Abstract

It is shown that density fluctuations obey a scaling law in an open Friedmann universe. In a flat universe, the fluctuations are not scale-invariant. We compute the growth rate of adiabatic scale-invariant density fluctuations in flat, open and inflationary universes. We find that, given a sufficiently long time, the density perturbations decay away in the Einstein-de-Sitter universe. On the contrary, the rapid growth of the density instabilities makes an open universe inhomogeneous in a time scale comparable to the age of our universe. We also find that the fluctuations grow exponentially in a flat inflationary universe.
I. INTRODUCTION

Friedmann-Robertson-Walker solution of the Einstein field equation describes a homogeneous and isotropic universe and forms the basis of standard cosmology. The strongest support for this model is provided by the cosmic microwave background radiation which is isotropic to about a part in $10^4$ on angular scale from $10''$ to $180^\circ$. However, the model also faces many open questions among which are the horizon problem, the flatness problem and the absence of antibaryons and monopoles.

With the availability of more and more reliable three-dimensional catalogues in the last few years, the standard cosmology has been confronting new challenging problems [1]. The observations of the red shift surveys indicate the existence of voids and filaments up to large scales in the universe [1] and raise doubts about the hypothesis of homogeneity, which is at the foundation of the Friedmann cosmology. The precise correlation length at which the matter distribution, given in these catalogues, smooths down and becomes homogeneous is the subject of strong controversy [2–5]. It is, however, unanimously agreed that the observed inhomogeneities point towards a scale-invariant distribution at least up to a certain scale $^1$. Since the Friedmann cosmology is inappropriate for describing a non-homogeneous universe, we seek for more general solutions of the Einstein field equation.

The general solution of the Einstein equation, not requiring the homogeneity assumption, was given by Tolman for spherically symmetric dust flows in comoving coordinates [6]. Starting from the Tolman solution, it was shown that there exist spherically symmetric inhomogeneous cosmological models which approach Friedmann universe at sufficiently large times, but whose initial density is an arbitrary function of the radial variable [7]. In more recent works, Tolman solution was used to model a hierarchical (fractal) universe which would accommodate the results of the red shift surveys [8].

On the non-relativistic front, advances have been made in describing the non-homogeneous distribution of matter at small scales [9]. Recently, it has been shown that the observed inhomogeneous matter distribution in the interstellar medium can be explained on the basis of Newtonian self-gravity by a purely field theoretical model [10]. Using this model, it was further shown that, at small scales, up to 100 pc, the density-density correlator of the interstellar medium obeys a scaling law [10].

At higher scales, similar analytical works on the dynamics of an inhomogeneous matter

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$^1$This scale ranges from $5$ Mpc, the lowest set by the supporters of the standard cosmology, [2,5] to $1000$ Mpc, the highest claimed by its critics [3].
distribution in the universe, are rather scarce [7,8]. A theoretical framework explaining
the origin of the inhomogeneity of the galaxy and cluster distributions, observed in the
three-dimensional catalogues, is not yet available.

In this work, we employ a perturbative method to explain how a small scale-invariant
fluctuation, seeded in a homogeneous background, evolves into an inhomogeneous universe.
We start, not with the Tolman dust solution, but with a general cosmological model which
allows non-vanishing pressure. We obtain our cosmological model by perturbing the Fried-
mann scale factor and allowing it to be radial-coordinate dependent through a general fluc-
tuation term. We compute this fluctuation by assuming that the universe consists of a
gas of point particles which interact through Einstein gravity. Using the grand canonical
ensemble of these particles, we show that, at all scales, as for the interstellar medium, the
density-density correlator obeys a scaling law in an open universe. We also show that in a
flat universe, the correlator does not scale.

Having shown that the fluctuation obeys a scaling law, we subsequently perturb the
energy-momentum tensor by a scale-invariant adiabatic perturbation and reduce the Einstein
field equations to a second-order hypergeometric equation. The hypergeometric solutions
are used to obtain the ratio of the perturbation to the background homogeneous density.

We evaluate the growth rate of the perturbation during the matter and radiation-
dominated eras up to the present time. We show that, in the lifetime of our universe,
any scale-invariant adiabatic inhomogeneity seeded in the open Friedmann universe grows
to dominate over the homogeneous background. On the contrary, in a flat universe, an
adiabatic scale-invariant perturbation decays away in this time scale. These results are in
complete agreement with those obtained for the Tolman dust solution [8]. We extend our
results to the inflationary universe and show that, given a sufficiently large time, the expo-
nential growth of the adiabatic perturbations makes the matter distribution inhomogeneous.
Finally, we obtain the thermodynamical conditions required for the growth of fluctuation in
a universe with an arbitrary pressure.

