Dualities in all order finite $N = 1$ gauge theories

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Abstract

We search for dual gauge theories of all-loop finite, $N = 1$ supersymmetric gauge theories. It is shown how to find explicitly the dual gauge theories of almost all chiral, $N = 1$, all-loop finite gauge theories, while several models have been discussed in detail, including a realistic finite $SU(5)$ unified theory. Out of our search only one all-loop, $N = 1$ finite $SO(10)$ theory emerges, so far, as a candidate for exhibiting also S-duality.

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1. Introduction

Finiteness is an essential conceptual ingredient in the various theoretical frameworks that hopefully will lead to a deeper understanding of nature. There exist many arguments to believe that the divergencies of ordinary field theory are not of fundamental nature but rather they are the manifestation of the existence of new physics at higher scales. Therefore when unification of all interactions has been achieved the theory might be completely finite. This is one of the main motivations and aims of string theory, of non-commutative geometry and of other approaches, which try to include also gravity in the unification of the interactions.

It is well established that there exist finite theories without the inclusion of gravity, i.e. just with gauge interactions. For example it is well known that all $N = 4$ and some $N = 2$ supersymmetric gauge theories are free from ultraviolet divergencies at all orders in perturbation theory. Of particular interest is the fact that there exist $N = 1$ supersymmetric gauge theories, which are finite to all orders in perturbation theory [1]. The 1- and 2-loop finiteness of these theories is guaranteed by, first, choosing the particle content such that the 1-loop gauge $\beta$-function vanishes, and, second, by adding a superpotential such that all 1-loop matter field anomalous dimensions are zero. Moreover, the all-loop finiteness requires that the relations between the Yukawa and gauge couplings, obtained by imposing the vanishing of the 1-loop anomalous dimensions, should be unique solutions of the reduction equations, i.e. to be possible to be uniquely determined to all orders. It is worth noting that along these lines there exist a realistic finite $SU(5)$ GUT which successfully has predicted, among others, the top quark mass [2].

The discussion about finite $N = 1$ gauge theories was so far limited to perturbative aspects, but non-perturbative problems like the bound state spectrum of these theories were not investigated. However, during the recent years a lot of progress in the understanding of supersymmetric gauge theories at strong coupling was achieved. The basic principle that allows to address non-perturbative problems is electric-magnetic, strong-weak coupling duality. $N = 4$ supersymmetric gauge theories are now believed to exhibit an exact Montonen-Olive duality [3]. The low-energy effective action of $N = 2$ supersymmetric
theories [4] can be even solved using duality and holomorphy. $N = 1$ supersymmetric
gauge theories, which have richer dynamics and are much better candidates to describe
the real world, as compared to the others, exhibit a weaker version of the electric-magnetic
duality symmetry [5], namely gauge theories differing in the ultraviolet can establish a
universality class, since they flow in the infrared to the same interacting fixed point of the
renormalization group. This means that two different, but dual $N = 1$ supersymmetric
field theories with different gauge groups and different numbers of interacting particles
nevertheless describe the same physics at low energies.

The main point in the present paper is to find in a systematic way duals of $N = 1$
finite theories in the sense just described. Our search covers most of the chiral $N = 1$
finites gauge theories including the realistic finite $SU(5)$ model. In the simplest case, for
a finite vectorlike $N = 1$ $SU(3)$ gauge theory with $N_f = 3N_c = 9$ fundamental plus
antifundamental quark superfields, the magnetic dual model is given by a non-finite,
asymptotically free $N = 1$ model with gauge group $SU(6)$, and again with 9 fundamental
matter fields [6]. However for more involved models with different kind of gauge groups
and several types of matter representaion the structure of the dual magnetic models is
apriori not determined. Indeed, we will find that the dual of a finite $N = 1$ theory is
almost never finite, but it will be either asymptotically free or non-free. However for
one particular model, namely $SO(10)$ with matter fields in $N_f = 8$ vector and $N_q = 8$
spinor representations, the dual theory has also vanishing one-loop $\beta$-function. Adding a
superpotential to this model, both the electric as well as the magnetic theories are hoped
to be all order finite. Therefore this model may serve as the first step in searching for
$S$-duality in $N = 1$ theories analogous to the one appearing in $N = 4$ theories.

The presence of the duality between seemingly unrelated theories most probably in-
dicates that they are parts of a more fundamental underlying theory. Indeed, studying
non-perturbative duality symmetries in string theory [7], many of the fields theory duali-
ties find a very beautiful, geometrical explanation. In particular, many (non)-perturbative
problems in gauge theories can be studied by considering the world volume dynamics of
various intersecting p-brane configurations, in the low-energy limit, where the gravita-
tional fluctuations are frozen, and only the low-dimensional gauge degrees of freedom on
the world volume are relevant [8]. In this context, it has been proposed that configurations
of branes in type II string theory are a natural framework to study four dimensional N=1
supersymmetric gauge dynamics [9]. The N=1 duality relates theories which in the brane
construction can be connected by a continuous path of rearranging the brane configura-
tions in the moduli space of vacua, along which the gauge symmetry is completely broken
and the infrared dynamics is weak. We examine several examples of brane constructions
in which the resulting low energy field theory is a finite gauge theory.

Our paper is organized as follows. In the next section we will review those aspects of
the $N = 1$ duality of [5] which will be relevant for our further study. Next, in section
three, we will recall the all order finiteness theorems for $N = 1$ supersymmetric gauge
theories. In section four we will search for duals of finite $N = 1$, $SU(N_c)$ gauge theories,
namely first for a simple $SU(3)$ gauge group with 9 vectorlike matter fields [6], second
for vectorlike finite $SU(N_c)$ models with two additional adjoint matter fields and third
for chiral finite $SU(N_c)$ gauge theories with one symmetric, one antisymmetric plus one
adjoint matter field, where in addition a special type of superpotential is required [10,11].
In section five we investigate an $SO(10)$ model with eight vector and spinor matter fields
[12], where both the initial as well as the dual magnetic theory are supposed to be all
order finite. In section six we will extend the search for duals of finite models by using
the deconfinement method [13], which allows us to generically include also an arbitrary
number of matter fields which transform under tensorial representations of the original
theory, namely symmetric and antisymmetric tensors of $SU(N_c)$. As a particular example
within this class we are able to construct the dual of a realistic $SU(5)$ finite gauge theory.
Finally, in section seven, we will construct finite $N = 1$ gauge theories and their duals
from brane configurations.

2. Dualities in $N = 1$ gauge theories

2.1. Supersymmetric QCD

Consider the dynamics of supersymmetric QCD (SQCD) with $N_f$ flavors of quarks in the
$(N_c + \bar{N}_c)$ representation of the gauge group [14]. The corresponding chiral superfields
will be denoted by $Q_i$ and $\bar{Q}_i$, $i = 1, \ldots, N_f$. So let's start with the case $N_f < N_c$. At the classical level the global symmetry group is

$$G_f = SU(N_f)_L \times SU(N_f)_R \times U(1)_B \times U(1)_A \times U(1)_R,$$

where the indices $B, A, R$ in the $U(1)$'s refer to baryon number, axial flavour and axial-$R$ symmetries. All $U(1)$'s are conserved classically but the two axial $U(1)$'s are broken quantum mechanically due to anomalies. However the two anomalous $U(1)$'s can be combined to form an anomaly free $R$-symmetry, say $R_{AF}$, whose charge is

$$Q_{AF} = Q_R + Q_A(N_f - N_c)/N_f$$

with $Q_A, Q_R$ referring to the two anomalous $U(1)$ transformations.

In the absence of mass terms, there is a classical moduli space of vacua\(^2\) which can be described in a gauge invariant way by the expectation values of meson superfields $M_{ij} = Q_i \bar{Q}_j$. The low energy dynamics of SQCD can be represented by an effective Lagrangian which is built out of the gauge invariant chiral superfields and would be a generalization of the corresponding effective Lagrangian of ordinary QCD. If this Lagrangian is built out of gauge invariant combinations of $Q_i$ and $\bar{Q}_i$, it must be a function of $M_{ij}$. The non-perturbative superpotential of this effective Lagrangian can be constructed\(^{15,16}\) by searching for a function of $M$ which is invariant under the non-anomalous global symmetry group (and has charge 2 under $Q_{AF}$). Factors of $\Lambda$ should then be supplied to provide the effective superpotential with dimension 3. It is

$$W_{\text{eff}} = c(\Lambda^{(3N_c - N_f)}/\det M)^{1/(N_c - N_f)}.$$  

This superpotential can be further constrained by considering various limits. For instance for large $M_{ij}$ it can be shown that the gauge symmetry is spontaneously broken

$$SU(N_c) \longrightarrow SU(N_c - N_f).$$

\(^1\)Antichiral superfields will be denoted by dagger.

\(^2\)Every time there exists a continuous family of solutions to the conditions for unbroken supersymmetry, there exist a manifold of degenerate vacuum states with zero energy. This manifold will be typically be parametrized by the vevs of chiral superfields, i.e. it will be a complex Kähler manifold, which is usually called moduli space.
Probably the most important characteristic of the superpotential is that it leads to a squark potential which tends to zero as $\det M$ tends to infinity. Therefore, the quantum theory does not have a ground state.

The above superpotentials are linked to each other by holomorphic decoupling. Then, if this superpotential is known explicitly for a particular $N_f$, one can compute its coefficient for all $N_f$, $N_c$. In fact in [15] it was shown that there is a direct derivation of the superpotential for the case $N_f = N_c - 1$. In this case the vevs of $Q_i$ and $\bar{Q}_i$ break the $SU(N_c)$ gauge symmetry completely.

The next step is to see what happens for larger values of $N_f$. Let us start with the case $N_f = N_c$. The naive intuition that it would just be a smooth extrapolation of the previous cases fails. Clearly the superpotential (2.3) is singular in this case. However a very interesting new feature now is the fact that is the first case that it is possible to build gauge invariant chiral fields with the quantum numbers of baryons. There exist two such terms

$$B = \epsilon_{j_1...j_N} \epsilon_{\alpha_1...\alpha_N} Q_{j_1}^{\alpha_1} ... Q_{j_N}^{\alpha_N},$$  \hspace{1cm} (2.5)

$$\bar{B} = \epsilon_{j_1...j_N} \epsilon_{\alpha_1...\alpha_N} \bar{Q}_{j_1}^{\alpha_1} ... \bar{Q}_{j_N}^{\alpha_N},$$  \hspace{1cm} (2.6)

where the down, up indices denote flavour, colour respectively. The classical moduli space has again a gauge invariant description in terms of the vevs of mesons $M_{ij}$ and baryons $B$ and $\bar{B}$, subject to the classical constraint

$$\det M - \bar{B}B = 0,$$  \hspace{1cm} (2.7)

which follows from Bose statistics of $B$, $\bar{B}$. One might think that the low energy dynamics of this theory is being described by the fields $M$, $B$, and $\bar{B}$ which are fluctuating subject to the constraint (2.7). However it was argued [17] that this manifold of vacua is distorted by non-perturbative effects. In fact there is no symmetry which prohibits the modification of the constraint (2.7) to

$$\det M - \bar{B}B = \Lambda^{2N_c},$$  \hspace{1cm} (2.8)

as was argued in [17].

The case with $N_f = N_c$ provides a first example of a theory with a moduli space (manifold of vacuum states). Note that the origin $M = B = \bar{B} = 0$ is not on the
moduli space due to (2.8). This in turn implies that the quantum dynamics necessarily break the anomaly free chiral symmetry. In fact different points on the quantum moduli space exhibit different patterns of chiral symmetry breaking. Moreover since there are no singularities on the quantum moduli space, the only massless particles are the moduli, the fluctuations of $M, B, \bar{B}$, preserving (2.8). In the semiclassical limit it is appropriate to think of the theory as being in the Higgs phase. On the other hand near the origin, since the theory is smooth in terms of mesons and baryons, it is appropriate to think of the theory as being in the confining phase. There is a smooth transition among the two regions.

Since the spectrum contains massless composite fermions one has to check whether 't Hooft's anomaly matching conditions [18] are satisfied. At a generic point in the moduli space, the original global symmetry of the model eq.(1) (note that $U(1)_R$ and $U(1)_{AF}$ coincide in this case) is broken down to $U(1)_R$ by the vevs of the fields $M, B$ and $\bar{B}$. However there are certain special points of maximal symmetry, where a large part of $G$ remains unbroken. At these points, the 't Hooft's conditions are especially strong. It is very encouraging that the conditions can be satisfied in the cases that $G$ is broken to

$$SU(N_f)_V \times U(1)_B \times U(1)_R$$

and to

$$SU(N_f)_L \times SU(N_f)_R \times U(1)_R.$$  \hspace{1cm} (2.9)

Therefore the above picture of the behaviour of SQCD for $N_f = N_c$ passes a highly non trivial consistency check.

