Black Hole Thermodynamics
from the point of view of Superstring Theory

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Abstract

In this review we try to give a pedagogical introduction to the recent progress in the
resolution of old problems of black hole thermodynamics within superstring theory. We
start with a brief description of classical black hole dynamics. Then, follow with the
consideration of general properties of supesymmetric black holes. We conclude with
the review of the statistical explanation of the black hole entropy and string theory
description of the black hole evaporation.

1. Introduction

Being just classical solutions in General Relativity, black holes [1] behave as if they possess
entropy [2] and temperature [3]. It was argued that this fact causes some puzzles which we
describe in the subsections 1.1, 1.2, 1.3. They can not be resolved, at least naively, if gravity
is considered as a classical background for quantum theory [2, 3, 4, 5]. Then, quantum
gravity is required. We believe that any reasonable, i.e. self-consistent, quantum theory of
gravity would not have these puzzles [6]. At present we have at our disposal only one such a
theory which is that of superstrings [7]. Therefore, a good exercise is to check if superstring
theory really has no puzzles under consideration.

From the modern point of view General Relativity and low energy string theory live at
different points of some space of parameters of fundamental M-theory [8, 9] which is yet to
be formulated [10]. In fact, one gets General Relativity from the fundamental theory when
all scales in the latter are big in comparison with the string one $\sqrt{\alpha'}$. So that all string
theory corrections to the General Relativity are small. While the string gas description is
valid when the string coupling constant $g_s$ tends to zero.

Although one still has to work to get a phenomenologically reasonable theory, that of
superstrings is considered as the most promising self consistent theory which quantizes
gravity and, even more, unifies it with Yang-Mills fields. As we discuss in this paper,
superstring theory also provides an explanation of the black hole thermodynamics. However,
our today’s knowledge about this theory is limited. Therefore, our answers also are not quite
complete.

Let us proceed now with the brief discussion of the puzzles and of the progress we are
going to review.

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1.1. Entropy

Black holes in General Relativity serve as attractors – objects present in the configuration spaces of some non-linear dynamical systems. We mean that independently of initial conditions before a collapse, there is a static solution characterized by a few parameters, after it, as \( t \to \infty \). In other words, the static black hole solutions do not "remember" initial conditions. This fact manifests itself in the so called No Hair theorem [11]. It establishes that one can characterize the static black hole solutions only by their masses, angular momenta and different charges corresponding to local symmetries. Qualitatively, argumentation goes as follows [11]: first, semiclassically a remote observer can see nothing behind an event horizon; second, gravity is sensitive only to the energy-momentum tensor; third, the frequency of quanta, emitted by an object falling into the black hole, tends to zero when measured by a distant observer [12]. Hence, after the collapse, as \( t \to \infty \), one can feel different internal black hole charges, corresponding to local symmetries, only via static fields.

All this means that the static black holes possess a degeneracy over initial values of any parameters (like multi-pole moments, global quantum numbers and etc.) in a theory containing General Relativity. Exceptions are those which appear in the statement of the No Hair theorem. Manifestation of this degeneracy is the appearance of the entropy \( S \) which is attributed to the black holes themselves [2]. It is proportional to the area \( A \) of the event horizon [2]: \( S = A/4\Gamma_N \), where \( h = c = 1 \) and \( \Gamma_N \) is the Newton constant.

This entropy was calculated via the use of the laws of classical black hole dynamics only [2, 3]. Thus, one needs to find a statistical explanation of the entropy. In principle to solve the problem one can calculate the degeneracy corresponding to the internal states of a black hole. This way one finds [13] the proportionality \( S \sim A \). If this is not satisfactory for some reasons, for example, because of the possibility of the information loss [4], one might consider fluctuations of the event horizon [6, 14]. Keeping the black hole mass and charges fixed, we find that the number of such "fluctuation states" is proportional to exponent of the horizon area. Hence, again one can qualitatively recover the proportionality\(^2\) \( S \sim A \) [14].

Thus, the real question is how to explain the coefficient of this proportionality. For example, in the just described reasoning, the coefficient is divergent. That is why, the correct statistical derivation of the entropy should include counting of the black hole internal states in quantum gravity. In this case the black hole degeneracy can be lifted.

As we have already said, the most promising candidate for quantum theory of gravity is that of superstrings. Within this theory the black holes of General Relativity correspond to some quantum string states. In fact, when the string coupling constant \( g_s \) tends to zero, the black hole horizon shrinks [16, 17] and becomes smaller than the string scale \( \sqrt{\alpha'} \). At this point the geometric description by General Relativity is not appropriate and one has a pure string state. Thus, we can calculate the degeneracy of the latter and compare its log with the black hole horizon area.

Why should one expect an agreement between these two numbers? To answer on this question we consider the four-dimensional Schwarzschild black hole as in ref. [17]. Its mass is equal to \( M_{bh} = r_0/2\Gamma_N \), where \( r_0 \) is the radius of the event horizon. We want to equate the latter with the mass of a string state at the excitation level \( N \), which is \( M_s^2 \sim N/\alpha' \) at zero string coupling \( g_s \). The Newton constant is related to the string coupling \( g_s \) and \( \alpha' \) by \( \Gamma_N \sim g_s^2\alpha' \). So it is clear that the mass of the black hole cannot be equal to the string mass for all values of \( g_s \). If we want to equate them, we have to decide at what value of the string coupling they should be equal. Clearly, the natural choice is to let \( g_s \) be the value at which

\(^2\)In three dimensions one even can recover the proper coefficient of the proportionality [15].
the string forms a black hole, which is when the horizon is of the order of the string scale $\sqrt{\alpha'}$. Setting the masses $M_{bh}^2$ and $M_s^2$ equal when $r_0^2 \sim \alpha'/\Gamma_N^2 \sim N/\alpha'$, which happens when $g_s \sim N^{-\frac{1}{4}}$. The black hole entropy is then $S_{bh} \sim r_0^2/\Gamma_N \sim \alpha'/\Gamma_N \sim \sqrt{N}$. At the same time, because the string state is a composition of the oscillator states its degeneracy could be evaluated as follows. It is equal to the number of splittings $N = \sum j_n j_m$ of $N$ into the numbers $j$, each appearing $n_j$ times in the sum. Simple combinatorial exercise\(^3\) shows that in the large $N$ limit the degeneracy is proportional to $e^{\sqrt{N}}$. Thus, we see a qualitative agreement between $S_{bh}$ and $S_s$ for the black hole. This qualitative reasoning can be generalized to any dimension and to charged black holes [17].

To find an agreement between $S_{bh}$ and $S_s$ up to the factor $1/4\Gamma_N$, we need some special circumstances [18]. For not much is yet known about string theory. The circumstances are as follows. Theorists use such black holes which have the entropy independent of the string coupling constant. There are solitons of this kind in superstring theory. Being supersymmetric (SUSY) charged extreme black holes, they saturate so called Bogomolni-Prasad-Sommerfield (BPS) bound. These solitons survive quantum corrections because they belong to some specific representations of the SUSY algebra. In fact, during a smooth variation of the coupling constants in a theory representations of symmetry algebras can not change [19]. Among the BPS black holes one should consider those, which are regular on the event horizon. Because singular event horizons lead to that string theory corrections for the event horizon area become strong.

That is why theorists are forced to use some kind of a tricky tuning of parameters of the black hole solutions in the supergravity (SUGRA) theories. Therefore, following [20], we consider a black hole which corresponds to the bound state of D-branes [21], as $g_s \rightarrow 0$. The latter are string states described as manifolds on which strings can terminate. The black hole in question carries both electric and magnetic charges under several electromagnetic like gauge fields and obeys the BPS bound. The presence of several charges helps to have a BPS solution with singularities of all fields, defining the solution, shifted from the event horizon. As we review below, in this case string theory perfectly explains the black hole entropy including the numerical factor $1/4\Gamma_N$.

One comment is in order at this point. The extreme black holes are fictitious from the thermodynamic point of view. While having non-zero entropy, they have zero temperature and, hence, do not evaporate. Therefore, we continue by considering slightly non-extreme black holes.

### 1.2. Evaporation

The entropy, discussed in the previous subsection, is a response function on variations of the temperature. It happens because the black holes behave as if they have the temperature [3]. In this subsection we try to give a qualitative explanation of this fact.

The black hole evaporation is just an ordinary decay process of a state in quantum theory combined with quantum tunneling. In fact, the black holes radiate because there can be a pair creation in the strong gravitational field [23]. During such a process one of the created particles falls down the black hole while the other escapes to infinity. This should be the

\[^3\text{One considers the generation function of those numbers} [7]: G(\omega) = \text{tr} \left( \omega^N \right) = \sum_{n=0}^{\infty} d_n \omega^n, \text{ where} \; \hat{N} = \sum_{m=1}^{\infty} a^+_m a_m \text{ is the operator of number of particles. We are looking for the number } d_N. \text{ At the same time} \; \text{tr}(\hat{N}) = \prod_{m=1}^{\infty} \text{tr}(\omega^{a^+_m a_m}) = \prod_{m=1}^{\infty} (1 - \omega^m) = \exp \left( - \sum_{m=1}^{\infty} \ln(1 - \omega^m) \right) = \exp \left( \sum_{m,n=1}^{\infty} \frac{\omega^m}{m} \right) \sim \exp \left( \frac{1}{1-\omega} \sum_{m=1}^{\infty} \frac{1}{m^2} \right) = \exp \left( \frac{\pi^2}{6(1-\omega)} \right). \text{ Therefore, } d_N = \frac{1}{4\pi} \int \frac{G(\omega) \omega^2}{1-\omega} \sim \exp \left( \text{const} \cdot \sqrt{N} \right).\]
case due to the energy conservation law: one of the particles acquires negative while the other equal positive energy.

Whether for the evaporation in the stationary situation\textsuperscript{4} one should have a black hole or just a very massive object, can be checked as follows. The first (very qualitative) argument is: the energy of two created particles should be equal to the modulus of the bending energy of one of them with the evaporating body. We mean that $\Gamma^{(4)}_{N} m_{1} r_{1} = m_{1} + m_{2}$, where $M$ is the mass of the evaporating body, $m_{1,2}$ are masses of the created particles and $r_{1}$ is the radius at which the particle creation is possible. Thus, the radius $r_{1}$ is always smaller than the Schwarzschild one which is equal to $\Gamma^{(4)}_{N} M$. Hence, in the stationary situation only the objects possessing event horizons can evaporate.

The second argument goes as follows: for such an evaporation to occur, space-time should be geodesic incomplete \textsuperscript{5}, hence singular. Therefore, it should contain an event horizon. In fact, geodesic incompleteness means, almost by definition, that there might be particles whose history has a beginning or end at a finite proper time. Thus, particle creation might happen only in the presence of a black hole \textsuperscript{22}.

It was argued \textsuperscript{4, 22} that due to the presence of the event horizon one should average over all ”invisible” states behind its surface. Doing this, one gets a mixed quantum state. However, this is not a whole story. The main fact about black holes, as we have already mentioned, is that one gets the thermal mixed state. For the static black hole solutions this thermal nature can be shown by considering quantum field theory in their background. After the Wick rotation from the Minkovski space to the Euclidean one, we should transform to the coordinates in which the metric is regular on the event horizon. This way one gets a well defined theory in which the time coordinate is necessarily an angular variable \textsuperscript{22}. Thus, obtained theory is thermal, with the length of the compact time direction being inverse temperature. Due to the No Hair theorem we have a finite number of the static solutions and can check the thermal behavior of all of them. However, these considerations are semiclassical \textsuperscript{31} and applicability of the semiclassics in this case is arguable \textsuperscript{6}.

Thus, without knowledge of quantum gravity, one can have the following ”puzzle” \textsuperscript{4, 22}: before a collapse there is a pure quantum state, after it, there is the mixed thermal one. Being a non-unitary transition it is a forbidden process in quantum theory. To resolve this puzzle we should find some unitary description of the black hole radiation. We believe that in any quantum theory of gravity the radiation is described by a unitary process \textsuperscript{6}. However, now at our disposal we have only superstring theory. Hence, if it pretends to be a quantum gravity it should give such a process.

Therefore, in this paper we study, following ref. \textsuperscript{24, 25}, the evaporation of slightly non-extreme black holes. First we consider the process in the black hole and then in the D-brane pictures. For the calculation of the decay rate in the first case, one can use, for example, quantum tunneling method. While in the second case, it is necessary to find a process which is a counterpart of the black hole radiation. For this reason one should consider some non-BPS excitations of a D-brane bound state. In our case, these excitations are given by a gas of non-BPS open (attached to the D-branes) strings. Two such strings can collide to form a closed one, escaping to infinity. Via standard quantum theory methods one can calculate the amplitude of this process. The radiation is thermal due to the canonical distribution

\textsuperscript{4}For an evaporation in the non-stationary situation the presence of the event horizon is not necessary \textsuperscript{23}. However, in this case the evaporation is not thermal.

\textsuperscript{5}One comment is in order here. There might be non-singular but geodesically incomplete space-times. For example, one can consider a collapse of a matter with pressure rather than that of the cosmic dust. However, this is non-stationary situation which evaporates non-thermally and eventually will turn into some static black hole.
of the strings in the gas [25]. As we review below, when one has a slight deviation from the extremity, there is a perfect agreement between the decay rates under consideration. Therefore, we know the unitary string theory description of the black hole radiation.

1.3. Information loss

We have already seen that in the semiclassical approximation one naively loses information about states felt inside a black hole. This was the reason for the averaging over intrinsic states. One can pose this "puzzle" in another way [22]. We shall skip all obvious assumptions in the description below.

Suppose that we have created somehow a pair of extreme black holes. After that, one of them can absorb a particle, carrying some conserved global charge, and evaporate back to the extremity. In the semiclassics the No Hair theorem establishes that one can not test such a global charge inside any static black hole. Moreover, the thermal evaporation does not carry any information about the black hole intrinsic state. Therefore, it seems that after such a process, the pair of black holes can annihilate back, loosing information about the quantum number under consideration. Hence, one can argue that there is an information loss in the black hole presence.

This is not quite true, however. For in the above consideration we have implicitly supposed that the semiclassical No Hair theorem is valid also on the quantum level. It is even possible to give arguments [22] in favor of this fact. Without possibility to check such an information loss process experimentally, this assumption might be reasonable if one could construct self-consistent non-unitary quantum theory. However, nobody yet was able to construct such a theory. At the same time, there is a self-consistent unitary theory which gives a quantum description of the black holes. They are given by D-brane bound states for some of them. In any case, in superstring theory one can measure the quantum state of any black hole. Hence, within superstring theory, in the described above process black holes will annihilate if and only if they are equivalent. It happens if and only if information is taken away by the radiation.

One may argue, keeping in mind that black hole and D-brane pictures are valid at different values of string theory coupling constants, that our arguments are wrong [26]. In fact, a neutron star has no event horizon and, hence, does not contradict unitarity. While if one will slightly change the value of the Newton constant, the neutron star would collapse to form a black hole which can contradict unitarity.