The plan of this article is as follows. We start section II with Einstein lagrangian and
derive the density-density correlator for a perturbed Friedmann universe. In Section III,
we perturb the Einstein field equation and the energy-momentum conservation equation
and obtain a second-order hypergeometric differential equation. In section IV, we solve this
equation for a flat universe and show that the perturbation decays in a time scale comparable
to the age of our universe. In Section V, we show that the scale-invariant fluctuation grows
exponentially in an inflationary universe. In Section VI, we discuss the hypergeometric
solution of the Einstein equation for a non-flat universe with an arbitrary pressure. The
consequences of interchanging the polar with the radial pressure in solving the Einstein field
II. SCALING BEHAVIOUR OF THE DENSITY FUNCTION

In this section, we evaluate the density-density correlator for flat and open universes. We use the field theoretical method developed in ref. [10] to obtain the infrared, i.e. large distance, correlator from the grand partition function of a gas of particles interacting through Einstein gravitational potential.

The dynamics of our universe is described by the Einstein lagrangian,

$$\mathcal{L} = \sqrt{-g} \left( g^{\mu\nu} T_{\mu\nu} - \frac{1}{16\pi G} R \right).$$  \hfill (1)

A homogeneous and isotropic universe is obtained by using the Friedmann-Robertson-Walker metric

$$\text{diag}(g_{\mu\nu}) = \left( -1, \frac{R_0^2}{1 - kr^2}, R_0^2 r^2, R_0^2 r^2 \sin^2 \theta \right)$$  \hfill (2)

and a diagonal energy-momentum tensor $T_{\mu
u}$ of a perfect fluid in the lagrangian (1).

We perturb the scale factor $R_0$ by a small perturbation term $\delta R$. The fluctuation in the Einstein lagrangian, for a matter-dominated universe is $^2$,

$$\delta \mathcal{L} = \delta RO \delta R + \frac{\rho_1 r^2}{\sqrt{1 - kr^2}} \delta R$$  \hfill (3)

where

$$O = \frac{\partial}{\partial r} \left( \frac{r^2 \sqrt{1 - kr^2}}{8\pi G R_0} \frac{\partial}{\partial r} \right) + \frac{3R_0^2 r^2 \rho}{\sqrt{1 - kr^2}},$$ \hfill (4)

$$\rho_1 = 3R_0^2 \rho + \frac{3k}{8\pi G}.$$ \hfill (5)

Our aim is to find the two body hamiltonian which describes the interaction of the point objects in the universe. We shall neglect all higher-order interactions and only work with

$^2$For the purpose of evaluating the density correlator, it suffices to consider a static problem where all the time derivatives are set to zero.
the quadratic terms. We also keep distances large. To obtain an operator for the two-body interaction, we substitute the equation of motion of the fluctuation $\delta R$ back into the lagrangian and obtain the effective lagrangian $^3$

$$\mathcal{L}_{\text{eff}} = -\rho_1 A^{-1} \rho_1,$$  

(6)

where the interaction operator is

$$A^{-1} = \frac{r^2}{2\sqrt{1 - kr^2}} O^{-1} \frac{r^2}{2\sqrt{1 - kr^2}}. \quad (7)$$

Next, we write the grand canonical ensemble for the universe. We consider the universe to be composed of a gas of particles of mass $m$ interacting through the gravitational field, as given by the two body lagrangian, (6), and the interaction operator, (7), at an equilibrium temperature $T$. The grand partition function, allowing for the variation of the number of particles $N$, is

$$Z = \sum_{N=0}^{\infty} \frac{z^N}{N!} \int \cdots \int \prod_{l=1}^{N} \frac{dp_l}{2\pi} dr_l e^{-H_N/T}, \quad (8)$$

where $z$ is the fugacity and the Hamiltonian $H_N$ is

$$H_N = \sum_{l=1}^{N} \frac{p_l^2}{2m} + \int dr dr' \left( \rho_1(r) A^{-1} \rho_1(r') \right). \quad (9)$$

Inserting this in the partition function and integrating over the momenta, we obtain

$$Z = \sum_{N=0}^{\infty} \frac{1}{N!} \left( \frac{z \sqrt{mT}}{2\pi} \right)^N \prod_{l=1}^{N} \int dr_l \exp \left[ \frac{-1}{T} \int \rho_1(r) A^{-1} \rho_1(r') dr dr' \right]. \quad (10)$$

The last integral can be expressed as a functional integral, $i.e.$,

$$\exp \left[ \frac{-1}{T} \int \rho_1(r) A^{-1} \rho_1(r') dr dr' \right] = \int \mathcal{D}\zeta e^{-\int dr (-\zeta A\zeta + 2\rho_1 \zeta/\sqrt{T})}. \quad (11)$$

At this point we define the density as,

$$\rho_1 = \sum_{l=1}^{N} \delta(r - r_l) \quad (12)$$

$^3$Since we are considering a static problem, the hamiltonian, $H = P_R \dot{R} - \mathcal{L}$, only differs from the lagrangian by a minus sign.
so that the partition function can be rewritten as 

\[
\mathcal{Z} = \int \mathcal{D}\zeta e^{-\int dr \left( -\zeta A\zeta - z\sqrt{mT/2\pi}e^{-2\zeta/\sqrt{T}} \right)},
\]

(13)

which describes an exponential self-interaction for the \(\zeta\) field.

