda \right) h_1 

+ \left. 2(1 - \gamma) h_1 \partial_i \left( \partial_i + \frac{4}{3} \Lambda \right) h + h \left( \partial_i - \frac{4}{3} \Lambda_0 + \frac{1}{3} \gamma \partial_i \right) h \right] - \frac{1}{4} \sigma^2 h^2. \quad (323)$$

Here

$$\Lambda_0 = \frac{1}{2} k^2 \sigma^2, \quad (324)$$

$$\bar{h}_{\mu\nu} = \bar{h}_{\mu\nu}^{ij} + 2 \nabla_\mu \xi_\nu^i + \nabla_\nu \xi_\mu^i - \frac{1}{4} g_{\mu\nu} \partial_i h_1, \quad (325)$$

$$h_{\mu\nu} = \bar{h}_{\mu\nu} + \frac{1}{4} g_{\mu\nu} h_1, \quad (326)$$

$$g^{\mu\nu} \bar{h}_{\mu\nu} = 0, \quad (327)$$

$$\nabla^\mu \bar{h}_{\mu\nu} = 0, \quad (328)$$

$$\nabla_\mu \xi_\nu^{i+} = 0, \quad (329)$$

$$\Delta_i (X) = 1_i (-\partial_i + X), i = 0, 1, 2, \quad (330)$$

where $1$ is the unit in the space of the corresponding fields.

Integrating over quantum fields $\bar{h}_{\mu\nu}^{ij}, \xi_\nu^{i+}, h_1, h$, taking account of the corresponding Jacobians and ghost contributions, we have in the limit $\gamma \rightarrow \infty$ (we omit the details of this straightforward calculation)

$$\Gamma_{h_{\mu\nu}} = \frac{1}{2} \text{Sp} \ln \Delta_2 \left( \frac{8}{3} \Lambda - k^2 \sigma^2 \right) - \frac{1}{2} \text{Sp} \ln \Delta_1 (-\Lambda) - \text{Sp} \ln \Delta_0 (-2\Lambda)$$

$$+ \frac{1}{2} \text{Sp} \ln \Delta_0 \left[ \frac{5}{2} \sigma^2 k^2 - \Lambda + \left( \Lambda^2 + k^2 \sigma^2 \Lambda + \frac{25}{4} k^2 \sigma^4 \right)^{1/2} \right]$$

$$+ \frac{1}{2} \text{Sp} \ln \Delta_0 \left[ \frac{5}{2} \sigma^2 k^2 - \Lambda - \left( \Lambda^2 + k^2 \sigma^2 \Lambda + \frac{25}{4} k^2 \sigma^4 \right)^{1/2} \right]. \quad (331)$$

The explicit expression for $\frac{1}{2} \text{Sp} \ln \Delta_i (X), i = 0, 1, 2$ is obtained in Ref. [98] in frames of $\zeta$-regularization [30]

$$\frac{1}{2} \text{Sp} \ln \Delta_i (X) = \frac{1}{2} B_i' \ln \left( \frac{\Lambda}{3\mu^2} \right) - \frac{1}{6} (2i + 1) \Gamma_i'(b_i), \quad (332)$$

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where \( b_i = 9/4 + i - 3X/\Lambda \), \( B_4^i \) is the well-known De Witt \( B_4 \) coefficient for the corresponding operator, \( F_4^i(b_i) \) is given by Eq. (2.36) in Ref. [98]. Using (332) in (331), one can easily obtain the explicit form for \( \Gamma_{h_{\mu\nu}} \). The final expression is very complicated, so we do not write it here. It is enough for our purposes to have \( \Gamma_{h_{\mu\nu}} \) in the form (331).

Let us now calculate the gravitino contribution to the effective action. We choose the gauge:

\[
\mathcal{L}_\psi = \frac{1}{2} \gamma_1 k \tilde{\chi} \chi,
\]

where \( \chi = \nabla^\mu \psi_\mu \). Then the Faddeev-Popov determinant is \( M = 1 \Box = \tilde{\nabla} \tilde{\nabla} + \Lambda \). Represent [98] \( \psi_\mu = \phi_\mu + \gamma_\mu \psi/4 \), \( \gamma^i \phi_\mu = 0 \), \( \phi_\mu = \phi^\perp_\mu + (\nabla_\mu - \gamma_\mu \tilde{\nabla}/4) \zeta \), \( \nabla^\mu \phi^\perp_\mu = 0 \), \( \tilde{\nabla} \equiv \gamma^i \nabla_\mu \) in the bilinear part of the gravitino Lagrangian (319) taking account of \( \mathcal{L}_\psi \). Integrating over the fields \( \phi^\perp_\mu, \zeta, \psi \) taking account of the corresponding Jacobians and ghost contribution we get in the limit \( \gamma_1 \to \infty \):

\[
\Gamma_{\psi_\mu} = -\frac{1}{4} \text{Sp} \ln \triangle_{3/2}(m^2) - \frac{1}{2} \text{Sp} \ln \left( \left(-\frac{1}{3} \nabla + 2m \right) \triangle_{1/2} \left(-\frac{4}{3} \Lambda \right) \right) - \frac{1}{9} \triangle_{1/2}(0)(\tilde{\nabla} + 2m) + \text{Sp} \ln \triangle_{1/2}(-\Lambda),
\]

where \( m = -\sqrt{\Pi k} \sigma \). One can write the explicit expression for \( \triangle_{1/2}(X), \triangle_{3/2}(X) \) using relations such as (332) (see Ref. [98]). We do not write this complicated expression here.