However, the above mentioned process can be considered as a quantum mechanics on the two black hole moduli space [26]. The latter is the space of, invisible at infinity, parameters defining a solution. In our case these are just black holes positions. The main point here is that the topology of the moduli spaces does not change during smooth variations of coupling constants [26]. At the same time, the moduli space of a D-brane bound state is the product of their individual moduli spaces. This space should be factored over the permutation group, acting on the D-brane positions, if and only if they are equivalent. Therefore, the same is true in the black hole case and our above arguments are correct. This fact shows that there is no the information loss "paradox" within superstring theory.
1.4. Content of the review

Our review is organized as follows. In the second section we briefly review the classical black hole dynamics. The third section includes a discussion of the BPS states in low dimensional SUSY theories, for simplicity of the presentation. In this context we describe properties of the extreme SUSY black holes in four- and five- dimensional SUGRA theories.

In the section four we show what kind of the ten-dimensional SUGRA solutions correspond to the black holes in dimensions smaller than ten. After that we argue which string excitations should quantize certain charged black hole solutions of the SUGRA theories.

In the section five we calculate thermodynamic quantities for the extreme black holes which are regular on the event horizon. We find that they have zero temperature and non-zero entropy. Then in this section we calculate the degeneracy of the quantum string excitations corresponding to these black holes. Logs of the obtained numbers perfectly coincide with the black hole areas. Also in the section five we continue with the review of the results on the radiation of the particular non-extreme black holes. Here we also find a perfect agreement between the string theory and black hole calculations.

Conclusions are given in the section six. Also to make the review self-contained as much as possible we included the discussion of some basics of superstring theory and of the superstring duality in the Appendices.

2. Black Hole Physics

In this section we present some basic facts concerning classical General Relativity solutions themselves and their dynamics.

2.1. Black Hole solutions

The simplest black hole solution is that due to Schwarzschild in four dimensions [1]. It is spherically symmetric solution of the free Einstein-Hilbert equations with asymptotically flat boundary conditions. It corresponds to the line element:

\[
\begin{align*}
    ds^2 &= -\left(1 - \frac{2M'}{r}\right)dt^2 + \left(1 - \frac{2M'}{r}\right)^{-1}dr^2 + r^2d\Omega^2_2, \\
    d\Omega^2_2 &= d\theta^2 + \sin^2(\theta)d\phi^2,
\end{align*}
\]

(1)

where \(M' = \Gamma^{(4)}_N M\) and \(M\) is the black hole mass measured by asymptotics of the energy-momentum tensor at infinity.

The Schwarzschild solution has the future event horizon at \(r = 2M'\). Semiclassically one can not see anything behind this surface. In the highly simplified context of spherically symmetric and static space-time geometries the definition of the event horizon is as follows\(^6\). This is a null surface, i.e. the surface to which a normal vector has zero norm, from behind of which a particle can not escape to infinity without exceeding speed of light.

The solution (1) has also the past event horizon which is the time reverse of the first one: a surface which is impossible to get behind. However, one can construct a black hole solution which appeared as the result of a collapse and, hence, has no past event horizon [27].

\(^6\)While for general space-times there are many subtleties involved [27].
As can be seen, the metric (1) has the singularity at the surface \( r = 2M' \). It is referred to as a *coordinate singularity* because the invariant of the curvature tensor is regular there. Hence, the singularity can be avoided by some coordinate change. Also there is a *curvature singularity* at the point \( r = 0 \), which can not be avoided by any coordinate change. In fact, the invariant of the curvature tensor is singular there.

There are also rotating (Kerr), charged (Reissner-Nordstrom) and both charged and rotating (Kerr-Newman) black hole solutions in General Relativity [1]. Below we will not discuss rotating black holes. But all methods we use can be also applied in the latter case [28].

The Reissner-Nordstrom black hole is spherically symmetric solution of the Einstein-Hilbert plus Maxwell equations of motion with asymptotically flat boundary conditions. In four dimensions the metric of the solution looks as follows:

\[
\begin{align*}
\ ds^2 &= -\Delta dt^2 + \Delta^{-1} dr^2 + r^2 d\Omega_2^2, \\
\Delta &= \left(1 - \frac{r_+}{r}\right)\left(1 - \frac{r_-}{r}\right), \quad r_+ \geq r_-.
\end{align*}
\] (2)

In proper units its mass and charge are:

\[
M = \frac{1}{2\Gamma_N^{(4)}(r_+ + r_-)}, \quad Q = \frac{1}{\Gamma_N^{(4)}} \sqrt{r_+ r_-}.
\] (3)

The Reissner-Nordstrom black hole has the coordinate singularity at the *outer* (event) horizon \( r = r_+ \). After the definition of quantum theory in the black hole background, the regularity of the energy-momentum tensor quantum average on the event horizon is required. Which leads to the energy momentum tensor divergence at \( r = r_- \) and produces the curvature singularity at the *inner* horizon \( r = r_- \). Also, as can be seen from (3), always \( M \geq Q \) [29]. Solutions with \( M = Q \) are referred to as *critical* or *extreme*. They have peculiar features, discussed in the following sections.

### 2.2. Black hole thermodynamics

There are two main laws of classical black hole dynamics [2, 30]. The first one can be derived, for example, from the expression for the black hole horizon area

\[
A = 4\pi\Gamma_N^{(4)} \left[2M^2 - Q^2 + 2M\sqrt{M^2 - Q^2}\right]
\] (4)

through its charge and mass from eq. (3). One gets that variations of these quantities are related to each other as:

\[
dM = \frac{\kappa}{8\pi\Gamma_N^{(4)}} dA + \Phi dQ,
\] (5)

where \( \kappa = \frac{2\pi(r_+ - r_-)}{A} \) is referred to as *surface gravity* and \( \Phi = \frac{4\pi Q r_+}{A} \) is the electromagnetic potential of the event horizon.

The second law establishes that the area of a black hole event horizon never decreases [30]. In fact, as can be shown [22], the null geodesic segments, which generate the event horizon, can not be converging.
At this point one can grasp an analogy with the first and second laws of thermodynamics, respectively! This might be thought just as a coincidence. But in a moment we will see that black holes really behave as thermodynamic objects.

Let us explain how the Schawrzschild black hole radiates [23, 3]. We gave already a qualitative explanation of the black hole radiation in the Introduction. Let us give another one in this subsection [3, 31].

One should define quantum field theory in the curved background of a black hole which was created by the collapse of a matter. Question of prime importance is what one should consider as vacuum. For this reason, one needs to find a time direction to define what we mean by the energy itself, by the lowest energy state and by asymptotic states. However, the metric of a black hole created by a collapse is time dependent. Hence, there are different time-like Killing vectors (that which define time direction) in different parts of space-time. Therefore, in different parts of space-time one has different vacua. Transformations between them are of the Bogolubov type, i.e. as that in the Bardeen-Couper-Shrifer theory of superconductors. These transformations lead to the generation of matter from geometry. The mixed state appears after an averaging over the particles gone behind the horizon. This way one gets the following formula for the Schwarzschild black hole decay rate (only due to a neutral particle radiation):

$$d\Gamma = \sigma_{gb}(k_0) \left( \frac{1}{\exp \frac{2\pi k_0}{\kappa} - 1} \right) \frac{d^3k}{(2\pi)^3}.$$  (6)

Here $k_0$ and $k$ are the energy and wave vector of an escaping particle, $\kappa = \frac{1}{8\pi(\mu M)^{1/2}}$ and $\sigma_{gb}$ is the so called graybody factor. It is equal to the absorption cross section of the black hole\(^7\). Looking at the formula (6), we can recognize the thermal behavior of the black hole, with $T = \frac{\kappa}{2\pi}$ being its temperature. Moreover, comparing (5) (when $Q = 0$) with the second law of thermodynamics when $T = \frac{\kappa}{2\pi}$, we can equate the entropy with $\frac{1}{4\pi(\mu M)^{1/2}} A$.

Due to this thermal radiation, the black holes loose their entropy and mass. The first fact leads to the generalized second law of thermodynamics: ”entropy of a matter outside a black hole plus $\frac{1}{4\pi(\mu M)^{1/2}} A$ times the area of the black hole event horizon never decreases”. While because of the second fact the Schwarzschild black hole will evaporate until $M \rightarrow 0$. Size of such a limiting black hole is of the order of the Plank scale and there will be strong quantum gravitational effects.

The decay rate for the Reissner-Nordstrom black hole is the same as in (6) but with:

$$T = \frac{(r_+ - r_-)}{A}.$$  (7)

Therefore, if one considers only neutral particle emission, the Reissner-Nordstrom black hole will evaporate until $T = 0$ when $M = Q (r_+ = r_-)$. The end point of such an evaporation is a stable extreme black hole (massless black hole is also considered as an extreme $M = Q = 0$) with the minimal allowed mass as discussed below eq. (3). In the next section we will see how SUSY explains this stability.

3. Supersymmetric black holes

The main tool which gives to superstring theory control over the low energy dynamics and over the dynamics of the extreme black holes is the SUSY [32] in the target space. The reason

\(^7\)We discuss $\sigma_{gb}$ for a particular solution in the subsection 5.3.
why one can control the dynamics is that SUSY gives strong restrictions on the effective low energy action. This is due to the non-renormalization theorems [33]. Also among all the excitations of the SUSY theories there are remarkable ones. They are referred to as BPS states. We will discuss them first in the context of the black hole physics below. In this section we explain what are these BPS states and their relation to the extreme black holes.

Below we consider only \( N \geq 2 \) SUSY and SUGRA theories. In this case one has almost complete control over the low energy dynamics. We start with the discussion of the BPS states. Then we consider peculiar features of the four- and five-dimensional extreme black holes in the \( N \geq 2 \) SUGRA theories.

The \( d \)-dimensional \( N \) extended SUSY algebra looks as follows [32]:

\[
\left\{ Q^I_\alpha, Q^J_\beta \right\} = \delta^{IJ} (C_{\gamma^\mu})_{\alpha\beta} P_\mu + \sum_{p=0,1,\ldots} (C_{\gamma^{\mu_1\ldots\mu_p}})_{\alpha\beta} Z_{\mu_1\ldots\mu_p}^{IJ}.
\]

Here \( Q^I_\alpha, I = 1, \ldots, N, \) being Dirac fermions, are generators of the SUSY algebra and \( \alpha \) are spinor indexes; \( P_\mu \) is the \( d \)-momentum; \( C \) is the charge conjugation operator and \( \gamma^{\mu_1\ldots\mu_p} \) is the anti-symmetric product of \( p \) \( d \)-dimensional Dirac \( \gamma \)-matrices; \( Z_{\mu_1\ldots\mu_p}^{IJ} \) are so called central charges or central extensions. Not all of them are independent. For example, all pairs \( Z_{\mu_1\ldots\mu_p}^{IJ} \) and \( Z_{\mu_1\ldots\mu_{d-p}}^{IJ} \) are related to each other through the absolutely anti-symmetric tensor in \( d \)-dimensions.

Now, as an example let us show how one can derive BPS bound for a particle state. We will consider the case of the four-dimensional \( N = 2 \) SUSY algebra. In the Weyl representation for the SUSY generators this algebra looks as follows

\[
\left\{ Q^I_\alpha, \bar{Q}^J_\dot{\alpha} \right\} = \sigma^\mu_{\alpha\dot{\beta}} P_\mu \delta^{IJ},
\]

\[
\left\{ Q^I_\alpha, Q^J_\beta \right\} = \epsilon_{\alpha\beta} Z^{IJ}, \quad \left\{ \bar{Q}^\dot{I}_\dot{\alpha}, \bar{Q}^{J\dot{\beta}} \right\} = \epsilon_{\dot{\alpha}\dot{\beta}} \bar{Z}^{IJ}.
\]

Therefore, from the positive semi-definiteness of the anti-commutator of \( (\sigma^\mu_{\alpha\beta} P_\mu Q^I_\alpha - Z \bar{Q}^I_{\dot{\beta}}) \) and its hermitian conjugate \( (\sigma^\mu_{\alpha\beta} P_\mu \bar{Q}^I_{\dot{\beta}} - \bar{Z} Q^I_\alpha) \) the BPS inequality follows [19]:

\[
-P_\mu^2 = M^2 \geq |Z|^2,
\]

where \( M \) is the mass of a particle state and \( Z^{IJ} = \epsilon^{IJ} Z \) is the central charge in the corresponding representation of the \( N = 2 \) SUSY algebra. The central charge is equal to a linear combination of the electric and magnetic charges of the particle state [19].

The BPS bound is saturated on the BPS states \( M^2 = |Z|^2 \). This happens when the equality [19] \( (\sigma^\mu_{\alpha\beta} P_\mu Q^I_\alpha - Z \bar{Q}^I_{\dot{\beta}}) \text{BPS} >= 0 \) holds. Hence, the BPS states obey kind of a "chirality" condition. Vise versa, if some state is annihilated by any linear combination of the supercharges, it saturates the BPS bound. As a result of this chirality less amount of terms is present in the Taylor expansion of the corresponding superfield [32] over the anticommuting variables. That is why such states compose so called ultra-short representation of the SUSY algebra.

Now we come to a crucial point. It is believed that there are no quantum corrections which can change the representation of any algebra. Surely if the symmetry under this algebra persists on the quantum level. In the case of SUSY it means that if some state is BPS semiclassically, it is BPS at any value of different coupling constants of the theory! Therefore, such a state survives quantum corrections. One also can control
the renormalization of its mass due to the equality $M = |Z|$. Moreover, if there is enough SUSY, this mass does not even renormalize. As well, a BPS state is stable if the process of its decay into lighter BPS states is forbidden by the energy conservation law. Hence, we can calculate the dimension of the corresponding representation at any convenient value of the couplings in a theory. One can be sure that such a calculation is true for any other their values. Similar fact in superstring theory [34] will be important below.

The generalization of all that to the $N = 4$ and $N = 8$ SUSY is straightforward. Now the central charge $Z^{IJ}$ is a matrix. Therefore, one can have different kinds of BPS states. A state composes the ultra-short multiplet if its mass is equal to all eigen-values of the central charge. While if the mass of a state is equal only to the biggest eigen-value(s) of the central charge, then the state is in a short BPS multiplet. The latter break more than half of the SUSY transformations. Hence, they are bigger than the ultra-short one.

In the presence of gravity, certain BPS solutions of the classical equations of motion in SUGRA theories have line elements of the extreme black holes. In this case, the BPS condition is equivalent to the extremity one. Thus, stability of the BPS states explains somewhat "magic" stability of the critical black holes.

Now let us briefly discuss, following [35], some properties of the black hole entropy in the $N \geq 2$ SUGRA theories. In the presence of the $N \geq 2$ SUSY, one might expect that the black hole entropy would be expressed (in addition to their mass, charge and momentum) through the vacuum expectation values (vev) of different scalar fields in the theory\(^8\). However, the BPS black holes do not depend on them. The scaler fields behave as attractors in the space of moduli. Starting from their value at infinity, moduli evolve, approaching the event horizon, until they run into a fixed point near the horizon. Therefore, by the time moduli reach the horizon they lose completely the information about the initial conditions. This is a kind of a generalization of the No Hair theorem for the SUSY BPS black holes.

The explanation of this phenomenon comes from the following fact. The black hole line element interpolates between the flat geometry at infinity and so called Bertotti-Robinson (BR) geometry at the horizon. As it appears, even if the BPS black hole solution preserves only some part of SUSY transformations, these two boundary solutions (with corresponding gauge fields) preserve all of it [36]. This restricts the boundary values of the moduli fields.