With the above experience one can go on and discuss the case with $N_f = N_c + 1$. The classical moduli is again described by the gauge invariant superfields mesons $M$ and baryons

$$B_i = \epsilon_{ij_1...j_{Nc}} \epsilon_{\alpha_1...\alpha_{Nc}} Q^{a_1}_{j_1} ... Q^{a_{Nc}}_{j_{Nc}},$$

$$\bar{B}_i = \epsilon_{ij_1...j_{Nc}} \epsilon_{\alpha_1...\alpha_{Nc}} \bar{Q}^{a_1}_{j_1} ... \bar{Q}^{a_{Nc}}_{j_{Nc}},$$

where the $j_i, \alpha_i$ are flavour, colour indices respectively. The fields $B_i, \bar{B}_i$ transform according to $(\tilde{N}_f, 1), (1, N_f)$, respectively, under $SU(N_f) \times SU(N_f)$. For this case it was
proposed [17] that the system is described by the superpotential

\[ W = \frac{1}{\Lambda^{2N_c-1}}(\det M - B_iM^{ij}\tilde{B}_j), \]  

(2.13)

which is invariant under the global symmetry of the model and has charge 2 under the anomaly free R-symmetry.

Some of the characteristic features of this case are the following. Unlike the previous case here the quantum moduli space is the same as the classical one [17]. The spectrum at the origin of field space consists of massless composite mesons and baryons and the chiral symmetry of the theory is unbroken there. Therefore there is confinement without chiral symmetry breaking. Again there exists a smooth transition among the Higgs and confining phases. Finally, the ’t Hooft’s conditions are again met.

### 2.2. Seiberg’s Duality

The picture described above has an obvious generalization in for higher values of \( N_f \). The gauge invariant chiral fields of the theory are the mesons \( M_{ij} \) and the baryons \( B_{ij\ldots k} \), \( \tilde{B}_{ij\ldots k} \). The \( B_{ij\ldots k} \) is built as a product of \( N_c \) quark superfields which contain all flavours except \( i, j, \ldots, k \) and \( \tilde{B} \) is defined similarly. An SU(\( N_f \)) × SU(\( N_f \)) invariant superpotential is given by

\[ W \sim (\det M - B_{ij\ldots k}M^{i\tilde{i}}M^{j\tilde{j}}\ldots M^{k\tilde{k}}\tilde{B}_{ij\ldots k}). \]  

(2.14)

However this superpotential does not have \( R \) charge 2, and the multiplet of fields \( M, B, \tilde{B} \) does not satisfy the ’t Hooft’s anomaly conditions. In fact the mismatch grows with each successive number of flavours.

To solve this puzzle, observe that the baryon superfields in (2.14) have

\[ \tilde{N}_c = N_f - N_c \]  

(2.15)

uncontracted indices. Making this observation, Seiberg’s idea [5] now is to regard these fields as bound states of new superfields \( q_i \) and \( \tilde{q}_i \), \( i = 1, \ldots, N_f \), which transform under the dual gauge group SU(\( \tilde{N}_c \)) according to the fundamental, antifundamental representations respectively. Then the baryon superfields would have the dual description

\[ B_{ij\ldots k} = \epsilon_{\alpha_1\ldots\alpha_{\tilde{N}_c}}q_i^{\alpha_1}q_j^{\alpha_2}\ldots q_k^{\alpha_{\tilde{N}_c}} \]  

(2.16)
and similarly for $\bar{B}$.

The complete proposal of Seiberg is that SQCD with gauge group $SU(N_c)$ and with $N_f$ flavours for $N_f > N_c + 1$, is dual to, i.e. can be described by, a supersymmetric gauge theory based on the group $SU(\tilde{N}_c)$ coupled to the elementary chiral superfields $q_i, \bar{q}_i$, $i = 1, ... N_f$, as well as to an elementary additional superfield $M^{ij}$, which is gauge singlet. The $M$ couples to $q, \bar{q}$ via the tree level superpotential

$$\tilde{W} = q M \bar{q}. \quad (2.17)$$

Without the superpotential the theory would have an additional $U(1)$ global symmetry acting on $M$. One can explicitly check that the superpotential preserves the anomaly free $R$-symmetry. Therefore the newly constructed theory has the same global symmetry as the original SQCD. Seiberg [5] refers to the relation among this theory and the original as non-Abelian electric-magnetic duality.

### 2.3. Fixed points and superconformal invariance

Consider the exact $\beta$-function$^3$ of SQCD with gauge group $SU(N_c)$ coupled to $N_f$ fundamental plus antifundamental matter fields:

$$\beta(g) = -\frac{g^3}{16\pi^2} \frac{3 N_c - N_f + N_f \gamma(g^2)}{1 - N_c g^2/8\pi^2}, \quad (2.18)$$

$$\gamma(g^2) = -\frac{g^2}{8\pi^2} \frac{N_c^2 - 1}{N_c} + O(g^4). \quad (2.19)$$

There is a non-trivial zero of the $\beta$-function for $N_f = (3 - \epsilon)N_c$, $N_f, N_c >> 1$. In this regime the $\beta$-function becomes

$$\beta(g) = -\frac{g^3}{16\pi^2} \frac{\epsilon - g^2/8\pi^2 N_c N_f}{1 - N_c g^2/8\pi^2}. \quad (2.20)$$

Therefore, at order $\epsilon$ the fixed point $g^2_*$ is given by

$$N_c g^2_* = \frac{8\pi^2}{3} \epsilon. \quad (2.21)$$

$^3$For a more detailed discussion on the exact beta-function see section 3.2.
In fact it was argued in [5] that such a fixed point exists in the range $\frac{3}{2}N_c \leq N_f \leq 3N_c$.\(^4\)

In SQCD the key observation is that the superconformal theory at the fixed point has a dual (magnetic) description in terms of a different gauge theory, $SU(N_f - N_c)$.

Let us discuss a little bit more the structure of the superconformal theory at the fixed point. Recall that in supersymmetric theories, there exists a vector supermultiplet of gauge invariant operators, called supercurrent [22] $J_{sup}$, which contains the classically conserved currents associated with $R$-invariance, supersymmetry and translation invariance, i.e. the $R$-current $J^\mu_R$, the supersymmetry current $Q^\mu_\alpha$ and the energy-momentum tensor $T^{\mu\nu}$, respectively. It is worth noting that the two latter currents remain conserved to all orders contrary to the first one. Also the $R$-current $J^\mu_R$ must be the current of the anomaly free $R$-symmetry, while the anomalous is the one which obtains quantum corrections only at one loop according to the nonrenormalization theorem [23], which holds also in the supersymmetric case [24].

There exist a second, chiral supermultiplet, containing, among others, the anomalies of the $R$-current $J^\mu_R$ as well as the trace anomalies of the supersymmetry current $Q^\mu_\alpha$ and the energy-momentum tensor $T^{\mu\nu}$. This supermultiplet is called supertrace anomaly and can be proven that is proportional to the full $\beta$-function $\beta_g$. Therefore at the infrared fixed point of SQCD it holds that

$$T^{\mu} = 0, \quad \sigma^{\mu}_{\alpha\beta} \bar{Q}^{\beta} = 0, \quad \partial_\mu J^\mu_R = 0.$$ \hfill (2.22)

The superconformal algebra gives restrictions on the eigenvalues of these operators. In particular the scaling dimension of a field is bounded by its $R$ charge

$$d \geq \frac{3}{2}|R|,$$ \hfill (2.23)

which holds strictly for gauge invariant operators and the inequality is saturated for chiral and antichiral superfields [25]. This has important consequences. For instance consider the operator product of two chiral operators $O_1(x)O_2(0)$. All the the operators in the resulting expansion have $R = R(O_1) + R(O_2)$ and therefore $d \geq d(O_1) + d(O_2)$. Thus there is no singularity in the expansion at $x = 0$ and we can define the product of the

\(^4\)Such a behavior was already conjectured to hold in ordinary QCD by [19]. For a recent discussion on the status of this speculation see [20] and for recent findings in lattice studies see e.g. [21].
two operators by simply taking the limit of $x$ to zero. If this limit does not vanish, it leads to a new chiral operator $O_3$ whose dimension is $d(O_3) = d(O_1) + d(O_2)$. Since the basic objects of the description so far are chiral superfields, one can work out their scaling dimensions from their R charge. In particular

$$Q\bar{Q} = M \quad \text{has} \quad d = \frac{3}{2} R(Q\bar{Q}) = \frac{3}{2} (N_f - N_c)/N_f$$

(2.24)

$$q\bar{q} = U \quad \text{has} \quad d = \frac{3}{2} R(q\bar{q}) = 3(N_c/N_f).$$

(2.25)

In addition the superpotential $\tilde{W} = qM\bar{q}$ has $R = 2$, as it should in order to preserve $R$-symmetry and $d = 3$, which is the correct value for a marginal perturbation. In SQCD the one loop coefficient of the $\beta$-function vanishes at $N_f = 3N_c$. At this point, the bilinear $U$ has $d = 1$ which is the dimension of a free field. Similarly, the dimension of the bilinear $M$ reaches 1 at $N_f = 3N_c/2$. This value of $N_f$ coincides with the value in which the one loop coefficient of the dual $\beta$-function vanishes and below which the dual theory becomes asymptotically non-free.

In summary, the behavior of SQCD with gauge group $SU(N_c)$ and $N_f$ fundamental plus antifundamental matter fields as a function of the parameter $N_f$ can be characterized as follows. The electric coupling becomes stronger for smaller $N_f$, whereas the magnetic coupling becomes stronger for larger $N_f$. Within the range $\frac{3}{2}N_c \leq N_f \leq 3N_c$ there is a superconformal fixed point for finite coupling constant. Under Seiberg’s duality this range is mapped to itself with self dual point $N_f = 2N_c$; an electric theory with $N_f = 3N_c$ and vanishing one-loop $\beta$-function gets mapped to an asymptotically free magnetic theory with $\tilde{N}_f = \frac{3}{2}\tilde{N}_c$, and vice versa. We will see in the following that for general gauge groups with several matter representations the magnetical (electric) dual of a finite electric (magnetic) theory is not necessarily asymptotically free, but can by either asymptotically free or asymptotically non-free or, in a particular case, again finite, i.e. has vanishing $\beta$-function.

In our search in the present paper we shall examine theories exactly at the point in which the one loop gauge $\beta$ function vanishes. Moreover as we shall see the anomalous dimensions of the superfields will also vanish at one loop which in turn will permit us to meet the conditions necessary to have vanishing $\beta$-functions to all loops. Since we will consider $SU(N_c)$ gauge theories not only with matter fields in the fundamental represen-
tation, and also other gauge groups than $SU(N_c)$, let us give the general expression for the one-loop $\beta$-function in terms of $\mu_{\text{gauge}} = C_2(G)$ and $\mu_{\text{matter}} = \sum_i l(R_i)$, where $C_2(G)$ is the quadratic Casimir of the adjoint representation of the gauge group $G$ and $l(R_i)$ is the Dynkin index of the matter representation $R_i$ [26]:

$$\beta^{(1)}_g = \frac{dg}{dt} = -\frac{g^3}{16\pi^2} (3\mu_{\text{gauge}} - \mu_{\text{matter}}) = \frac{g^3}{16\pi^2} \left[ \sum_i l(R_i) - 3C_2(G) \right].$$

For $\mu_{\text{matter}} < \mu_{\text{gauge}}$ there is an effective superpotential analogous to eq.(3), and at the point $\mu_{\text{matter}} = \mu_{\text{gauge}}$ there is a quantum smoothed moduli space of the form eq.(8). For $\mu_{\text{matter}} < 3\mu_{\text{gauge}}$ ($\mu_{\text{matter}} > 3\mu_{\text{gauge}}$) the theory is asymptotically (infrared) free, and at $\mu_{\text{matter}} = \mu_{\text{gauge}}$ one can construct finite gauge theories (see next chapter). Let us stress that the $\beta$-function of the magnetic dual of a finite electric theory is not universally determined (i.e. we do not know a priori which value of $\mu_{\text{matter}}$ gets mapped to a finite magnetic theory). Also, the self-dual point is non-universal. As discussed before, SQCD with $G = SU(N_c)$ and $N_f$ matter fields in the fundamental plus antifundamental representations has $\mu_{\text{gauge}} = N_c$ and $\mu_{\text{matter}} = N_f$. Adding chiral matter fields in the adjoint representation of $SU(N_c)$, they contribute with $\mu_{\text{matter}}(\text{adj}) = N_c$ to the $\beta$-function; chiral fields in the symmetric tensor representation of dimension $\frac{1}{2}N_c(N_c + 1)$ contribute with $\mu_{\text{matter}}(\text{sym}) = \frac{N_c}{2} + 1$, whereas antisymmetric tensor field matter fields contribute with $\mu_{\text{matter}}(\text{antisym}) = \frac{N_c}{2} - 1$. For $G = SO(10)$ we have that $\mu_{\text{gauge}} = 8$, $\mu_{\text{matter}}(\text{vector}) = 1$ and $\mu_{\text{matter}}(\text{spinor}) = 2$.

