Excitation of a Kaluza-Klein mode by parametric resonance

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Abstract

In this paper we investigate a parametric resonance phenomenon of a Kaluza-Klein mode in a $D$-dimensional generalized Kaluza-Klein theory. As the origin of the parametric resonance we consider a small oscillation of a scale of the compactification around a today’s value of it. To make our arguments definite and for simplicity we consider two classes of models of the compactification: those by $S_d$ ($d = D - 4$) and those by $S_{d_1} \times S_{d_2}$ ($d_1 \geq d_2$, $d_1 + d_2 = D - 4$). For these models we show that parametric resonance can occur for the Kaluza-Klein mode. After that, we give formulas of a creation rate and a number of created quanta of the Kaluza-Klein mode due to the parametric resonance, taking into account the first and the second resonance band. By using the formulas we calculate those quantities for each model of the compactification. Finally we give conditions for the parametric resonance to be efficient and discuss cosmological implications.

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I. INTRODUCTION

Many unified theories require spacetime dimension higher than 4. We call them generalized Kaluza-Klein theories. For example the superstring theory predicts spacetime dimension $D = 10$ [1], the M-theory $D = 11$ [2]. On the contrary we can see only a 4-dimensional part of the spacetime by experiments. Hence the extra $(D - 4)$-dimensional part is thought to be compactified so that it cannot be seen by us. One may be able to give a 4-dimensional theory describing our 4-dimensional universe from a generalized Kaluza-Klein theory by the compactification of the extra dimensions. However, there are a plenty number of ways of compactification and there are the same number of the corresponding 4-dimensional theories for a $D$-dimensional theory. Thus we need criterion to judge which compactification must be selected. For example, in the case of the superstring theory it is well known that a ‘good’ compactification must be realized by a Calabi-Yau manifold in order for the $N = 1$ supersymmetry to remain in the corresponding 4-dimensional theory [1]. In this paper we give a suggestive argument toward a cosmological criterion for compactification.

Kolb and Slansky [3] examined a model of the compactification by $S_1$ in the 5-dimensional theory with a metric field described by an Einstein-Hilbert action and a massless scalar field described by a Klein-Gordon action. They investigated a cosmological evolution of energy density of a mode of the scalar field with a non-zero momentum in the direction of the compactified extra space $S_1$ assuming that there is no entropy production. The mode is usually called a Kaluza-Klein mode and the quantum of the mode is called a pyrgon. They showed that if a scale of the compactification is of order of the 4-dimensional Planck length and energy density of pyrgons is comparable with energy density of radiation at an early epoch of the universe, then at the recombination epoch the former energy density terribly exceeds the critical density of the universe. The physical reason of their result is essentially the momentum conservation in the direction of the compactification: a pyrgon with a positive momentum in the direction cannot decay without meeting with another pyrgon with a negative momentum. For this reason energy density $\rho_{KK}$ of the Kaluza-
Klein mode decreases as $a^{-3}$, where $a$ is a scale factor of the universe, while energy density $\rho_{\text{rad}}$ of radiation decreases as $a^{-4}$. As a result the ratio $\rho_{\text{KK}}/\rho_{\text{rad}}$ evolves as $a$. Since the momentum conservation in the direction of the compactification does always hold, their result is expected to hold in a wider class of models of the compactification. Thus, if entropy production is negligible, energy density of pyrgons must not be comparable with energy density of radiation at any early epoch of the universe in a wider class of models of the compactification. On the other hand, if pyrgons are suitably created at an early epoch and they are diluted by entropy production after their creation (e.g., reheating after inflation) then the pyrgons may become a dark matter at today. Anyway, we have to analyze whether pyrgons can be created in early universe or not in order to obtain a knowledge of the compactification from a cosmological point of view. In this paper, motivated by these considerations, we investigate a catastrophic creation of quanta of a Kaluza-Klein mode in a $D$-dimensional generalized Kaluza-Klein theory.

As the origin of the catastrophic creation of quanta of the Kaluza-Klein mode we consider a small oscillation of a scale of the compactification at an early epoch of the universe. In general it is expected that frequency of the oscillation is of order of inverse of the scale of the compactification and mass of the Kaluza-Klein mode is of the same order. Hence the mass of the Kaluza-Klein mode oscillates with a frequency of order of the mass itself. Such an oscillation of the mass causes a catastrophic creation of quanta when frequency and amplitude of the oscillation are in a band in their parameter space. This sort of catastrophic creation of quanta is called a parametric resonance phenomenon [4]. In this paper we apply a theory of parametric resonance $^1$ to a Kaluza-Klein mode and judge whether the catastrophic creation occurs efficiently or not.

In Sec. II we calculate 4-dimensional actions for two classes of simple models of the

$^1$The theory of parametric resonance was applied to reheating (or preheating) process after inflation by many authors [5,6].
compactification. In Sec. III a Kaluza-Klein mode is investigated quantum mechanically on a classical background of a 4-dimensional Friedmann universe with a small oscillation of the scale of the compactification. We show that parametric resonance of the Kaluza-Klein mode can occur. After that we intend to judge whether the resonance is efficient to produce sufficient quanta of the Kaluza-Klein mode. Sec. IV is devoted to summarize this paper.

II. 4-DIMENSIONAL ACTION

Our starting point in this paper is a $D$-dimensional Einstein-Hilbert action with a cosmological constant $\bar{\Lambda}$:

$$I_{EH} = \frac{1}{2\kappa^2} \int d^D x \sqrt{-\bar{g}} \left( \bar{R} - 2\bar{\Lambda} \right), \tag{2.1}$$

where $\kappa$ is a positive constant, $\bar{g}$ and $\bar{R}$ are a determinant and a Ricci scalar of a $D$-dimensional metric tensor $\bar{g}_{MN}$. In addition to the gravitational field described by the Einstein-Hilbert action, we consider other massless fields, too. For the present, we include a real massless scalar field $\bar{\phi}$ described by a $D$-dimensional Klein-Gordon action:

$$I_{KG} = -\frac{1}{2} \int d^D x \sqrt{-\bar{g}} \bar{g}^{MN} \partial_M \bar{\phi} \partial_N \bar{\phi}. \tag{2.2}$$

As stated in the previous section, some candidates of the unified theory require $D > 4$ while the spacetime dimension we can see by experiments is 4. Hence we have to compactify the extra $(D-4)$-dimensional space so that it can not be seen by experiments. In this paper to make arguments definite we consider two classes of simple models of the compactification. Unfortunately, there are a great number of other models of the compactification and there is

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no conclusive first principle to determine which model should be taken. However, qualitative
properties shown in this paper for simple models are expected to be common in a wider class
of models of the compactification.

A. Compactification by $S_d$

First we consider a compactification by a $d$-dimensional sphere $S_d$ ($d = D - 4$). In this
case the metric tensor $\bar{g}_{MN}$ can be written as

$$\bar{g}_{MN}dx^Mdx^N = \hat{g}_{\mu\nu}dx^\mu dx^\nu + b^2\Omega^{(d)}_{ij}dx^idx^j,$$

(2.3)

where $\hat{g}_{\mu\nu}$ is a 4-dimensional metric depending only on the 4-dimensional coordinates $x^\mu$ ($\mu = 0, 1, 2, 3$), $b$ is a scale of the compactification (a radius of the $d$-dimensional sphere) depending only on the 4-dimensional coordinates, and $\Omega^{(d)}_{ij}dx^idx^j$ is a line element of a unit $d$-sphere. The scalar field $\bar{\phi}$ is expanded as follows:

$$\bar{\phi} = b_0^{-d/2}\sum_{l,m} \phi_{lm}(x^\mu)Y^{(d)}_{lm}(x^i),$$

(2.4)

where the constant $b_0$ is a today’s value of $b$. In the expression, $Y^{(d)}_{lm}$ ($l = 1, 2, \cdots$; and $m$ denotes a set of $d - 1$ numbers that are need in order for a set of all $Y^{(d)}_{lm}$ to be a complete set of $L^2$ functions on the $d$-sphere) are real harmonics on the $d$-sphere satisfying

$$\frac{1}{\sqrt{\Omega^{(d)}}}\partial_i \left( \sqrt{\Omega^{(d)}}\Omega^{(d)ij}\partial_j Y^{(d)}_{lm} \right) + l(l + d - 1)Y^{(d)}_{lm} = 0,$$

(2.5)

and $\phi_{lm}(x^\mu)$ is a real function depending only on the 4-dimensional coordinates $x^\mu$. The Einstein-Hilbert action and the Klein-Gordon action in this case are

$$I_{EH} = \frac{1}{2\kappa^2} \int d^dx\sqrt{-\hat{g}} \left( \frac{b}{b_0} \right)^d \left[ \hat{R} + d(d - 1)\hat{g}^{\mu\nu}(\partial_\mu \ln b)(\partial_\nu \ln b) + d(d - 1)b^{-2} - 2\Lambda \right],$$

$$I_{KG} = -\frac{1}{2} \sum_{l,m} \int d^dx\sqrt{-\hat{g}} \left( \frac{b}{b_0} \right)^d \left[ \hat{g}^{\mu\nu}\partial_\mu\phi_{lm}\partial_\nu\phi_{lm} + \frac{l(l + d - 1)}{b^2}\phi_{lm}^2 \right],$$

(2.7)
where $\kappa$ is a positive constant defined by $\kappa^2 = \bar{\kappa}^2/(2^D b_0^D \pi)$ and $\hat{R}$ is a Ricci scalar of the 4-dimensional metric tensor $\hat{g}_{\mu\nu}$. Since we shall consider the case when the scale $b$ of the compactification is time dependent, the factor $(b/b_0)^D$ before the Ricci scalar $\hat{R}$ and before the kinetic term $\hat{g}^{\mu\nu} \partial_{\mu} \phi_{lm} \partial_{\nu} \phi_{lm}$ of the $\phi_{lm}$ field is not a constant. Hence those expressions of the Einstein-Hilbert action and the Klein-Gordon action are quite different from the usual form of the 4-dimensional Einstein-Hilbert action and the 4-dimensional Klein-Gordon action.