Briefly, it works as follows. The explicit form of the BR metric is taken as a limit near the horizon $r \to 0$ of the black hole metric:

$$ds^2 = -h^2 dt^2 + h^{-2} d\bar{r}^2,$$

\[\Delta h^{-1} = 0,\]  \hspace{1cm} (12)

where: $h^{-2} = \frac{A}{4\pi\Gamma_N r^2} = \frac{M_{BR}^2}{\Gamma_N r^2}$ and the BR mass is defined by the black hole area of the horizon:

\(^8\)One refers to these scalar fields as moduli. Their vevs parametrize the vacuum moduli space or flat directions in the theory. In the $N \geq 2$ SUSY theories, containing Yang-Mills fields, there are scalar potentials of the following type:

$$V(\phi) = \frac{1}{2} r (|\phi^+\phi|)^2.$$

They appear after the integration over the D-fields [32]. Here $\phi$, being the superpartner of a gauge field, is the moduli type field. The potential under consideration vanishes when the $\phi$ field lives in the maximal diagonal or abelian subalgebra of the gauge symmetry algebra. Hence, arbitrary complex number, standing on the diagonal of the field $\phi$, parametrizes the moduli space of vacua.
\[ M_{BR}^2 = \frac{A}{4\pi^2}. \]  
(13)

It can be straightforwardly checked that the geometry (12), with corresponding gauge fields\(^9\), preserves all SUSY \([35, 36]\). From the formula (13) one can find the expression for the entropy through the corresponding central charge which is equal to the BPS \(M_{BR}\) mass. Hence, the area of the horizon is proportional to the square of the central charge of the SUSY algebra. The latter is computed at the point where it is extremized in the moduli space \([35]\). Hence,

\[ S = \frac{A}{4\Gamma_N^{(4)}} = \pi |Z|_{\text{fix}}^2 \]  
(14)
in four dimensions. Similar calculation gives \([35]\):

\[ S = \frac{A}{4\Gamma_N^{(5)}} \sim |Z|_{\text{fix}}^{3/2} \]  
(15)
for five-dimensional BPS black holes.

Computing the extreme value of the central charge, one finds that the entropy is expressed through some ratio of the black hole charges. Thus, it is some kind of topological quantity: it depends only upon discrete parameters in the theory. This observation will appear useful for the discussion in the section five.

### 4. SUGRA solitons

In the previous section we have discussed the BPS solutions in the low dimensional SUGRA. Among them there are non-perturbative excitations. At the same time, from its definition \([7]\) (see also the Appendix A) string theory might be thought as intrinsically perturbative. In fact, it is defined as the sum over perturbative corrections.

This is not quite true, actually, because there do exist non-perturbative excitations in the superstring theory. Now we are ready to discuss variants of these fluctuations which are among massive SUGRA solitons. Concretely, this section is devoted to the general discussion of how one constructs the classical solutions of the ten-dimensional Type II SUGRA equations of motion \([37]\). Then we consider compactifications of the ten-dimensional SUGRA to diverse dimensions. This way one can get particle-like black holes from these SUGRA solitons. At the end we follow with the consideration of the quantum counterparts of some black holes in superstring theory.

Below we use only Type II superstring theories \([7]\). Let us sketch what are their massless excitations. For the details one can consult the Appendix B. For our purposes we need to know only bosonic excitations of these theories. They are subdivided into two types. First type is that of \(NS - \bar{NS}\), which is common for both of the Type II theories. Moreover, it contains the same fields as in the bosonic string theory. These are the graviton \(G_{\mu\nu}\), antisymmetric tensor \(B_{\mu\nu}\) and dilaton \(\varphi\). Also one can add the dual or "magnetic" variant of the \(B_{\mu\nu}\) field in the following ten-dimensional sense:

\[ H_{\mu_1\ldots\mu_7} = e^{-2\varphi} \epsilon_{\mu_1\ldots\mu_{10}} H^{\mu_8\mu_9\mu_{10}} , \]  
(16)

where \(H_{\mu_1\mu_2\mu_3}\) is the field strength of the antisymmetric tensor field \(B\) and \(H_{\mu_1\ldots\mu_7}\) is that of the dual field with six indexes.

\(^9\)Expressions for these gauge fields will be given below for particular solutions.
There is also $R - \tilde{R}$ sector. Fields from this sector in the Type IIA and IIB theories are different. We present them separately for both of these theories.

**In the Type IIA theory:**

$A_{\mu}$ is the 1-form potential; $A_{\mu\nu\alpha}$ is the 3-form potential. Also there are dual, ”magnetic”, fields:

\[
F_{\mu_1...\mu_8} = \epsilon_{\mu_1...\mu_{10}} F^{\mu_9\mu_{10}} \\
F_{\mu_1...\mu_6} = \epsilon_{\mu_1...\mu_{10}} F^{\mu_7...\mu_{10}}.
\]

(17)

where $F$ are the field strengths of the corresponding gauge potentials $A$.

Superstring theory contains at low energies that of SUGRA [7]. For example, the bosonic part of the low energy SUGRA action of the Type IIA theory looks as follows:

\[
S_{IIA} = \frac{1}{16\pi \Gamma_N^{(10)}} \int d^{10} x \sqrt{-G} \left\{ e^{-2\varphi} \left[ R + 4(\nabla_{\mu}\varphi)^2 - \frac{1}{3} H_{(3)}^2 \right] - \alpha' \left( \nabla_{\mu}\chi \right)^2 - \frac{\alpha'}{3} F_{(2)}^2 - \alpha' \frac{\alpha'}{288} \epsilon^{\mu_1...\mu_{10}} F_{\mu_1...\mu_4} F_{\mu_5...\mu_8} B_{\mu_9\mu_{10}} \right\} + O(\alpha'),
\]

(18)

Here $\Gamma_N^{(10)} = 8\pi^6 g_s^2 \alpha'$ is the ten-dimensional Newton constant. $g_s$ and $\alpha'$ are the string coupling constant and the inverse string tension respectively; $O(\alpha')$ represents the $\alpha'$ corrections (see the Appendix A). From now on we usually will suppress indexes of the tensor forms and write only their rank as a subscript; $F_{(4)} = F_{(4)} + 2A_{(1)} \wedge H_{(3)}$.

The action (18) correctly reproduces the low energy string amplitudes for the massless modes. There are also various dual forms of the action (18) expressed through the dual fields (17). The latter fact is also true for the Type IIB theory.

**In the Type IIB theory:**

$\chi$ is the axion pseudo-scalar; $A_{\mu\nu}$ is the 2-form potential; $A_{\mu\nu\alpha\beta}$ is the self-dual 4-form potential. Their dual ”magnetic” fields are:

\[
F_{\mu_1...\mu_6} = \epsilon_{\mu_1...\mu_{10}} \partial_{\mu_{10}} \chi \\
F_{\mu_1...\mu_7} = \epsilon_{\mu_1...\mu_{10}} F^{\mu_8\mu_9\mu_{10}} \\
F_{\mu_1...\mu_5} = \epsilon_{\mu_1...\mu_{10}} F^{\mu_6...\mu_{10}}.
\]

(19)

The bosonic part of the low energy SUGRA action of the Type IIB theory looks as follows [7]:

\[
S_{IIB} = \frac{1}{16\pi \Gamma_N^{(10)}} \int d^{10} x \sqrt{-G} \left\{ e^{-2\varphi} \left[ R + 4(\nabla_{\mu}\varphi)^2 - \frac{1}{3} H_{(3)}^2 \right] - \alpha' (\nabla_{\mu}\chi)^2 - \frac{\alpha'}{3} F_{(3)}^2 - ... \right\} + O(\alpha')
\]

(20)

Unfortunately, we do not known any canonical way of writing an action for the self-dual field-strength $F_{(5)}$ of the potential $A_{(4)}$. This does not cause any problems because one knows equations of motions in this Type IIB SUGRA theory [7]. That is enough for the consideration of classical solutions.

Now we continue with SUGRA solitons. These solitons correspond, from the string world-sheet point of view, to some approximate ($\frac{\alpha'}{\ell_s^2} \to 0$ where $R$ is a characteristic size
of the solution) superconformal field theories. We will concentrate first on the solutions that preserve some SUSY, i.e. on the charged BPS solutions. They are generalizations of the extreme Reissner-Nordstrom solution to the case when curvature singularities live on multi-dimensional world-volumes. The latter are surrounded by multi-dimensional event horizons.

These solutions will be extended \textit{p-branes} of \textit{p} spatial dimensions. They can carry either "electric" charge under the $A_{(p+1)}$ form or "magnetic" one under the $A_{(7-p)}$ form. It is these charges which are present in the ten-dimensional SUSY algebra (8) as the central extensions with tensor indexes.

A p-brane solution carries the charge under a $A_n$ or its dual field if the p-brane world-sheet theory includes either $\int d^n x \epsilon^{a_1...a_n} A_{\mu_1...\mu_n} \partial_{a_1} x^{\mu_1} ... \partial_{a_n} x^{\mu_n}$ or $\int d^n x \epsilon^{a_1...a_n} \epsilon_{\mu_1...\mu_{10-n}} A^{\mu_1...\mu_{10-n}} \partial_{a_1} x^{\mu_9-n} ... \partial_{a_n} x^{\mu_{10}}$ terms, respectively. For example, a particle (0-brane) carries the charge under an $A_\mu$ gauge potential if its world-sheet theory includes the following term $\int dx_\mu A_\mu$.

A string (1-brane) carries the charge under a $B_{\mu \nu}$ tensor potential if its world-sheet theory includes the term: $\int d^2 \sigma \epsilon^{ab} B_{\mu \nu} \partial_a x^\mu \partial_b x^\nu$.

It is worth mentioning that no perturbative string state can be charged with respect to any $R - \tilde{R}$ field. In fact, closed string vertex operators contain only gauge invariant $F$'s rather than corresponding potentials $A$ [7] (see also the end of the Appendix B).

\section*{4.1. Solitons in ten dimensions}

To get a static p-brane solution one starts with the ansatz for the string metric [37]:

$$ds^2 = h \left[ f_p^{-1} \left( -dt^2 + dx_1^2 + ... + dx_9^2 \right) + dx_{p+1}^2 + ... + dx_9^2 \right].$$  \hspace{1cm} (21)

It breaks the ten-dimensional Lorentz group $SO(9,1)$ down to the $SO(p,1) \times SO(9-p)$. Hence, the p-brane should be dynamical to maintain the invariance under the full group. Here $f_p$ and $h$ are some functions of the transverse coordinates $x_{p+1}, ..., x_9$ which should be found. Also one allows the dilaton $\varphi$ and the component $A_{0...p}$ of the corresponding $R - \tilde{R}$ field to be non-zero, setting all other fields to zero.

It is also possible to consider a 1-brane charged soliton, placed, for example, in the 9-th direction and charged with respect to the $NS - \bar{NS}$ tensor field $B_{\mu \nu}$ [37]. This can be done using (21) for the case of $p = 1$ and keeping non-zero the $B_{09}$ field instead of $A_{0...p}$. Such a soliton would be a massive fundamental superstring excitation. It is in the perturbation spectrum because its tension is of the order of one in the string units. One calls it the F-string, where F comes from the "Fundamental".

In a similar way one can construct an object which is "magnetically" charged with respect to the field $B_{\mu \nu}$. This is a so called \textit{solitonic NS 5-brane} [37]. Which has the mass per unit volume proportional to $\frac{1}{g_s}$ [37]. Hence, the solitonic 5-brane is a non-perturbative excitation.

In this paper we mostly will be interested in the $R - \tilde{R}$ solitons, i.e. charged with respect to the $R - \tilde{R}$ tensor fields. Because they, unlike the F-string and NS 5-brane, have an exact conformal field theory treatment (see below). We mean that one knows what kind of objects in string theory quantize these solutions.

The main distinction of the $R - \tilde{R}$ solitons from the F-string and NS 5-brane comes from the fact that their mass per unit volume in the string units is of the order of $\frac{1}{g_s}$ [37]. This is due to the unusual power of the exponential of the dilaton in front of the kinetic terms (18),(20) for the $R - \tilde{R}$ potentials [7].

The p-brane $R - \tilde{R}$ solution (metric (21), $\varphi$ and $A_{0...p}$ fields) becomes BPS if it preserves some part of the SUSY transformations in the corresponding SUGRA theory [37]. It means
that the part of the SUSY transformations does not move the fields under consideration. Hence, to obey this condition one should insist on the vanishing of the SUSY transformations for the gravitino and dilatino\textsuperscript{10} (which were taken to be zero) in the corresponding Type II SUGRA theory \cite{37}. This forces the dilaton, $R - \tilde{R}$ tensor field and metric to be related to each other and to take the form:

$$
\begin{align*}
    ds^2 &= f_p^{-\frac{1}{2}} \left(-dt^2 + dx_1^2 + \ldots + dx_p^2\right) + f_p^{\frac{1}{2}} \left(dx_{p+2}^2 + \ldots + dx_9^2\right), \\
    e^{-2\varphi} &= f_p^{\frac{p-3}{2}}, \\
    A_{0\ldots p} &= -\frac{1}{2} \left(f_p^{-1} - 1\right),
\end{align*}
$$

(22)

where $p = 0, 2, 4, 6, 8$ in the Type IIA and $p = 1, 3, 5, 7$ in the Type IIB theories \cite{37} (see the Appendix B and previous subsection).

All these solutions are BPS and for any function $f_p$ they preserve only half of the SUSY because of the conditions:

$$
\eta_r = \gamma^0 \ldots \gamma^p \eta_l
$$

(23)

where $\eta_r$ and $\eta_l$ are the right and left chiral parts of the infinitesimal local SUSY generator. The equations of motion of the theory (related to the closure of the SUSY algebra) imply \cite{37} that $f_p$ should be a harmonic function $\Delta^{(p)} f_p = 0$. Here $\Delta^{(p)}$ is the flat Laplasian in the directions $p + 1, \ldots, 9$. From (22) one gets the extreme limit of the charged black p-brane when the harmonic function is

$$
    f_p = 1 + \frac{nc_{10}^p}{r^{7-p}},
$$

(24)

where $r = \sqrt{x_{p+1}^2 + \ldots + x_9^2}$; $c_{10}^p$ is related to the minimum charge of the p-brane and will be accounted for particular solutions in the section five. The corresponding charge is defined through a kind of the integral Maxwell equation for a tensor gauge potential:

$$
    Q_p \sim \int e^{2\varphi} \ast F_{(p+2)}
$$

(25)

where $F$ is the field strength of the $A_i$; $\ast$ is the Hodge dual operation which is done through the absolutely anti-symmetric tensor in ten dimensions. The integral is taken over a hypersurface which surrounds the p-brane.

In the eq. (24) $n$ is an integer because of the Dirac type quantization of charges \cite{37}:

$$
    Q_p Q_{6-p} = 2\pi k, \quad k \in \mathbb{Z}
$$

(26)

where $Q_{6-p}$ is the charge of the dual brane soliton with respect to the dual field.

Since all BPS solutions treated so far depend on some harmonic function $f_p$, one can construct multiple brane solutions. It is done by taking $f_p = 1 + \sum_i \frac{c_{10}^p}{|\vec{r}_i - \vec{r}|^{7-p}}$ which describes a set of branes at positions $\vec{r}_i$ \cite{37}. Such solutions similarly break only half of the SUSY. Moreover, they are in the static equilibrium because the force between parallel BPS solutions with the same parity of charges is always zero \cite{37}. In fact, the gravitational attraction between the BPS solitons is compensated by the repulsion through the $R - \tilde{R}$ tensor fields.