3. Finite $N = 1$ Gauge Theories

There exist two known claims on the way how all loop finiteness can be achieved, which we shall address in the present chapter.

3.1. All order finiteness theorem

Let us consider a chiral, anomaly free, $N = 1$ globally supersymmetric gauge theory based on a group $G$ with gauge coupling constant $g$. The superpotential of the theory is given
by
\[ W = \frac{1}{2} m_{ij} \phi_i \phi_j + \frac{1}{6} C_{ijk} \phi_i \phi_j \phi_k \, , \quad (3.27) \]

where \( m_{ij} \) and \( C_{ijk} \) are gauge invariant tensors and the matter field \( \phi_i \) transforms according to the irreducible representation \( R_i \) of the gauge group \( G \).

The \( N = 1 \) non-renormalization theorem [27] ensures that there are no mass and cubic-interaction-term infinities. As a result the only surviving possible infinities are the wave-function renormalization constants \( Z_i \), i.e., one infinity for each field. The one-loop \( \beta \)-function of the gauge coupling \( g \) is given in eq.(2.26). The \( \beta \)-functions of \( C_{ijk} \), by virtue of the non-renormalization theorem [27], are related to the anomalous dimension matrix \( \gamma_j^i \) of the matter fields \( \phi_i \) as:
\[ \beta_{ijk} = \frac{dC_{ijk}}{dt} = C_{ijl} \gamma^l_k + C_{ikl} \gamma^l_j + C_{jkl} \gamma^l_i \, . \quad (3.28) \]

At one-loop level \( \gamma_j^i \) are [26]
\[ \gamma_j^{(1)i} = \frac{1}{32\pi^2} \left[ C^{ikl} C_{jkl} - 2 g^2 C_2(R_i) \delta^i_j \right] , \quad (3.29) \]

where \( C_2(R_i) \) is the quadratic Casimir of the representation \( R_i \), and \( C^{ijk} = C_{ijk}^* \).

Therefore, all the one-loop \( \beta \)-functions of the theory vanish if \( \beta_g^{(1)} \) and \( \gamma_j^{(1)i} \) vanish, i.e.
\[ \sum_i \ell(R_i) = 3C_2(G) \, , \quad (3.30) \]

\[ C^{ikl} C_{jkl} = 2\delta^i_j g^2 C_2(R_i) \, , \quad (3.31) \]

A very interesting result is that the conditions (3.30,3.31) are necessary and sufficient for finiteness at the two-loop level [26].

The one- and two-loop finiteness conditions (3.30,3.31) restrict considerably the possible choices of the irreps. \( R_i \) for a given group \( G \) as well as the Yukawa couplings in the superpotential (3.27). Note in particular that the finiteness conditions cannot be applied to the supersymmetric standard model (SSM), since the presence of a \( U(1) \) gauge group is incompatible with the condition (3.30), due to \( C_2[U(1)] = 0 \). This naturally leads to the expectation that finiteness should be attained at the grand unified level only, the SSM being just the corresponding, low-energy, effective theory.
A natural question to ask is what happens at higher loop orders. The answer is contained in a theorem [28] which states the necessary and sufficient conditions to achieve finiteness at all orders. Before we discuss the theorem let us make some introductory remarks. The finiteness conditions impose relations between gauge and Yukawa couplings. To require such relations which render the couplings mutually dependent at a given renormalization point is trivial. What is not trivial is to guarantee that relations leading to a reduction of the couplings hold at any renormalization point.\(^5\) The necessary, but also sufficient, condition for this to happen is to require that such relations are solutions to the RE’s

\[ \beta_g \frac{dC_{ijk}}{dg} = \beta_{ijk} \tag{3.32} \]

and hold at all orders [30]. Remarkably the existence of all-order solutions to (3.32) can be decided at the one-loop level [30].

Let us now turn to the all-order finiteness theorem [28], which states under which circumstances a \(N = 1\) supersymmetric gauge theory can become finite to all orders in the sense of vanishing \(\beta\)-functions, that is of physical scale invariance. It is based on (a) the structure of the supercurrent in \(N = 1\) supersymmetric gauge theories [22,31,32], and on (b) the non-renormalization properties of \(N = 1\) chiral anomalies [28,24]. Details on the proof can be found in refs. [28] and further discussion in refs. [24,33,34]. Here, we will just discuss the main points of this theorem.

One-loop finiteness, i.e. vanishing of the \(\beta\)-functions at one-loop, implies that the Yukawa couplings \(C_{ijk}\) must be functions of the gauge coupling \(g\). To find a similar condition to all orders it is necessary and sufficient for the Yukawa couplings to be a formal power series in \(g\), which is solution of the RGE’s (3.32).

We can now state the theorem for all-order vanishing \(\beta\)-functions.

**Theorem:**

Consider an \(N = 1\) supersymmetric Yang-Mills theory, with simple gauge group. If the following conditions are satisfied

1. There is no gauge anomaly.

\(^5\)For a discussion of dualities in connection with the reduction of couplings see [29].
2. The gauge $\beta$-function vanishes at one-loop

$$\beta_g^{(1)} = 0 = \sum_i l(R_i) - 3C_2(G). \quad (3.33)$$

3. There exist solutions of the form

$$C_{ijk} = \rho_{ijk} g, \quad \rho_{ijk} \in \mathbb{C}$$

(3.34)

to the conditions of vanishing one-loop matter fields anomalous dimensions

$$\gamma_i^{(1)} = 0 = \frac{1}{32\pi^2} \left[ C^{ikl} C_{jkl} - 2 g^2 C_2(R_i) \delta_{ij} \right]. \quad (3.35)$$

4. these solutions are isolated and non-degenerate when considered as solutions of vanishing one-loop Yukawa $\beta$-functions:

$$\beta_{ijk} = 0. \quad (3.36)$$

Then, each of the solutions (3.34) can be uniquely extended to a formal power series in $g$, and the associated supersymmetric gauge theories models depend on the single coupling constant $g$ with a $\beta$ function which vanishes at all-orders.

It is important to note a few things: The requirement of isolated and non-degenerate solutions guarantees the existence of a formal power series solution to the reduction equations (3.32). The vanishing of the gauge $\beta$-function at one-loop, $\beta_g^{(1)}$, is equivalent to the vanishing of the R current anomaly. The vanishing of the anomalous dimensions at one-loop implies the vanishing of the Yukawa couplings $\beta$-functions at that order. It also implies the vanishing of the chiral anomaly coefficients. This last property is a necessary condition for having $\beta$ functions vanishing at all orders.

### 3.2. The exact $\beta$-functions

The second method of searching for all loop finite theories was suggested some time ago in [35] and was reviewed recently in [6]. It is based on the all orders relation among the gauge $\beta$-function $\beta_g$ and the anomalous dimensions of the superfields $\gamma$ which was first derived using instanton calculus [36] given by

$$\beta_g^{NSVZ} \propto \left[ \sum_i l(R_i) - 3C_2(G) - \sum_i l(R_i) \gamma_i \right], \quad (3.37)$$
which is claimed also to hold non-perturbatively [6]. In addition the second method is based on the general relation among the $\beta$-functions for the Yukawa couplings and anomalous dimensions of the superfields given in (3.28) in accordance with the non-renormalization theorem [37].

Now observe that both $\beta$-functions, the gauge $\beta$-function eq.(3.37) and the Yukawa $\beta$-function eq.(3.28), are linear functionals of the anomalous dimensions of the matter superfields. These anomalous dimensions now are complicated functions of the couplings, however the relations among $\beta$’s and $\gamma$’s are very simple.

The criterion for having a finite theory is that all $\beta$-functions eqs.(3.37) and (3.28) vanish simultaneously. This puts $n$ constraints on the $n$ couplings $g, C_{ijk}$. In the case that these constraints are linearly independent, then one expects that their solutions are isolated points in the space of couplings. However if the one-loop gauge $\beta$-function vanishes, i.e. $\sum_i l(R_i) - 3C_2(G) = 0$, and if only $p$ constraints are linearly independent, then one is led to an $n - p$ dimensional manifold of fixed points. Thus, if some of the $\beta$-functions are linearly dependent, and there is at least one generic fixed point somewhere in the space of couplings, then the theory will have a manifold of fixed points. The theories having couplings chosen to lie on this manifold of fixed points are interacting and finite in the sense that all the $\beta$ functions, which are the physically relevant quantities, vanish. Of course it is always possible that the constraints have no solutions at all, as for instance when they put contradictory conditions on the anomalous dimensions.

In practice, when searching for all loop finite theories the second method gives a faster answer, since it does not require calculation of the the anomalous dimensions even at one-loop. However in general one has to be careful. The first method is certainly better for real calculations in a given theory and gives unambiguous results, but requires more work.

Before closing our reference to the second method it is worth mentioning the relation among the $\beta_{g}^{NSVZ}$ and the $\beta_{g}$ and $\gamma$, when calculated in the DRED scheme. The relationship between $\beta_{g}^{NSVZ}$ and $\beta_{g}^{DRED}$ has been explored recently [38], with the conclusion that there exist an analytic redefinition of $g$, $g \rightarrow g'(g, C)$ which connects them, however the two schemes start to deviate at three loops.
4. Duals of $SU(N)$ finite models with vectors and tensors

In the first examples on duality transformations in four dimensional $N = 1$ supersymmetric gauge theories, presented in [5,39,40], the dual pairs were gauge theories based on the classical groups $SU(N)$, $SO(N)$, $Sp(N)$ containing a single group factor and matter fields in the fundamental plus anti-fundamental irreps for $SU(N)$, vector for $SO(N)$ and fundamental for $Sp(N)$, i.e. with matter fields in the defining representation of the corresponding gauge group. Among the possible generalizations of these constructions interesting dual pairs have been found containing tensor matter fields [11],[41], paving the way to search for duals of realistic GUTs. A quite complete list of dual pairs has been presented in [10] containing matter fields in two-index tensor representations in addition to the fields in the defining representation of the above gauge groups. A common feature of these constructions is the fact that they relate pairs of theories of similar type. In addition they all contain gauge singlet mesons in the dual superpotentials and in all the self-dual points there exist marginal operators, which take the form of meson mass terms.

The above classification of dual pairs allows us to examine all one loop finite chiral models based on the $SU(N)$ gauge group with matter fields in one conjugate symmetric, one antisymmetric and one adjoint representations in addition to fields transforming according to the fundamental or anti-fundamental ones in the prospect of making them all loop finite. In general, to do so we have to add in the superpotential, which in [10] is restricted in gauge invariant renormalizable terms made out of the tensor fields also the corresponding ones involving combination of tensors and fundamentals. In the following, first we briefly recall SQCD with only fundamental matter fields. Then we apply the methods of [10] in a vector-like one loop finite $SU(N)$ model with two fields in the adjoint representation, and third we present representative examples of chiral models as well as the basic features of the general $SU(N)$ models with the above particle content. Any chiral model of this class, listed in [42], can be examined in a straightforward way following our examples.
4.1. A simple example

Let us first discuss the application of the method described in section 3.2 to a simple example, namely SQCD. SQCD has played a major role in the recent developments of duality and other exact results in $N = 1$ supersymmetric gauge theories, as reviewed in section 2. Since we are looking for a model which allows for gauge invariant Yukawa couplings, so we are restricted to $N_c = 3$ where the baryons are cubic and can be added to the superpotential. Then the condition that the one-loop $\beta$ function vanishes yields $N_f = 9$. This model was analysed in [6]. Let us sumarize the results.

The model under consideration has the superpotential

$$W = h (Q^1 Q^2 Q^3 + Q^4 Q^5 Q^6 + Q^7 Q^8 Q^9 + \bar{Q}_1 \bar{Q}_2 \bar{Q}_3 + \bar{Q}_4 \bar{Q}_5 \bar{Q}_6 + \bar{Q}_7 \bar{Q}_8 \bar{Q}_9) . \quad (4.38)$$

One can use the exact $\beta$-function (3.37) to show that this is finite. First note that the above superpotential preserves a global $[SU(3)^3 \ltimes S_3]^2 \ltimes Z_2$ subgroup of the $[SU(9)]^2 \ltimes Z_2$ flavour symmetry. Since the $Q$ and $\bar{Q}$ still form irreducible representations of this global symmetry group, it is ensured that they all have the same anomalous dimension. Recalling the exact formulae for the gauge and Yukawa $\beta$-functions (3.37), (3.28) we obtain that in this case

$$\beta_{gauge} = \beta_g \propto 3/2\gamma \quad (4.39)$$

$$\beta_{Yuk} = \beta_h \propto 3/2\gamma. \quad (4.40)$$

The eqs.(4.39), (4.40) are obviously linearly dependent and hence according to the discussion of [6] reviewed in 3.2 the theory is finite, for some relation among the couplings $h$ and $g$. To actually find the required relation among $h$ and $g$ one would have to calculate the $\gamma$ order by order.