In this paper we shall examine excitations of the $\phi_{lm}$ fields due to a time dependence of the compactification scale $b$. For this purpose we have to analyze a quantum dynamics of the $\phi_{lm}$ fields. However, as seen in the previous paragraph, the action of the $\phi_{lm}$ fields looks quite different from the usual 4-dimensional Klein-Gordon action and is not a useful form for the analysis of the quantum dynamics. Thus we perform the following conformal transformation in order to obtain a useful expression.

$$
\hat{g}_{\mu\nu} = \left( \frac{b}{b_0} \right)^{-d} g_{\mu\nu}.
$$

(2.8)

By this conformal transformation we obtain

$$
I_{EH} = \int d^4 x \sqrt{-\hat{g}} \left[ \frac{1}{2\kappa^2} \hat{R} - \frac{1}{2} \hat{g}^{\mu\nu} \partial_{\mu} \sigma \partial_{\nu} \sigma - U_0(\sigma) \right],
$$

$$
I_{KG} = -\frac{1}{2} \sum_{l,m} \int d^4 x \sqrt{-g} \left[ g^{\mu\nu} \partial_{\mu} \phi_{lm} \partial_{\nu} \phi_{lm} + M_l^2(\sigma) \phi_{lm}^2 \right],
$$

(2.9)

where $R$ is a Ricci scalar of the 4-dimensional metric tensor $g_{\mu\nu}$, the field $\sigma$ is defined by

$$
\sigma = \sigma_0 \ln \left( \frac{b}{b_0} \right),
$$

$$
\sigma_0 = \sqrt{\frac{d(d+2)}{2\kappa^2}},
$$

(2.10)

the mass $M_l(\sigma)$ of the 4-dimensional scalar field $\phi_{lm}$ is given by

$$
M_l^2(\sigma) = \frac{l(l+d-1)}{b_0^2} e^{-(d+2)\sigma/\sigma_0},
$$

(2.11)

and the potential $U_0(\sigma)$ of $\sigma$ is
\[ U_0(\sigma) = \frac{\bar{\Lambda}}{\kappa^2} e^{-d\sigma/\sigma_0} - \frac{d(d-1)}{2\kappa^2 b_0^2} e^{-(d+2)\sigma/\sigma_0}. \] (2.12)

Here we mention that a mode with non-zero \( l \) has a mass of order \( 1/b_0 \) and is called a Kaluza-Klein mode.

The expression (2.9) can be understood as the 4-dimensional Einstein-Hilbert action plus the 4-dimensional Klein-Gordon action with a potential. However, it can be easily confirmed that \( U_0(\sigma) \) has no local minimum. It means that there is no stable compactification by \( S_d \) if the model is not modified any more. Therefore we have to take into account some mechanism in order to stabilize the potential of the \( \sigma \) field. In this paper as the mechanism we consider a 1-loop effective action (an action for so-called Casimir effects) contributed by all 4-dimensional matter fields. The 1-loop effective action is of the following form [7]:

\[ I_{1\text{loop}} = -\int d^4x \sqrt{-g} V_{1\text{loop}}(\sigma), \] (2.13)

where \( V_{1\text{loop}} \) is an effective potential, which in turn is a sum over all contributing fields 4.

\[ V_{1\text{loop}}(\sigma) = \sum_i c_i V(M_i), \] (2.14)

where \( i \) distinguishes 4-dimensional fields contributing to the 1-loop effective action, \( M_i \) is a mass of the field specified by \( i \), and \( c_i \) is a numerical factor depending on what spin the field \( i \) has 5. In the expression the function \( V(M) \) is given by

\[ V(M) = -\frac{1}{2} \lim_{n \to 4} (4\pi)^{-n/2} \Gamma(-n/2) M^n. \] (2.15)

Since squared masses of 4-dimensional fields deduced from the massless \( D \)-dimensional theory is proportional to \( e^{-(d+2)\sigma/\sigma_0} \) as the squared mass (2.11) of the \( \phi_{lm} \) fields, the 1-loop effective action is of the following form in general.

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4Strictly speaking, we have to restrict the dimension \( D \) to be odd since otherwise the conformal anomaly arises and the arguments become complicated.

5For example, \( c_i = 1 \) for spin 0, \( c_i = -4 \) for spin 1/2 [7].
$V_{\text{1loop}}(\sigma) = Ae^{-2(d+2)\sigma/\sigma_0}$, \hfill (2.16)

where $A$ is a constant. We could calculate the constant $A$ in principle if we knew details of the $D$-dimensional theory. However, we do not know it. Thus we shall determine the constant $A$ phenomenologically. We require that the total potential $U_0(\sigma) + V_{\text{1loop}}(\sigma)$ of the field $\sigma$ has an extremum at $\sigma = 0$ (or at $b = b_0$, where $b_0$ is the today’s value of $b$) and the extremum (or the 4-dimensional cosmological constant) is zero. Thus the constant $A$ and the renormalized value of the $D$-dimensional cosmological constant $\bar{\Lambda}$ can be expressed in terms of $b_0$, and the total potential of $\sigma$ is

$$U_1(\sigma) = \alpha \left[ \frac{2}{d+2} e^{-2(d+2)\sigma/\sigma_0} + e^{-d\sigma/\sigma_0} - \frac{d + 4}{d+2} e^{-(d+2)\sigma/\sigma_0} \right],$$ \hfill (2.17)

where $\alpha$ is a constant defined by

$$\alpha = \frac{d(d - 1)(d + 2)}{2(d + 4)\kappa^2 b_0^2}. \hfill (2.18)$$

This form of the total potential of the field $\sigma$ is used in many literature [8]. Its second derivative at $\sigma = 0$ is given by

$$U''_1(0) = \frac{2(d - 1)}{b_0^2}, \hfill (2.19)$$

and is positive for $d \geq 2$. Thus, for $d \geq 2$, the potential $U_1$ has a local minimum at $\sigma = 0$.

Finally the 4-dimensional action for the fields $g_{\mu\nu}$, $\sigma$ and $\phi_{lm}$ in this model is

$$I = \int d^4x \sqrt{-g} \left[ \frac{1}{2\kappa^2} R - \frac{1}{2} g^{\mu\nu} \partial_{\mu} \sigma \partial_{\nu} \sigma - U_1(\sigma) \right]$$

$$- \frac{1}{2} \sum_{l,m} \int d^4x \sqrt{-g} \left[ g^{\mu\nu} \partial_{\mu} \phi_{lm} \partial_{\nu} \phi_{lm} + M_i^2(\sigma) \phi_{lm}^2 \right]. \hfill (2.20)$$

**B. Compactification by $S_{d_1} \times S_{d_2}$**

Next we consider a compactification by a direct product $S_{d_1} \times S_{d_2}$ of a $d_1$-dimensional sphere and a $d_2$-dimensional sphere ($d_1 + d_2 = D - 4$). We assume that $d_1 \geq d_2 \geq 1$. In this case the metric tensor $\tilde{g}_{MN}$ can be written as
\[ \tilde{g}_{MN} dx^M dx^N = \hat{g}_{\mu\nu} dx^\mu dx^\nu + b_1^2 \Omega_{ij}^{(d_1)} dx^i dx^j + b_2^2 \Omega_{pq}^{(d_2)} dx^p dx^q, \]  

(2.21)

where \( \hat{g}_{\mu\nu} \) is a 4-dimensional metric depending only on the 4-dimensional coordinates \( x^\mu \) \((\mu = 0, 1, 2, 3)\), \( b_1 \) and \( b_2 \) are radii of the \( d_1 \)-dimensional sphere and the \( d_2 \)-dimensional sphere depending only on the 4-dimensional coordinates, and \( \Omega_{ij}^{(d_1)} \) \( dx^i dx^j \) and \( \Omega_{pq}^{(d_2)} \) \( dx^p dx^q \) are line elements of a unit \( d_1 \)-sphere and a unit \( d_2 \)-sphere, respectively. The scalar field \( \phi \) is expanded as follows:

\[ \phi = b_1^{-d_1/2} b_2^{-d_2/2} \sum_{l_1,l_2,m_1,m_2} \phi_{l_1l_2m_1m_2} (x^\mu) Y_{l_1m_2}^i (x^i) Y_{l_2m_2}^p (x^p), \]  

(2.22)

where \( b_{10} \) and \( b_{20} \) are today’s values of \( b_1 \) and \( b_2 \), \( Y_{l_1m_1}^{(d_1)} \) and \( Y_{l_2m_2}^{(d_2)} \) are real harmonics on the \( d_1 \)-sphere and the \( d_2 \)-sphere, respectively, and \( \phi_{l_1l_2m_1m_2} (x^\mu) \) is a real function depending only on the 4-dimensional coordinates \( x^\mu \). The Einstein-Hilbert action and the Klein-Gordon action in this case are

\[
I_{EH} = \int d^4 x \sqrt{-g} \left[ \frac{1}{2\kappa^2} R - \frac{1}{2} g^{\mu\nu} \partial_\mu \sigma_+ \partial_\nu \sigma_+ - \frac{1}{2} g^{\mu\nu} \partial_\mu \sigma_- \partial_\nu \sigma_- - U_0(\sigma_1, \sigma_2) \right],
\]

\[
I_{KG} = -\frac{1}{2} \sum_{l_1,l_2,m_1,m_2} \int d^4 x \sqrt{-g} \left[ g^{\mu\nu} \partial_\mu \phi_{l_1l_2m_1m_2} \partial_\nu \phi_{l_1l_2m_1m_2} + M^2_{l_1,l_2}(\sigma_1, \sigma_2) \phi_{l_1l_2m_1m_2}^2 \right],
\]

(2.23)

where we have performed the conformal transformation

\[ \tilde{g}_{\mu\nu} = \left( \frac{b_1}{b_{10}} \right)^{-d_1} \left( \frac{b_2}{b_{20}} \right)^{-d_2} g_{\mu\nu}. \]  

(2.24)

In the expressions, \( \kappa \) is a positive constant defined by \( \kappa^2 = \bar{\kappa}^2 (2^{d_1+d_2} b_{10} b_{20} \pi^2) \), \( R \) is a Ricci scalar of the 4-dimensional metric tensor \( g_{\mu\nu} \). The 4-dimensional fields \( \sigma_{1,2} \) are defined by

\[
\sigma_{1,2} = \sigma_{10,20} \ln \left( \frac{b_{1,2}^{1,2}}{b_{10,20}} \right),
\]

\[
\sigma_{10,20} = \sqrt{\frac{d_1(1 + d_1)}{2\kappa^2}},
\]

(2.25)

and \( \sigma_\pm \) are linear combinations of \( \sigma_{1,2} \):

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6See the comments before (2.8).
where \( a = \sqrt{\frac{d_1 d_2}{(d_1+2)(d_2+2)}} \). The mass \( M_{l_1 l_2}(\sigma_1, \sigma_2) \) of the 4-dimensional scalar field \( \phi_{l_1 l_2 m_1 m_2} \) is given by

\[
M_{l_1 l_2}^2(\sigma_1, \sigma_2) = e^{-d_1 \sigma_1 / \sigma_{10} - d_2 \sigma_2 / \sigma_{20}} \left[ \frac{l_1(l_1 + d_1 - 1)}{b_{10}^2} e^{-2 \sigma_1 / \sigma_{10}} + \frac{l_2(l_2 + d_2 - 1)}{b_{20}^2} e^{-2 \sigma_2 / \sigma_{20}} \right],
\]

(2.27)

and the potential \( U_0(\sigma_1, \sigma_2) \) is

\[
U_0(\sigma_1, \sigma_2) = e^{-d_1 \sigma_1 / \sigma_{10} - d_2 \sigma_2 / \sigma_{20}} \left[ \frac{\bar{\Lambda}}{\kappa^2} - \frac{d_1(d_1 - 1)}{2 \kappa^2 b_{10}^2} e^{-2 \sigma_1 / \sigma_{10}} - \frac{d_2(d_2 - 1)}{2 \kappa^2 b_{20}^2} e^{-2 \sigma_2 / \sigma_{20}} \right].
\]

(2.28)

Here we mention that a mode with non-zero \( l_1 \) or non-zero \( l_2 \) has a mass of order \( 1/b_{10} \) or \( 1/b_{20} \) and is called a Kaluza-Klein mode.