The equations of motion, obtained from the expressions (10), (11) and (13) are

\[
\frac{\partial \mathcal{Z}}{\partial \zeta} = \langle 2A\zeta - 2\frac{\rho_1}{\sqrt{T}} \rangle = \langle 2A\zeta - 2z\sqrt{\frac{m}{2\pi}}e^{-2\zeta/\sqrt{T}} \rangle = 0,
\]

(14)

where the angled brackets stand for the average values. Thus, the average density is given by,

\[
\langle \rho_1(r) \rangle = z\sqrt{\frac{mT}{2\pi}} \langle e^{-2\zeta/\sqrt{T}} \rangle.
\]

(15)

This expression can also be obtained by introducing an external source for \(\rho_1\) in (11). Analogous procedure leads to the two-point function

\[
\langle \rho_1(r) \rho_1(r') \rangle = \frac{z^2mT}{2\pi} \langle e^{-2\zeta/\sqrt{T}}e^{-2\zeta/\sqrt{T}} \rangle = \frac{z^2mT}{2\pi} e^{\frac{4}{T} \langle \zeta(r) \zeta(r') \rangle},
\]

(16)

where the last step is valid because the field \(\zeta\) is one-dimensional in this time-independent problem.

The Green function,

\[
\mathcal{G} = \langle \zeta(r) \zeta(r') \rangle = -A^{-1},
\]

(17)

can be computed for both flat and open universes in the limit of large radial distances.

For a flat universe, \(k = 0\), the infrared Green function satisfies the differential equation,

\[
\frac{\partial}{\partial r} \left( \frac{1}{r^2} \frac{\partial \mathcal{G}}{\partial r} \right) = \delta(r - r')
\]

(18)

and is given by

\[
\mathcal{G} = \frac{\epsilon(r - r')}{6} \left[ r^3 - r'^3 \right] = \frac{1}{6} \left| r^3 - r'^3 \right|
\]

(19)

where \(\epsilon(r - r')\) is the step function. Substituting this Green function together with the relation (5) in (16), we obtain the density-density correlator

\[\text{Note that in the exponential term the action is unbounded from below, which is due to the attractiveness of the gravitational interaction. The divergence can be easily eliminated by a short distance cut-off. For a comprehensive discussion see ref. [10]}
\]
\[
\langle \rho(r) \rho(r') \rangle_{k=0} \sim \exp \left[ -\frac{4\pi G R_0(t)}{3T} r^3 - r'^3 \right],
\]
which shows that for a flat universe the correlator does not scale and therefore, the assumption of homogeneity is valid.

For an open universe, the Green function obeys the differential equation
\[
\frac{\partial}{\partial r} \left( r \frac{\partial G}{\partial r} \right) = \delta(r - r')
\]
leading to
\[
\langle \zeta(r) \zeta(r') \rangle_{k=1} = \epsilon(r - r') \ln \left( \frac{r}{r'} \right),
\]
which on substitution in (16) and using (5) yields
\[
\langle \rho(r) \rho(r') \rangle_{k=1} \sim \left| \frac{r'}{r} \right|^{8\pi G \epsilon R_0/T}.
\]
Thus, in an open universe, for large distances, the density correlator scales and the assumption of the homogeneity of the universe has to be revised.

III. EINSTEIN EQUATIONS

In the previous section, we have perturbed the Friedmann scale factor by a general fluctuation term and have shown that in an open universe the fluctuation obeys a scaling law. In this section, we study the dynamics of scale-invariant fluctuations. We perturb the energy-momentum tensor and Einstein field equation around the background homogeneous Friedmann universe by scale-invariant adiabatic fluctuations. We show that Einstein equations reduce to a second-order hypergeometric equation. Subsequently, the density fluctuations can be obtained from the hypergeometric solutions.