The dual is an $SU(6)$ gauge theory with 9 flavors, 81 meson singlets $M$ and superpotential

$$\tilde{W} = M^r_s q_r \tilde{q}^s + (B^{123} + B^{456} + B^{789} + \tilde{B}_{123} + \tilde{B}_{456} + \tilde{B}_{789}) \quad (4.41)$$

where $B^{123} \propto q_4 q_5 q_6 q_7 q_8 q_9$. This is an asymptotically free theory and hence strongly coupled in the infrared. In the infrared the original finite theory and its asymptotically free dual describe the same physics. Note that by inclusion of the superpotential the formerly border case of $N_f = 3N_c$ now has interesting infrared physics, too.
We see that the dual theory of a finite gauge theory is not necessarily finite. Nevertheless it would be much more interesting to find a dual which is also finite. In this case one would expect the usual $N = 1$ duality to be a strong-weak coupling S-duality valid at all scales.

4.2. A vector-like finite $SU(N_c)$ gauge theory with two adjoints

Let us consider a vector-like $N = 1$ theory, based on the gauge group $SU(N_c)$ with two fields in the adjoint representation. For this model to have vanishing gauge $\beta$-function we have to add equal number of left- and right-handed matter fields in the fundamental representation and moreover, according to (2.26) the number of flavours, $N_f$ should be equal to the number of colours, $N_c$. The model with this superfield content has symmetry group

$$G = SU(N_c)_{\text{local}} \times [SU(N_c)_L \times SU(N_c)_R \times U(1)_B \times U(1)_R]_{\text{global}}$$

under which the superfields transform as

$$Q \sim (N_c; N_c, 1; 1, 1/2),$$

$$\bar{Q} \sim (\bar{N}_c; 1, \bar{N}_c; -1, 1/2),$$

$$X \sim (N_c^2 - 1; 1, 1; 0, 1),$$

$$Y \sim (N_c^2 - 1; 1, 1; 0, 1/2).$$

The tree level superpotential is

$$W = TrX^3/3 + TrXY^2 + f_1 \bar{Q}XQ + f_2 \bar{Q}YQ.$$  (4.47)

where the global symmetries are explicitly broken for non-vanishing $f_1$, $f_2$.

The theory with the above superpotential, having vanishing one-loop beta-function can become all orders finite requiring first that the anomalous dimensions of all fields vanish and then checking if the uniqueness requirement of the theorem of section 3.1 is satisfied. We find that the theory is finite to all orders only if $f_1 = g^2$ and $f_2 = 0$. The dual of this model is based on the symmetry group

$$\tilde{G} = SU(5N_c)_{\text{local}} \times [SU(N_f)_L \times SU(N_f)_R \times U(1)_B \times U(1)_R]_{\text{global}}$$

(4.48)
The matter superfield content of the dual theory transforms as follows

\[ q \sim (5N_c; \tilde{N}_c, 1; 1/2, 0), \]  
\[ \bar{q} \sim (5N_c; N_c, -1/2, 0), \]  
\[ X' \sim (25N_c^2 - 1; 1, 1; 0, 1), \]  
\[ Y' \sim (25N_c^2 - 1; 1, 1; 0, 1/2). \]  

The dual theory with the above particle content is asymptotically free. The dual superpotential has the form

\[ \tilde{W} = \text{Tr}X'^3/3 + \text{Tr}X'Y'^2 + M_0qX'Y'^2\bar{q} + M_1qX'Y'\bar{q} + M_2qX'\bar{q} \]  
\[ + N_0qY'^2\bar{q} + N_1qY'\bar{q} + N_2q\bar{q} + f_1N_0 + f_2M_1 \]  

where \( M_0, M_1 \) and \( M_2 \) are singlet fields in the dual theory and correspond to the operators \( QQ, QY\bar{Q} \) and \( QY'^2\bar{Q} \) of the original theory respectively. In the same way \( QX\bar{Q}, QXY\bar{Q} \) and \( QXY'^2\bar{Q} \) get mapped to \( N_0, N_1 \) and \( N_2 \). For the finite theory \( f_1 \neq 0, f_2 = 0 \), the dual gauge theory is broken to \( SU(2N_c) \). The resulting theory has 2 adjoints and after Higgsing a total of \( 8N_c \) fundamentals and antifundamentals. Now the theory with the above particle content is asymptotically non-free. Note that the dual theory is not finite.

### 4.3. Chiral finite \( SU(N_c) \) gauge theories

Next let us consider chiral theories finite theories based on the \( SU(N) \) gauge group and containing matter fields in the adjoint, antisymmetric and conjugate symmetric tensors in addition to those in the fundamental and anti-fundamental representations. Specifically consider an \( SU(N_c), N_c > 3 \) gauge theory with the following symmetry group

\[ G = SU(N_c)_{\text{local}} \times [SU(N_f)_L \times SU(N_f')_R \times U(1)_Y \times U(1)_B \times U(1)_R]_{\text{global}} \]  

and particle content transforming according to

\[ Q \sim (N_c; N_f, 1; \frac{6}{N_f} - 1, \frac{1}{N_c}, 1 - \frac{N_c + 6k}{N_f(k + 1)}), \]  

where \( k \) is an integer.
The theory is chiral and for $N_f = N_c + 4$ and $N'_f = N_c - 4$ the theory has vanishing one-loop $\beta$-function. The superpotential is

$$W = TrX^{(k+1)}/(k + 1) + TrXYY' + f_1\bar{Q}XQ + f_2\bar{Q}\bar{Q}Y + f_3QQY'$$.

Again the full global symmetries are explicitly broken for nonzero $f_1$, $f_2$, $f_3$. In order the theory to be renormalizable $k$ has to be 1 or 2. Then the theory with the above superpotential can become all loop finite in the usual way.

The dual of this theory is an $SU(3k(N_f+ N'_f)/2 - N_c)$ gauge theory with the following full symmetry

$$\tilde{G} = SU(\tilde{N}_c)_{local} \times [SU(N_f) \times SU(N'_f) \times U(1)_Y \times U(1)_B \times U(1)_R]_{global}$$

and particle content transforming according to

$$q \sim (\tilde{N}_c; N_f; 1; -\frac{6}{N_f} + 1, \frac{1}{N_c}, 1 - \frac{\tilde{N}_c + 6k}{N_f(k + 1)})$$,

$$\bar{q} \sim (\tilde{N}_c; 1, \tilde{N}'_f; -\frac{6}{N'_f} - 1, -\frac{1}{N_c}, 1 - \frac{\tilde{N}_c - 6k}{N'_f(k + 1)})$$,

$$\tilde{X} \sim (\tilde{N}_c^2 - 1; 1, 1; 0, 0, \frac{2}{k + 1})$$,

$$\tilde{Y} \sim (\tilde{N}_c(\tilde{N}_c - 1)/2; 1, 1; -1, \frac{2}{N_c}, \frac{2}{k + 1})$$,

$$\tilde{Y}' \sim (\tilde{N}_c(\tilde{N}_c + 1)/2; 1, 1, 1; 1, -\frac{2}{N_c}, \frac{2}{k + 1})$$.

In addition there are the meson fields $N$, $P$, $\tilde{P}$ and $M$.

The dual theory with the above gauge group and particle content is asymptotically free and has the following superpotential

$$\tilde{W} = Tr\tilde{X}^{(k+1)}/(k + 1) + Tr\tilde{X}\tilde{Y}\tilde{Y}'$$.
\[ + \sum_{j=0}^{k-1} \left\{ N_j \bar{q} \bar{X}^{k-j-1} q + P_j q \bar{X}^{k-j-1} \bar{Y}' + \tilde{P} q \tilde{Y} \bar{X}^{k-j-1} \bar{q} + M_j \bar{q} \bar{X}^{k-j-1} \bar{Y} \tilde{Y}' q \right\} + f_1 M_1 + f_2 \tilde{P}_0 + f_3 P_0 \]

Let’s examine the theory when \( k = 2 \). In that case the dual theory is an \( SU(5N_c) \) gauge theory and when all \( f_i, i = 1, 2, 3 \) are non-vanishing the theory can break to \( SU(3N_c - 2) \) at most. In the latter case the theory contains \( 4N_c + 8 \) superfields in the fundamental representation, \( 4N_c \) in the anti-fundamental, one superfield in the adjoint and one in the conjugate symmetric representations, while there is no remaining superfield in the antisymmetric rep. With this particle content the theory is asymptotically non-free.

Next let us discuss a couple of specific models based on \( SU(N_c) \), one with \( N_c \) odd and the other with \( N_c \) even which, as we shall see, distinguishes further these models. All other models can then be examined similarly.

Let us consider an \( SU(5) \) gauge theory containing the chiral supermultiplets \((5, \bar{5}, 24, 10, \bar{15})\) with multiplicities \((9, 1, 1, 1, 1)\). The model with this particle content has vanishing one-loop \( \beta \)-function. The superpotential for \( k = 1 \) becomes

\[
W = (Tr 24^3)/3 + Tr 24 10 \bar{15} + \sum_{i=1}^{5} f_1 \bar{5}_i 24 \bar{5}_i 5 + \sum_{i} f_2 \bar{5}_i 5_i 10 + \sum_{i} f_3 5_i 5_i \bar{15} \tag{4.68}
\]

This model can become all loop finite. Its dual is an \( SU(25) \) gauge theory, which contains the chiral supermultiplets \((25, 25, 224, 105, 120)\) with multiplicities \((9, 1, 1, 1, 1)\). The dual theory then looks from first sight rather unattractive. The superpotential can be read from eq.(4.67) and from the matching of the gauge invariant operators by straightforward substitution of the corresponding fields. The theory becomes simpler after considering its spontaneous breakdown. Then the vevs \( < \bar{25}_i 224 25_i > \), which are enforced by the equations of motion from the dual superpotential, break the gauge group of the dual theory by one unit each, the vevs \( < \bar{25}_i 105 25_i > \) break the gauge group by two units all together (since the antisymmetric tensor field is absorbed via the Higgs mechanism) and the vevs \( < 25, 120 25_i > \) break the gauge group by one unit each [43]. Therefore the dual theory eventually is an \( SU(13) \), which contains one adjoint, one antisymmetric and one conjugate symmetric tensors and 20 superfields in the fundamental and 28 in the antifundamental reps. The theory is asymptotically non-free.
As a second concrete example let us consider an $SU(4)$ gauge theory. The theory with particle content in the representations $(4, 15, 6, 10)$ with multiplicities $(8, 1, 1, 1)$ has vanishing one loop $\beta$-function. A similar analysis as was done in the previous example is straightforward. What is more interesting here is that the model belongs to the class of models with $N_c = kn$. Then with $k = 2$ the theory has flat directions [10], which break the group to $SU(2) \times SU(2)$. The dual gauge group breaks from $SU(3k(N_f + N_f^/) / 2 - N_c)$ down to $SU(3(N_f + N_f^/ / 2 - 2)^k$, i.e. in the present case to $SU(8) \times SU(8)$.

5. Duals of Chiral Finite Models: Vectors and Spinors in $SO(10)$

5.1. The method and an instructive example

This method of searching for duals applies to $N = 1$ supersymmetric gauge theories with gauge group $SO(10)$ and matter fields in $N_q$ spinor representations and $N_f$ vector representations [12]. Its dual counterpart is a chiral model with semisimple gauge group $SU(N_f + 2N_q - 7) \times Sp(2N_q - 2)$.

It is instructive to consider first a relatively simple example consisting of a $N = 1$ supersymmetric $SO(10)$ model with two spinors and $N_f$ vector matter fields. The model has symmetry group:

$$G = SO(10)_{local} \times [SU(N_f) \times SU(2) \times U(1)_Y \times U(1)_R]_{global}$$ (5.69)

under which the chiral superfields transform as

$$V_{ij}^{\mu} \sim (10; \square, 1; -4, \frac{N_f - 4}{N_f + 4})$$ (5.70)

$$Q_i^a \sim (16; 1, 2; N_f, \frac{N_f - 4}{N_f + 4})$$ (5.71)

and has vanishing superpotential. With the above hypercharge and R-charge assignements for the matter fields the model is free of global anomalies and is asymptotically free for $N_f < 20$.