As in the case of the compactification by \( S_d \), it can be easily confirmed that \( U_0(\sigma_1, \sigma_2) \) has no local minimum, and that there is no stable compactification by \( S_{d_1} \times S_{d_2} \) if the model is not modified any more. In order to stabilize the potential we consider a 1-loop effective action (an action for so-called Casimir effects) contributed by all 4-dimensional matter fields. The 1-loop effective action is of the following form [7]:

\[
I_{\text{1-loop}} = - \int d^4x \sqrt{-g} V_{\text{1-loop}}(\sigma_1, \sigma_2),
\]

(2.29)

where \( V_{\text{1-loop}} \) is an effective potential, which in turn is a sum over all contributing fields \(^7\).

\[
V_{\text{1-loop}}(\sigma_1, \sigma_2) = \sum_i c_i V(M_i),
\]

(2.30)

where \( i \) distinguishes 4-dimensional fields contributing to the 1-loop effective action, \( M_i \) is a mass of the field specified by \( i \), and \( c_i \) is a numerical factor depending on what spin the field \( i \) has \(^8\). In the expression the function \( V(M) \) is given by (2.15). Since, in the present

\(^7\)See the footnote soon after (2.13).

\(^8\)See the footnote soon after (2.14).
model, squared masses of 4-dimensional fields deduced from the $D$-dimensional massless theory is a sum of a term proportional to $e^{-(d_1+2)\sigma_1/\sigma_{10}-d_2\sigma_2/\sigma_{20}}$ and a term proportional to $e^{-d_1\sigma_1/\sigma_{10}-(d_2+2)\sigma_2/\sigma_{20}}$ as the squared mass (2.27) of the $\phi_{lm}$ fields, the 1-loop effective action is of the following form in general.

$$V_{\text{loop}}(\sigma_1, \sigma_2) = e^{-2d_1\sigma_1/\sigma_{10}-2d_2\sigma_2/\sigma_{20}} \left(A_{11}e^{-4\sigma_1/\sigma_{10}} + 2A_{12}e^{-2\sigma_1/\sigma_{10}-2\sigma_2/\sigma_{20}} + A_{22}e^{-4\sigma_2/\sigma_{20}} \right),$$

(2.31)

where $A_{11}$, $A_{12}$ and $A_{22}$ are constants. We require that the total potential $U_0(\sigma_1, \sigma_2) + V_{\text{loop}}(\sigma_1, \sigma_2)$ of the fields $\sigma_1$ and $\sigma_2$ has an extremum at $\sigma_1 = \sigma_2 = 0$ (or at $b_1 = b_{10}$, $b_2 = b_{20}$, where $b_{10}$ and $b_{20}$ are the today’s values of $b_1$ and $b_2$, respectively) and the extremum (or the 4-dimensional cosmological constant) is zero. Thus $A_{11}$, $A_{22}$ and $\bar{\Lambda}$ can be expressed in terms of $b_{10}$, $b_{20}$ and the unknown constant $A_{12}$, and the total potential of $\sigma_1$ and $\sigma_2$ is

$$U_1(\sigma_1, \sigma_2) = \alpha_1 U_1^{(1)}(\sigma_1, \sigma_2) + \alpha_2 U_1^{(2)}(\sigma_1, \sigma_2)$$

$$-A_{12}e^{-2d_1\sigma_1/\sigma_{10}-2d_2\sigma_2/\sigma_{20}} \left(e^{-2\sigma_1/\sigma_{10}} - e^{-2\sigma_2/\sigma_{20}} \right)^2,$$

(2.32)

where the constants $\alpha_{1,2}$ and the functions $U_1^{(1,2)}(\sigma_1, \sigma_2)$ are defined by

$$\alpha_1 = \frac{d_1(d_1-1)(d_1+d_2+2)}{2(d_1+d_2+4)\kappa^2b_{10}^2},$$

$$\alpha_2 = \frac{d_2(d_2-1)(d_1+d_2+2)}{2(d_1+d_2+4)\kappa^2b_{20}^2},$$

$$U_1^{(1)}(\sigma_1, \sigma_2) = \frac{d_2}{2(d_1+d_2+2)}e^{-2d_1\sigma_1/\sigma_{10}-2d_2\sigma_2/\sigma_{20}} + \frac{d_1}{d_1+d_2+2}e^{-(d_1+2)\sigma_1/\sigma_{10}-(d_2+2)\sigma_2/\sigma_{20}},$$

$$U_1^{(2)}(\sigma_1, \sigma_2) = \frac{d_1}{2(d_1+d_2+2)}e^{-2d_1\sigma_1/\sigma_{10}-(d_2+2)\sigma_2/\sigma_{20}} + \frac{d_1}{d_1+d_2+2}e^{-(d_1+2)\sigma_1/\sigma_{10}-(d_2+2)\sigma_2/\sigma_{20}}.$$  

(2.33)

Finally the 4-dimensional action for the fields $g_{\mu\nu}$, $\sigma_1$, $\sigma_2$ and $\phi_{l_1l_2m_1m_2}$ in this model is
\[ I = \int d^4x \sqrt{-g} \left[ \frac{1}{2\kappa^2} R - \frac{1}{2} g^{\mu\nu} \partial_\mu \sigma_+ \partial_\nu \sigma_+ - \frac{1}{2} g^{\mu\nu} \partial_\mu \sigma_- \partial_\nu \sigma_- - U_1(\sigma_1, \sigma_2) \right] \\
- \frac{1}{2} \sum_{l_1, l_2, m_1, m_2} \int d^4x \sqrt{-g} \left[ g^{\mu\nu} \partial_\mu \phi_{l_1l_2m_1m_2} \partial_\nu \phi_{l_1l_2m_1m_2} + M_{l_1l_2}^2(\sigma) \phi_{l_1l_2m_1m_2}^2 \right]. \quad (2.34) \]

Unfortunately, the potential \( U_1(\sigma_1, \sigma_2) \) depends on the unknown constant \( A_{12} \). In order to make our arguments independent of the value of \( A_{12} \) we take the following ansatz:

\[ \frac{b_1}{b_{10}} = \frac{b_2}{b_{20}}. \quad (2.35) \]

Physically, this condition means that the shape of the extra \((D - 4)\)-dimensional space \( S_{d_1} \times S_{d_2} \) is fixed and only its volume changes. In this case let us introduce a field \( \sigma \) by

\[ \sigma/\sigma_0 = \sigma_1/\sigma_{10} = \sigma_2/\sigma_{20}, \quad \sigma_0 = \sqrt{\frac{(d_1 + d_2)(d_1 + d_2 + 2)}{2\kappa^2}}. \quad (2.36) \]

The action (2.34) is simplified as

\[ I = \int d^4x \sqrt{-g} \left[ \frac{1}{2\kappa^2} R - \frac{1}{2} g^{\mu\nu} \partial_\mu \sigma_\sigma \partial_\nu \sigma - U_1(\sigma) \right] \\
- \frac{1}{2} \sum_{l_1, l_2, m_1, m_2} \int d^4x \sqrt{-g} \left[ g^{\mu\nu} \partial_\mu \phi_{l_1l_2m_1m_2} \partial_\nu \phi_{l_1l_2m_1m_2} + M_{l_1l_2}^2(\sigma) \phi_{l_1l_2m_1m_2}^2 \right], \quad (2.37) \]

where the mass \( M_{l_1l_2}(\sigma) \) of the Kaluza-Klein mode \( \phi_{l_1l_2m_1m_2} \) is given by

\[ M_{l_1l_2}^2(\sigma) = \left[ \frac{l_1(l_1 + d_1 - 1)}{b_{10}^2} + \frac{l_2(l_2 + d_2 - 1)}{b_{20}^2} \right] e^{-(d_1 + d_2 + 2)\sigma/\sigma_0}, \quad (2.38) \]

and the potential \( U_1(\sigma) \) of the \( \sigma \) field is

\[ U_1(\sigma) = (\alpha_1 + \alpha_2) \left[ \frac{2}{d_1 + d_2 + 2} e^{-2(d_1 + d_2 + 2)\sigma/\sigma_0} + e^{-(d_1 + d_2)\sigma/\sigma_0} \right. \\
\left. \quad - \frac{d_1 + d_2 + 4}{d_1 + d_2 + 2} e^{-(d_1 + d_2 + 2)\sigma/\sigma_0} \right]. \quad (2.39) \]

Note that the potential \( U_1(\sigma) \) is independent of the unknown constant \( A_{12} \). The second derivative of \( U_1(\sigma) \) at \( \sigma = 0 \) is

\[ U_1''(0) = \frac{2}{d_1 + d_2} \left[ \frac{d_1(d_1 - 1)}{d_{10}^2} + \frac{d_2(d_2 - 1)}{d_{20}^2} \right]. \quad (2.40) \]

Hence, if \( d_1 \geq 2 \), the extremum of \( U_1(\sigma) \) at \( \sigma = 0 \) is a local minimum and the compactification is stable at least locally.
III. PARAMETRIC RESONANCE

In the previous section we have shown that the compactification by $S_d$ ($d \geq 2$) is stable, and the compactification by $S_{d_1} \times S_{d_2}$ ($d_1 \geq 2$) is also stable at least in the sub-configuration-space defined by (2.35). Thus, we consider the $S_d$ compactification with $d \geq 2$ and the $S_{d_1} \times S_{d_2}$ compactification with $d_1 \geq 2$. In this section we judge whether a catastrophic creation of quanta of the Kaluza-Klein mode $\phi_{lm}$ (or $\phi_{l_1l_2m_1m_2}$) can occur due to an oscillation of the $\sigma$ field around the local minimum $\sigma = 0$. We treat the metric tensor $g_{\mu\nu}$ and the field $\sigma$ classically, and the fields $\phi_{lm}$ (or $\phi_{l_1l_2m_1m_2}$) quantum mechanically.