The Einstein field equation corresponding to the lagrangian (1) is
\[
R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = -8\pi G T_{\mu\nu}.
\]
We expand the scale factor, density and pressure around the background homogeneous functions \( R_0, \rho_0 \) and \( P_0 \) respectively, i.e.,
\[
R(t, r) = R_0(t) + X(t)r^{-\sigma},
\]
\[
g_{tt} T^{tt} = \rho(t, r) = \rho_0(t) + \lambda(t)r^{-\gamma},
\]
\[
g_{rr} T^{rr} = P_r(t, r) = P_0(t) + \pi(t)r^{-\gamma},
\]
\[
g_{\theta\theta} T^{\theta\theta} = P_0(t, r) = P_0(t) + \pi_\theta(t)r^{-\gamma}.
\]
where the fractal codimension $\gamma$ is approximately 1 [11] and the comoving radial coordinate has been used; i.e., $X(t) = q(t)R_0^{-\gamma}(t)$ etc..

By expanding the energy-momentum conservation equations,

$$\frac{\partial}{\partial t} \left( R^3 \rho \right) - \dot{R} R^2 (R_r + 2P_0) + \frac{1}{r^2} k r \frac{\partial}{\partial r} \left( T_{r0} r^2 R \right) - kr RT_{r0} = 0$$

around the homogeneous background, we find that to first-order in power of $\gamma$ and for very large $r$.

$$\gamma = \sigma = \gamma'$$

This is substituted in the expressions (25-28) which are subsequently used in the Einstein field equation (24) to yield

$$6 \frac{\dot{R}_0}{R_0} \left( \frac{\dot{X}}{R_0} - X \frac{\dot{R}_0}{R_0} \right) + 2k \frac{\dot{X}}{R_0^3} (\gamma + 1) (\gamma - 3) = -8\pi G \lambda(t)$$

$$\frac{\ddot{X}}{R_0} - X \frac{\ddot{R}_0}{R_0^2} + \frac{\dot{X}}{R_0} - k (\gamma + 1) \frac{X}{R_0^3} = -4\pi G \pi_r(t)$$

$$\frac{\dot{R}_0}{R_0} \left( \frac{\dot{X}}{R_0} - X \frac{\dot{R}_0}{R_0} \right) + \frac{\ddot{X}}{R_0} - X \frac{\ddot{R}_0}{R_0^2} + \frac{k(\gamma + 1)(\gamma - 2)}{2R_0^3} \dot{X} = -4\pi G \pi_\theta(t)$$

A linear relationship between the homogeneous pressure $P_0$ and the density $\rho_0$, i.e. $P_0 = \omega \rho_0$, is usually taken as the homogeneous equation of state [12,13]. It is most practical to use an adiabatic perturbation where $\pi_r = \alpha \lambda$ and the proportionality constant $\alpha$ takes a value very close to that of $\omega$. The coupled equations (32, 33) now reduce to the differential equation,

$$\ddot{X} - \left( 3\alpha - 1 \right) \frac{\dot{R}_0}{R_0} \dot{X} + \left( 3\alpha - 1 \right) \frac{\dot{R}_0^2}{R_0^2} - \frac{\ddot{R}_0}{R_0} - k(1 + \gamma)(1 + \alpha \gamma - 3\alpha) \frac{X}{R_0^3} = 0$$

The first-derivative term is eliminated, by the redefinition

$$X = R_0^{3\alpha - 1} Y, \quad \alpha'$$

to yield the second-order differential equation

$^{5}$The off-diagonal component of energy-momentum tensor is expected to arise for all types of inhomogeneous universes, including the Tolman universe [8]. This component is also expanded as,

$$T_{0r}(t,r) = \tau(t)r^{-\gamma'-1}.$$
\[
\dot{Y} + \left(6\pi G(\alpha - 1)(\alpha \rho_0 - P_0) - A_3 \frac{k}{R_0^2}\right) Y = 0, \tag{37}
\]

where

\[
A_3 = ((1 + \gamma)(1 + \alpha(\gamma - 3)) - \frac{3}{4}(\alpha - 1)(3\alpha - 1). \tag{38}
\]

In the remaining part of this work, we solve this differential equation for different types of universes in different thermodynamical setups. From the solutions, the perturbation coefficient \(X(t)\) and subsequently, by using the Einstein equation (32), the coefficient of the density fluctuation \(\lambda(t)\) can be found. Using these results, we establish whether, in the lifetime of our universe, the fluctuation grows to overrule the homogeneous background or else is overruled by it and the universe remains homogeneous.

### IV. FLAT UNIVERSE

In this section, we solve the hypergeometric equation (37) for Einstein-de-Sitter universe. We show that, given a sufficiently long time, any scale-invariant adiabatic fluctuation decays away in a flat universe. This confirms the result of Section II: that the density fluctuation does not scale in a flat universe.