\textsuperscript{10}Superpartners of the graviton and dilaton respectively.
One also can construct BPS bound states of $p$-branes with different $p$’s obeying some conditions [38]. As we explain in the subsection 4.3, such a construction already breaks more than half of the SUSY. The conditions on $p$’s are needed to preserve at least some smaller part of it. We use this construction below when consider multicharged black holes.

4.2. Black Holes from Black Branes

Now let us proceed with the Type II theory compactified to $d$ dimensions on a torus $T^{10−d}$ [7]. We identify coordinates by

$$x_i \sim x_i + 2\pi R, \quad i = d, \ldots, 9,$$

and choose the periodic boundary conditions on this $10−d$ dimensional "box". Fields that vary over the box will acquire masses of the order of $1/R$ where $R$ is the typical compactification size\(^\text{11}\). Thus, if we are interested in the **low energy** physics in $d$ extended dimensions, fields might be taken to be independent of the internal coordinates of the torus.

One can observe that if we have any solution in ten dimensions which is periodic under $x_i \rightarrow x_i + 2\pi R$, then it will also be a solution of the compactified theory. For any $p$-brane, the solution is automatically translation invariant in the directions parallel to the brane.

We will be interested in solutions where a brane is completely wrapped along the internal directions. Its volume is represented as a part of the torus $T^{10−d}$. Therefore, from the point of view of an observer in $d$ dimensions one has a localized, spherically symmetric solution. The latter can carry a charge with respect to a gauge field which is the remnant

$$A_\mu = A_{\mu i_1 \ldots i_p}$$

of the corresponding $R − \tilde{R}$ tensor field. Here $\mu$ is the $d$ dimensional index and $i_n$ lie in compact directions.

The BPS solutions of the kind under consideration correspond to the extreme limits of the charged black holes. The final result is that the $d$-dimensional solutions are given again by (22) with $p = 0$ but now in terms of $d$-dimensional harmonic functions. Hence, when we are in the $d$-dimensional theory, the only way we have to tell that a black hole contains a particular type of the $p$-branes is by looking at the gauge fields that it excites.

4.3. $R − \tilde{R}$ solitons as $D$-branes

In this section we show what kind of objects in superstring theory quantize the $R − \tilde{R}$ black $p$-branes and, hence, the black holes with corresponding kind of charges.

Originally it was thought that the only allowed open string theory is the non-orientable Type I theory in ten dimensions with the gauge group $SO(32)$ [7]. This open string theory has the **Neumann (N)** boundary conditions on all coordinates:

$$n^a \partial_a x_\mu = 0, \quad \psi^\mu = \pm \tilde{\psi}^\mu, \quad \mu = 0, \ldots, 9$$

where $n^a$ is a vector normal to the boundary; $\psi^\mu$ are the world-sheet superpartners of the coordinates $x_\mu$. The latter represent positions of the string in the target space.

\(^{11}\)Which usually is taken to be big in comparison with $\sqrt{\alpha'}$ to maintain the $\alpha'$ corrections to be small (see the Appendix A).
However, one actually can make open string sectors in the closed Type II superstring theory [21]. As it appears, the open strings in these sectors should also have the Dirichlet (D) boundary conditions on some part of the string coordinates:

\[ n^a \partial_a x_m = 0, \quad \psi^m = \pm \tilde{\psi}^m, \quad m = 0, \ldots, p \quad (N) \]

\[ x^i = C^i, \quad \psi^i = \mp \tilde{\psi}^i \quad i = p + 1, \ldots, 9 \quad (D), \]

(30)

where \( C^i \) are arbitrary, fixed numbers. Therefore, there are open strings with their boundaries allowed to lie only on \( p + 1 \)-dimensional sub-manifolds of the ten-dimensional target space. While in the bulk of this space one has only ordinary Type II closed strings. These sub-manifolds are referred to as \( \text{D}p \)-\( \text{branes} \). Here \( p = 0, 2, 4, 6, 8 \) for the Type IIA and \( p = 1, 3, 5, 7 \) for the Type IIB theories [21].

The \( \text{D}p \)-branes have several features which are interesting for us. First, they break the Lorentz invariance in the target space. Hence, to maintain it, one should consider these \( \text{D}p \)-branes as dynamical excitations. Second, because of the boundary conditions (30), the \( \text{D}p \)-branes break a half of the SUSY transformations\(^{12}\). Thus, the \( \text{D}p \)-branes are BPS states. Third, the closed Type II strings are charged with respect to the \( B_{\mu\nu} \) field (see the Appendix A) so that the two-dimensional theory is invariant under the transformations:

\[ B_{\mu\nu} \rightarrow B_{\mu\nu} + \partial[\mu\rho\nu]. \]

When we break the string, however, there is a boundary term appearing after such a transformation. Hence, to maintain the invariance of the theory one has to accept that the boundary is charged with respect to some abelian gauge field \( A_m \). In this case the boundary term under consideration can be compensated by a shift \( A_m \rightarrow A_m - \rho_m \). (This shift is different from the ordinary gauge transformation \( A_m + \partial_m \lambda \) of the field \( A_m \).) Therefore, there should be the gauge fields living on the \( \text{D}p \)-branes. Below we give other arguments in favor of their appearance.

The presence of a \( \text{D}p \)-brane in the Type II superstring theory can be described in particular by the superconformal field theory [40] with additional boundary terms:

\[ S_{2d} = \frac{1}{2\pi\alpha'} \int d^2 \sigma \left( G^{\mu\nu} \partial_\sigma x_\mu \partial^\sigma x_\nu + \epsilon^{ab} B^{\mu\nu} \partial_\sigma x_\mu \partial^\sigma x_\nu \right) + S_{\text{boundary}} + \ldots, \]

\[ S_{\text{boundary}} = \int ds \left[ \sum_{m=0}^{p} A_m(x_0, \ldots, x_p) \partial_t x^m + \sum_{i=p+1}^{9} \phi_i(x_0, \ldots, x_p) \partial_n x^i \right], \]

(32)

where dots in the first equation stand for the fermionic superpartners; \( s \) is some parameter on the boundary; \( \partial_t \) and \( \partial_n \) are the tangential and normal derivatives to the boundary respectively; \( A_m \) is the above mentioned gauge field on the world-volume of the \( \text{D}p \)-brane and \( \phi_i \) are coordinates which determine positions of the \( \text{D}p \)-brane in the bulk.

The low energy dynamics of the \( \text{D}p \)-branes is governed by low energy Born-Infeld action [40]:

\[ S_{\text{BI}} = -M_p \int d^{p+1} \xi e^{-\varphi} \left[ \sqrt{\det \left( g_{mn} + b_{mn} + 2\pi F_{mn} \right)} - 1 \right] + \ldots \]

\[ F_{mn} = \partial_{[m} A_{n]}, \quad g_{mn} = G_{ij} \partial_m \phi_i \partial_n \phi_j + G_{im} \partial_n \phi_j + G_{mn}, \]

\[ b_{mn} = B_{ij} \partial_m \phi_i \partial_n \phi_j + B_{i[m} \partial_n] \phi_j + B_{mn} \]

(33)

\(^{12}\)As in the case of the Type I superstring theory, the SUSY generators are related to each other on the boundary [7]
which correctly reproduces the low energy string amplitudes for the massless modes. Here again we skipped fermionic terms. One can calculate the mass $M_p$ per unit volume, observing that there is a coupling of the Dp-brane to the metric from the bulk space. This coupling is another evidence in favor of the fact that Dp-branes are dynamical excitations. One finds that 

$$M_p = \frac{\pi}{g_s} (4\pi^3 \alpha')^{\frac{2-p}{2}}$$

which is the first indication that the Dp-branes are $R - \tilde{R}$ solitons. Actually, one can prove [39] that the Dp-branes do carry charges with respect to the $R - \tilde{R}$ tensor gauge fields. In fact, the force between any two equivalent Dp-branes vanishes [21], because they are BPS solitons. This is an indication that there should be a repulsion, due to some tensor fields, compensating the gravitational attraction.

What physics underlies the connection between "flat" Dp-branes and "curved" $R - \tilde{R}$ p-brane solitons? The $R - \tilde{R}$ black p-branes which were considered in the previous section are solutions of the SUGRA equations of motion as $\frac{1}{R^2} \to 0$. Here $R$ is a characteristic size of the solution. At the same time the Dp-branes can be exactly described as the manifolds on which strings can terminate. The description is valid when $g_s \to 0$. In the latter limit the size of the event horizon becomes smaller than the string characteristic scale $\sqrt{\alpha'}$. Thus, the Dp-branes give a microscopic, string theory, description of the $R - \tilde{R}$ solitons. It is believed that when $g_s$ is not zero, the Dp-branes are dressed and, by curving space-time, form an event horizon.

What about the low energy theory for the Dp-branes? At low energies the fields in the action (33) are week and in the flat target-space we have:

$$S_{BI} = \frac{1}{4} F^2_{mn} + |\partial \phi|^2 + ...$$

where dots stand for the fermionic terms. Therefore, the theory for the low energy excitations of a Dp-brane is described by the compactification [41] of the ten-dimensional SUSY QED to $(p + 1)$ dimensions. In fact, in diverse dimensions the scalars $\phi_i$ can be treated as the components $A_i$ of the ten-dimensional vector field $A_\mu$ along compact directions.

The low energy action (34) also can be found from another point of view [41]. In fact, at low energies (or as $\alpha' \to 0$) the string which terminates on the brane looks as the massless vector excitation - the lowest energy excitation of the open string [7]. At this point one easily finds that the low energy theory for such an excitation is SUSY QED – the only supersymmetric and gauge-invariant action containing smallest number of the derivatives.

The last point of view helps to understand the low energy theory describing a bound state of the Dp-branes. Let us consider $n$ parallel Dp-branes with the same $p$. In this case, in addition to the strings which terminate by both their ends on the same Dp-brane, there are strings stretched between different ones. The strings of the first kind give familiar massless vector excitation on each Dp-brane. While the strings of the second kind give massive (mass proportional to a distance between corresponding Dp-branes) vectors. They are charged with respect to the gauge fields on the both Dp-branes. Therefore, these latter vector excitations are similar to the $W^\pm$-bosons. They acquire masses through a kind of the Higgs mechanism (splitting of Dp-branes) and become massless when Dp-branes approach each other. Hence, one finds that the world-volume theory on the bound state of $n$ Dp-branes is nothing but $U(n) = SU(n) \times U(1)$ maximally supersymmetric SUSY Yang-Mills (SYM) theory [41]. While all possible positions $\phi^i$ of the Dp-branes are the Higgs vacuum expectation values.

In this language one also can understand how to construct bound states of the D-branes with different $p$'s [38]. For example, SYM theory on a Dp-brane has BPS excitations which
break half of the SUSY on the world-volume of the Dp-brane. In four dimensions \((p = 3)\)
these are instantons. While when one has \(p \geq 3\) the instantons acquire extra (more than 0)
longitudinal dimensions. They become monopoles at \(p = 4\), strings at \(p = 5\) and etc.. One
interprets the Dp-branes with such BPS excitations as Dp-brane bound states. The latter
break more SUSY than ”mother” brane and have given above relation between their values
of \(p\)'s. One also can use other constructions, with other relations between \(p\)'s, which we do
not need in this review.

Also it is possible to understand, in analogy with the case of strings, what kinds of branes
can terminate on each other [42]. In fact, the strings terminate on the Dp-branes because
their boundaries can be considered as particles living on the Dp-brane. These particles carry
electric charges with respect to the gauge fields which also live on the Dp-branes. Similarly,
if there is a tensor field \(A_{m_1 \ldots m_n}\) living on a Dp-brane then a D\((n+1)\)-brane can terminate
on it. For example, in the Type IIA theory the solitonic NS 5-brane contains the \(B_{mn}\) field
in its world-volume theory. Therefore, a D2-brane can terminate on this 5-brane. In this
case the D2-brane boundary is a string living on the NS 5-brane and charged with respect
to the \(B_{mn}\) field. We use such a construction when consider four-dimensional black holes in
the section five.

5. Microscopic black hole entropy

In this section we consider a concrete extreme black hole solution in a compactification of the
Type II superstring theory. We calculate the semiclassical value of the area of its horizon.
After that we identify what Dp-brane bound state quantizes this black hole. Then we count
the degeneracy of the state and find a complete equivalence between log of this number and
the horizon area.

We need to work with the extreme BPS solutions which have a singularity of any field
defining the solution shifted from the event horizon. If this is not the case, then the \(\alpha'\)
corrections to the black hole area are strong. As it appears [20], to have a non-singular event
horizon, one needs less parameters for five-dimensional black holes than for four-dimensional
ones. Therefore, for simplicity, we start with a five-dimensional extreme black hole. Then we
follow with the calculation of the entropy and decay rate of a five-dimensional non-extreme
black hole. After that we sketch similar calculations for a four-dimensional extreme black
hole.

5.1. The five-dimensional black hole

In the following three subsections we closely follow the presentation in the ref. [20]. We
consider the Type IIB string theory compactified on a torus \(T^5\). The low energy theory is
the maximally supersymmetric SUGRA theory in five dimensions: it has 32 components of
the SUSY generators. The theory contains 27 abelian gauge fields, appearing as in (28) from
the metric and various antisymmetric tensor fields. The full string theory contains charged
objects that couple to each combination of these fields. Due to the Dirac condition these
charges should be quantized in integer multiples of the elementary units.

Thus, we consider the black hole solution which, in the extreme limit, is a bound state
of the \(Q_5\) \(R - \bar{R}\) 5-branes wrapped on the \(T^5\) and of the \(Q_1\) \(R - \bar{R}\) 1-branes wrapped on
a \(S^1\) (we choose it as the direction 9) carrying quantized momentum\(^{13}\) \(P = \frac{N}{R_9}\) along the

\(^{13}\) \(N\) is the charge with respect to the gauge field \(G_{\mu 9}\).
compact direction of the $R - \tilde{R}$ 1-brane. Such a bound state of the $R - \tilde{R}$ p-branes should preserve a half of the SUSY transformations for each kind of the gauge charges. Hence, this BPS solution preserves $\frac{1}{8}$ of all 32 components of the SUSY generators.