A. The hamiltonians for $\phi_{lm}$ and $\phi_{l_1l_2m_1m_2}$

We assume that the 4-dimensional metric $g_{\mu\nu}$ is of the Friedmann universe and that the field $\sigma$ is homogeneous on this background. The following equation of motion for the field $\sigma$ is derived from the 4-dimensional action (2.20) or (2.37):

$$\ddot{\sigma} + 3H \dot{\sigma} + U_1'(\sigma) = 0,$$

(3.1)

where $H = \dot{a}/a$ ($a$ is a scale factor of the universe) and the overdot denotes derivation with respect to the cosmological time $t$. Hereafter we assume that

$$H \ll \omega,$$

(3.2)

where $\omega$ is defined by

$$\omega = \frac{1}{2} \sqrt{U_1''(0)}.$$

(3.3)

Hence for a small deviation from the local minimum $\sigma = 0$ ($|\sigma/\sigma_0| \ll 1/D$), a solution $\sigma(t)$ of the equation (3.1) can be written as

$$\frac{\sigma(t)}{\sigma_0} = \tilde{\sigma}(t) \cos[2\omega(t - t_0)],$$

(3.4)

where the coefficient $\tilde{\sigma}(t)$ ($|\tilde{\sigma}(t)| \ll 1/D$) is a slowly varying function of $t$.
\[ \frac{\dot{\sigma}}{\sigma} \ll \omega, \quad (3.5) \]

and \( t_0 \) is an arbitrary constant. In order to determine a time dependence of the function \( \tilde{\sigma}(t) \), we introduce energy density \( \rho_\sigma \) of the \( \sigma \) field by

\[ \rho_\sigma \equiv \frac{1}{2}\dot{\sigma}^2 + \frac{1}{2}(2\omega)^2\sigma^2. \quad (3.6) \]

From (3.1) the following equation is easily derived:

\[ \rho_\sigma = -3H\dot{\sigma}^2. \quad (3.7) \]

By averaging this equation from \( t-\pi/(2\omega) \) to \( t+\pi/(2\omega) \), we obtain the following approximate equation.

\[ \rho_\sigma \simeq -\frac{3}{2}H(2\omega)^2(\sigma_0\dot{\sigma})^2. \quad (3.8) \]

Since in our situation the value of \( \rho_\sigma \) is approximately given by

\[ \rho_\sigma \simeq \frac{1}{2}(2\omega)^2(\sigma_0\dot{\sigma})^2, \quad (3.9) \]

the time dependence of \( \tilde{\sigma} \) is derived from the averaged equation (3.8) as

\[ \tilde{\sigma}(t) \propto a^{-3/2}. \quad (3.10) \]

In the remaining of this section we investigate a quantum dynamics of the Kaluza-Klein mode \( \phi_{lm} \) (for the \( S_d \) case) and \( \phi_{l_1l_2m_1m_2} \) (for the \( S_{d_1} \times S_{d_2} \) case) on the classical background of the Friedmann universe and the field \( \sigma \) given by (3.4) and (3.10). For this purpose we first go to the momentum representation by

\[ \phi_{lm}(x, t) = a^{-3/2} \int \frac{\sqrt{2d^3k}}{(2\pi)^3} \left[ \phi_{lm}^{(1)}(k) \cos(k \cdot x) + \phi_{lm}^{(2)}(k) \sin(k \cdot x) \right], \quad (3.11) \]

\[ \phi_{l_1l_2m_1m_2}(x, t) = a^{-3/2} \int \frac{\sqrt{2d^3k}}{(2\pi)^3} \left[ \phi_{l_1l_2m_1m_2}^{(1)}(k) \cos(k \cdot x) + \phi_{l_1l_2m_1m_2}^{(2)}(k) \sin(k \cdot x) \right], \quad (3.12) \]
where $\phi^{(1,2)k}(t)$ and $\phi^{(1,2)k}_{l_1l_2m_1m_2}(t)$ are real functions of the cosmological time $t$, the vector $\mathbf{x}$ denotes the comoving coordinates. Next we obtain the Hamiltonian corresponding to the time $t$ from the action (2.20) and (2.37). The result is summarized as follows:

$$H_{S_d} = \sum_{l,m} \int d\mathbf{k} \left( H^{(1)k}_{lm} + H^{(2)k}_{lm} \right)$$

(3.13)

for the $S_d$ case, and

$$H_{S_d \times S_d} = \sum_{l_1,l_2,m_1,m_2} \int d\mathbf{k} \left( H^{(1)k}_{l_1l_2m_1m_2} + H^{(2)k}_{l_1l_2m_1m_2} \right)$$

(3.14)

for the $S_d \times S_d$ case. In the expressions the partial hamiltonians $H^{(1,2)k}_{lm}$ and $H^{(1,2)k}_{l_1l_2m_1m_2}$ are of the following form.

$$H = \frac{1}{2} \left[ P^2 + \Omega^2(t)Q^2 \right]$$

(3.15)

where $Q$ denotes $\phi^{(1,2)k}_{lm}$ for $H^{(1,2)k}_{lm}$ and $\phi^{(1,2)k}_{l_1l_2m_1m_2}$ for $H^{(1,2)k}_{l_1l_2m_1m_2}$, and $P$ denotes its conjugate momentum. The function $\Omega(t)$ is a positive function of $t$ given by

$$\Omega^2(t) = \omega^2 \left[ A + B e^{-\epsilon \cos 2\omega t} \right]$$

(3.16)

where the parameters $A$, $B$, $\epsilon$ and $\omega$ are given by

$$A = \frac{2b_0^2 (k/a)^2}{d-1} + \Delta A,$$

$$B = \frac{2}{d-1} l(l+d-1),$$

$$\epsilon = (d+2)\tilde{\sigma},$$

$$\omega = \frac{1}{b_0} \sqrt{\frac{d-1}{2}}$$

(3.17)

To obtain the representation, for concreteness, we assume that the Friedmann universe is flat one. If we assume the open or closed Friedmann universe then the corresponding representation is changed. However, the resulting partial hamiltonians are of the same form as (3.15), provided that $k^2$ is replaced by the corresponding eigen value.

We can set $t_0 = 0$ without loss of generality.
for the $S_d$ case \textsuperscript{11}, and

$$A = \frac{2(d_1 + d_2)b_{10}^2(k/a)^2}{d_1(d_1 - 1) + d_2(d_2 - 1)(b_{10}/b_{20})^2} + \Delta A,$$

$$B = \frac{2(d_1 + d_2)[l_1(l_1 + d_1 - 1) + l_2(l_2 + d_2 - 1)(b_{10}/b_{20})^2]}{d_1(d_1 - 1) + d_2(d_2 - 1)(b_{10}/b_{20})^2},$$

$$\epsilon = (d_1 + d_2 + 2)\bar{\sigma},$$

$$\omega = \sqrt{\frac{1}{2(d_1 + d_2)\left[\frac{d_1(d_1 - 1)}{b_{10}^2} + \frac{d_2(d_2 - 1)}{b_{20}^2}\right]}}$$

(3.18)

for the $S_{d_1} \times S_{d_2}$ case \textsuperscript{12}, respectively. In both expressions of $A$, the term $\Delta A$ denotes

$$\Delta A = -\omega^{-2}\left(\frac{9}{4}H^2 + \frac{3}{2}\dot{H}\right).$$

(3.19)

In the remaining of this paper we consider the case when the condition (3.2) and the following condition hold.

$$\dot{H} \ll \omega^2.$$

(3.20)

Hence we can neglect the term $\Delta A$ in $A$. In general the parameters $A$ and $\epsilon$ appearing in the function $\Omega$ have time dependence even when we neglect the term $\Delta A$. However, since $A \propto a^{-2}$ and $\epsilon \propto a^{-3/2}$, it is shown from (3.2) that their change is extremely slow:

$$\frac{|\dot{A}|}{A} = 2H \ll \omega,$$

$$\frac{|\dot{\epsilon}|}{\epsilon} = \frac{3}{2}H \ll \omega.$$  

(3.21)

B. The resonance band and the number of created quanta

A system with a Hamiltonian like (3.15), when it is canonically quantized, has the property that a catastrophic creation of quanta (parametric resonance) can occur if the param-

\textsuperscript{11}If $l = 0$, then $B = 0$ and the parametric resonance does not arise. Thus, hereafter we consider a mode with $l \geq 1$.

\textsuperscript{12}If $l_1 = l_2 = 0$, then $B = 0$ and the parametric resonance does not arise. Thus, hereafter we consider a mode with $l_1 \geq 1$ or $l_2 \geq 1$. 

16
eters $A$, $B$ and $\epsilon$ is in a band in their parameter space. Our task now is to derive the band and a number of created quanta of the Kaluza-Klein mode $\phi_{lm}$ or $\phi_{l_1l_2m_1m_2}$. First we expand (3.16) both in power of $\epsilon$ and in Fourier series. For this purpose we use the identity
\[ e^{-\epsilon \cos 2z} = I_0(\epsilon) + 2 \sum_{n=1}^{\infty} (-1)^n I_n(\epsilon) \cos 2nz, \] (3.22)
where $I_n(x)$ is the modified Bessel function and is expanded as
\[ I_n(x) = \sum_{s=0}^{\infty} \frac{(x/2)^{n+2s}}{s!(n+s)!}. \] (3.23)
The obtained expression is
\[ \Omega^2(t) = \omega^2 \left[ h - \sum_{s=1}^{\infty} \sum_{n=-\infty}^\infty e^s g_n^{(s)} e^{inz} \right] \] (3.24)
where $z = \omega t$. In the expression the constants $h$ and $g_n^{(s)}$ are given by
\[ h = A + BI_0(\epsilon), \]
\[ g_{n+2s}^{(n+2s)} = \frac{(-1)^{n+1}B}{2^{n+2s}s!(n+s)!}, \] (3.25)
where $n = 1, 2, \cdots; s = 0, 1, \cdots$, and other $g_n^{(s)}$ are all zero. For example,
\[ g_{\pm2}^{(1)} = \frac{B}{2}, \quad g_n^{(1)} = 0 \quad (n \neq \pm2), \]
\[ g_{\pm4}^{(2)} = -\frac{B}{8}, \quad g_n^{(2)} = 0 \quad (n \neq \pm4). \] (3.26)