For a flat homogeneous universe, the density function is given by \([13]\)

\[
\rho_0 \sim R_0^{3(1+\omega)}; \quad R_0 \sim t^{\frac{2}{3(1+\omega)}}. \tag{39}
\]

Substituting these in the differential equation (37) gives

\[
\dot{Y} - \frac{(\alpha - 1)(\alpha - \omega)}{(1 + \omega)^2} \frac{Y}{t^2} = 0 \tag{40}
\]

which can be solved exactly to yield

\[
X = R_0^{3\alpha - 1} Y = t^{-\frac{3\alpha + 1}{3(1+\omega)}} + \frac{1}{2} \frac{\omega - 1 - 2\alpha}{2(2\omega + 1)}, \tag{41}
\]

for the fluctuation coefficient of the scale factor (see equation (36)). Thus, we find that \(X\) either dominates over \(R_0\) or is of the same order as \(R_0\) \(^6\). The latter is true when \((\alpha + 1)\) and \((\omega + 1)\) have the same sign and \(2|\alpha + 1| \geq |\omega + 1|\). However, since the observations probe the structure of the matter distribution, it is more appropriate to compare the density

\(^6\)We take the positive exponent in (41) for large times.
coefficients, $\rho_0$ and $\lambda R_0^2$, than the scale factors. The asymptotic expression for $\lambda$, obtained by substituting for $X$, (41), in the Einstein equation, (32), is

$$\lambda \sim \begin{cases} t^{(2\alpha - \omega - 3)/(1+\omega)} \\ 0 \end{cases}$$

while for $\rho_0$ is \[12,13\]

$$\rho_0 \sim t^{-2}.$$ (43)

The growth parameter

$$\zeta = \frac{\ln (\lambda R_0^2 / \rho_0)}{\ln t} = \frac{-2\alpha + \omega - 1 + \frac{2}{3} \gamma}{1 + \omega},$$ (44)

can be used to evaluate the rate of growth of the fluctuation with respect to the background, given by,

$$\eta = \left(\frac{t}{t_0}\right)^\zeta,$$ (45)

where $t_0$ and $t$ mark the beginning and the end of the thermodynamical era under consideration. Since the growth parameter is negative for a matter-dominated universe (i.e., $\omega = \alpha = 0$), the perturbation decays. For the Einstein-de-Sitter universe, the perturbation decreases by a factor of $10^{-2}$, from the beginning of the matter-dominated era to the present time (i.e., for $t/t_0 \approx 10^5$). Thus, the homogeneity hypothesis of the standard cosmology remains valid in this universe.

V. INFLATIONARY UNIVERSE

In this section, we solve the differential equation (37) for an inflationary universe. We show that in all flat inflationary universes the density fluctuations grow exponentially, leading to an inhomogeneous universe.

At sufficiently large times, all flat, open and closed inflationary universes are represented by an exponential scale factor

$$R_0 \sim e^{Ht}$$ (46)

and a constant matter distribution $^7$

$^7$We assume that the Hubble parameter is constant. However, this is only true for a flat universe.
\[ \rho_0 = -\frac{3H^2}{8\pi G}. \]  

(47)

The coefficient of the \( Y \) term in the differential equation (37) is now constant, leading to the exponential solutions

\[ Y \sim e^{\pm \frac{3}{2}(\alpha - 1)Ht} \]  

(48)

and, on using (36), to

\[ X \sim \begin{cases} 
& e^{Ht} \\
& e^{(3\alpha - 2)Ht} 
\end{cases} \]  

(49)

Substituting this result in the Einstein equation (32) we find the coefficient of the density fluctuation,

\[ \lambda \sim -\frac{9H^2(\alpha - 1)}{4\pi G}e^{-3(\alpha + 1)Ht}. \]  

(50)

Since the homogeneous background density is constant for an inflationary universe, the perturbation decays only if \( \alpha > (\gamma/3 - 1) \). For an adiabatic perturbation, this range of \( \alpha \) is not allowed and therefore the fluctuation grows exponentially and leads to an inhomogeneous universe.

**VI. OPEN UNIVERSE**

In this section, we solve the differential equation (37) for an open universe. We show that in an open universe a scale-invariant adiabatic perturbation grows to dominate over the background homogeneous distribution of matter. Thus, we confirm the result of Section II: that the density-density correlator is scale-invariant in an open universe.