Let us discuss first its confining phase. Assuming the following hierarchy of successive
spontaneous symmetry breakings

\[ SO(10) \xrightarrow{2^{(16)}} G_2 \xrightarrow{10} SU(3) \xrightarrow{10} SU(2) \xrightarrow{10} 1, \]  

(5.72)

one can easily count the gauge invariant operators needed to act as moduli space coordinates for small numbers of vector flavours. These gauge invariant operators appear in the effective low energy description of the theory. In table (5.1) the number of partonic degrees of freedom and the generic unbroken gauge subgroup \( H_{local} \) as a function of \( N_f \) are given. The dimension of the coset space \( G_{local}/H_{local} \) coincides with the number of matter fields eaten by the Higgs mechanism. The number of remaining partons listed in the table equals the number of independent hadrons which label flat directions in the effective theory.

<table>
<thead>
<tr>
<th>( N_f )</th>
<th>Parton DOF</th>
<th>Unbroken Subgroup</th>
<th>Eaten DOF</th>
<th>Hadrons</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>32</td>
<td>( G_2 )</td>
<td>45 - 14 = 31</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>42</td>
<td>( SU(3) )</td>
<td>45 - 8 = 37</td>
<td>5</td>
</tr>
<tr>
<td>2</td>
<td>52</td>
<td>( SU(2) )</td>
<td>45 - 3 = 42</td>
<td>10</td>
</tr>
<tr>
<td>3</td>
<td>62</td>
<td>1</td>
<td>45</td>
<td>17</td>
</tr>
<tr>
<td>4</td>
<td>72</td>
<td>1</td>
<td>45</td>
<td>27</td>
</tr>
<tr>
<td>5</td>
<td>82</td>
<td>1</td>
<td>45</td>
<td>37</td>
</tr>
</tbody>
</table>

Table 1: Number of gauge invariant operators which are the coordinates of moduli space

In order to construct explicitly the hadron fields we recall the tensor product

\[ 16 \times 16 = 10_s + 120_A + 126_s = [1]_s + [3]_A + [5]_s, \]  

(5.73)

where \([n]\) denotes an antisymmetric rank-\(n\) tensor, the ‘S’ and ‘A’ subscripts indicate symmetry and antisymmetry under spinor exchange and the tilde over the last term implies that the rank-5 irrep is complex self-dual. One can then produce the following gauge invariant composites

\[ K = Q_T^r(\sigma_3\sigma_2)_{IJ} \Gamma^\mu CQ_J K' \Gamma_{\mu} CQ_L \sim (1; 1, 1; 4N_f, 4\frac{N_f - 4}{N_f + 4}), \]  

(5.74)
\[ M^{ij} = (V^T)^{\mu}_{\nu} V^\nu_j \sim (1; 0, 1; -8, 2 \frac{N_f - 4}{N_f + 4}), \quad (5.75) \]

\[ N^i_x = Q^i_I (\sigma \sigma_2) i,j \sim \gamma^{ij} C Q_J \sim (1; 0, 1; 3, 2N_f - 4, 3 \frac{N_f - 4}{N_f + 4}), \quad (5.76) \]

\[ P^{ijk} = \frac{1}{3!} Q^I_I (\sigma \sigma_2)_I,J \sim \gamma^{ijk} C Q_J \sim (1; 0, 1; 2N_f - 12, 5 \frac{N_f - 4}{N_f + 4}), \quad (5.77) \]

\[ R^{ijkl} = \frac{1}{4!} Q^I_I (\sigma \sigma_2)_I,K \sim \gamma^{ijkl} C Q_L \sim (1; 0, 1; 4N_f - 16, 8 \frac{N_f - 4}{N_f + 4}), \quad (5.78) \]

\[ T^{ijklm}_x = \frac{1}{5!} Q^I_I (\sigma \sigma_2)_I,J \sim \gamma^{ijklm} C Q_J \sim (1; 0, 3; 2N_f - 20, 7 \frac{N_f - 4}{N_f + 4}), \quad (5.80) \]

where Greek, small Latin and large Latin letters respectively denote \( SO(10) \), \( SU(N_f) \) and \( SU(2) \) spinor indices.

Comparing the number of composite operators with the number of independent flat directions as a function of \( N_f \), one finds that \( K,M,N \) and \( P \) account for all massless fields in the \( SO(10) \) model up to three flavours. For \( N_f \) larger or equal to 4 an increasing number of constraints is also needed, which can be imposed in the superpotential form using Lagrange multipliers.

Turning to the search of the dual counterpart of this model we should recall that there exist a class of dual pairs [44] in which one member is a \( N = 1 \) supersymmetric model with gauge group \( G_2 \) with \( N_f \) matter fields in the fundamental representation and its dual is based on \( SU(N_f - 3) \) gauge group. Since the \( SO(10) \) model with two spinors reduces to the above \( G_2 \) model along a flat direction, where both spinors acquire vev, it is natural to start searching for its dual by looking for extensions of the \( SU(N_f - 3) \), which is dual to \( G_2 \). After exploring several possibilities the authors in [12] concluded that the dual of \( SO(10) \) is not a simple group. The simplest generalization found was that the dual has symmetry group

\[ \tilde{G} = [SU(N_f - 3) \times Sp(2)]_{local} \times \gamma [SU(N_f) \times SU(2) \times U(1)_Y \times U(1)_R]_{global} \quad (5.83) \]
with superfield content

\[ q_i^\alpha \sim (\Box, 1; \Box, 1; 2 \frac{N_f - 6}{N_f - 3}, \frac{5(N_f - 4)}{(N_f - 3)(N_f + 4)}) \]  
\[ q_i^{\alpha\dot{\alpha}} \sim (\Box, 2; 1, 2; - \frac{2N_f}{N_f - 3}, \frac{(N_f + 2)(N_f - 4)}{(N_f - 3)(N_f + 4)}) \]  
\[ q_\alpha^X \sim (\Box, 1; 1; - 2N_f \frac{N_f - 4}{N_f - 3}, \frac{(N_f + 2)(N_f - 4)}{(N_f - 3)(N_f + 4)}) \]  
\[ s_{\alpha\beta} \sim (\Box, 1, 1; 4 \frac{N_f - 4}{N_f + 4}) \]  
\[ t_i^{\dot{\alpha}} \sim (1, 2; 1, 2; 2N_f, 2N_f - 4) \]  
\[ m^{(ij)} \sim (1, 1; \Box, 1; -8, 2 \frac{N_f - 4}{N_f + 4}) \]  
\[ n_i^X \sim (1, 1; \Box, 3; 2N_f - 4, 3 \frac{N_f - 4}{N_f + 4}) \]

and tree level superpotential

\[ \tilde{W} = \frac{1}{\mu_1} m^{(ij)} q_i^{\alpha\dot{\alpha}} s_{\alpha\beta} q_j^\beta + \frac{1}{\mu_2} n_i^X q_i^X q_\alpha^X + \lambda_1 \epsilon_{\alpha\beta\dot{\alpha}\dot{\beta}} s_{\alpha\beta} q_i^{\alpha\dot{\alpha}} s_{\beta\dot{\beta}} q_j^{\beta\dot{\beta}} + \lambda_2 \epsilon_{\alpha\beta\dot{\alpha}\dot{\beta}} q_i^{\alpha\dot{\alpha}} (\sigma_\times \sigma_2)^{(i)} q_j^{\beta\dot{\beta}}. \]  

Then the above construction was generalized to $N = 1$ supersymmetric $SO(10)$ gauge theory with arbitrary $N_q$ spinor representations and $N_f$ vector representations.

### 5.2. The dual of a finite $SO(10)$ gauge theory

Next let us apply the generalized method to an $SO(10)$ finite model. Consider a $N = 1$ supersymmetric model based on the gauge group $SO(10)$ with matter fields in $N_f = 8$ vector and $N_q = 8$ spinor representations. The model with this superfield content has vanishing one-loop gauge $\beta$ function and symmetry group

\[ G = SO(10)_{local} \times SU(8) \times SU(8) \times U(1)_Y \times U(1)_R_{global} \]

under which the superfields transform as

\[ V_{i,\mu} \sim (10, 8, 1; -16, 2/3) \]  
\[ Q_{A,I} \sim (16; 1, 8, 8, 2/3). \]
Then its dual is found to be based on the gauge symmetry group

\[
\tilde{G} = [SU(17) \times Sp(14)]_{\text{local}} \times [SU(8) \times SU(2) \times U(1)_Y \times U(1)_R]_{\text{global}}
\] (5.96)

It is worth noting that only an \(SU(2)\) subgroup of the global \(SU(8)\) that rotates the spinors in the original theory is realized in the ultraviolet of its dual. The \(SU(2)\) subgroup is embedded inside \(SU(8)\) so that the fundamental 8-irrep of the latter is mapped on the 8-dimensional irrep of the former. The full \(SU(8)\) global symmetry is realized in the dual theory only at long distances.

The matter content of the dual theory has the following transformation properties

\[
q^i = (17, 1; \bar{8}, 1; 160/17, -14/51), \quad (5.97)
\]

\[
q'^I = (17, 14; 1, 2; -112/17, 20/51), \quad (5.98)
\]

\[
\bar{q}^a_{(I-1...I-2N-q-2)} = (17, 1; 1, 15; -160/17, 14/51), \quad (5.99)
\]

\[
s_{\alpha,\beta} = (153, 1; 1, 1; 224/17, 62/51), \quad (5.100)
\]

\[
t^\alpha_{(I-1...I-13)} = (1, 14; 1, 14; 16, 4/3), \quad (5.101)
\]

\[
m^{(ij)} = (1, 1; 36, 1; -32, 4/3), \quad (5.102)
\]

\[
n^i_{(I-1...I-14)} = (1, 1; 8, 15; 0, 2). \quad (5.103)
\]

The tree level superpotential of the dual theory is

\[
\tilde{W} = \frac{1}{\mu_1^2} mqsq + \lambda_1 q' s q' + \frac{1}{\mu_2^2} qn\bar{q} + \lambda_2 q' \bar{q} t \quad (5.104)
\]

The coefficients \(\mu_{1,2}\) and \(\lambda_{1,2}\) represent dimensionful and dimensionless couplings. The mesons \(m\) and \(n\) are gauge singlets of the dual gauge group corresponding to the gauge invariant operators \(VV\) and \(VQQ\) respectively of the original gauge group.

There are some remarks to be made concerning this dual. The first is that it satisfies all necessary anomaly checks including the anomaly matching conditions associated with the common \(SU(8) \times SU(2) \times U(1)_Y \times U(1)_R\) global symmetry group. The second concerns the \(\beta\) functions of the dual product gauge groups. The first factor has \(\beta\) function coefficient \(b_0^{(1)} = 16\) and therefore it is asymptotically free, while the second vanishes i.e. is finite at one loop.
Now in order to make the original theory finite to all orders we have to add the superpotential

\[ W = h \sum_{i=1}^{8} Q_i Q_i V_i, \]  

(5.105)

\((i\) labeling the 8 flavors\) which is the simplest choice having a universal Yukawa coupling. This can easily be seen to be all-loop finite. One way to see this is to apply the all-loop finiteness theorem of 3.1. The 1-loop \(\beta\) function for the Yukawa coupling reads

\[ \beta_h^{(1)} = h(14|h|^2 - \frac{63}{2}g^2) \]  

(5.106)

and the 1-loop \(\gamma\) functions are

\[ \gamma_{10}^{(1)} = [4|h|^2 - 9g^2] \]  

(5.107)

and

\[ \gamma_{16}^{(1)} = [5|h|^2 - \frac{45}{4}g^2] \]  

(5.108)

which obviously vanish simultaneously for

\[ |h|^2 = \frac{9}{4}g^2 \]  

(5.109)

and satisfy the criteria of the theorem 3.1.

Finiteness can also be established according to the method of [6] by noting that the gauge and the Yukawa \(\beta\) functions

\[ \beta_g \propto 8\gamma_V + 16\gamma_Q \]  

(5.110)

\[ \beta_h \propto \gamma_V + 2\gamma_Q \]  

(5.111)

(5.112)

are linearly dependent.