We start by presenting the non-extreme ten-dimensional solution [43]:

$$e^{-2(\varphi - \varphi_0)} = \left(1 + \frac{r_0^2 \text{sh}^2(\gamma)}{r^2}\right) \left(1 + \frac{r_0^2 \text{sh}^2(\alpha)}{r^2}\right)^{-1}$$

(35)

where $\varphi_0$ is the vev of the dilaton $\varphi$ and

$$ds^2_{\text{str}} = \left(1 + \frac{r_0^2 \text{sh}^2(\gamma)}{r^2}\right)^{-\frac{1}{2}} \left(1 + \frac{r_0^2 \text{sh}^2(\alpha)}{r^2}\right)^{-\frac{1}{2}} \times$$

$$\times \left[-dt^2 + dx_9^2 + \frac{r_0^2}{r^2} (\text{ch}(\sigma)dt + \text{sh}(\sigma)dx_9)^2 + \right. $$

$$+ \left. \left(1 + \frac{r_0^2 \text{sh}^2(\alpha)}{r^2}\right) (dx_5^2 + \ldots + dx_8^2) \right] + $$

$$+ \left(1 + \frac{r_0^2 \text{sh}^2(\gamma)}{r^2}\right)^{\frac{1}{2}} \left(1 + \frac{r_0^2 \text{sh}^2(\alpha)}{r^2}\right)^{\frac{1}{2}} \left[\left(1 - \frac{r_0^2}{r^2}\right)^{-1} dr^2 + r^2 d\Omega_3^2 \right].$$

(36)

Here the subscript ”str” means that the line element is written in the string frame (see the discussion below the eq. (71) in the Appendix A); $d\Omega_3^2$ represents the angular part of the metric for the polar angles of the vector $\vec{r}$, where $r = \sqrt{x_1^2 + \ldots + x_4^2}$. Also components of the $F_{\mu\nu}$ are non-zero since the solution carries the 1- and 5-brane charges. The last charge is dual to the first one due to the relation (19).

This solution parametrized by four independent quantities: $\alpha, \gamma, \sigma, r_0$. There are also two extra parameters which enter through the charge quantization conditions. These parameters are the radius of the 9-th dimension $R_9$ and the product of the radii in the other four compact directions $V = R_5 R_6 R_7 R_8$. The three charges of the black brane can be most easily viewed when the ten-dimensional solution (36) compactified to six dimensions:

$$Q_1 = \frac{V}{4\pi^2 g_s} \int e^{2\varphi_6} * F_{(3)} = \frac{V r_0^2}{2g_s} \text{sh}(2\alpha)$$

$$Q_5 = \frac{1}{4\pi^2 g_s} \int F_{(3)} = \frac{r_0^2}{2g_s} \text{sh}(2\gamma)$$

$$N = \frac{R_9^2 V r_0^2}{2g_s^2} \text{sh}(2\sigma),$$

(37)

where ”*” is the Hodge duality operation in six dimensions $t, \ldots, x_5$. The integrals are taken over a three-dimensional sub-manifold surrounding the black hole; $\varphi_6$ is the six-dimensional dilaton. It differs from the ten-dimensional one by a linear combination of logs of components of the ten-dimensional metric along compact directions. For simplicity we set from now on $\alpha' = 1$. All the charges are normalized to be integers.

Now reducing (36) to five dimensions, using the standard dimensional reduction procedure [44], the solution takes simple and symmetric form:

$$ds^2_5 = -\Delta^{-\frac{1}{2}} h dt^2 + \Delta^{\frac{1}{2}} \left(h^{-1} dr^2 + r^2 d\Omega_3^2 \right)$$

(38)
where

\[
\Delta = \left(1 + \frac{r_1^2}{r_2^2}\right) \left(1 + \frac{r_5^2}{r_2^2}\right) \left(1 + \frac{r_n^2}{r_2^2}\right)
\]

\[
h = 1 - \frac{r_0^2}{r_2^2}
\]

\[
r_1^2 = r_0^2 \text{sh}^2(\alpha), \quad r_5^2 = r_0^2 \text{sh}^2(\gamma), \quad r_n^2 = r_0^2 \text{sh}^2(\sigma).
\] (39)

This is just the five-dimensional Schwarzschild metric with the time and space components rescaled by different powers of the function \(\Delta\). The event horizon is at the surface \(r = r_0\). Note that the five-dimensional Reissner-Nordstrom solution corresponds to the case of \(\alpha = \gamma = \sigma\).

Why did we choose this solution? The reason is that three charges, in the case under consideration, is the minimal number needed to have a non-singular solution at the event horizon [20]. In fact, as we have seen in the section 4.1, a black p-brane produces the dilaton field of the form \(e^{-2\varphi} = f_p^{	ext{ch}^2}\), with \(f_p\) being a harmonic function. A superposition of the black branes produces the product of such functions and one sees how 1-branes can cancel 5-branes in their effect on the dilaton (35). A similar thing is true for the compactification volume. For any p-brane, the string metric is such that as we get closer to the brane the volume parallel to it shrinks, due to the brane tension. While the volume perpendicular to it expands, due to the pressure of the electric field lines. It is easy to see how superposing the 1- and 5-branes can stabilize the volume in the directions 5,6,7,8, since they are perpendicular to the 1-brane and parallel to the 5-brane. The volume in the direction 9 would still seem to shrink, due to the tension of the branes. This is indeed why we put the momentum along the 1-branes, to balance the tension and produce a stable radius in the 9-th direction.

Also there are \(U\)-duality [9] transformations which exchange (see the Appendix C) the above mentioned 27 gauge fields with each other and invert different couplings (such as \(g_s, R_9\) and \(V\)) in the theory. Therefore, via these \(U\)-duality transformations one can get from our solution another one which is charged with respect to any three of that 27 charges.

One can calculate thermodynamic quantities corresponding to this solution [20]. For example, the mass is equal to

\[
M = \frac{R_9 V r_0^2}{2 g_s^2} \left(\text{ch}(2\alpha) + \text{ch}(2\gamma) + \text{ch}(2\sigma)\right).
\] (40)

While the entropy is:

\[
S = \frac{A_{10}}{4 \Gamma_{10}^{(5)}} = \frac{A_5}{4 \Gamma_5^{(5)}} = \frac{2\pi R_9 V r_0^3}{g_s^2} \text{ch}(\alpha)\text{ch}(\gamma)\text{ch}(\sigma)
\] (41)

where the five-dimensional Newton constant is \(\Gamma_5^{(5)} = \frac{\pi g_s^2}{4V R_9^5}\). And the temperature of this non-extreme black hole is equal to:

\[
T = \frac{1}{2\pi r_0 \text{ch}(\alpha)\text{ch}(\gamma)\text{ch}(\sigma)}
\] (42)

Let us take a look at the formulae (40), (41) and (42). It is possible to trade the six parameters of the general solution for the six quantities \((N_1, N_1, N_5, N_5, N_l, N_r)\) which are the "numbers" of the 1-branes, anti-1-branes, 5-branes, anti-5-branes, left-moving momentum and right-moving momentum respectively. This is accomplished by equating the total
mass, charges and the entropy of the black hole with those of a collection of the numbers 
\( (N_1, N_{\bar{1}}, N_5, N_{\bar{5}}, N_r, N_l) \) non-interacting ”constituent” branes, anti-branes and momentum. By non-interacting we mean that the mass is simply the sum of the masses of the constituents. We take the \( N \)'s to be

\[
N_1 = \frac{V r_0^2}{4 g_s} e^{2\alpha}, \quad N_{\bar{1}} = \frac{V r_0^2}{4 g_s} e^{-2\alpha};
\]

\[
N_5 = \frac{r_0^2}{4 g_s} e^{2\gamma}, \quad N_{\bar{5}} = \frac{r_0^2}{4 g_s} e^{-2\gamma};
\]

\[
N_r = \frac{r_0^2 R^2}{4 g_s^2} e^{2\sigma}, \quad N_l = \frac{r_0^2 R^2}{4 g_s^2} e^{-2\sigma}.
\]

These \( N \)'s reduce to the numbers of branes, anti-branes and momentum in certain limits where those concepts are well defined [20]. In terms of them the charges are simply

\[
Q_1 = N_1 - N_{\bar{1}}, Q_5 = N_5 - N_{\bar{5}}, N = N_r - N_l,
\]

the total energy is:

\[
M = \frac{R_9}{g_s} (N_1 + N_{\bar{1}}) + \frac{R_9 V}{g_s} (N_5 + N_{\bar{5}}) + \frac{1}{R_9} (N_r + N_l),
\]

and the volume and radius are

\[
V = \left( \frac{N_1 N_{\bar{1}}}{N_5 N_{\bar{5}}} \right)^{\frac{1}{2}}, \quad R_9 = \left( \frac{g_s^2 N_r N_l}{N_1 N_{\bar{1}}} \right)^{\frac{1}{4}}.
\]

Of course there seems to be no reason for neglecting interactions between collections of branes and momentum modes composing a highly non-extreme black hole at strong or intermediate coupling. Hence, the conditions (43) would seem to be inappropriate for describing a generic black hole. However, the utility of these definitions can be seen when we reexpress the black hole entropy (41) in terms of the \( N \)'s. It takes the remarkably simple form

\[
S = 2\pi \left( \sqrt{N_1 + N_{\bar{1}}} \right) \left( \sqrt{N_5 + N_{\bar{5}}} \right) \left( \sqrt{N_r + N_l} \right)
\]

and appears, being U-duality invariant, to be useful and transparent for the understanding of physics below.

The extreme limit corresponds to taking \( r_0 \to 0 \) and \( \alpha, \gamma, \sigma \to \infty \), keeping the charges (37) finite. Thus, in terms of (43), we include either branes or anti-branes rather than both of them. The extreme solution preserves only \( \frac{1}{8} \) of the SUSY transformations and corresponds to the short supermultiplet. In the extreme limit the entropy becomes:

\[
S = 2\pi \sqrt{N Q_1 Q_5}
\]

and the temperature vanishes.

Note that the entropy of the BPS solution is independent, as we discussed in the section three, of any continuous parameters. Therefore, it is independent of \( \varphi_0 \) and one says that the curved black hole description is valid at \( g_s = e^{\varphi_0} \sim 1 \) (see the Appendix A), otherwise (if \( g_s \to 0 \) or \( g_s \to \infty \)) the string perturbation or flat Dp-brane description would be applicable\(^\dagger\). At the same time in the black hole region all sizes should be big in comparison

\(^\dagger\)If \( g_s \to \infty \), then perturbation theory of the dual string would be applicable, because as we sketched in the Appendix C all the known superstring theories are related to each other through a strong-week \( (g_s \to \frac{1}{g_s}) \) or other kinds of duality.
with the string scale to make the $\alpha'$ corrections small! Thus, in the above calculation, using the low energy theory of the Type IIB string, we adopted the following approximation ($\alpha' = 1$):

$$g_s Q_1 \gg 1, \quad g_s Q_5 \gg 1, \quad g_s^2 N \gg 1.$$  \hspace{1cm} (48)

Here $g_s Q_1$, $g_s Q_5$, and $g_s^2 N$, as one can see from the metric (36), set characteristic size of the black hole, i.e. the area of event horizon and its radii.

It is worth mentioning that the formula (47) for the entropy is U-duality invariant: after any U-duality transformation the value of the charges will remain the same, but we would have a bound state of different kinds of branes. This U-duality invariance and independence upon $\varphi_0$ is the indication of the fact that one is able to find a complete agreement between calculations of the entropy at different values of the couplings in the theory.

5.2. D-branes and the extreme black hole

We continue with the Type IIB string theory on the torus $T^5 = T^4 \times S^1$. We consider a configuration of the $Q_5$ D5-branes wrapping the whole $T^5$, $Q_1$ D-strings wrapping the $S^1$ and momentum $\frac{N}{R_9}$ along 9-th direction. All charges are integers. This configuration of the Dp-branes corresponds (by comparison of the charges and of the mass) to the above described bound state of the $R - \tilde{R}$ solitons.

Now we use open string perturbation theory, i.e. we consider another vacuum background for the Type IIB perturbation theory. Then one should take the effective coupling constants\(^{15}\) $g_s Q_1$, $g_s Q_5$ and $g_s^2 N$ to be much smaller than one! This should be contrasted with the approximations (48) adopted in the previous subsection.

The total mass of the system is

$$M = \frac{Q_5 R_9 V}{g_s} + \frac{Q_1 R_9}{g_s} + \frac{N}{R_9}$$  \hspace{1cm} (49)

and saturates the corresponding BPS bound. We will calculate the degeneracy of this state. Such a calculation was first done in the ref. [45].

As the Dp-branes are invariant under the Lorentz transformations along the directions parallel to their volume, they can not carry the momentum $\frac{N}{R_9}$ just moving rigidly. Our task would be to identify the excitations which carry this momentum. The BPS mass formula for the whole system implies that these excitations have to be massless and moving along the $S^1$. In fact, the excitation energy, defined as the total mass of the system minus the mass of the 1-branes and 5-branes, is equal to the momentum. If any excitation fails to be massless it would contribute more to the energy than to the momentum and the BPS mass formula would be violated.

Excitations of the branes are described by massless open strings. There are many types of the open strings to consider: those that go from one 1-brane to another 1-brane, which we denote as $(1, 1)$ strings, as well as the corresponding $(5, 5)$, $(1, 5)$ and $(5, 1)$ strings (the last two being different because the strings are oriented). We want to excite these strings and make them carry the momentum in the direction of the $S^1$. However, exciting some of them makes others massive [20]. Therefore, we have to find how to excite the strings so that maximum number of them remains massless. This configuration will have the highest entropy.

\(^{15}\) $Q_1$, $Q_5$ and $N$ appear in the effective couplings of the open string perturbation theory as the multiplicities of the ends of the strings.
We have already said that the low energy Lagrangians for the \((1,1)\) and \((5,5)\) strings are dimensional reductions of the \(d=10, N=1\) SYM to two or six dimensions for the D1- or D5-branes, respectively. However, because of the additional braking of the SUSY due to the composition of different types of the Dp-branes the situation is changed. One has the \(N=4\) rather than \(N=8\) SYM theory on the world-volume of the D1-brane and \(N=1\) rather than \(N=2\) SYM on the D5-brane.

The \((1,1)\) strings represent vector multiplets containing \(A_{m}^{(1)}\) gauge field with four scalars \(\phi_{i}^{(1)}\) and hypermultiplets in the adjoint representation containing four scalars \(\varphi_{j}^{(1)}\). These scalars represent positions of the D1-branes in the bulk. At the same time the \((5,5)\) strings form the vector multiplets containing only \(A_{n}^{(5)}\) gauge field and the hypermultiplets in the adjoint representation containing four scalars \(\varphi_{j}^{(5)}\). The latter represent positions of the D5-branes in the bulk.

One also can show (see, for example, [20]) that the \((1,5)\) and \((5,1)\) strings form together the hypermultiplets. They transform as the products of the fundamental representation of the \(U(Q_{1})\) and the anti-fundamental of the \(U(Q_{5})\) and their complex conjugate. These hypermultiplets contain four scalars \(\chi_{i}\) which represent relative positions of the D1- and D5-branes.

Thus, knowing the multiplet content, one can easily derive low energy actions on the D1- and D5-branes with the mentioned numbers of SUSY. The interaction Lagrangian for the mentioned fields is fixed largely by the SUSY. The only allowed coupling between the vector and hypermultiplets is the gauge one. The SUSY requires, however, some potential for the scalars. It arises, in our case, from the three \(D\)-fields [32] (do not mix with D-branes) for each gauge generator \(t^{a}\).

Let us count the number of massless bosonic excitations, the number of fermionic excitations being equal to the bosonic ones due to the SUSY. The BPS states under consideration have only left moving excitations. Classically one can view these states as traveling waves propagating along the \(S^{1}\). In order to have the traveling wave solutions mass terms have to vanish exactly. If we set all fields in the Lagrangian to zero then we can have the traveling waves for any field. However, if we have a wave for one field the mentioned potential terms generate effective mass terms for other waves.