For a non-vanishing \( k \), the differential equation (37) is complicated. We solve the equation in three steps of increasing difficulty. We start with a matter-dominated universe where the fluctuation makes a zero contribution to the pressure. We then consider a matter-dominated universe with a very small pressure caused by fluctuations. Finally, we generalize our results to include the radiation-dominated universe. We derive the thermodynamical conditions necessary for the growth of perturbation in a universe with a non-vanishing arbitrary pressure.
A. Matter-dominated open universe

\[ P = 0 ; \quad P_0 = 0 , \quad \alpha = 0 \]

In this case, the homogeneous density and scale factor are given parametrically in terms of an angle \( \theta \) as

\[ \rho_0 \sim R_0^{-3} ; \quad R_0 = \frac{dt}{d\theta} ; \quad t = \frac{\Omega_0}{2H_0(\Omega_0 - 1)^{3/2}}(\theta - \sin \theta) \]  

(51)

where \( \theta \) is real for a closed and imaginary for an open universe and \( \Omega \) is the ratio of the present to the critical density \([13]\). Substituting these in the differential equation (37), we obtain

\[ \frac{d}{d\theta} \left( \frac{1}{1 - \cos \theta} \frac{dY}{d\theta} \right) - \left( \gamma + \frac{1}{4} \right) \frac{Y}{(1 - \cos \theta)} = 0. \]  

(52)

By making the substitution \( 2z = 1 - \cos \theta \), we re-write the above equation in the standard form of a hypergeometric equation,

\[ z(1 - z) \frac{d^2Y}{dz^2} - \frac{1}{2} \frac{dY}{dz} - \left( \gamma + \frac{1}{4} \right) Y = 0. \]  

(53)

The solutions can be found by using the Gauss relation for hypergeometric functions \( F(a, b, c; z) \) \([14]\). That is,

\[ F(a, -1 - a, -\frac{1}{2}; z) = 2(a + \frac{1}{2})F(a, -a - 1, \frac{1}{2}; z) - 2aF(a + 1, -a - 1, \frac{1}{2}; z), \]  

(54)

where for our problem, \( a = (-1 \pm \sqrt{-4\gamma})/2 \) and \( z = \sin^2(\theta/2) \). The two tabulated solutions of (53) are \([14]\),

\[ Y_I = \cos \frac{\theta}{2} \cosh(\sqrt{\gamma} \theta) - 2\sqrt{\gamma} \sin \frac{\theta}{2} \sinh(\sqrt{\gamma} \theta), \]  

(55)

\[ Y_{II} = \sqrt{\gamma} \sin \frac{\theta}{2} \cosh(\sqrt{\gamma} \theta) - \cos \frac{\theta}{2} \sinh(\sqrt{\gamma} \theta). \]  

(56)

Asymptotically, for an open universe, \( R_0 \) varies linearly with time, whereas, using (36) and taking the asymptotic limit of either (55) or (56) we obtain,

\[ X \sim t^{\sqrt{\gamma}}. \]  

(57)

Thus, the perturbative corrections to the homogeneous scale factor are insignificant for large times. On the other hand, for the matter distribution, we obtain, by substituting for \( X \) in the Einstein equation (32),
\[ \lambda \sim t^{\gamma - 3}. \] (58)

The growth parameter of the density fluctuation

\[
\Re \left( \frac{\ln(\lambda R_0^3/\rho_0)}{\ln t} \right) = \gamma
\] (59)

shows that, for \( \gamma \approx 1 \), the fluctuation evolves to dominate over the homogeneous background density. In fact, using expression (45), we find that the perturbation grows by a factor of \( 10^5 \) in the matter-dominated era.

**B. Matter-dominated open universe with a small fluctuation pressure**

\[ P_0 = 0, \, \alpha \neq 0 \]

In this subsection, we study a matter-dominated open universe, where the fluctuation makes a very small \(^8\), but nevertheless nonvanishing, contribution to the pressure.

The homogeneous density and scale factor are the same as those given in (51). The differential equation (37) can be written as

\[
d^2Y \left( \frac{1}{\sin \theta} \right) d^2Y - \left( \frac{1}{1 - \cos \theta} \right) dY + \left( -\frac{9}{2} \alpha (\alpha - 1) \frac{1}{1 - \cos \theta} - A_3 \right) Y = 0.
\] (60)

As in the previous subsection, we make the substitution \( 2z = 1 - \cos \theta \) and rewrite the above differential equation as

\[
d^2Y \left( \frac{1/2}{z} - \frac{1/2}{z - 1} \right) \frac{dY}{dz} - \left( -\frac{9\alpha (\alpha - 1)}{4z} - A_3 \right) Y = 0.
\] (61)

However, unlike the equation (53), this expression needs a further transformation to become hypergeometric. We make the substitution

\[ Y = z^\zeta (z - 1)^\eta \Psi, \] (62)

where \( \zeta = 3\alpha/2 \) or \( 3(1 - \alpha)/2 \) and \( \mu = 0 \) or \( \mu = 1/2 \). Since each of the parameters \( \zeta \) and \( \eta \) takes two values, there are four solutions to the differential equation (61). However, only the following two solutions:

\[ Y_I = z^{3\alpha/2} F \left( \frac{3\alpha}{2} - \frac{1}{2} + \frac{\sqrt{1 - 4A_3}}{2}, \frac{3\alpha}{2} - \frac{1}{2} - \frac{\sqrt{1 - 4A_3}}{2}, 3\alpha - \frac{1}{2}; z \right) \] (63)