Now let us study the finiteness on the dual side. The complete dual superpotential is

\[ \tilde{W} = \frac{1}{\mu_1^2}mqsq + \lambda_1 q'sq' + \frac{1}{\mu_2^2}nq\bar{q} + \lambda_2 q'\bar{q}t + \tilde{h}n \]  

(5.113)

The equations of motion for \(N\) enforce \(<q\bar{q}>\) to be nonzero and of rank 8 since we included the sum over all 8 flavors in the original theory. This breaks the dual theory to
while eating $8$ $q$, $\bar{q}$ flavors via the Higgs mechanism. It is easy to work out the matter content charged under the unbroken gauge group. The remaining symmetry group is

$$\tilde{G} = [SU(9) \times Sp(14)]_{local} \times [SU(8) \times SU(2) \times U(1)_R \times U(1)_Y]_{global}$$

with the charged (under $SU(9) \times Sp(14)$) fields transforming like

$$q_1' \sim (1, 14; 8, 2; 2/3, -16),$$
$$q_2' \sim (9, 14; 1, 2; 4/27, 16/9),$$
$$\bar{q} \sim (\bar{9}, 1; 1, 7; 14/27, -160/9),$$
$$s_1 \sim (\mathbf{45}, 1; 1, 1; 46/27, -32/9),$$
$$s_2 \sim (\bar{9}, 1; \bar{8}, 1; 32/27, 128/9),$$
$$t_1 \sim (1, 14; 1, 6; 4/3, 16),$$
$$t_2 \sim (1, 14; 1, 8; 4/3, 16),$$

where $q_1'$ and $q_2'$ come from $q'$ in eq.(5.98), $s_1$ and $s_2$ from $s$ in eq.(5.100) and finally $t_1$ and $t_2$ from $t$ in eq.(5.101). They interact via a superpotential

$$\tilde{W} = \lambda_1 (q_2's_1q_2' + q_1's_2) + \lambda_2 (q_2'\bar{q}t_1 + \bar{q}q_2't_2)$$

The resulting product gauge group now has vanishing one-loop $\beta$ function for both gauge factors! This is very interesting, since it seems to indicate that we this way created a duality between two finite theories. In addition to the above charged fields, there will be in general also gauge singlet fields. These gauge singlets will introduce new interactions in the superpotential which are quite cumbersome to be worked out. Note however that these gauge singlet fields would spoil finiteness. So we propose that indeed the charged sector from above describes the full dual theory and all the gauge singlets actually decouple.

The following facts support this scenario:

- With the superpotential (5.122) the magnetic theory is all-loop finite. It is easy to verify this with the approach of [6]. The gauge and Yukawa $\beta$ functions are

$$\beta_{SU(9)} \sim 28 \gamma_q + 7 \gamma_{\bar{q}} + 19 \gamma_s,$$
\[ \beta_{Sp(14)} \sim 34\gamma q' + 14\gamma t, \quad (5.124) \]
\[ \beta_{\lambda_1} \sim 2\gamma q' + \gamma s, \quad (5.125) \]
\[ \beta_{\lambda_2} \sim \gamma q' + \gamma q + \gamma t. \quad (5.126) \]

Since

\[ 2\beta_{SU(9)} + \beta_{Sp(14)} - 38\beta_{\lambda_1} - 14\beta_{\lambda_2} = 0 \]

all \( \beta \)-functions are linearly dependent, and hence according to [6] we got a finite theory in the sense that all \( \beta \)-functions vanish.

- The anomalies for \( U(1)^3, U(1), U(1)^3_R, U(1)^3_{R'}, U(1)^2_RU(1), U(1)_RU(1)^2 \) match those in the original theory. This is a highly non-trivial check.\(^6\)

6. Searching for duals of FUTs with the Deconfinement Method

6.1. The method and an instructive example

This method of determining duals of given gauge theories can be used in all theories containing superfields in tensor representations.

The basic idea of the method [13] is to assume that the tensors of the gauge theory under consideration are composite fields made out of elementary fields transforming according to the fundamental representation of another gauge group that confines and reproduces the tensorial spectrum of the original theory. Thus the latter gauge group is supposed to describe the fundamental theory and the former is only an effective theory. Since it is known how to ‘deconfine’ an arbitrary two-index tensor, the method can be applied to all chiral models in the list of [42], with the exception of \( E_6 \) models and \( SO(10) \) with spinors.

\(^6\)For the non-abelian symmetries things are much more complicated. Already in the unbroken theory of [12] the issue of accidental symmetries [45] complicates the discussion of the non-abelian global symmetries. Therefore we did not work out the anomaly matching with respect to the non-abelian global symmetries.
The method has the advantage that it is straightforward, and the obvious disadvantage that it introduces a new ‘fundamental’ gauge group for every tensor superfield present in the original theory. Therefore the dual theories obtained by applying this method are usually rather complicated.

In the following first we present a simple example following [46], in order to demonstrate how the method works in practice, then we apply it in a simple chiral one-loop finite $SU(3)$ model and then we find the dual of the realistic finite $SU(5)$ GUT. The method can then applied in all other models of [42] that contain tensor superfields with the exception already mentioned.

Let us first consider an $SU(N_c)$, $N = 1$ supersymmetric gauge theory with the following matter superfields: $N_f$ in the fundamental, $N'_f$ in the anti-fundamental and one in the antisymmetric tensor representations. The $SU(N_c)^3$ anomaly cancellation requires $N_f = N'_f - N_c + 4$. The full symmetry group of the theory is then

$$G = SU(N_c)_{\text{local}} \times [SU(N_f)_L \times SU(N'_f)_R] \times U(1)_Y \times U(1)_B \times U(1)_R]_{\text{global}}$$

and the superfields are transforming according to

$$Q \sim (N_c; N_f, 1; 1, N_c - N_f, 2 - 6/N_c),$$

$$\bar{Q} \sim (\bar{N_c}; 1, N'_f; -N_f/(N_c + N_f - 4), m_f, 6/N_c),$$

$$Y \sim (N_c(N_c + 1)/2; 1, 1; 0, -2N_f, -12/N_c).$$

To see how the method works consider a $N = 1$ gauge theory based on the $Sp((N_c - 3)/2)$ group ($N_c$ odd) containing $N_c + 1$ matter fields in the fundamental representation say, $y, z$ and $N_c$ singlets $\bar{P}_i, i = 1, \ldots, N_c$. Therefore the theory has the following symmetry

$$G' = Sp((N_c - 3)/2)_{\text{local}} \times [SU(N_f)_L \times SU(N'_f)_R]_{\text{global}}$$

and the matter fields transform according to

$$y, z \sim ((N_c - 3)/2; N_c + 1, 1),$$

$$\bar{P}_i \sim (1; 1, N_c).$$

This is a confining theory and provides a superpotential [40]

$$W = y^N z = A^{((N-1)/2)} P,$$
for the gauge invariant fields $A_{ij} = y_i y_j$ and $P_i = z y_i$. Then one adds a mass term for $P_i$ and $\bar{P}_i$ in the superpotential in the form of a coupling $z y_i \bar{P}_i$, which breaks the global flavour symmetry

$$SU(N_c + 1) \times SU(N_c) \rightarrow SU(N_c) \times U(1).$$

Integrating out the superfields $P_i$, $\bar{P}_i$ one obtains a theory with $N_c(N_c - 1)/2$ singlet superfields $A_{ij}$ without superpotential, i.e. without constraints on the light superfields $A_{ij}$.

Next consider gauging the $SU(N_c)$ flavour symmetry. Clearly under this symmetry the superfield $A$ transforms as an antisymmetric tensor. Then in order to cancel the $SU(N_c)^3$ anomaly one has to introduce more $Sp((N_c - 3)/2)$ singlets say, $Q$ and $\bar{Q}$, which transform according to the fundamental and anti-fundamental reps of $SU(N_c)$.

Therefore the above constructed $SU(N_c) \times Sp((N_c - 3)/2)$ theory is equivalent to the original theory. The charge assignments in this expanded theory can be determined from the superpotential and the definitions of the composite fields and its full resulting symmetry is

$$G'' = [SU(N_c) \times Sp((N_c - 3)/2)]_{local} \times [SU(N_f)_L \times SU(N_f)_R \times U(1)_Y \times U(1)_B \times U(1)_R]_{global}$$

The matter fields transform under the above symmetry group according to

- $y \sim (N_c, (N_c - 3)/2; 1, 1; 0, -N_f, -6/N_c)$, 
- $z \sim (1, (N_c - 3)/2; 1, 1; 0, N_f N_c, 8)$, 
- $\bar{P} \sim (\bar{N}_c, 1; 1, 1; 0, N_f(1 - N_c), -6 + 6/N_c)$, 
- $Q \sim (N_c, 1; N_f, 1; 1, N_c - N_f, 2 - 6/N_c)$, 
- $\bar{Q} \sim (\bar{N}_c, 1; 1, N_c + N_f - 4; -N_f/(N_c + N_f - 4), N_f, 6/N_c)$.

The $SU(N_c)$ gauge group is in a non-Abelian Coulomb phase for $N_f \geq 5$, and therefore can be dualized by the $SU(N_c)$ duality prescription of [5]. The resulting dual model is a gauge theory based on the

$$SU(N_f - 3) \times Sp((N_c - 3)/2)$$
group with the following symmetry

\[ \tilde{G} = [SU(N_f - 3) \times Sp((N_c - 3)/2)]_{local} \times [SU(N_f)_L \times SU(N_f)_R]_{global} \]  

(6.143)

with matter content transforming according to

\[
\begin{align*}
\tilde{x} & \sim (N_f - 3), (N_c - 3)/2, 1, 1, \\
p & \sim (N_f - 3, 1; 1, 1), \\
\bar{q} & \sim \left( (N_f - 3), 1; \tilde{N}_f, 1 \right), \\
q & \sim (N_f - 3, 1; 1, (\tilde{N}_c - N_f - 4)), \\
\bar{l} & \sim (1, (N_c - 3)/2; 1, N_c + N_f - 4),
\end{align*}
\]

(6.144)

\[
\begin{align*}
M & = Q\bar{Q} \sim (1, 1; N_f, N_c + N_f - 4), \\
B_1 & = QA^{(N_c - 1)/2} \sim (1, 1; N_f, 1),
\end{align*}
\]

(6.149)

where \( M = Q\bar{Q} \) are mesons and \( B_k = Q^k A^{((N-k)/2)} \) are baryons. The dual theory has superpotential

\[ \tilde{W} = Mq\bar{q} + B_1p\bar{q} + \bar{l}\tilde{x}q. \]  

(6.151)

The gauge group \( Sp((N_c - 3)/2) \) for \( N_f \geq 5 \) is in a non-abelian Coulomb phase and can be dualized by the \( Sp \) duality according to [40]. The resulting dual theory after integrating out the massive fields a gauge theory based on the group \( SU(N_f - 3) \times Sp(N_f - 4) \) with full symmetry

\[ \tilde{G}' = [SU(N_f - 3) \times Sp(N_f - 4)]_{local} \times [SU(N_f)_L \times SU(N_f)_R \times U(1)_Y \times U(1)_B \times U(1)_R]_{global} \]  

(6.152)

and matter fields transforming according to

\[
\begin{align*}
x & \sim (N_f - 3, N_f - 4; 1, 1; -N_f/(N_f - 3), 0, -1) \\
p & \sim (N_f - 3, 1; 1, 1; -N_f/(N_f - 3), N_cN_f, 6) \\
\bar{\alpha} & \sim \left( (N_f - 3)(N_f - 4)/2, 1, 1; 2N_f/(N_f - 3), 0, 4 \right) \\
\bar{q} & \sim \left( (N_f - 3), 1; \tilde{N}_f, 1; 3/(N_f - 3), -N_c, 0 \right)
\end{align*}
\]

(6.153)

(6.154)

(6.155)

(6.156)
\[ l \sim (1, N_f - 4; 1, (N_c + N_f - 4); N_f/(N_c + N_f - 4), 0, 1) \] (6.157)
\[ M \sim (1, 1; N_f, N_c + N_f - 4; (N_c - 4)/(N_c + N_f - 4), N_c, 2) \] (6.158)
\[ H \sim (1, 1; 1, (N_c + N_f - 4)(N_c + N_f - 5)/2; -2N_f/(N_c + N_f - 4), 0, 0) \] (6.159)
\[ B_1 \sim (1, 1; N_f, 1; 1, N_c(1 - N_f), -4) \] (6.160)

and superpotential
\[ \tilde{W}' = M \bar{q}l x + Hll + B_1p\bar{q} + \bar{\alpha}x^2, \] (6.161)
where
\[ B_k = Q_k A^{((N-k)/2)} \] (6.162)
are baryons. This is the most general superpotential allowed by the symmetries, holomorphy and smoothness near the origin in field space.

Armed with the above explicit construction we can first apply the method to a chiral finite model based on \( SU(3) \) gauge group and then to a realistic chiral finite \( SU(5) \) GUT.

### 6.2. The dual of a finite chiral \( SU(3) \)

One of the simplest chiral finite models that can be treated with the deconfinement method is a \( SU(3) \) gauge group with matter transforming symmetric tensor representation and an appropriate number of fundamentals and anti-fundamentals to make it one-loop finite and to cancel the anomaly, that is 1 symmetric tensor \( S \), 3 fundamentals \( Q \) and 10 antifundamentals \( \bar{Q} \). The deconfinement of theories with symmetric 2-index tensors has been discussed in great detail in [41].