Next let us introduce a parameter $\Delta$ by
\[ h = \left( \frac{p}{q} \right)^2 + \epsilon \Delta, \] (3.27)
where $p$ and $q$ are mutually prime positive integers. In appendix A and B it is shown that the parametric resonance does occur if the parameter $\Delta$ is in the following band:
\[ \Delta_0 - |\delta| < \Delta < \Delta_0 + |\delta|, \] (3.28)
where $\Delta_0$ is a combination of the coefficients $g_n^{(s)}$ being of order $O(\epsilon)$, and $\delta$ is defined by
\[ \delta = g_{2p/q}^{(1)} + \epsilon \left[ \sum_{n \neq 0,2p/q} \frac{g_n^{(1)} g_{2p/q-n}^{(1)}}{n(2p/q - n)} + g_{2p/q}^{(2)} \right]. \] (3.29)
Thus the existence of non-trivial interval of the resonance band requires that $\delta \neq 0$, which in turn requires that $p/q$ is 1 or 2. Physically the first resonance corresponds to a decay of a quantum of the field $\sigma$ with energy $2\omega$ into two quanta of the Kaluza-Klein mode with each energy $\omega$, and the second resonance corresponds to a decay of a quantum of $\sigma$ into one quantum of the Kaluza-Klein mode with energy $2\omega$. The parameters $\Delta_0$ and $\delta$ in our case are given as follows:

\[
\Delta_0 = -\frac{B^2}{32}\epsilon, \\
\delta = \frac{B}{2} 
\]  

(3.30)

for the $p/q = 1$ resonance; and

\[
\Delta_0 = \frac{B^2}{24}\epsilon, \\
\delta = \frac{B(B - 2)}{16}\epsilon 
\]  

(3.31)

for the $p/q = 2$ resonance. Next the Floquet index $\mu_+$, which represents a ‘strength’ of the resonance, is given by the formula (A26) and can be expressed as

\[
\mu_+ = \left| \frac{\epsilon \delta}{2p/q} \right| \sqrt{1 - x^2} (1 + O(\epsilon)),
\]

(3.32)

where $x$ is defined by $x = (\Delta - \Delta_0)/\delta$.

As shown in appendix B, by using the form of $\mu_+$, a number of excited quanta is given by the following formula:

\[
N(t) \simeq \sinh^2 \left( \int \mu_+ dz \right). 
\]

(3.33)

We now estimate the integral in this formula. If $B < (p/q)^2$ then

\[
\lim_{a \to +0} x = \pm \infty, \quad \lim_{a \to \infty} x = \mp \infty, 
\]

(3.34)

and thus we can use the following approximation:

\[
\int \mu_+ dz \simeq \left| \frac{\epsilon \delta}{2p/q} \right| \left| \frac{dz}{dx} \right|_{\Delta = \Delta_0} \int_{-1}^{1} \sqrt{1 - x^2} dx = \left| \frac{\epsilon \delta}{4p/q} \right| \left| \frac{dz}{dx} \right|_{\Delta = \Delta_0}. 
\]

(3.35)

The quantity $|dz/dx|_{\Delta = \Delta_0}$ can be calculated from (3.30) or (3.31) as follows:

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13To obtain the expressions we have used the fact that $A \propto a^{-2}$ and that $\epsilon \propto a^{-3/2}$.  

18
\[ \left| \frac{dz}{dx} \right|_{\Delta = \Delta_0} = \frac{B}{4(B-1)} \cdot \frac{|\epsilon|}{H\omega^{-1}} \left( 1 + O(\epsilon^2) \right) \] (3.36)

for the $p/q = 1$ resonance with $B < 1$; and

\[ \left| \frac{dz}{dx} \right|_{\Delta = \Delta_0} = \frac{B(B-2)}{32(B-4)} \cdot \frac{\epsilon^2}{H\omega^{-1}} \left( 1 + O(\epsilon^2) \right) \] (3.37)

for the $p/q = 2$ resonance with $B < 4$. If $B = 1$ then $x$ for the $p/q = 1$ resonance satisfies

\[ \lim_{a \to \infty} x = 0, \] (3.38)

and thus the integral is estimated as

\[ \int \mu_+ dz \simeq \frac{\omega}{4} \int_{t_*}^{\infty} \epsilon dt, \] (3.39)

where $t_*$ is the time when the mode enters the $p/q = 1$ resonance band. If $B = 4$ then $x$ for the $p/q = 2$ resonance satisfies

\[ \lim_{a \to \infty} x = \infty, \] (3.40)

and thus the integral $\int \mu_+ dz$ is given by (3.35) at an order estimate, where $|dz/dx|_{\Delta = \Delta_0}$ is given by

\[ \left. \frac{dz}{dx} \right|_{\Delta = \Delta_0} = \frac{3}{2} \cdot \frac{1}{H\omega^{-1}} \left( 1 + O(\epsilon^2) \right). \] (3.41)

Therefore the number of created quanta by the $p/q = 1$ resonance is given by

\[ N \simeq \sinh^2 \left[ \frac{B^2 \pi}{2^5(B-1)} \cdot \frac{\epsilon^2}{H\omega^{-1}} \right] \] (3.42)

for $B < 1$; and

\[ N \simeq \sinh^2 \left[ \frac{\omega}{4} \int_{t_*}^{\infty} \epsilon dt \right] \] (3.43)

for $B = 1$. Note that the integral in the expression (3.43) can be performed when a time dependence of the scale factor $a$ is specified. The number of created quanta by the $p/q = 2$ resonance is

19
\[ N \simeq \sinh^2 \left[ \frac{B^2(B-2)^2 \pi}{2^{12}(B-4)} \cdot \frac{\epsilon^4}{H \omega^{-1}} \right] \]  
(3.44)

for \( B < 4 \); and

\[ N \simeq \sinh^2 \left[ O(1) \cdot \frac{\epsilon^2}{H \omega^{-1}} \right] \]  
(3.45)

for \( B = 4 \).

Finally we mention that in order for the parametric resonance to be efficient to produce the quanta of the Kaluza-Klein mode the following condition is necessary and sufficient:

\[ N \gg 1, \quad \Gamma \gtrsim 3H, \]  
(3.46)

where \( \Gamma \) is a creation rate of the quanta at the center of the resonance band given by

\[ \Gamma = 2\omega \mu_+|_{x=0}. \]  
(3.47)

Using the formula (3.32), \( \Gamma \) can be written explicitly as follows:

\[ \Gamma = \frac{B}{2} \epsilon \omega \]  
(3.48)

for \( B \leq 1 \); and

\[ \Gamma = \frac{B|B-2|}{32} \epsilon^2 \omega \]  
(3.49)

for \( B \leq 4 \).

### C. The \( S_d \) case

We now apply the results obtained in the previous subsection to each model of the compactification. First, in this subsection we consider the \( S_d \) case with \( d \geq 2 \). In this case it can be seen from (3.17) that \( B > 1 \) for \( l \geq 1 \)\(^{14}\). Hence the resonance of \( p/q = 1 \) does not occur. On the other hand, the resonance of \( p/q = 2 \) can occur for \( l = 1 \) since \( B|_{l=1} = 4 \) for

\(^{14}\)See the footnote soon after (3.17).
$d = 2$ and $2 < B_{l=1} < 4$ for $d \geq 3$. For $l \geq 2$, the resonance of $p/q = 2$ does not arise since $B > 4$.

For $d = 2$ and $l = 1$, the number $N$ of created quanta and the creation rate $\Gamma$ are given by (3.45) and (3.49) with $B = 4$. Hence the parametric resonance is efficient to produce the quanta of the corresponding Kaluza-Klein mode $\phi_{1m}$ when

$$\frac{\rho_\sigma}{\rho_0} \gtrsim O(1) \cdot \frac{\omega}{H}. \quad (3.50)$$

where $\rho_\sigma$ is energy density of the oscillation of $\sigma$, which is approximately given by (3.9), and $\rho_0$ is so called critical density of the universe defined by

$$\rho_0 = \frac{3H^2}{\kappa^2}. \quad (3.51)$$

For $d \geq 3$ and $l = 1$, $N$ and $\Gamma$ are given by (3.44) and (3.49) with $B = 2d/(d - 1)$. Hence the parametric resonance is efficient to produce the quanta of the corresponding Kaluza-Klein mode $\phi_{1m}$ when

$$\frac{\rho_\sigma}{\rho_0} \gtrsim O(1) \cdot \left( \frac{\omega}{H} \right)^{3/2}. \quad (3.52)$$

Note that, because of the assumption (3.2), the condition (3.50) or (3.52) requires

$$\frac{\rho_\sigma}{\rho_0} \gg 1. \quad (3.53)$$

Thus in our situation ($H \ll \omega$ and $|\sigma/\sigma_0| \ll 1/D$) we can conclude that the parametric resonance do not overproduce the quanta of the Kaluza-Klein mode in this model.

D. The $S_{d_1} \times S_{d_2}$ case

Following the $S_d$ case, we consider the $S_{d_1} \times S_{d_2}$ case. In this model the parameter $B$ is given by (3.18).

For $l_1$ and $l_2$ satisfying $B < 1$, the number $N$ of created quanta of the corresponding mode and the creation rate $\Gamma$ are given by (3.42) and (3.48), respectively. Hence, if $B < 1$
and \(|B - 1| = O(1)|\), the parametric resonance is efficient to produce the quanta of the Kaluza-Klein mode \(\phi_{l_1l_2m_1m_2}\) only when

\[
\frac{\rho_\sigma}{\rho_0} \gtrsim O(1) \cdot \frac{\omega}{H}. \tag{3.54}
\]

If \(B < 1\) and \(|B - 1| = O(\epsilon)|\) then the condition for efficient production becomes

\[
\frac{\rho_\sigma}{\rho_0} \gtrsim O(1). \tag{3.55}
\]

For \(l_1\) and \(l_2\) satisfying \(B = 1\), \(N\) and \(\Gamma\) are given by (3.43) and (3.48) with \(B = 1\). In order to estimate the integral in the expression of \(N\) we have to specify a time dependence of the scale factor \(a\). If \(a \propto e^{Ht}\) then

\[
\int_{t_\ast}^{\infty} \epsilon dt = 2 \cdot \frac{\epsilon}{3H} \bigg|_{t=t_\ast}. \tag{3.56}
\]

If \(a \propto t^n\) \((n > 2/3)\) then

\[
\int_{t_\ast}^{\infty} \epsilon dt = \frac{2n}{3n-2} \cdot \frac{\epsilon}{H} \bigg|_{t=t_\ast}. \tag{3.57}
\]

If \(a \propto t^n\) \((n \leq 2/3)\) then

\[
\int_{t_\ast}^{\infty} \epsilon dt = \infty. \tag{3.58}
\]

Anyway, the parametric resonance is efficient to produce the quanta of the corresponding Kaluza-Klein mode \(\phi_{l_1l_2m_1m_2}\) when

\[
\frac{\rho_\sigma}{\rho_0} \gtrsim O(1). \tag{3.59}
\]

For \(l_1\) and \(l_2\) satisfying \(1 < B < 2\) or \(2 < B < 4\), \(N\) and \(\Gamma\) are given by (3.44) and (3.49), respectively. Hence in this case, if \(|B - 4| = O(1)|\), the parametric resonance is efficient to produce the quanta of the corresponding Kaluza-Klein mode \(\phi_{l_1l_2m_1m_2}\) only when

\[
\frac{\rho_\sigma}{\rho_0} \gtrsim O(1) \cdot \left(\frac{\omega}{H}\right)^{3/2}. \tag{3.60}
\]

For \(l_1\) and \(l_2\) satisfying \(B = 4\), \(N\) and \(\Gamma\) are given by (3.45) and (3.49) with \(B = 4\). Hence the parametric resonance is efficient to produce the quanta of the corresponding Kaluza-Klein mode \(\phi_{l_1l_2m_1m_2}\) when
\[
\frac{\rho_\sigma}{\rho_0} \gtrsim O(1) \cdot \frac{\omega}{H}.
\]

Finally for \(l_1\) and \(l_2\) satisfying \(B = 2\) or \(B > 4\), \(N\) and \(\Gamma\) are zero in our treatment. Thus, in this case, there is no regime when the the parametric resonance is efficient to produce the quanta of the corresponding Kaluza-Klein mode \(\phi_{l_1,l_2m_1m_2}\).