\(^8\)For \( P = 0 \), the parameter \( \alpha \) has to be very small to keep the perturbation adiabatic.
and
\[ Y_{II} = z^{3(1-a)/2} F \left( 1 - \frac{3}{2} \alpha + \frac{\sqrt{1-4A_3}}{2}, 1 - \frac{3}{2} \alpha - \frac{\sqrt{1-4A_3}}{2}, \frac{5}{2} - 3\alpha; z \right), \tag{64} \]
are independent.

We use the linear transformation formula [14],
\[
F(a, b, c; z) = \frac{\Gamma(c)\Gamma(b-a)}{\Gamma(b)\Gamma(c-a)} (-z)^{-a} F \left( a, 1 - c + a, 1 - b + a; \frac{1}{z} \right) + \frac{\Gamma(c)\Gamma(a-b)}{\Gamma(a)\Gamma(c-b)} (-z)^{-b} F \left( b, 1 - c + b, 1 - a + b; \frac{1}{z} \right) \tag{65}
\]
for hypergeometric functions, where asymptotically \( z \sim t \), to obtain
\[ Y \sim t^{\frac{3}{2}+\sqrt{1-4A_3}} \tag{66} \]
for very large times. Subsequently, for an open universe at large times, when \( R_0 \sim e^{-i\theta} \sim t \) and \( \rho_0 \sim t^{-3} \), we obtain, using (36) and (32),
\[ X \sim t^{x} \quad ; \quad x = \frac{3}{2} \alpha + \frac{1}{2} \sqrt{1-4A_3} \tag{67} \]
\[ \lambda \sim t^{-\frac{3}{2}\alpha-3+\frac{1}{2} \sqrt{1-4A_3}}. \tag{68} \]

For complex values of the square-root term in \( x \), the matter distribution becomes inhomogeneous only when \( \alpha \leq \frac{2\gamma}{3} \). On the other hand, for real values of the square root, the inhomogeneity is enhanced. This occurs, for \( \gamma \approx 1 \), if \( \alpha \gtrsim 0.8 \) or \( \alpha \lesssim -0.4 \). Putting these results together, we see that also for \( \alpha \gtrsim 1 \) the scale-invariant perturbation dominates over the homogeneous background. The inhomogeneity grows by a factor of \( 10^{3/2} \) for \( \alpha \approx 1/2 \) during the matter-dominated era (see eqn. (45)).

C. Open universe with non-vanishing pressure

\[ P_0 \neq 0 \quad , \quad \alpha \neq 0 \]

In this subsection, we consider the most general equation of state for an open universe where neither the homogeneous background nor the perturbation contributions to the pressure vanish.

The homogeneous density \( \rho_0 \), obtained by using the equation of state \( P_0 = \omega \rho_0 \) in the energy-momentum tensor equation (30), is
\[ \rho_0 = C_1 R_0^{3(\omega-1)} \tag{69} \]
where $C_1$ is an integration constant. By substituting for $\rho_0$ in the homogeneous Einstein equations \(^9\), we obtain

$$\frac{dR_0}{dt} = \sqrt{1 - \frac{8\pi G}{3} C_1 R_0^{3\omega - 1}}. \quad (70)$$

The solution is the hypergeometric equation \(^{10}\)

$$t = \frac{-2\sqrt{C_2}}{3(\omega - 1)} R_0^{\frac{1}{2}(1-\omega)} F \left( \frac{1}{2}, \frac{3(\omega - 1)}{2(3\omega - 1)}, \frac{9\omega - 5}{2(3\omega - 1)}, -C_2 R_0^{1-3\omega} \right). \quad (71)$$

Thus, since the exact dependence of $R_0$ on $t$ is not known, the differential equation (37) cannot be solved exactly. However, asymptotically, for large times, $R_0$ varies linearly with time and thus the differential equation (37) becomes

$$\frac{d^2Y}{dt^2} + Ct^{-3\omega - 3}Y + A_3t^{-2}Y = 0, \quad (72)$$

where $C$ is a constant. Next, we substitute

$$Y = t^{\frac{1}{2}(1+\sqrt{1-4A_3})} (\frac{2}{2+\omega+1}) F \left( \frac{1}{3}, \frac{2}{2+\omega+1}, \tau \right), \quad (73)$$