First consider the theory without a tree-level superpotential. We will add the superpotential that is necessary to make the theory finite later on. The symmetry group is
\[ G = SU(3)_{local} \times [SU(3) \times SU(10) \times U(1)_1 \times U(1)_2U \times (1)_R]_{global} \] (6.163)
where the matter fields transform as:
\[ Q \sim (3; 3, 1, 1, 0, 4), \] (6.164)
\[ \bar{Q} \sim (\bar{3}; 1, 10, -3/10, 3, 0), \] (6.165)
\[ S \sim (6; 1, 1, 0, -6, 0). \] (6.166)
The symmetric tensor $S$ can be deconfined with the help of a $SO(8)$ theory. Now this $SO(8)$ has only fundamental matter and can be dualized according to [39]. After dualizing also the $SU(3)$ (which by the virtue of the deconfinement has only fundamental matter), we obtain the following net dual with symmetry group

$$\tilde{G} = [SU(8) \times SO(14)]_{\text{local}} \times [SU(3) \times SU(10) \times U(1)_1 \times U(1)_2 \times U(1)_R]_{\text{global}} \quad (6.167)$$

and matter content

\[
\begin{align*}
\bar{q} & \sim (\bar{8}; 1, 3, 1, -\frac{5}{8}, -3, -\frac{5}{2}) , \quad (6.168) \\
p & \sim (8; 1, 1, 1, -\frac{3}{8}, 9, -\frac{7}{2}) , \quad (6.169) \\
u & \sim (1, 1, 1, 1, 0, -18, 6) , \quad (6.170) \\
M & \sim (1, 1, 3, 10, \frac{7}{10}, 3, 4) , \quad (6.171) \\
N & \sim (1, 1, 3, 1, 1, -6, 8) , \quad (6.172) \\
x & \sim (8; 14, 1, 1, -\frac{3}{8}, 0, -\frac{1}{2}) , \quad (6.173) \\
\bar{C} & \sim (1; 14, 1, 10, \frac{3}{10}, 0, 1) , \quad (6.174) \\
\bar{S} & \sim (\bar{36}; 1, 1, 1, \frac{3}{4}, 0, 3) , \quad (6.175) \\
H & \sim (1; 1, 1, 45, -\frac{3}{5}, 0, 0) . \quad (6.176)
\end{align*}
\]

and with the superpotential

$$\tilde{W} = M x \bar{C} \bar{q} + N p \bar{q} + S p^2 u + \bar{S} x^2 + H \bar{C}^2 . \quad (6.177)$$

With the above particle content both gauge factors are asymptotically free.

Next in order to make the theory finite we add an tree-level superpotential to the original theory,

$$W = \lambda_1 Q_1 Q_2 Q_3 + \lambda_2 \left( \sum_{i=1}^{10} S \bar{Q}_i \bar{Q}_i \right) . \quad (6.178)$$

The superpotential $W$ breaks explicitly $U(1)_1$ and $U(1)_R$ but leaves $U(1)_2$ unbroken. However a linear combination of $U(1)_1$ and $U(1)_R$ is preserved. This is an R-symmetry with $Q$, $\bar{Q}$ and $S$ having charges $2/3$, 1 and 0. The non-abelian global $SU(3)$ is preserved by the superpotential, the global $SU(10)$ is broken to its $SO(10)$ subgroup. For simplicity
we apply the method of [6] to show that this is finite. Due to the global symmetries we only have 3 independent gammas, \( \gamma_Q, \gamma_{\bar{Q}} \) and \( \gamma_S \). The relevant \( \beta \) functions are

\[
\begin{align*}
\beta_{\text{gauge}} & \sim 3\gamma_q + 10\gamma_{\bar{Q}} + 5\gamma_S \\
\beta_{\lambda_1} & \sim 3\gamma_q \\
\beta_{\lambda_2} & \sim \gamma_S + 2\gamma_{\bar{Q}}
\end{align*}
\] (6.179)  
(6.180)  
(6.181)

and hence

\[
\beta_{\text{gauge}} \sim \beta_{\lambda_1} + 5\beta_{\lambda_2}. \] (6.182)

The conditions of vanishing of all \( \beta \) functions are hence linearly dependent. Therefore we obtain a fixed line passing through the origin of coupling space. Along the fixed line we obtain interacting theories which are finite in the sense that all \( \beta \) functions, which are the physically relevant quantities, are zero.

On the dual side \( SQ\bar{Q} \) becomes the singlet field \( H \). Adding this to the superpotential forces \( \bar{C} \) to get a non-zero vev and breaks the \( SO(14) \) to \( SO(6) \). On the other hand \( Q^3 \) is mapped to \( \bar{C}^{10}W_{SO(14)}^2 \), where \( W_{SO(14)} \) denotes the chiral field strength of the \( SO(14) \) gauge group [41]. This term is highly non-renormalizable. Moreover the effect of adding it to the superpotential is not easy to study. Clearly the dual theory is far from being finite. It definitely looks much more complicated than the theory we started with. Nevertheless it is interesting to see that the dual theory can be obtained.

### 6.3. The dual of a realistic \( SU(5) \) finite gauge theory

We have already observed that deconfinement leads to ‘ugly’ dualities. In addition to the criticism mentioned above we have seen that in general some of the superpotential terms necessary to make the original theory finite are mapped in the dual theory to operators which are quartic or of even higher order leading to a highly non-renormalizable field theory. Nevertheless it is interesting to see that using the deconfinement method it is in principle possible to construct duals even for realistic models like the one presented in [2].
6.3.1. The realistic finite unified SU(5) theory

Let us recall the main features of the finite unified model based on SU(5). From the classification of theories with vanishing one-loop $\beta$ function for the gauge coupling [42], one can see that using SU(5) as gauge group there exist only two candidate models which can accommodate three fermion generations. These models contain the chiral supermultiplets $\mathbf{5}, \mathbf{\bar{5}}, \mathbf{10}, \mathbf{\bar{10}}, \mathbf{24}$ with the multiplicities $(6, 9, 4, 1, 0)$ and $(4, 7, 3, 0, 1)$, respectively. Only the second one contains a 24-plet which can be used for spontaneous symmetry breaking (SSB) of SU(5) down to SU(3) $\times$ SU(2) $\times$ U(1). (For the first model one has to incorporate another way, such as the Wilson flux breaking to achieve the desired SSB of SU(5) [2]). Therefore, we would like to concentrate only on the second model.

To simplify the situation, one neglects the intergenerational mixing among the lepton and quark supermultiplets and consider the following SU(5) invariant cubic superpotential for the (second) model:

$$W = \sum_{i=1}^{3} \sum_{\alpha=1}^{4} \left[ \frac{1}{2} g^{u}_{i\alpha} \mathbf{10}_{i} \mathbf{10}_{\alpha}, H_{\alpha} + g^{d}_{i\alpha} \mathbf{10}_{i} \mathbf{\bar{5}}_{\alpha} H_{\alpha} \right]$$

$$+ \sum_{\alpha=1}^{4} g^{f}_{\alpha} H_{\alpha} \mathbf{24} H_{\alpha} + \frac{g^{\lambda}}{3} (\mathbf{24})^{3}, \text{ with } g^{u,d}_{i\alpha} = 0 \text{ for } i \neq \alpha , \quad (6.183)$$

where the $\mathbf{10}_{i}$’s and $\mathbf{\bar{5}}_{i}$’s are the usual three generations, and the four ($\mathbf{5} + \mathbf{\bar{5}}$) Higgses are denoted by $H_{\alpha}, \mathbf{\bar{H}}_{\alpha}$. The superpotential is not the most general one, but by virtue of the non-renormalization theorem, this does not contradict the philosophy of the coupling unification by the reduction method (a RG invariant fine tuning is a solution of the reduction equation). In the case at hand, however, one can find a discrete symmetry that can be imposed on the most general cubic superpotential to arrive at the non-intergenerational mixing [2]. This is given in table (2).

Given the superpotential $W$, the $\beta$ functions of the model can be computed and were found to be [47]

$$\beta^{(1)}_{g} = 0,$$

$$\beta^{u(1)}_{i\alpha} = \frac{1}{16\pi^{2}} \left[ -\frac{96}{5} g^{2} + 6 \sum_{\beta=1}^{4} (g^{u}_{i\beta})^{2} + 3 \sum_{j=1}^{3} (g^{u}_{j\alpha})^{2} + \frac{24}{5} (g^{f}_{\alpha})^{2} \right]$$
Table 2: The charges of the $Z_7 \times Z_3 \times Z_2$ symmetry

<table>
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<th>10_1</th>
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<tr>
<td>Z_7</td>
<td>1</td>
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<td>4</td>
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<tr>
<td>Z_3</td>
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Then regarding the gauge coupling $g$ as the primary coupling one can solve the reduction equations with the power series ansatz. It was found that the power series,

$$
\beta_{i\alpha}^{d(1)} = \frac{1}{16\pi^2} \left[ -\frac{84}{5} g^2 + 3 \sum_{\beta=1}^{4} (g_{i\beta}^u)^2 + \frac{24}{5} (g_{f\alpha}^l)^2 + 4 \sum_{j=1}^{3} (g_{j\alpha}^d)^2 \right] + 6 \sum_{\beta=1}^{4} (g_{i\beta}^d)^2 g_{i\alpha}^u ,
$$

$$
\beta_{\lambda(1)} = \frac{1}{16\pi^2} \left[ -30 g^2 + \frac{63}{5} (g_{\lambda})^2 + 3 \sum_{\alpha=1}^{4} (g_{\alpha}^l)^2 \right] g_{\lambda} ,
$$

$$
\beta_{\alpha}^{l(1)} = \frac{1}{16\pi^2} \left[ -\frac{98}{5} g^2 + 3 \sum_{i=1}^{3} (g_{i\alpha}^u)^2 + 4 \sum_{i=1}^{3} (g_{i\alpha}^d)^2 + \frac{48}{5} (g_{f\alpha}^l)^2 \right] + \sum_{\beta=1}^{4} (g_{f\beta}^l)^2 + \frac{21}{5} (g_{\lambda})^2 g_{\alpha}^f ,
$$

exists uniquely, where $\ldots$ indicates higher order terms and all the other couplings have to vanish. One can verify that the higher order terms can be uniquely computed.

Consequently, all the one-loop $\beta$ functions of the theory vanish. Moreover, all the one-loop anomalous dimensions for the chiral supermultiplets,

$$
\gamma_{10i}^{(1)} = \frac{1}{16\pi^2} \left[ -\frac{36}{5} g^2 + 3 \sum_{\beta=1}^{4} (g_{i\beta}^u)^2 + 2 \sum_{\beta=1}^{4} (g_{i\beta}^d)^2 \right] ,
$$

exists uniquely, where $\ldots$ indicates higher order terms and all the other couplings have to vanish. One can verify that the higher order terms can be uniquely computed.
\begin{align*}
\gamma_{5t}^{(1)} &= \frac{1}{16\pi^2} \left[ \frac{-24}{5} g^2 + 4 \sum_{\beta=1}^{4} (g_{\beta\beta}^t)^2 \right], \\
\gamma_{H_t}^{(1)} &= \frac{1}{16\pi^2} \left[ \frac{-24}{5} g^2 + 3 \sum_{i=1}^{3} (g_{\alpha\alpha}^u)^2 + \frac{24}{5} (g_{\alpha\alpha}^d)^2 \right], \\
\gamma_{H_d}^{(1)} &= \frac{1}{16\pi^2} \left[ \frac{-24}{5} g^2 + 4 \sum_{i=1}^{3} (g_{\alpha\alpha}^d)^2 + \frac{24}{5} (g_{\alpha\alpha}^d)^2 \right], \\
\gamma_{24}^{(1)} &= \frac{1}{16\pi^2} \left[ -10 g^2 + \sum_{\alpha=1}^{4} (g_{\alpha\alpha}^f)^2 + \frac{21}{5} (g_{\alpha\alpha}^\lambda)^2 \right],
\end{align*}

(6.186)

also vanish in the reduced system. As it has already been mentioned in section 3.1, these conditions are necessary and sufficient for finiteness to all orders in perturbation theory.

In most of the previous studies of the present model [48,49], however, the complete reduction of the Yukawa couplings, which is necessary for all-order-finiteness, was ignored. They have used the freedom offered by the degeneracy in the one- and two-loop approximations in order to make specific ansätze that could lead to phenomenologically acceptable predictions. In the above model, a diagonal solution for the Yukawa couplings was found, with each family coupled to a different Higgs. However, the fact that mass terms do not influence the RG functions in a certain class of renormalization schemes was used in order to introduce appropriate mass terms that permit to perform a rotation in the Higgs sector such that only one pair of Higgs doublets, coupled to the third family, remains light and acquires a non-vanishing vev [49]. Note that the effective coupling of the Higgs doublets to the first family after the rotation is very small avoiding in this way a potential problem with the proton lifetime [50]. Thus, effectively, we have at low energies the Minimal Supersymmetric Standard Model (MSSM) with only one pair of Higgs doublets satisfying at \( M_{\text{GUT}} \) the following boundary conditions

\[ g_{t}^2 = \frac{8}{5} g^2 + O(g^4), \quad g_{b}^2 = g_{\tau}^2 = \frac{6}{5} g^2 + O(g^4), \]

(6.187)

where \( g_i, (i = t, b, \tau) \) are the top, bottom and tau Yukawa couplings of the MSSM, and the other Yukawa couplings should be regarded as free.