A natural question now is about the presence of the negative modes in the effective action \( \Gamma = \Gamma_{h_{\mu\nu}} + \Gamma_{\psi_\mu} \). The spectra of all operators in (331) and (334) are given in Ref. [98]. Using the result of Ref. [98] for \( \Gamma_{h_{\mu\nu}} \) (331) one can show that the operator \( \triangle_{3/2}(8\Lambda/3 - k^2 \sigma^2) \) does not include the negative modes if \( 1 \leq k^2 \sigma^2/2\Lambda \leq 8/3 \). The three last terms in (331) contain the negative modes for arbitrary values \( \sigma \) and \( \Lambda \) \((\Lambda > 0)\). Therefore, \( \Gamma_{h_{\mu\nu}} \) contains an imaginary part for all \( \sigma \) and \( \Lambda \). Then \( \Gamma \) also contains an imaginary part. (Remember that the structure of the fermion sector does not influence the bosonic sector.) Thus the effective equations also contain an imaginary part. It indicates a vacuum instability of the considered background and the absence of dynamical supersymmetry breaking at the 1-loop level.

It is interesting that one can easily obtain the effective action in \( N = 1 \) supergravity in flat spacetime using the effective action (331) and (334). This effective action also contains an imaginary part for all \( \sigma \).

Thus, we found the effective action for composite gravitino in \( N = 1 \) supergravity and discussed its properties.

In summary, in the present section the use of four-fermion type models in the theories of quantum gravity has been discussed. Few particular examples presented here indicate that four-fermion models could find wider applications than it is usually believed.

### 10 Summary and outlook

In the present paper we discussed four-fermion models in an external gravitational field. As is well-known these models are extremely useful for analytical study of dynamical symmetry breaking. Our main purpose was to investigate the influence of the external gravitational field to dynamical symmetry breaking.

Using 1/N-expansion and working in the leading order of this expansion we calculated the effective potential in four-fermion models in \( D \)-dimensional spacetime \((2 \leq D \leq 4)\) for a variety of backgrounds. In particular, we considered the class of spacetimes where derivative expansion maybe applied. Here the effective potential for the composite operator \( \psi \psi \) has been calculated taking into account terms linear on curvature. We also evaluated the effective potential on the background with constant curvature; Minkowski space, de Sitter background, anti-de Sitter background and Einstein universe. In all these cases an external curvature has been taken into account exactly.
Phase structure of the effective potential has been carefully studied using analytical and numerical methods. The possibility of chiral symmetry breaking due to curvature effects and curvature-induced phase transitions has been discussed in detail. The phase diagrams for dynamically generated fermion mass have been constructed. For example, for negative curvature chiral symmetry is always broken down.

Similar questions have been addressed for the extensions of four-fermion models; higher derivative four-fermion model and gauged NJL model. In particular, for gauged NJL model (which may play the role of SM or GUT without elementary scalars) the technique of renormalization group improvement has been applied. The effective potential equivalent to Schwinger-Dyson approach has been found analytically. Chiral symmetry breaking has been estimated.

We also discuss the combined influence of two effects (non-trivial topology and external curvature, non-vanishing magnetic field and external curvature) to phase structure of four-fermion model. The effective potential has again been calculated for few specific backgrounds. Phase structure has been analyzed. In particular we found that in some case two external effects (for example, magnetic field and gravitational field) may compensate each other. That leads to the restoration of chiral symmetry, which was originally broken.

We presented the number of examples where an analytical study of composite states is possible even in the presence of non-trivial gravitational background. Having in mind the cosmological applications, it is straightforward now to use the effective potential obtained above for the construction of inflationary universe where the role of inflaton will be played by a composite state of fermion fields.[6]

In our calculation we limited ourselves to case of weakly curved or constant curvature spacetimes. In this case the effective action is reduced to the effective potential except for the volume factor. However, for more general backgrounds one has to calculate the effective action in four-fermion models. It is expected that instabilities (due to creation of instanton-antiinstanton condensation) may appear in such situation. That may completely change the phase structure of the theory. Such an extremely hard task as the calculation of effective action on general curved background is quite natural extension of the presented results.

Another very interesting line of research is related with the study of realistic model of elementary particles in the early universe. One can start from some GUT scenarios which maybe reformulated as gauged four-fermion model in the same way as the model of §5.1. Then, the role of gravitational background to phase transitions maybe discussed in all detail, even in non-perturbative regime.

Note also that methods developed in the present paper maybe applied to other theories. For example, in supergravities one can study the effective potential of composite gravitino (see §9.3), for gauge theories one can study the effective potential for composite vectors [7] and so on.

It is interesting to note also that above approach maybe well extended to supersymmetric theories. For example, in §5.3 we discussed supersymmetric NJL model in curved spacetime and found the possibility of dynamical chiral symmetry breaking due to gravitational effects.

It would be also extremely interesting to analyze the next-to-leading corrections to effective potential of four-fermion model on curved background. Such multi-loop calculations may clarify the phase structure in many cases when the creation of multi-critical points is possible.

Finally, it is interesting to note that in realistic situation (like early universe) the combination of few external effects maybe quite natural. Hence, it could be interesting to study phase structure of four-fermion model in curved spacetime with finite temperature,[67] magnetic field, chemical potential and non-trivial topology.

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