After all we can conclude that if there is a mode such that \(B \leq 1\) and \(|B - 1| = O(\epsilon)\) then the quanta of the Kaluza-Klein mode can be overproduced by the parametric resonance. Otherwise, at least in our situation \((H \ll \omega\) and \(|\sigma/\sigma_0 \ll 1/D\)) the parametric resonance is so mild that the overproduction does not occur.

**IV. SUMMARY AND DISCUSSION**

In this paper we have investigated a catastrophic creation of quanta of a Kaluza-Klein mode in a \(D\)-dimensional generalized Kaluza-Klein theory. As the origin of the catastrophic creation we have considered a small oscillation of a scale of compactification around a today’s value of it. To make our arguments definite and for simplicity we have considered two classes of models of the compactification: those by \(S_d\) \((d = D - 4)\) and those by \(S_{d_1} \times S_{d_2}\) \((d_1 \geq d_2, d_1 + d_2 = D - 4)\), then have shown that the compactification by \(S_d\) can be stable for \(d \geq 2\) and the compactification by \(S_{d_1} \times S_{d_2}\) can be stable for \(d_1 \geq 2\). For these stable models we have given a hamiltonian for a Kaluza-Klein mode. A 4-dimensional metric and a scale of the compactification have been treated classically, and the Kaluza-Klein mode quantum mechanically. The form of the hamiltonian for the Kaluza-Klein mode shows that a so-called parametric resonance phenomenon can occur: quanta of the Kaluza-Klein mode can be excited catastrophically when frequency and amplitude of the oscillation of the scale of the compactification are in a band in their parameter space. We have given formulas of a creation rate and a number of created quanta of the Kaluza-Klein mode due to the parametric resonance, taking into account the first \((p/q = 1)\) and the second \((p/q = 2)\) resonance band. After that by using the formulas we have calculated those quantities for each model of the compactification.
We have shown that for the model of the compactification by $S_d$ ($d \geq 2$) the first resonance cannot occur but the second resonance can occur for the $l = 1$ mode. For a model of the compactification by $S_{d_1} \times S_{d_2}$, if there exists a set of positive integers $l_1$ and $l_2$ satisfying $B \leq 1$ then the first resonance can also occur for the corresponding mode $\phi_{l_1 l_2 m_1 m_2}$, where $B$ is given by (3.18). Finally we have given conditions for the parametric resonance to produce sufficient quanta of the Kaluza-Klein mode for each model of the compactification. Our conclusion is as follows: (1) in the model of the compactification by $S_d$ the parametric resonance does not overproduce the quanta of the Kaluza-Klein mode; (2) in the model of the compactification by $S_{d_1} \times S_{d_2}$, if there is a mode satisfying $B \leq 1$ and $|B - 1| = O(\epsilon)$, then the quanta of the Kaluza-Klein mode can be overproduced by the parametric resonance; (3) otherwise in the $S_{d_1} \times S_{d_2}$ model, the parametric resonance is so mild that the overproduction does not occur.

As mentioned in Sec. I, if entropy production is negligible, there must not be a catastrophic creation of quanta of a Kaluza-Klein mode. This condition can be regarded as a cosmological criterion for compactification. Thus from the results obtained in this paper, assuming that entropy production is negligible, we can conclude that the model of the compactification by $S_{d_1} \times S_{d_2}$ is ruled out if there is a set of positive integers $l_1$ and $l_2$ satisfying the condition $B \leq 1$ and $|B - 1| = O(\epsilon)$, where $B$ is defined by (3.18). One may expect that inflation followed by reheating can alter the conclusion and save the model. However, maybe it is not the case. In general it is natural that the field $\sigma$, which represents a scale of the compactification, couples to the inflaton field. Hence at the end of the slow rolling phase of the inflation we can expect that energy density $\rho_{\sigma}$ of coherent oscillation of the $\sigma$ is of the same order as energy density of oscillation of the inflaton. In this case the parametric resonance of a Kaluza-Klein mode occurs after inflation, if our 4-dimensional universe is obtained by the compactification with the compact space $S_{d_1} \times S_{d_2}$ and if there is a set of positive integers $l_1$ and $l_2$ satisfying the condition $B \leq 1$ and $|B - 1| = O(\epsilon)$. The energy density $\rho_{KK}$ of the created quanta of the Kaluza-Klein mode is expected to be of the same order as $\rho_{\sigma}$. Hence $\rho_{KK}/\rho_{\text{rad}} = O(1)$ is expected after the reheating. Thus, naively, the
inflation cannot save the model. However, we have to keep in mind that there are, as usual in arguments on early universe, loopholes in the above arguments: (1) a late time inflation may save the model if there is no enough time after the late time inflation for the ratio $\rho_{KK}/\rho_{rad}$ to grow sufficiently; (2) if compactification radius is small compared with experimental scale but large enough compared with Planck scale then there may be a non-trivial range of allowed parameters. To fill up the loopholes is one of the future works we have to do.

Our next question now is whether or not the model of the compactification by $S_d$ and the model by $S_{d_1} \times S_{d_2}$, if there is no integers $l_1$ and $l_2$ satisfying $B \leq 1$ and $|B - 1| = O(\epsilon)$, do safely pass the criterion. We can say that the models are not ruled out from our analysis. However, unfortunately, we cannot answer this question definitely from only the results obtained in this paper since there is a possibility that the structure of the resonance may alter seriously when $\epsilon \gtrsim O(1)$. In this paper we have considered only the case that the scale of the compactification oscillates with a very small amplitude ($\epsilon \ll 1$). However, supposing that our 4-dimensional universe was realized by a dynamical mechanism of compactification from a higher-dimensional spacetime, it is natural that there is an epoch when $\epsilon \gtrsim O(1)$ before the epoch we have considered. The former epoch is called broad resonance regime and in this regime it is expected that the resonance becomes broad and more efficient. In fact, in a simple model of reheating after inflation, it is the case [5]. Thus we have to analyze the former epoch in future works in order to answer the above question.

Although we have investigated only two classes of models of the compactification, qualitative properties of parametric resonance of a Kaluza-Klein mode are expected to be common in a wider class of models of the compactification. It is because mass of a Kaluza-Klein mode and frequency of an oscillation of a scale of the compactification are both expected to be of order of the Planck mass in a wider class of models. Hence it is valuable to analyze a parametric resonance phenomenon of a Kaluza-Klein mode in more realistic models of the compactification with other compact manifolds (eg. Calabi-Yau manifolds) or by other ways of stabilization of the compactification (eg. non-trivial configuration of anti-symmetric
fields, gaugino condensation, etc.).

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APPENDIX A: FORMULA OF THE FLOQUET INDEX

In this appendix we consider the following ordinary differential equation for a real function $Y$ of $z$.

$$\frac{d^2 Y}{dz^2} + h Y = g(z) Y,$$

(A1)

where $h$ is a real constant and $g(z)$ is a real periodic function of $z$ with the period $2\pi$. We assume that $g(z) = g(-z)$. In this case this equation is called a Hill’s equation. For a set of linearly independent two solutions $Y_1(z)$ and $Y_2(z)$, $Y_1(2\pi + z)$ and $Y_2(2\pi + z)$ are linearly independent, too. Hence, by the uniqueness of the solution of the equation, the later two solutions are linear combinations of the former two solutions:

$$\begin{pmatrix} Y_1(2\pi + z) \\ Y_2(2\pi + z) \end{pmatrix} = U \begin{pmatrix} Y_1(z) \\ Y_2(z) \end{pmatrix},$$

(A2)

where $U$ is a $2 \times 2$ regular matrix with all components being real constants. Therefore there are independent linear combinations $\tilde{Y}_1(z)$ and $\tilde{Y}_2(z)$ of $Y_1(z)$ and $Y_2(z)$ such that

$$\tilde{Y}_1(2\pi + z) = \phi_1 \tilde{Y}_1(z),$$

$$\tilde{Y}_2(2\pi + z) = \phi_2 \tilde{Y}_2(z),$$

(A3)

where $\phi_1$ and $\phi_2$ are eigenvalues of $U$ and are not zero by definition. By introducing a constant $\mu$ by $\phi_1 = e^{2\mu \pi}$, $\tilde{Y}_1(z)$ can be expressed by a periodic function $\phi(z)$ with the period $2\pi$ as

\[\text{Note that in general the function } \phi(z) \text{ is complex. When } \mu \text{ is real, } \phi(z) \text{ is a real function.}\]
\[ \tilde{Y}_1(z) = e^{\mu z} \phi(z). \]  
(A4)

Since (A1) is invariant under the replacement \( z \rightarrow -z \), \( \tilde{Y}_1(-z) \) is also a solution. Moreover, if \( \Re \mu \neq 0 \) or \( 2 \Im \mu \) is not an integer then \( \tilde{Y}_1(-z) \) is linearly independent of \( \tilde{Y}_1(z) \). Thus, in this case, \( \phi_2 = e^{-2\mu \pi} \) and the general solution of (A1) can be written as

\[ Y(z) = c_1 e^{\mu z} \phi(z) + c_2 e^{-\mu z} \phi(-z), \]  
(A5)

where \( c_1 \) and \( c_2 \) are complex constants such that both \( Y \) and \( dY/dz \) are real at \( z = z_0 \). The constant \( \mu \) is called a Floquet index.