If we give some expectation value to the scalars from the \((1,5)\) and \((5,1)\) strings, then we are effectively separating the D1- and D5-branes and we expect a small number of the massless particles (proportional to \(Q_{1}+Q_{5}\)). On the contrary a configuration with the large number of the massless particles is achieved by exciting all hypermultiplets, from the \((1,1), (5,5)\) and \((1,5), (5,1)\) strings. This gives masses to the scalars from the vector multiplets describing the transverse motion of the branes. The total number of components of the hypermultiplets is \(4Q_{1}^{2}+4Q_{5}^{2}+4Q_{1}Q_{5}\). Conditions of vanishing of the scalar potential impose \(3Q_{1}^{2}+3Q_{5}^{2}\) constraints. In addition we should identify gauge equivalent configurations. The number of possible gauge transformations is \(Q_{1}^{2}+Q_{5}^{2}\). This implies that the remaining number of bosonic massless degrees of freedom is \(4Q_{1}Q_{5}\). The counting, as we have done, is correct for the large charges up to possible subleading corrections.

In order to calculate the entropy we notice that we have the gas of the left moving particles with \(N_{F,B}=4Q_{1}Q_{5}\) bosonic and fermionic species on the compact one-dimensional space of the length \(L=2\pi R_{9}\). Its energy is \(E=\frac{N}{R_{9}}\). In the string theory the degeneracy of such a state (as we sketched in the Introduction) is [46]: \(n \sim \exp\left(\sqrt{\pi(2N_{B}+N_{F})EL/6}\right)\) where under the square root in the exponent there is the number of oscillator states. Then the entropy of the state is equal to:
\[ S = \log n = 2\pi \sqrt{Q_1 Q_5 N} \] 

which is in the perfect agreement, including numerical coefficient, with (47).

### 5.3. D-branes and the non-extreme black hole

In this subsection we discuss the non-extreme black hole in the D-brane picture [43, 24, 47]. We are working in the dilute gas regime

\[ r_0, r_n << r_1, r_5, \]  

where the \( r \)'s are defined in the eq. (39). Under these conditions one is very close to a configuration of the extreme D1- and D5-branes and SUSY non-renormalization arguments do indeed help us [48] (see also the second paper in [26]) and explain the agreement that we are going to find. This is the simplest case of the near extreme black hole, if we want to consider more general ones, one has to include other excitations besides the right movers.

The mass of the state is equal to:

\[
M = \frac{\pi}{4 \Gamma_N^{(5)}} \left[ r_1^2 + r_5^2 + \frac{r_0^2 \cosh(2\sigma)}{2} \right] = \\
= \frac{1}{g_s^2} \left[ R_9 g_s Q_1 + R_9 V g_s Q_5 + \frac{g_s^2 N}{R_9} + \frac{VR_9 r_0^2 e^{-2\sigma}}{2} \right]
\]

Its excitations are approximately described by transverse oscillations generated by open strings attached to the D-branes. These oscillations carry the momentum \( N \) and are described by the gas of the both left and right movers on the strings. Equating the energy of this gas to \( \frac{N}{R_9} + \frac{VR_9 r_0^2 e^{-2\sigma}}{2g_s^2} \) and its momentum to \( \frac{N}{R_9} \) we can determine the total energy carried by the right and left movers.

The entropy calculation proceeds as in the extreme case and one finds a perfect agreement with the black hole answer (44) and (46) in the approximation (51):

\[
S = \frac{2\pi^2 r_1 r_5 r_0 \cosh(\sigma)}{4 \Gamma_N^{(5)}} = 2\pi \left( \sqrt{Q_1 Q_5 N_1} + \sqrt{Q_1 Q_5 N_r} \right).
\]

Moreover, one finds also an agreement for the decay rate of this non-BPS state computed in the different (black hole and string) pictures [49, 25]. The basic process in the D-brane picture is when a right moving string with some quantized momentum \( P_9 \) collides with a left moving one of the opposite momentum. They give a closed string state of the energy equal twice the momentum which can escape to infinity. If the moments are not exactly opposite the outgoing string carries some momentum in the 9-th direction which means that it is charged and very massive particle from the five-dimensional point of view. This last emission is suppressed at low temperature, therefore, we will not consider it.

Let us sketch the derivation of the decay rate of the D-branes [25]. From the action (33) one can find the vertex

\[ A_D \sim G_{67} \partial x^6 \partial x^7 \] 

(54)
for the interaction of the string with the component $G_{67}$ of the graviton - the state of the massless escaping closed string. This component of the metric tensor looks as a scalar from the five-dimensional point of view because the 6 and 7 are compact directions.

The decay rate is given by

$$d\Gamma(p, q, k) = \frac{(2\pi)^2}{8\pi R_9} \delta(p + q - k) \frac{|A_D|^2}{p_0 q_0 k_0 V_4} \frac{V_4}{(2\pi)^4}$$

(55)

where $p, q$ and $k$ are the wave vectors of the colliding strings and of the scalar respectively; $p_0, q_0, k_0$ are their energy components and $V_4$ denotes the volume of the spatial noncompact four dimensions. After the averaging over the canonical ensembles of the left and right gases of the strings attached to the D-branes one finds that the decay rate is equal to [25]:

$$d\Gamma(k) = \frac{\pi^3 g_s^2}{V} Q_1 Q_5 \omega \frac{1}{e^{\frac{k_0}{2T_l}} - 1} \frac{1}{e^{\frac{k_0}{2T_r}} - 1} (2\pi)^4$$

(56)

where

$$\frac{1}{T} = \frac{1}{2} \left( \frac{1}{T_l} + \frac{1}{T_r} \right),$$

$$T_l = \frac{1}{\pi} \frac{r_0 e^\sigma}{2r_1 r_5}, \quad T_r = \frac{1}{\pi} \frac{r_0 e^{-\sigma}}{2r_1 r_5}$$

(57)

are the "effective" temperatures in the almost non-interacting left and right sectors respectively [25]. They, being equal to $T_l^{-1} = T^{-1}(1 \pm \mu)$, are some natural combinations of the temperature and the chemical potential $\mu$, which gives the gas some net momentum.

Obtained answer for the decay rate should be compared with that from the eq. (6) for the five-dimensional black hole. For this reason one should calculate the graybody factor $\sigma_{gb}(k_0)$. This calculation proceeds as follows [24]. One considers the five-dimensional Klein-Gordon equation with the Laplacean defined through the metric in (38). One looks for the absorption cross section of the barrier created by this curved background. This is just what we call $\sigma_{gb}(k_0)$. The answer is the following

$$\sigma_{gb} = \pi^3 r_1^2 r_5^2 \omega \frac{e^{\frac{\kappa_0}{r_5}} - 1}{(e^{\frac{k_0}{2T_l}} - 1) (e^{\frac{k_0}{2T_r}} - 1)}$$

(58)

which gives the perfect agreement between the variant of the formula (6) in five dimensions and (56).

### 5.4. The four-dimensional black hole

In a similar way as we did for the five-dimensional black hole, one can construct the four-dimensional one. Now we are going to work in the Type IIA rather than Type IIB theory. For in this case it is easier to construct wanted four-dimensional black hole.

The solution, which we are interested in, consists of a configuration of the $R - \tilde{R}$ 2-, 6-branes and momentum that we had in the $d = 5$ case and putting all this on the $T^6$ [50]. In the extreme limit we obtain the black hole solution which preserves $\frac{1}{4}$ of the SUSY. However, this black hole has a singular geometry at the horizon. The reason is that some of the scalar...
fields are unbalanced, for example, we can see from (22) that the dilaton field will not go to a constant as we approach the horizon, \( e^{-2\phi} = f_2^{-\frac{1}{2}} f_6^{\frac{3}{2}} \).

It is interesting that one can put an additional type of charge without breaking any additional SUSY. This charge has to be the solitonic NS 5-brane, it is the only one allowed by the SUSY [50]. It also has the virtue of balancing all scalars, for example, the dilaton now behaves as \( e^{-2\phi} = f_2^{-\frac{1}{2}} f_6^{\frac{3}{2}} f_8^{-1} \). In order to be more precise let us say that our torus is \( T^6 = T^4 \times S'_1 \times S_1 \) and we have the D6-brane wrapping over all \( T^6 \), D2-brane wrapping over the \( S'_1 \times S_1 \) (directions 4,9), solitonic 5-branes wrapping over the \( T^4 \times S_1 \) (directions 5,6,7,8,9) and momentum flowing along the \( S_1 \) (direction 9).

Let us start with presenting the solution for the non-extreme black hole, i.e. the "bound state" of the both \( R - \tilde{R} \) branes and anti-branes. After doing the dimensional reduction to four dimensions, the Einstein frame metric (see the Appendix A) reads:

\[
\begin{align*}
    ds^2 &= -f^{-\frac{1}{2}}(r) \left( 1 - \frac{r_0}{r} \right) dt^2 + f^{\frac{1}{2}}(r) \left[ \left( 1 - \frac{r_0}{r} \right)^{-1} dr^2 + r^2 \left( d\theta^2 + \sin^2(\theta) d\phi^2 \right) \right] \\
    f(r) &= \left( 1 + \frac{r_0 s h^2(\alpha_2)}{r} \right) \left( 1 + \frac{r_0 s h^2(\alpha_5)}{r} \right) \left( 1 + \frac{r_0 s h^2(\alpha_6)}{r} \right) \left( 1 + \frac{r_0 s h^2(\alpha_p)}{r} \right)
\end{align*}
\]  

This metric is parametrized by five independent quantities \( \alpha_2, \alpha_5, \alpha_6, \alpha_p \) and \( r_0 \). The event horizon lies at \( r = r_0 \). The special case \( \alpha_2 = \alpha_5 = \alpha_6 = \alpha_p \) corresponds to the Reissner-Nordstrom metric. Hence, we see that General Relativity solution is among the cases studied.

The overall solution contains three additional parameters which are related to the asymptotic values of the three scalars. From the ten-dimensional point of view, these are the product of the radii of \( T^4 \), \( V = R_5 R_6 R_7 R_8 \), and the radii of the \( S^1 \) and \( S'^1 \), \( R_9 \) and \( R_4 \), and they appear in the quantization condition for the charges.

There are, in addition, \( U(1) \) gauge fields excited, corresponding to the four physical charges. One is the gauge field coming from the component \( G_{\mu9} \) of the metric, which is responsible for the "momentum charge" \( N \). Then we have the \( R - \tilde{R} \) gauge field coming from the component \( A_{\mu9} \) of the three form potential which is responsible for the D2-brane charge, \( Q_2 \). The D6-brane charge, \( Q_6 \), appears as the magnetic charge for the one form \( R - \tilde{R} \) potential \( A_\mu \), and finally the solitonic 5-brane charge, \( Q_5 \), also appears as the magnetic charge for the gauge field coming from the NS antisymmetric tensor with one index along the direction 4, \( B_{\mu4} \). The physical charges are expressed in terms of these quantities as:

\[
\begin{align*}
    Q_2 &= \frac{r_0 V}{g_s} sh(2\alpha_2) \\
    Q_5 &= \frac{r_0 R_4}{g_s} sh(2\alpha_5) \\
    Q_6 &= \frac{r_0}{g_s} sh(2\alpha_6) \\
    N &= \frac{r_0 V R_5^2 R_4}{g_s^2} sh(2\alpha_p)
\end{align*}
\]  

where again we have set \( \alpha' = 1 \) and the four-dimensional Newton constant becomes \( \Gamma_N^{(4)} = \frac{g_s^2}{8V R_5 R_6} \).

The mass of the solution is equal to

26
\[
M = \frac{r_0 V R_4 R_9}{g_s^2} (ch(2\alpha_2) + ch(2\alpha_5) + ch(2\alpha_6) + ch(2\alpha_p))
\] (61)

while the entropy is
\[
S = A_4 \frac{8\pi r_0^3 V R_4 R_9}{4\Gamma_N^{(4)}} ch(\alpha_2) ch(\alpha_5) ch(\alpha_6) ch(\alpha_p)
\] (62)

The extreme limit corresponds to the \(r_0 \to 0, \alpha_i \to \pm \infty\) with \(Q_i\) fixed. In this limit the entropy becomes equal to
\[
S = 2\pi \sqrt{Q_2 Q_5 Q_6 N},
\] (63)

which is, as (47), is U-dual and independent of the moduli.

Now we will use the D-brane methods to recover the entropy (63) (see \[50\]). We have already seen how one can construct the BPS state from the D-branes. The calculation of the degeneracy in this case is almost the same as for the five-dimensional black hole. However, there are some new ingredients due to the solitonic 5-brane. The 5-branes intersect two kinds of the D-branes along the \(S_1\). Different 5-branes will be at different positions along the \(S_1'\). The D-branes can break and the ends separate in \(T^4\) when it crosses the 5-brane \[42\] (see also the end of the section 4.3). Hence, the 5\(Q_2\) toroidal D2-branes break up into the \(Q_2Q_5\) cylindrical D2-branes, each of which is bounded by a pair of 5-branes. The momentum-carrying open strings now carry an extra label describing which pair of the 5-branes they lie in between. The number of species becomes \(N_{B,F} = 4Q_2Q_5Q_6\). The number of BPS saturated states of this system as a function of \(Q_2, Q_5, Q_6\) and \(N\) follows from the same reasoning as in the five-dimensional case:
\[
S = 2\pi \sqrt{\frac{(2N_B + N_F)ER_9}{12}} = 2\pi \sqrt{Q_2Q_5Q_6 N},
\] (64)

which indeed reproduces the classical result.

6. Conclusions

Thus, we see that superstring theory is able to give an explanation of black hole thermodynamics. It is true at least for some kind of solutions which are close to the extremity and regular on their horizons. For arbitrary solutions, calculations presented here are not applicable. However, in the situation when we can apply above mentioned methods, one finds a perfect explanation of black hole thermodynamics by the means of superstring theory. The main tool (and the only restriction) which underlies this explanation is the presence of the \(N \geq 2\) SUSY\textsuperscript{16}.

Much more remains to be understood. For example, a universal statistical explanation of the black hole entropy remains elusive (see, however, \[53\] for the most recent attempts). Also main problem which is left is to understand the black hole dynamics with the SUSY breaking.

At the end it is worth mentioning that we do not know what is the string counterpart of the baby universes and warm holes. Therefore, we do not know how to apply string theory

\textsuperscript{16}Even if we worked above with the four and five-dimensional \(N = 8\) SUSY, everything can be straightforwardly generalized to the five-dimensional \(N = 4\) case \[45\], four-dimensional \(N = 4\) case \[51\] and to the four-dimensional \(N = 2\) case \[52\].
methods to this situation. See, however, in the ref. [54] the discussion of the so called conifold transitions in superstring theory. They are topology change processes or, if you will, they are string theory analogues of the gravitational warm holes. See also [55] on the relation between black hole singularities and the topology change transitions. However, such a process describes a topology change in the internal (compact) space rather than external (our) space-time.

For the most recent developments within M-atrix theory [10] in the Schwarzschild black hole physics see [56].

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Appendix A. (Elements of String Theory)

In this appendix we give some basic definitions for objects appearing in superstring theory, which are needed for our discussion in the text. For a review of superstring theory see [7].