where $t = C^{-\frac{1}{1+3\omega}} \tau^{\frac{2}{1+3\omega}}$, in the differential equation (72) to obtain a confluent hypergeometric equation \([14]\). However, asymptotically, $F(1/\tau)$ approaches a constant value \([14]\) and we obtain, using (36),

$$X \sim t^{-3\alpha + \sqrt{1-4A_3}}, \quad (74)$$

whose behaviour does not depend on $\omega$. For the density perturbation, we obtain

$$\lambda \sim t^{[(3\alpha + \sqrt{1-4A_3})/2] - 3} \quad (75)$$

and subsequently the growth parameter of the density fluctuation is

$$\frac{\ln(\lambda R_0^3 / \rho_0)}{\ln t} \sim t^{(-\frac{3\omega}{3\omega + 1} + 3\omega)}. \quad (76)$$

Thus, the fluctuation dominates over the background homogeneous density for all positive values of $1 + 3\omega - 3\alpha/2$ or for large positive values of $\alpha$ and $\omega \gtrsim -0.3$.

\(^9\)The homogeneous Einstein equation are simply obtained by using the Friedmann metric (2) in the Einstein field equation (24).

\(^{10}\)The expression (51) is recovered for $\omega = 0$. 

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The growth rate of the perturbation, obtained by using the above growth parameter in equation (45), is much larger in the radiation-dominated era \( (t/t_0 \approx 10^8) \) than in the matter-dominated era \( (t/t_0 \approx 10^5) \). In the radiation-dominated period, the perturbation grows by a factor of about \( 10^{12} \) for \( \alpha \approx 1/3 \) and reduces to its lowest value at \( 10^4 \) for \( \alpha \approx 1 \). Thus, the homogeneous Friedmann model needs to be revised for an open universe.

VII. THE RÔLE OF THE POLAR PRESSURE

The differential equation (37) has been obtained by using the equation of state, \( \pi_r = \alpha \lambda \), to relate the radial pressure to the density via the parameter \( \alpha \). So far, we have used the time-time component (32) and the radial-radial component (33) of the Einstein equation. However, instead of relating the radial pressure to the density, we could equally relate the polar pressure to the density through a similar adiabatic equation of state,

\[
\pi_\theta = \alpha_\theta \lambda. \quad (77)
\]

We could also use the polar-polar component of the Einstein equation (34) instead of (33). In this section, we show that the results obtained so far, are left intact by this interchange.

Using equations (32) and (34) and the above adiabatic equation of state for the perturbations, we obtain the second-order differential equation

\[
\ddot{Y} - \left( 6\pi (\alpha_\theta - 1)(\alpha_\theta \rho - P_0) - A_{\theta 3} \frac{k}{R_0^2} \right) Y = 0 \quad (78)
\]

where

\[
A_{\theta 3} = (\gamma + 1) (\alpha_\theta (\gamma - 3) - \gamma + 2) - \frac{3}{4} (\alpha_\theta - 1)(3\alpha_\theta - 1) = A_3 + (1 - \gamma). \quad (79)
\]

The differential equation (78) is identical to the previously obtained equation (37), with \( \alpha \) and \( A_3 \) replaced by \( \alpha_\theta \) and \( A_{\theta 3} \), respectively. For a flat universe, \( k = 0 \), this interchange is immaterial since the coefficient \( A_{\theta 3} \) is eliminated from the differential equation (78) and we obtain the same results as those obtained in Sections IV and V.

For \( k \neq 0 \), the differential equation is modified because of the extra terms in \( A_{\theta 3} \) as compared to \( A_3, (38) \). However, the fractal codimension obtained from the observational results is approximately 1 [11], which leads to the equivalence of \( A_3 \) and \( A_{\theta 3} \). Thus, the radial and polar pressures play the same rôle in the evolution of density fluctuations and their interchange has no effect on our results.
VIII. CONCLUSION

We conclude that any density fluctuation in an open universe is scale-invariant. In a flat universe, the scaling law is not obeyed by density perturbations.

We have seen that a small seed of adiabatic inhomogeneity decays by a factor of 100 in the matter-dominated era. We have shown that, on the contrary, during this time, such a seed grows by a factor of $10^5$ in an open universe. Moreover, the fluctuation grows by a factor of $10^{12}$ in the radiation-dominated era. Thus, we conclude that any adiabatic scale-invariant perturbation would eventually make an open universe inhomogeneous. On the contrary, in a flat universe, the homogeneity hypothesis of the Friedmann cosmology remains valid. In a flat inflationary universe, scale-invariant adiabatic density fluctuations grow exponentially, leading to an inhomogeneous universe.

We also conclude that polar and radial pressures play the same rôle in the dynamics of perturbation growth. Their interchange leaves our results unaffected.

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