The equation (A1) is said to be stable when \( \mu \) is pure imaginary, while unstable when \( \mu \) has a non-zero real part [9]. The purpose of this appendix is to obtain a formula of the value of \( \mu \) and to give a criterion when the equation is unstable. For this purpose we use the method of averaging [10]. Let us suppose the case when the function \( g(z) \) is parameterized by a small parameter \( \epsilon \) and \( g(z) = O(\epsilon) \), and assume that \( g(z) \) can be expanded by \( \epsilon \). In this case by redefinition of the real constant \( h \) we can expand \( g(z) \) as

\[ g(z) = \sum_{s=1}^{\infty} \sum_{n=-\infty}^{\infty} \epsilon^s g_n^{(s)} e^{inz} = 2 \sum_{s=1}^{\infty} \sum_{n=1}^{\infty} \epsilon^s g_n^{(s)} \cos nz, \]  
(A6)

where \( g_n^{(s)} (s = 1, 2, \cdots; n = 0, \pm 1, \pm 2, \cdots) \) are real constants satisfying

\[ g_0^{(s)} = 0, \]
\[ g_n^{(s)} = g_{-n}^{(s)} \]  
(A7)

for all \( s \).

Let us introduce a parameter \( \Delta \) by

\[ h = \left( \frac{p}{q} \right)^2 + \epsilon \Delta, \]  
(A8)

\[ ^{16}\text{When} \mu \text{is real, the coefficients} \ c_1 \text{and} \ c_2 \text{are arbitrary real constants.} \]
where \( p \) and \( q \) are mutually prime positive integers. We consider a solution of the equation (A1) of the following form

\[
Y(z) = a \cos \left( \frac{p}{q} z + \theta \right) + \sum_{s=1}^{\infty} \epsilon^s u(s)(a, \theta, z),
\]

(A9)

where \( a \) and \( \theta \) are slowly varying real functions of \( z \):

\[
\frac{da}{dz} = \sum_{s=1}^{\infty} \epsilon^s A_s(a, \theta),
\]

\[
\frac{d\theta}{dz} = \sum_{s=1}^{\infty} \epsilon^s B_s(a, \theta),
\]

(A10)

and \( u(s) \) (\( s = 1, 2, \ldots \)) are functions satisfying

\[
u(s)(a, \theta, z) = u(s)(a, \theta + 2\pi, z) = u(s)(a, \theta, z + 2q\pi).
\]

(A11)

Substituting the ansatz (A9) into the Hill’s equation (A1), we obtain the following \( O(\epsilon) \)- and \( O(\epsilon^2) \)-equations.

\[
\sum_{n=-\infty}^{\infty} \frac{p^2 - n^2}{q^2} u_n^{(1)}(a, \theta) e^{inz/q} = a \left( \sum_{n=-\infty}^{\infty} g_n^{(1)} e^{inz} + \frac{2p}{q} B_1(a, \theta) - \Delta \right) \cos \left( \frac{p}{q} z + \theta \right)
\]

\[
\quad + \frac{2p}{q} A_1(a, \theta) \sin \left( \frac{p}{q} z + \theta \right),
\]

(A12)

\[
\sum_{n=-\infty}^{\infty} \frac{p^2 - n^2}{q^2} u_n^{(2)}(a, \theta) e^{inz/q} = \tilde{g}^{(2)}(a, \theta, z)
\]

\[
\quad + \left( \frac{2p}{q} A_2 + a \frac{\partial B_1}{\partial a} A_1 + a \frac{\partial B_1}{\partial \theta} B_1 + 2A_1 B_1 \right) \sin \left( \frac{p}{q} z + \theta \right)
\]

\[
\quad + \left( \frac{2p}{q} a B_2 - a \frac{\partial A_1}{\partial a} A_1 - a \frac{\partial A_1}{\partial \theta} B_1 + a B_1^2 \right) \cos \left( \frac{p}{q} z + \theta \right),
\]

(A13)

where the fourier components \( u_n^{(1)}(a, \theta) \) and \( u_n^{(2)}(a, \theta) \) of \( u^{(1)}(a, \theta, z) \) and \( u^{(2)}(a, \theta, z) \) have been introduced:

\[
u(s)(a, \theta, z) = \sum_{n=-\infty}^{\infty} u_n^{(s)}(a, \theta) e^{inz/q},
\]

(A14)

\[\text{Note that in order to obtain the formula of the value of } \mu \text{ we only have to seek a special solution of (A1) because of the form of the general solution (A5). So we concentrate on this form of the solution.}\]
and \( \tilde{g}^{(2)} \) is defined by

\[
\tilde{g}^{(2)}(a, \theta, z) \equiv \left( \sum_{n=-\infty}^{\infty} g_n^{(1)} e^{inz} - \Delta \right) u^{(1)}(a, \theta, z) + a \sum_{n=-\infty}^{\infty} g_n^{(2)} e^{inz} \cos \left( \frac{p}{q} z + \theta \right) - 2A_1(a, \theta) \frac{\partial^2 u^{(1)}(a, \theta, z)}{\partial a \partial z} - 2B_1(a, \theta) \frac{\partial^2 u^{(1)}(a, \theta, z)}{\partial \theta \partial z}.
\]

(A15)

From the \( O(\epsilon) \)-equation (A12), we obtain the following form of \( A_1, B_1 \) and \( u^{(1)}(n \neq \pm p) \):

\[
A_1(a, \theta) = -\frac{a}{2p/q} g_{2p/q}^{(1)} \sin 2\theta,
\]

\[
B_1(a, \theta) = \frac{1}{2p/q} \left[ \Delta - g_{2p/q}^{(1)} \cos 2\theta \right],
\]

(A16)

and

\[
u_n^{(1)}(a, \theta) = \frac{aq^2}{2(p^2 - n^2)} \left( g_{n/q-p/q}^{(1)} e^{i\theta} + g_{n/q+p/q}^{(1)} e^{-i\theta} \right).
\]

(A17)

Without loss of generality we can set \( u_{\pm p}^{(1)} \) to be zero by redefinition of \( a \) and \( \theta \). Thus

\[
u^{(1)}(a, \theta, z) = \sum_{n \neq \pm p} \frac{aq^2 g_{n/q-p/q}^{(1)}}{p^2 - n^2} \cos \left( \frac{n}{q} z - \theta \right).
\]

(A18)

Next, from the \( O(\epsilon^2) \)-equation (A13), we obtain

\[
A_2(a, \theta) = -\frac{1}{2p/q} \left( \frac{\partial B_1}{\partial a} A_1 + a \frac{\partial B_1}{\partial \theta} B_1 + 2A_1 B_1 \right) - \frac{i}{2p/q} \left( \tilde{g}_p^{(2)} e^{-i\theta} - \tilde{g}_{-p}^{(2)} e^{i\theta} \right),
\]

\[
B_2(a, \theta) = \frac{1}{a \cdot 2p/q} \left( \frac{\partial A_1}{\partial a} A_1 + a \frac{\partial A_1}{\partial \theta} B_1 - aB_1^2 \right) - \frac{1}{a \cdot 2p/q} \left( \tilde{g}_p^{(2)} e^{-i\theta} + \tilde{g}_{-p}^{(2)} e^{i\theta} \right),
\]

(A19)

where we have introduced the fourier component \( \tilde{g}_n^{(2)} \) of \( \tilde{g}^{(2)} \) by

\[
\tilde{g}^{(2)}(a, \theta, z) = \sum_{n=\infty}^{\infty} \tilde{g}_n^{(2)}(a, \theta) e^{inz/q}.
\]

(A20)

Substituting (A16) and (A18) into (A19), we obtain

\[\text{When } n \text{ is not an integer we define } g_n^{(s)} \text{ by } g_n^{(s)} \equiv 0.\]
\[ A_2(a, \theta) = -\frac{a}{2p/q} \left[ \sum_{n \neq 0, 2p/q} \frac{g_n^{(1)} g_{2p/q-n}^{(1)} - n(2p/q - n)}{n(2p/q - n)} + g_{2p/q}^{(2)} \right] \sin 2\theta, \]

\[ B_2(a, \theta) = \frac{1}{(2p/q)^2} \left( g_{2p/q}^{(1)2} - \Delta^2 \right) - \frac{1}{2p/q} \sum_{n \neq 0, 2p/q} \frac{g_n^{(1)2}}{n(2p/q - n)} \]

\[ -\frac{1}{2p/q} \left[ \sum_{n \neq 0, 2p/q} \frac{g_n^{(1)2} g_{2p/q-n}^{(1)} - n(2p/q - n)}{n(2p/q - n)} + g_{2p/q}^{(2)} \right] \cos 2\theta. \]  

(A21)

Finally we seek the formula of the value of \( \mu \) up to order \( O(\epsilon^2) \) by using the results obtained here. Let us introduce slowly varying functions \( x \) and \( y \) of \( z \) by

\[ x(z) \equiv a \cos \theta, \]

\[ y(z) \equiv a \sin \theta. \]  

(A22)

The following equation for \( x \) and \( y \) can be obtained from (A10), (A16) and (A21).

\[ \frac{d}{dz} \begin{pmatrix} x(z) \\ y(z) \end{pmatrix} = V \begin{pmatrix} x(z) \\ y(z) \end{pmatrix} + O(\epsilon^3), \]  

(A23)

where \( V \) is a \( 2 \times 2 \) matrix of the form

\[ V = \frac{\epsilon}{2p/q} \begin{pmatrix} 0 & -(\delta + \tilde{\Delta}) \\ -(\delta - \tilde{\Delta}) & 0 \end{pmatrix}. \]  

(A24)

The constants \( \delta \) and \( \tilde{\Delta} \) are defined by

\[ \delta = g_{2p/q}^{(1)} + \epsilon \left[ \sum_{n \neq 0, 2p/q} \frac{g_n^{(1)} g_{2p/q-n}^{(1)} - n(2p/q - n)}{n(2p/q - n)} + g_{2p/q}^{(2)} \right], \]

\[ \tilde{\Delta} = \Delta + \epsilon \left[ \frac{1}{(2p/q)^2} \left( g_{2p/q}^{(1)2} - \Delta^2 \right) - \sum_{n \neq 0, 2p/q} \frac{g_n^{(1)2}}{n(2p/q - n)} \right]. \]  

(A25)

Since up to order \( O(\epsilon^2) \) the Floquet index \( \mu \) can be understood as an eigenvalue of \( V \), it is given by

\[ \mu = \mu_{\pm} = \pm \frac{|\epsilon|}{2p/q} \sqrt{\delta^2 - \tilde{\Delta}^2} + O(\epsilon^3). \]  

(A26)

Thus the necessary and sufficient condition for the equation (A1) to be unstable is

\[ -|\delta| < \tilde{\Delta} < |\delta|. \]  

(A27)
This inequality can be solved with respect to $\Delta$ by using the fact that $\Delta = 0(1)$. The result is as follows up to $O(\epsilon)$.

$$\Delta_0 - |\delta| < \Delta < \Delta_0 + |\delta|,$$

where $\Delta_0$ is defined by

$$\Delta_0 = \epsilon \sum_{n \neq 0, 2p/q} \frac{g_n^{(1)2}}{n(2p/q - n)}.$$

**APPENDIX B: NUMBER OF QUANTA EXCITED BY PARAMETRIC RESONANCE**

In this appendix we derive a formula of number of quanta excited by the parametric resonance. Although the formula can be found in literature\(^\text{19}\), we derive it in this appendix for completeness.