We start with the bosonic string. The generation function of correlators in string theory is defined as follows:

$$Z = \sum_{g=0}^{\infty} Z_g = \sum_{g=0}^{\infty} \int [\mathcal{D}h_{ab}]_g \mathcal{D}x_\mu e^{-iS(x_\mu,h_{ab})}.$$  \hspace{1cm} (65)

Here the sum is over the genus $g$ of the string world-sheet – two-dimensional surface spaned inside a target space by the string during its time evolution. This sum is an expansion over the string loop corrections. If one considers closed strings then these are spheres with $g$ handles, otherwise they are discs with holes and handles of the total number $g$. $x_\mu(\sigma^1, \sigma^2), \mu = 0, ..., d - 1$ is an embedding coordinate of the string into a $d$-dimensional target space; $h_{ab}, a, b = 0, 1$ is the world-sheet metric. Also the measure $[\mathcal{D}h_{ab}]_g$ should be properly defined. Below we sketch how to do this for the case of $g = 0$.

We proceed with the closed bosonic string. In this case the action in (65) looks as follows:

$$S = \frac{1}{2\pi\alpha'} \int d^2\sigma \left\{ \sqrt{h}h^{ab}G_{\mu\nu}(x)\partial_\mu x^a\partial_\nu x^b + \epsilon^{ab}B_{\mu\nu}(x)\partial_\mu x^a\partial_\nu x^b + \alpha'\sqrt{h}R^{(2)}\varphi(x) \right\},$$ \hspace{1cm} (66)

where $\sigma^a, a = 1, 2$ are the coordinates on the string world-sheet; $\alpha'$ is the string scale or inverse string tension. It represents an expansion parameter of the nonlinear $\sigma$-model (66) perturbation theory. From the world-sheet point of view $x_\mu$ are scalars enumerated by $\mu$. At the same time they compose the vector from the target space point of view.

From the world-sheet point of view the $G_{\mu\nu}, B_{\mu\nu}$ and $\varphi$ (the graviton, antisymmetric tensor and dilaton respectively) represent coupling constants. While from the target space point of view the theory (66) represents the string in the background field of a "gas". The latter is composed of the massless excitations of the string (external sources in (65), (66)). In fact, as we will show at the end of the section, the string theory has the graviton, antisymmetric tensor and dilaton as the massless excitations in its spectrum. One can write
down the vertex operators for these excitations. Then it is possible to consider the string propagating in the flat target space (i.e. to use (65) with the action from the eq. (66) where \(G_{\mu\nu} = \eta_{\mu\nu}, B_{\mu\nu} = 0\) and \(\varphi = 0\) and interacting with these excitations. Summing over all tree level interactions with any number of the external legs, corresponding to these fields, one gets exponent of the interactions. This is the action from the eq. (66). To get correlation functions of these massless string excitations, one should differentiate the eq. (65) over them.

The vev \(\varphi_0\) of the dilaton gives the coupling constant for the mentioned genus expansion:

\[
Z_g \sim \exp \left\{ \frac{1}{2\pi} \int d^2\sigma \sqrt{h} R^{(2)} \varphi_0 \right\} = e^{2(\varphi - 1)\varphi_0} = g_s^{2(\varphi - 1)},
\]  

(67)

where \(R^{(2)}\) is the two-dimensional scalar curvature. Therefore, string theory has two expansion parameters the \(g_s = e^{\varphi_0}\) and \(\alpha'\).

Two-dimensional reparametrization invariance (general covariance on the world-sheet) of the action (66) leads to the conservation of the world-sheet energy-momentum tensor

\[
\partial_a T_{ab} = 0.
\]

(68)

After the gauge fixing of the reparametrization invariance one can use new flat complex two-dimensional coordinate \(z = e^{\sigma_1 + i\sigma_2}\) and its complex conjugate.

One also should insist on the dilatational invariance

\[
T_{aa} = 0
\]

(69)

required for the functional \(Z\) to be well defined. Both conditions (68) and (69) demand that the generator of the conformal transformations\(^{17}\) \(T(z, \bar{z}) = T_{11} - T_{22} + 2iT_{12}\) should be a holomorphic function of \(z\), i.e. \(\frac{\partial}{\partial\bar{z}} T(z) = 0\). Similarly there should be an anti-holomorphic generator \(\frac{\partial}{\partial z} \bar{T}(\bar{z}) = 0\).

The dilatational invariance of the \(\sigma\)-model, at the one loop level on the world-sheet, leads to the vanishing of the \(\beta\)-functions for the \(G\), \(B\) and \(\varphi\) couplings\(^{18}\):

\[
\beta^G_{\mu\nu} = R_{\mu\nu} - 2\nabla_\mu \nabla_\nu \varphi + \frac{1}{4} H_{\alpha\beta\mu} H_{\nu}^{\alpha\beta} + O(\alpha') = 0, \quad H_{\mu\nu\lambda} = \partial_{[\mu} B_{\nu\lambda]} \\
\beta^B_{\mu\nu} = \nabla^{\lambda} \left( e^{2\varphi} H_{\mu\nu\lambda} \right) + O(\alpha') = 0 \\
\beta^\varphi = \frac{d - 26}{48\pi} + \alpha' \left( R + \frac{1}{12} H^2_{\mu\nu\lambda} + (\nabla_\nu \varphi)^2 + 2\nabla_\mu \nabla_\nu \varphi \right) + O(\alpha'^2) = 0,
\]

(70)

where \(\nabla_\mu\) is the covariant derivative in the metric \(G_{\mu\nu}\) and \(O(\alpha')\) encodes the non-linear \(\sigma\)-model loop corrections.

Obtained equations for the string massless modes describe their low energy dynamics. It is possible to consider these equations as the Euler-Lagrange ones derived from the \(d = 26\)-dimensional action:

\[
S_{eff} \sim \int d^{26}x \sqrt{-G} e^{-2\varphi} \left[ R + 4(\nabla_\mu \varphi)^2 - \frac{1}{3} H^2_{\mu\nu\lambda} \right] + O(\alpha'),
\]

(71)

This action is written in the so called string units or string frame or by the use of the string metric. From the string metric one can pass to the ordinary Einstein metric (frame) trough the rescaling: \(G_E = G e^{4\varphi}\). Hence, one can see that the 26-dimensional Einstein-Hilbert

---

\(^{17}\)Which are holomorphic reparametrizations, acting on the \(z\).

\(^{18}\)Briefly, the reason for this is that a non-zero \(\beta\)-function leads (as in QCD) to the appearance of a mass scale in the theory which manifestly breaks the dilatational invariance.
action appears as a part of the low energy action for the massless string modes. It is also possible to get four dimensional gravity from string theory via compactification [7]. In this respect one says that string theory is quantum theory of gravity.

Let us discuss now the string spectrum. For this reason we can use the canonical quantization of the theory (66) (defined on the cylindrical world-sheet) in the flat target space. General solution of the string classical equation of motion, derived from the action (66) (defined on the cylindrical world-sheet) in the flat target space.

\[ x^\mu(z) = X^\mu_{\ell}(z) + P^\mu_{\ell} \log z + i \sqrt{\alpha^'} \sum_{m \neq 0} \frac{1}{2m} a^\mu_m z^{-m}; \quad \text{and} \quad x^\mu(z, \bar{z}) = x^\mu_{\ell}(z) + x^\mu_{\bar{r}}(\bar{z}), \quad (72) \]

where \( X^\mu_{\ell} = X^\mu_{\ell} \) and \( P^\mu_{\ell} = P^\mu_{\ell} \) represent the center of mass coordinate and the momentum of the string, respectively; \( a_{-m} = a^+_m \). The same expansion through the \( \tilde{a}^\mu_m \) is true for the case of the \( x_{\bar{r}}(\bar{z}) \).

To quantize the string, one uses the canonical commutation relations for the left and right string modes \( a^\mu_m \) and \( \tilde{a}^\mu_m \) respectively:

\[
[a^\mu_m, a^\nu_n] = [\tilde{a}^\mu_m, \tilde{a}^\nu_n] = im \delta_{m+n} \eta^{\mu\nu}; \quad [a^\mu_m, \tilde{a}^\nu_n] = 0; \quad [P^\mu, X^\nu] = i\eta^{\mu\nu}
\]

where \( P^\mu = P^\mu_{\ell} + P^\mu_{\bar{r}} \) and \( X^\nu = X^\nu_{\ell} + X^\nu_{\bar{r}} \) (73)

with the Minkovskian metric \( \eta^{\mu\nu} \). After the standard definition of the vacuum \(|0, 0\rangle\) (vacuum in the both left and right sectors) one defines the states as follows: \( a_{-n_1}^{\mu_1} \ldots \tilde{a}_{-m_1}^{\nu_1} \ldots |0, 0\rangle \), where \( \sum n_i = N_\ell \) and \( \sum m_i = N_{\bar{r}} \).

The Fourier components of the generator of the conformal invariance \( L_{-n} = \frac{1}{2\pi i} \int \frac{dz}{z^{n+1}} T(z) \)

can be expressed through the harmonics \( a^\mu_m \) as:

\[
L_0 = \frac{\alpha^'}{2} P^2 + \sum_m : a^\mu_{-m} \cdot a^\mu_m : \quad L_n = \frac{1}{2} \sum_m : a^\mu_{n+m} \cdot a^\mu_{-m} : \quad (74)
\]

Thus, \( L_0 \) is the string Hamiltonian and it looks like a composition of the oscillator Hamiltonians. The \( L_n \) generate, after the inclusion of the terms due to the reparametrization ghosts, the following Virasoro algebra:

\[
[L_n, L_m] = \left(n - m\right) L_{n+m} + \frac{d - 26}{12} (m^3 - m) \delta_{n+m,0} \quad (75)
\]

where the last term is the so called conformal anomaly. It appears from the normal ordering of the \( a_m \) and \( a^\mu_m = a_{-m} \) and of the corresponding ghost modes. The number \( c = d - 26 \) is referred to as central charge of the conformal field theory. It counts the number of degrees of freedom on the string world-sheet. In our case, \( d \) comes from the \( x_\mu \) and 26 is due to the ghosts. All that is also true for the case of the components \( \tilde{L}_n \) of the \( \tilde{T}(\bar{z}) \) expressed through the \( \tilde{a}^\mu_m \) modes.

As the result of the fixing of the reparametrization invariance one has the Virasoro constraints [7] (like the Gauss law in gauge theories and the Wheller-DeWitt equation in gravity theory) on physical states:

---

\[ \text{The equation takes this form only after the fixing of the conformal gauge } h_{zz} \sim \delta_{zz} \text{ by the use of the reparametrization invariance.} \]
\[ L_n |\psi> = 0, \quad \tilde{L}_n |\psi> = 0, \quad n \geq 0 \]
\[ (L_0 - \tilde{L}_0) |\psi> = 0; \quad (L_0 + \tilde{L}_0 - 2) |\psi> = 0. \]  
(76)

The last line of this formula leads to

\[ N_l = N_r, \quad \frac{\alpha'}{2} P^2 + N_l + N_r - 2 = 0. \]  
(77)

Hence, one has massless mode when \( N_l = N_r = 1 \). This is the tensor excitation \( a^\mu_0 \tilde{a}^\nu_0 |0,0> \) of the string. The symmetric, antisymmetric parts and the trace of this excitation correspond to the mentioned graviton, antisymmetric tensor and dilaton fields respectively.

The bosonic string theory also has a tachyon: \( N_l = N_r = 0 \), \( M^2 = -P^2 = -\frac{4}{\alpha'} \). This is the pathological excitation because its presence means that we have chosen a wrong (unstable) vacuum with imaginary energy.

**Appendix B. (Construction of the Type II Superstring Theories)**

To get a sensible string theory one should add SUSY on the two-dimensional world-sheet [7]. The SUSY is added by the aid of the anticommuting \( \psi_\mu \) fields which are world-sheet superpartners of the \( x_\mu \) and by the aid of the world-sheet metric superpartner. From the world-sheet point of view the fields \( \psi_\mu \) are fermions enumerated by \( \mu \). At the same time they compose a vector from the target space point of view.

In this appendix we discuss how one constructs the Type II superstring theories which we use in the main body of the text. One considers the \( N = 1 \) two-dimensional SUGRA. Due to the presence of the conformal symmetry this SUGRA symmetry is enhanced to the superconformal one. As we discuss below one needs to do some extra work to get SUGRA in the target space.

In the flat target space the superstring with a cylindrical world-sheet is described by the action:

\[ S = \frac{1}{4 \pi \alpha'} \int d^2 z \left( \partial_\sigma x^\mu \partial_\bar{\sigma} x_\mu + \bar{\psi}_\mu \partial_\mu \psi_\mu \right) \]  
(78)

where we have fixed the world-sheet metric to be \( h_{\bar{z}z} = \delta_{\bar{z}z} \) and get rid of its superpartner by the use of the reparametrization and superconformal invariances. We did not include the reparametrization ghost terms into this action.

There are two types of the boundary conditions on the left and right fermions\(^{20}\) on the closed string world-sheet: the Ramond (R) type;

\[ \psi_\mu (\sigma^2 + 2\pi) = \psi_\mu (\sigma^2) \]  
(79)

and the Neveu-Schwarz (NS) type;

\[ \psi_\mu (\sigma^2 + 2\pi) = -\psi_\mu (\sigma^2). \]  
(80)

\(^{20}\)Appearance of the both types of boundary conditions is demanded by the modular invariance - remnant of the conformal invariance on the world-sheets with higher topologies. Such an invariance exchanges with each other the sectors in string theory obeying these boundary conditions.
and the same for the $\tilde{\psi}_\mu$. Therefore, in addition to the (72), one also has two kinds of the mode expansion for the solutions of the free two-dimensional Dirac equation on the cylinder:

$$
\psi^\mu(z) = \psi^\mu_0 + \sum_n \frac{b^\mu_n}{z^n} \quad (R),
$$

$$
\psi^\mu(z) = \sum_n \frac{c^\mu_n + \frac{1}{2}}{z^{n+\frac{1}{2}}} \quad (NS)
$$

and similarly for the $\tilde{\psi}$ field.

Quantizing the superstring theory (78), one imposes the standard anti-commutation relations on these modes. For example, the zero modes $\psi^\mu_0$ generate the Clifford algebra (algebra of the Dirac $\gamma$-matrices):

$$
\{\psi^\mu_0, \psi'^\nu_0\} = \eta^{\mu\nu}.
$$

The superstring states are constructed by the multiplication of some state from the left sector to a state on the same level (because of the eq. (77)) from the right one. Thus, one has, depending on the relative boundary conditions in the left and right sectors, four kinds of states:

$$
\begin{align*}
NS - \bar{NS} &\quad NS - \bar{R} \\
R - \bar{NS} &\quad R - \bar{R}.
\end{align*}
$$

(83)

Let us discuss the left sector (discussion of the right one is similar). Vacuum states are:

$$
\begin{align*}
P^2_{\mu} = \frac{1}{\alpha'} |0> &\quad NS \\
\bar{P}^2_{\mu} = 0 &\quad R
\end{align*}
$$

(84)

where $|0>$, in the Ramond sector, is defined below. While $|0>$ in the Naveu-Schwarz sector is defined as a standard vacuum for fermions.

There is the tachyon in this spectrum. To get rid of it, one should project on the eigenstates of the operator $(-1)^f$ with the eigen-value $(-1)$, where $f$ counts the fermionic number of an operator. This is so called GSO projection which kills the tachyon $|0>$ and leaves the $c^\mu_- |0>$ state in the NS sector. It is this GSO projection which leads, after the account of the both left and right sectors, to the appearance of the SUSY in the target space. After that the anti-diagonal elements in (83) give the superpartners to the diagonal ones.