We consider a quantum mechanics of an oscillator described by the time dependent Hamiltonian

$$\hat{H} = \frac{1}{2} \left[ \hat{P}^2 + \Omega^2(t)\hat{Q}^2 \right],$$

where $\Omega(t)$ is a positive function of time $t$, $\hat{Q}$ and $\hat{P}$ are operators representing a coordinate of the oscillator and its conjugate momentum:

$$[\hat{Q}, \hat{P}] = i.$$

First let us define the time dependent operator $\hat{a}(t)$ by

$$\hat{a}(t) = \frac{e^{i \int \Omega dt}}{\sqrt{2\Omega}} \left( \Omega \hat{Q} + i\hat{P} \right).$$

\(^{19}\)For example, see Ref. [6]. However, their derivation is incomplete as will be stated later. The obtained formula of the number of quanta is exactly same in Ref. [6] and in this appendix.
The operator \( \hat{a}(t) \) and its hermitian conjugate \( \hat{a}^\dagger(t) \) have the standard commutation relation of creation and annihilation operators:

\[
[\hat{a}, \hat{a}^\dagger] = 1. \tag{B4}
\]

Since the hamiltonian can be expressed as

\[
\hat{H} = \Omega \left( \hat{a}^\dagger \hat{a} + \frac{1}{2} \right), \tag{B5}
\]

the time dependent vacuum state \( |0_t\rangle \), which is defined by \( \hat{a} |0_t\rangle = 0 \), minimizes the value of the Hamiltonian at a moment \( t \). Hence, the creation and the annihilation operators \( \hat{a}(t) \) and \( \hat{a}^\dagger(t) \) regard, so to speak, instantaneous quanta at the moment \( t \).

The equation of motion for \( \hat{Q} \) and \( \hat{P} \) is equivalent to the following differential equation for the operator \( \hat{a}(t) \) \(^{20}\):

\[
\dot{\hat{a}} = \frac{\dot{\Omega}}{2\Omega} e^{2i \int \Omega dt} \hat{a}^\dagger. \tag{B6}
\]

A general solution of this equation is written as

\[
\hat{a}(t) = \alpha(t) \hat{a}(0) + \beta^*(t) \hat{a}^\dagger(0), \tag{B7}
\]

where \( \alpha(t) \) and \( \beta(t) \) are complex functions satisfying the differential equation

\[
\dot{\alpha} = \frac{\dot{\Omega}}{2\Omega} e^{2i \int \Omega dt} \beta, \tag{B8}
\]

\[
\dot{\beta} = \frac{\dot{\Omega}}{2\Omega} e^{-2i \int \Omega dt} \alpha,
\]

and the initial condition

\[
\alpha(0) = 1, \ \beta(0) = 0. \tag{B9}
\]

The number of the instantaneous quanta at the moment \( t \) for the vacuum state \( |0_0\rangle \), which is annihilated by \( \hat{a}(0) \), can be expressed by \( \beta(t) \) as

\[^{20}\text{Hereafter the dot denotes the derivation with respect to the time coordinate } t.\]
\[ N(t) = \langle 0_0| \hat{a}^\dagger(t) \hat{a}(t) |0_0 \rangle = |\beta(t)|^2. \]  

By using (B3) and (B7), the operator \( \hat{Q} \) is expressed by the operator \( \hat{a}(0) \) and its hermitian conjugate \( \hat{a}^\dagger(0) \) as

\[
\hat{Q}(t) = Q^{(+)}(t)\hat{a}(0) + Q^{(-)}(t)\hat{a}^\dagger(0),
\]

where \( Q^{(+)}(t) \) and \( Q^{(-)}(t) \) are defined by

\[
Q^{(+)}(t) = \frac{1}{\sqrt{2\Omega(t)}} \left[ \alpha(t)e^{-i\int \Omega dt} + \beta(t)e^{i\int \Omega dt} \right],
\]

\[
Q^{(-)}(t) = \frac{1}{\sqrt{2\Omega(t)}} \left[ \beta^*(t)e^{-i\int \Omega dt} + \alpha^*(t)e^{i\int \Omega dt} \right],
\]

or equivalently,

\[
\alpha(t) = \frac{e^{i\int \Omega dt}}{\sqrt{2\Omega(t)}} \left[ \Omega(t)Q^{(+)}(t) + i\dot{Q}^{(+)}(t) \right],
\]

\[
\beta(t) = \frac{e^{-i\int \Omega dt}}{\sqrt{2\Omega(t)}} \left[ \Omega(t)Q^{(-)}(t) - i\dot{Q}^{(-)}(t) \right].
\]

Since \( \hat{a}(0) \) and \( \hat{a}^\dagger(0) \) do not commute with each other, the equation of motion of \( \dot{Q}(t) \) means

\[
\ddot{Q}^{(\pm)}(t) + \Omega^2 Q^{(\pm)}(t) = 0.
\]

It is shown that the initial condition (B9) is equivalent to

\[
Q^{(-)}(0) = \frac{1}{\sqrt{2\Omega(0)}}, \quad \Omega(0)Q^{(-)}(0) - i\dot{Q}^{(-)}(0) = 0,
\]

by using the relation

\[
\alpha(t) = \frac{2\Omega(t)}{\Omega(t)}e^{2i\int \Omega dt} \dot{\beta}(t) = \frac{e^{i\int \Omega dt}}{\sqrt{2\Omega(t)}} \left( \Omega(t)Q^{(-)}(t) + i\dot{Q}^{(-)}(t) \right).
\]

Thus, in order to calculate the number of instantaneous quanta \( N(t) \), what we have to do is to seek the solution \( Q^{(-)}(t) \) of the differential equation (B14) with the initial condition (B15), and to substitute it into (B10) with (B13). Note that the function \( Q^{(-)} \) is complex while the classical coordinate \( Q \) corresponding to the operator \( \dot{Q} \) is real.
Let us consider the case when the function $\Omega(t)\) is given by

$$\Omega^2(t) = \omega^2[h - g(\omega t)], \quad (B17)$$

where $\omega$ and $h$ are positive constants, $g(z)$ is a real function given by (A6). In this case the corresponding differential equation for $Q^{(-)}$ is the Hill’s equation (A1) with $Y$ replaced by $Q^{(-)}$, provided that

$$z = \omega t. \quad (B18)$$

In Appendix A we have obtained a real solution of the form (A9). We write the real part and the imaginary part of $Q^{(-)}$ in the form (A9), respectively:

$$Q^{(-)}(t) = (x_1 + ix_2)\cos\left(\frac{p}{q}z\right) - (y_1 + iy_2)\sin\left(\frac{p}{q}z\right) + O(\epsilon), \quad (B19)$$

where $x_{1,2}$ and $y_{1,2}$ are real functions satisfying the following differential equations.

$$\frac{d}{dz} \begin{pmatrix} x_{1,2}(z) \\ y_{1,2}(z) \end{pmatrix} = V \begin{pmatrix} x_{1,2}(z) \\ y_{1,2}(z) \end{pmatrix} + O(\epsilon^3), \quad (B20)$$

where the matrix $V$ is given by (A24). For the functions $x_{1,2}$ and $y_{1,2}$, the following initial condition in the lowest order of $\epsilon$ is derived from the initial condition (B15):

$$x_1(0) = y_2(0) = 1 \sqrt{\omega \cdot \frac{2p}{q}},$$

$$x_2(0) = y_1(0) = 0. \quad (B21)$$

Hence, in the lowest order, the solution of (B20) is given by

$$x_1 = \frac{1}{\sqrt{\omega \cdot \frac{2p}{q}}} \cosh \mu_+ z,$$

$$x_2 = \frac{1}{\sqrt{\omega \cdot \frac{2p}{q}}} \sqrt{\frac{\delta - \hat{\Delta}}{\delta + \hat{\Delta}}} \sinh \mu_+ z,$$

$$y_1 = -\frac{1}{\sqrt{\omega \cdot \frac{2p}{q}}} \sqrt{\frac{\delta + \hat{\Delta}}{\delta - \hat{\Delta}}} \sinh \mu_+ z,$$

$$y_2 = \frac{1}{\sqrt{\omega \cdot \frac{2p}{q}}} \cosh \mu_+ z. \quad (B22)$$
Since it is shown from (B13) and (B19) that
\[
|\beta(t)| = \frac{\sqrt{\omega \cdot 2p/q}}{2} \sqrt{(x_1 - y_2)^2 + (x_2 + y_1)^2 + O(\epsilon)}, \quad (B23)
\]
the number of quanta \(N(t)\) defined by (B10) is \(^{21}\)
\[
N(t) = \frac{1}{1 - \left(\tilde{\Delta}/\delta\right)^2} \sinh^2 \mu_+ z + O(\epsilon). \quad (B24)
\]

Finally we consider the case when the value of \(\delta\) and \(\tilde{\Delta}\) are not constants in time but they are given as functions of \(z\). We assume that their change is so slow that
\[
\left|\frac{d}{dz} \ln |\delta \pm \tilde{\Delta}| \right| \ll \mu_+. \quad (B25)
\]
In this case \(\mu_+ z\) in (B22) is replaced by \(\int \mu_+ dz\). Thus the number of quanta in this case is
\[
N(t) \simeq \sinh^2 \left(\int \mu_+ dz\right). \quad (B26)
\]

\(^{21}\)In Ref. [6] they used the equation \(\ddot{\beta} = \mu_+^2 \beta + O(\epsilon)\) to derive this formula of \(N(t)\). However, the term \(\mu_+^2 \beta\) is of order \(O(\epsilon^2)\) since \(\mu_+\) is of order \(O(\epsilon)\). Thus their derivation is incomplete.
REFERENCES