Let us discuss what happens in the $R$ sector. For this reason we change the basis of the zero modes $\psi^\mu_0$ to

$$
\begin{align*}
d^+_0 = \frac{1}{\sqrt{2}} \left( \psi^1_0 \mp \psi^0_0 \right) &\quad d^+_i = \frac{1}{\sqrt{2^i}} \left( \psi^{2i}_0 \pm \psi^{2i+1}_0 \right), \quad i = 1, \ldots, 4.
\end{align*}
$$

(85)

Then from (82) one gets:

$$
\{d^+_I, d^-_J\} = \delta_{IJ}, \quad I = 0, \ldots, 4.
$$

(86)

These $d^+_I$ generate $2^5 = 32$ Ramond ground states $|s> = |\pm \frac{1}{2}, \ldots, \pm \frac{1}{2} >$ where:
\[ d_I | -\frac{1}{2}, \ldots, -\frac{1}{2} > = 0, \quad d^+_I | -\frac{1}{2}, \ldots, -\frac{1}{2} > = | \ldots, s_I = +\frac{1}{2}, \ldots > \quad (87) \]

As in the bosonic string there are Super-Virasoro conditions on the physical states of the superstring; for example, \( P_\mu \psi^0 | 0 > = 0 \). So in the frame where \( P_\mu = (p^0, p^0, 0, \ldots, 0) \) one has that \( P_\mu \psi^0 = \sqrt{2} p^0 d^0 \) and, then, \( s_0 = +\frac{1}{2}, i = 1, \ldots, 4 \), i.e. 16 physical vacua: \( 8_s \) with the even number of \(-\frac{1}{2}\) and \( 8_c \) with the odd number of \(-\frac{1}{2}\). These \( 8_s \) and \( 8_c \) give different chirality spinor representations of the ten-dimensional Lorentz group. The GSO projection keeps one among these states and removes the other. At the same time one has an arbitrary choice for the vacuum:

\[ (-1)^f | -\frac{1}{2}, \ldots, -\frac{1}{2} > = \pm | -\frac{1}{2}, \ldots, -\frac{1}{2} > \quad (88) \]

Therefore, if one chooses the opposite signs for the vacua of the \( R \) and \( \tilde{R} \) sectors, one gets the non-chiral Type IIA theory. If the same, then one gets the chiral Type IIB.

Thus, in the \( R - \tilde{R} \) sector one has the bosonic tensor fields \( A_{\mu_1 \ldots \mu_n} \) which are related to the states obtained as a products of that from (84) in the left and right sectors. The vertex for an emission of such a string state is defined as:

\[ \hat{V} \sim Q^\alpha \left[ \tilde{C} \tilde{F} \right]_{\alpha \beta} \tilde{Q}^\beta, \quad \hat{F} = F_{\mu_1 \ldots \mu_n} \gamma^{[\mu_1 \ldots \gamma_{\mu_n}],} \]

\[ F_{\mu_1 \ldots \mu_n} = \partial_{[\mu_n} A_{\mu_1 \ldots \mu_{n-1}]}, \quad C\gamma_\mu C^{-1} = -\gamma^T_\mu, \quad (89) \]

where \( Q^\alpha \) and \( \tilde{Q}^\alpha \) are the two-dimensional fields [7] (compositions of the ghost fields and \( \psi \)'s) which generate the target space SUSY; \( F \) is the field strength of the \( R - \tilde{R} \) tensor potential \( A \). Due to the chirality properties of the \( Q \)'s, forced by the GSO projection, in the Type IIA theory there are only even rank \( F \) are present, while in the Type IIB only odd rank \( F \) are present.

The discussion of the other part of the spectrum and of the low energy actions in the Type II theories one can find in the section 4.

**Appendix C. (Elements of Duality)**

Besides SUSY there is another tool used to control the low energy dynamics in superstring theory. This tool is duality (for the review see [8]) which helps to get some insights about strong coupling dynamics. That is why here we explain a few facts about duality, which are necessary for our discussion in the main text.

The qualitative idea of duality came from the Fourier transformation\(^{21}\) which exchanges slowly varying functions with fast harmonics. In fact, very naively, duality transformations, acting on the functional integral of a theory, exchange fast quantum fluctuations with slowly varying semiclassical ones. Usually duality transformations are accompanied by the inversions of some coupling constants. Therefore, we can pass, via duality, from the week coupling region to the strong one. Thus, it is possible to find what are the dynamical degrees of freedom when microscopic ones are strongly coupled. Under such duality transformations some theories are self-dual while other are exchanged with each other. In the latter case some theory describes the strong coupling dynamics of another one.

\(^{21}\)In the case of free theories the duality is simply Fourier transformation acting on the functional integral.
In superstring theory theorists use several, related to each other, types of dualities. These are T-duality, S-duality, U-duality, and Mirror symmetry\(^ {22} \). Where, U-duality is the composition of the T- and S-dualities when they do not commute \([9]\).

Let us begin with T-duality and then follow with S-duality. For the review of T-duality see \([57]\), while we discuss it very qualitatively. We start with the closed bosonic string theory which is self-dual under T-duality. The zero modes in the expansion (72) from the Appendix A are:

\[
x^\mu(z, \bar{z}) = x^\mu_l(z) + x^\mu_r(\bar{z}) \sim -i \sqrt{\frac{\alpha'}{2}} (a^\mu_0 + \tilde{a}^\mu_0) \sigma^2 + \sqrt{\frac{\alpha'}{2}} (a^\mu_0 - \tilde{a}^\mu_0) \sigma^1 \tag{90}\]

where the momentum \(P^\mu\) in the eq. (72) is equal to

\[
P^\mu = P^\mu_l + P^\mu_r = \frac{1}{2\sqrt{\alpha'}} (a^\mu_0 + \tilde{a}^\mu_0). \tag{91}\]

Under the transformation \(\sigma^1 \sim \sigma^1 + 2\pi\), \(x^\mu(z, \bar{z})\) changes by \(2\pi \sqrt{\frac{\alpha'}{2}} (a^\mu_0 - \tilde{a}^\mu_0)\). Because for a non-compact spatial direction \(\mu\), \(x^\mu(z, \bar{z})\) is single valued, one has the equality

\[
a^\mu_0 = \tilde{a}^\mu_0 = \sqrt{\frac{\alpha'}{2}} P^\mu. \tag{92}\]

However, for a compact direction, say \(j\), of a radius \(R\), \(X^j\) has the period \(2\pi R\). Then, under the transformation \(\sigma^1 \sim \sigma^1 + 2\pi\), \(X^j(z, \bar{z})\) can change by \(2\pi m R\), where \(m\) counts the number of times our string wraps around the compact direction. At the same time, the momentum \(P^j\) can take the values \(\frac{n R}{\alpha'}\). This means that

\[
a^j_0 + \tilde{a}^j_0 = \frac{2n}{R} \sqrt{\frac{\alpha'}{2}},
\]

\[
a^j_0 - \tilde{a}^j_0 = mR \sqrt{\frac{2}{\alpha'}} \tag{93}\]

and so

\[
a^j_0 = (\frac{n}{R} + \frac{mR}{\alpha'}) \sqrt{\frac{\alpha'}{2}},
\]

\[
\tilde{a}^j_0 = (\frac{n}{R} - \frac{mR}{\alpha'}) \sqrt{\frac{\alpha'}{2}} \tag{94}\]

Turning to the mass spectrum, we have

\[
M^2 = -(P^\mu)^2 = \frac{2}{\alpha'} (a^\mu_0)^2 + \frac{4}{\alpha'} (N_l - 1) = \left(\frac{n}{R} + \frac{mR}{\alpha'}\right)^2 + \frac{4}{\alpha'} (N_l - 1)
\]

\[
= \frac{2}{\alpha'} (\tilde{a}^\mu_0)^2 + \frac{4}{\alpha'} (N_r - 1) = \left(\frac{n}{R} - \frac{mR}{\alpha'}\right)^2 + \frac{4}{\alpha'} (N_r - 1) \tag{95}\]

As one can see from this formula, the first condition from the eq. (77) is spoiled. Therefore, along the compact directions (in contrast to the non-compact ones), strings might have only either left or right moving modes, which we use in the section five.

\(^{22}\)In this paper we will not discuss the Mirror symmetry which is a kind of a generalization of T-duality.
The mass spectra of the theories at the radius equal to $R$ and $\frac{\alpha'}{R}$ are identical if we make the exchange $n \to m$. This indicates that the theories at $R$ and at $\frac{1}{R}$ are identical. Such a flipping of $R$ is referred to as T-duality transformation. Necessity for having winding modes $m$ for T-duality says that it is a transformation peculiar only for extended objects which can wrap around compact directions.

There is a more rigorous proof that T-duality is an exact symmetry of the bosonic string theory. It is valid for any $\alpha'$ at each order of the expansion over $g_s$ [57]. For example, one can show that the partition function (65) and correlators for the string are invariant under such a transformation. Thus, T-duality is the symmetry of the conformal non-linear $\sigma$-model.

T-duality exchanges the strong and the week coupling regions. In fact, if we take the string theory on a target space with a sufficiently large radius $R$ then the non-linear sigma model perturbation theory (over $\alpha'$) is valid. Hence, our ordinary space-time geometry receives only small corrections. However, as $R \to 0$, this perturbation theory spoils and one should use T-duality to pass to that of the dual theory at $R' \sim \frac{1}{R}$. In this situation $R^2 / \alpha'$ plays the role of the coupling constant in the theory.

In the modern language T-duality concept can be reformulated as follows [57]. One has the moduli space of conformal theories parametrized by the $R$. This is the space of the unitary (after account of the BRST symmetry) theories with the conformal central charge (do not mix with the SUSY one) $c = d - 26$. The latter is equal to zero in our case. It happens that in this moduli space there are theories which at the classical level (their actions) look different but their quantum theories (correlation functions) are identical. They are related to each other through the exchange $R \to \frac{1}{R}$ and the momentum modes are exchanged with the winding ones. Moreover, one can see that the momentum modes are charged with respect to the gauge field $G_{\mu j}$ (such as in (28)). The charge being equal to the momentum in the compact direction. While the winding modes are charged with respect to the gauge field $B_{\mu j}$. The charge being equal to the winding number. Therefore, the $G_{\mu j}$ and $B_{\mu j}$ are exchanged under T-duality transformations [57].

All that can be straightforwardly generalized to the case of the compactification on a torus $T^n$, $n \leq 10$ which has different radii in different directions $R_k$, $k = 1, \ldots, n$ [57]. T-duality identifies all theories which are obtained from the one on $T^n$ with $R_k$ by flipping any number of the $R_k$'s. The closed bosonic string theory is self-dual under such transformations.

In the case of the superstring theories the situation is slightly more complicated [57]. For example, under T-duality transformations the Type IIA and IIB theories are exchanged with each other. The reason for this is that under T-duality transformations the supercurrent in the right sector flips its chirality [21]. Therefore, one passes from the chiral Type IIB theory to the non-chiral Type IIA one or vice versa.

Let us proceed with S-duality which is more complicated than T-duality. In fact, T-duality, being non-perturbative from the point of view of the $\alpha'$ corrections, is perturbative from the point of view of the $g_s$ corrections. Moreover, the flat ($T^n$) compactifications do not receive quantum corrections. Therefore, one can find more or less rigorous proof of T-duality. While in favor of S-duality we have just evidences supported by consistency checks [8]. At this point the main difficulty is due to the fact that S-duality is non-perturbative from the point of view of the $g_s$ corrections (it exchanges $g_s \to \frac{1}{g_s}$). While one has no other than the perturbative definition of superstring theory.

All of the evidences in favor of S-duality are coming from the non-renormalization theorems for the low energy SUGRA actions and for the BPS states. The point is that, while there are no perturbative quantum corrections which deform these low energy SUGRA

\[23\] The last fact is due to that $G_{jj}$ in the eq. (66) is either $\sim R$ or $\sim \frac{1}{R}$. 

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actions, there might be non-perturbative quantum effects which do change them. One can "control" these changes, looking at different BPS states in these theories which are various p-brane solitons.

Thus, comparing BPS spectra and looking for the transformations of the low energy actions (in the string frame) under the exchange $g_s \to \frac{1}{g_s}$, one can find different relations between all superstring theories \[8, 9\]. Mainly this is due to the fact that in ten dimensions, because of different anomalies, only five superstring theories\[24\] can exist \[7\]. Also, all these superstring theories are related to M-theory \[8, 9\]. At low energies this theory is described by the 11-dimensional SUGRA one.

For example, the Type IIB theory is self-dual under S-duality \[9\]: one can unify the dilaton and the axion fields of this theory into a complex field $\rho = \chi + i e^\varphi$. It appears that the low energy SUGRA theory is invariant under the $SL(2, \mathbb{Z})$ transformations acting rationally on the $\rho$. At the same time the $NS-\bar{NS}$ field $B_{\mu\nu}$ and the $R-\bar{R}$ field $A_{\mu\nu}$ are mixed under such transformations. One can unite them into the two-dimensional representation of the $SL(2, \mathbb{Z})$. Therefore, under such a duality transformation the fundamental string soliton (charged with respect to the $B_{\mu\nu}$ field) exchanged with the $R-\bar{R}$ 1-brane (charged with respect to the $A_{\mu\nu}$ field) or with "dyonic" strings, carrying some amounts of both types of charges.

At the same time, the Type IIA theory in the strong coupling ($g_s \to \infty$) limit is described by M-theory \[9\]. Here eleventh dimension is compactified on a circle of the radius $R \sim g_s^{3/4}$ which tends to $\infty$ as $g_s \to \infty$. Therefore, in the strong coupling region the Type IIA theory acquires extra spatial dimension. At the same time, the F-string soliton becomes the 2-brane wrapped over the extra compact direction of the M-theory.

Thus, one can see that S-duality exchanges fundamental excitations with solitonic ones and also mixes corresponding tensor gauge fields. Presently one can not prove these facts but can make different consistency checks of these duality conjectures \[8\].

If one considers the compactification of the Type II theories on the $T^n$ with $n \geq 2$ then the T- and S-dualities do not commute\[9\]. They compose bigger U-duality. It exchanges different gauge fields which are remnants (in the sense of (28)) of the metric, antisymmetric tensor and different $R-\bar{R}$ fields.

Now it is believed \[9\] that all known superstring theories are related to each other and to the 11-dimensional SUGRA through different kinds of dualities. Hence, we have only one theory, usually referred to as M-theory, which is defined on a big moduli space of different parameters. Then if we go to an infinity in any direction in the moduli space, one gets either already known superstring theories and their compactifications or SUGRA theories in different dimensions.

References


\[24\]At the same time there are plenty of theories which can exist in $d \leq 9$ dimensions.


A. Polyakov, "Gauge Fields and Strings", (Springer-Verlag, 1987)

J. Polchinski, hep-th/9611050;  
P.S. Aspinwall, hep-th/9611137;


R. Sorkin, gr-qc/9508002.  
L.H. Ford and N.F. Svaiter, gr-qc/9704050.  
M. Maggiore, hep-th/9404172;  


S. Vongehr, hep-th/9709172.