The model predicts, among others, the top quark mass \( M_t = 183GeV \) subject to corrections of less than 4\% [51]. Another variant of the present model has been suggested in [52]. Adding soft breaking terms (which are supposed not to influence the \( \beta \) functions
beyond $M_{\text{GUT}}$), we can obtain supersymmetry breaking. The conditions on the soft breaking terms to preserve one-loop finiteness have been given already some time ago [53]. Recently, the same problem in two-loop orders has been addressed [54]. Even more recently a new solution to the two-loop finiteness conditions was found with very interesting phenomenological implications [52]. It is an open problem whether there exists a suitable set of conditions on the soft terms for all-loop finiteness.

6.3.2. Constructing the dual

To construct the dual of for the model describe above, we first consider a theory with the same gauge group and matter content, but without a tree level superpotential for the original theory. This model can be dualized. We will add the superpotential later in order to make the theory all loop finite and look for the corresponding effects on the dual side. Without the superpotential several non-abelian global symmetries are restored. This simplified model has the following symmetry group

$$G = SU(5)_{\text{local}} \times [SU(4) \times SU(7) \times SU(3) \times U(1)_5 \times U(1)_{10} \times U(1)_{24} \times U(1)_R]_{\text{global}}$$

(6.188)

and matter content

$$Q \sim (5; 4, 1, 1, 7, 9, 10, 0),$$

(6.189)

$$\bar{Q} \sim (\bar{5}; 1, 7, 1, -4, 0, 0, 1),$$

(6.190)

$$T \sim (10; 1, 1, 3, 0, -4, 0, 1),$$

(6.191)

$$H \sim (24; 1, 1, 1, 0, 0, -4, \frac{2}{7}) .$$

(6.192)

Deconfining the adjoint by a $SU(4)$ gauge group we get the following gauge theory with symmetry group

$$G_{\text{deconfined}} = [SU(5) \times SU(4)]_{\text{local}} \times [SU(2)_A \times SU(2)_B \times SU(2)_C \times SU(4) \times SU(7) \times$$

$$\times U(1)_5 \times U(1)_A \times U(1)_B \times U(1)_C \times U(1)_R]_{\text{global}}$$

(6.193)

and matter content

$$Q \sim (5; 1, 1, 1, 1, 4, 1, 7, 3, 3, 3, 3, 0),$$

(6.194)
\[ \tilde{Q} \sim (\bar{5}, 1, 1, 1, 1, 1, 7, -4, 0, 0, 0, 1), \quad (6.195) \]

\[ a \sim (5; 1, 2, 1, 1, 1, 1, 0, -2, 0, 0, 1/2), \quad (6.196) \]

\[ \alpha \sim (1; 1, 2, 1, 1, 1, 0, 10, 0, 0, -1/2) \quad (6.197) \]

\[ A \sim (\bar{5}, 1, 1, 1, 1, 1, 1, 0, -8, 0, 0, 2), \quad (6.198) \]

\[ b \sim (5; 1, 1, 2, 1, 1, 1, 0, 0, -2, 0, 1/2), \quad (6.199) \]

\[ \beta \sim (1; 1, 1, 2, 1, 1, 1, 0, 0, 10, 0, -1/2), \quad (6.200) \]

\[ B \sim (\bar{5}; 1, 1, 1, 1, 1, 1, 0, 0, -8, 0, 2), \quad (6.201) \]

\[ c \sim (5; 1, 1, 1, 2, 1, 1, 0, 0, 0, -2, 1/2), \quad (6.202) \]

\[ \gamma \sim (1; 1, 1, 1, 2, 1, 1, 1, 0, 0, 0, 10, -1/2), \quad (6.203) \]

\[ C' \sim (\bar{5}; 1, 1, 1, 1, 1, 1, 0, 0, 0, -8, 2), \quad (6.204) \]

\[ x \sim (5; 4, 1, 1, 1, 1, 1, 0, 0, 0, 0, 1/5), \quad (6.205) \]

\[ \tilde{x} \sim (\bar{5}; 4, 1, 1, 1, 1, 1, 0, 0, 0, 0, 1/5), \quad (6.206) \]

\[ p_2 \sim (1; 1, 1, 1, 1, 1, 1, 0, 0, 0, 0, 8/5), \quad (6.207) \]

\[ p_5 \sim (5; 1, 1, 1, 1, 1, 1, 0, 0, 0, 0, 6/5), \quad (6.208) \]

\[ \tilde{p}_5 \sim (\bar{5}, 1, 1, 1, 1, 1, 1, 0, 0, 0, 0, 6/5). \quad (6.209) \]

\[ (6.210) \]

The superpotential in this deconfined theory is

\[ W = a^5 \alpha + b^5 \beta + c^5 \gamma + A \alpha a + B \beta b + C \gamma c + p_2 x \tilde{x} + p_5 x^4 + \tilde{p}_5 \tilde{x}^4. \quad (6.211) \]

Now we can dualize the \( SU(5) \) factor to obtain a dual model. We show some details of this dual in the appendix.

Adding the tree-level superpotential required for finiteness both on the original and the dual side does not pose a problem, since it is known how the various operators map to each other under duality. While some of the cubic terms get mapped to mass terms in the dual theory, many of them get mapped to baryonic operators, which involve a product of ten fields. Adding these to the dual superpotential leaves us with a highly non-renormalizable theory.
7. Finite gauge theories and their duals from branes

Recently it has become clear, that field theory results can be derived by studying the
dynamics of branes in string theory. Here we will discuss some examples, which have an
easy realization in terms of a brane setup.

One important insight for understanding various duality symmetries in string theory
is that string theory contains a new kind of object, the Dirichlet (D)-brane [55], which
is some kind of topological defect, where open strings can end. With the help of the
D-branes many of the connections among dual theories have been clarified.

More specifically a D-brane is a hypersurface on space time on which open strings
are allowed to end. The open strings are then quantized in the usual way with the only
difference that the end points satisfy Dirichlet boundary conditions $X^\mu(0) = X^\mu(\pi) = x^\mu$
for the coordinates normal to the hypersurface, which makes sense for a hypersurface
of any number of dimensions, say p space and one time. The real novelty of the D-
brane appears when several parallel branes are brought into contact. Then open strings
stretching from one D-brane to another produce new states, that become massless when
the D-branes coincide. The discussion now is exactly as for the type I strings. In fact
the type I open strings can be regarded as associated with 32 D9-branes filling the space
time. The open string fields again become matrices, and are governed by a dimensionally
reduced supersymmetric gauge theory based on the group $U(N)$. The Lagrangian is

$$L = 1/g_s Tr F^2 + 1/(g_s \alpha')^2 Tr (DX)^2 + \bar{\psi} D \psi + 1/(g_s \alpha')^4 \sum Tr [X^i, X^j]^2.$$  \hspace{1cm} (7.212)

Separating the branes in space corresponds to giving a vev to the matrix $X^i$.

The promotion of space-time coordinates to matrices reminds very strongly on non-
commutative geometry, and the picture has indeed many similarities to the construction
of gauge theories in terms of the noncommutative geometry of a discrete bundle over
space-time [56]. Although the matrix nature of the D-brane coordinates $X^i$ is important
in describing the full dynamics, the moduli space of D-branes in flat space is $[X^i, X^j] = 0$.
Therefore in low energies, where the moduli space approximation is a good one, the non-
commutativity plays no role.

During the past year many interesting results, both in field theory and in string theory,
were discovered by studying the worldvolume dynamics of branes in string theories. A particular construction has been used in [8] in order to study $N = 4$ supersymmetric gauge theories in $2 + 1$ dimensions. Of particular interest for our purposes is the work in [9], where the authors following [8] found a rather simple description on the way the $N = 1$ duality of [5] arises in 4 dimensions. Later it was realized that many detailed properties of these field theories can be studied by lifting the setup to M-theory [57]. In the following we will discuss some examples of finite gauge theories which have an easy realization in terms of a brane setup. However there are two facts that limit the construction of such models. The first is that a prerequisite in the construction of finite theories is the existence of a tree order superpotetial. The second is the fact that all the gauge theories resulting from branes so far are vector like 7.

### 7.1. SQCD with one additional superfield in the adjoint

Let us consider a vector-like gauge theory based on the $SU(N_c)$ gauge group which contains $N_f$ superfields $Q_i, \bar{Q}_i$ transforming as in SQCD and an additional one $X$ transforming according to the adjoint representation.

The vanishing of the one-loop $\beta$ function requires that $N_f = 2N_c$. In order to upgrade the model to an all loop finite one, we have to add an appropriate superpotential. By adding the superpotential

$$\lambda TrX^3 + h\bar{Q}_iXQ_i,$$

the model is derivable from the brane setup of fig. 1. We consider several branes of IIA string theory in 10 flat dimensions. In addition to the Dirichlet branes mentioned earlier, there are also solitonic fivebranes which are called NS fivebranes. The only property of these NS fivebranes we will need in the following is that D4 branes can end on them. In fig. 1 one studies the worldvolume theory of the $N_c$ D4 branes stretching infinitely in the 0123 directions and a finite interval between NS fivebranes in the 6 direction, effectively compactifying their world volume gauge theory from 4+1 to 3+1 dimensions. The left NS branes stretch in the 012345 directions, while the right NS$'$ branes are rotated with respect to this and stretch in the 012389 directions. The boundary conditions on the NS

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7See however [58]
and NS’ branes project out several degrees of freedom and break supersymmetry down to $N = 1$. We remain with an $SU(N_c)$ gauge theory with adjoint fields $X$ and $X'$ coupling via a superpotential

$$W = \lambda Tr X^{k+1} + \lambda' Tr X'^{k'+1} + h \bar{Q}_i X Q_i. \quad (7.214)$$

Note that for $k = k' = 1$ the superpotential generates masses for the adjoint fields and the theory reduces to pure SYM at low energies. By adding $N_f$ D6 branes stretching in the 0123789 directions in between the 5 branes we add $N_f$ chiral matter multiplets in the fundamental and in the anti-fundamental representation.

For $k = 2, k' = 1$ we hence obtain an $SU(N_c)$ gauge theory with $N_f$ flavours and an additional adjoint field with the superpotential (7.213). Note that the superpotential (7.213) breaks the $SU(N_f)_L \times SU(N_f)_R$ global symmetry to its diagonal subgroup.

![Figure 1: Brane configuration corresponding to SQCD with additional adjoint matter](image)

However requiring vanishing of the one-loop anomalous dimensions of the superfields involved, i.e. $\gamma^{(1)}_Q, \gamma^{(1)}_{\bar{Q}}, \gamma^{(1)}_X$, in order to meet the all loop finiteness conditions we find

$$h = g, \quad \lambda = 0, \quad (7.215)$$

where $g$ is the gauge coupling.

Therefore the requirement of all loop finiteness reduces the model to the N=2 gauge theory, for which the appropriate dual description is known [4]. Moreover one should note that according to [59] the brane configuration with NS and NS’ branes only realizes the
situation where the couplings $\lambda, \lambda' \to \infty$. In order to adjust the values of the couplings one has to rotate the NS' branes in the 5689 plane. To obtain $h = 0$ we have to make a 90° rotation and hence turn the NS' branes into NS branes. This is known to be the brane setup for the $N = 2$ gauge theory [9].

7.2. SQCD with two additional superfields in the adjoint

Our next example is again a vector-like theory based on $SU(N_c)$ containing $N_f$ flavours $Q_i, \bar{Q}$ transforming as in SQCD but it has in addition two superfields $X, Y$ instead of one transforming according to the adjoint representation. The vanishing of the one-loop $\beta$ function requires that $N_f = N_c$. Now in order to make the theory all loop finite we add the following superpotential

$$W_c = s_x X^3 + s_y Y^3 + \bar{Q}YQ, \quad (7.216)$$

which again breaks the global flavor symmetry to its diagonal subgroup.

This gauge theory is again derivable from brane setup [59] of fig1, this time with $k = k' = 2$. Moving the NS branes to the right and thereby passing the D6 and NS' branes one obtains the situation shown in fig. 2. Since this move is thought to be irrelevant for the IR physics, the setup of fig. 2 describes the dual of the gauge theory we started with.

![Figure 2: Dual brane configuration. $k'$ NS' fivebranes are connected by $\bar{N}_c = kN_f - N_c$ fourbranes to $k$ NS fivebranes on their right, and by $k$ fourbranes to each of $N_f$ D sixbranes on their left](image)

The dual is an $SU(2N_f - N_c)$ gauge theory, again with two adjoints $x$ and $y$ and $N_f$ fundamentals $q$ and $\bar{q}$. In addition there are gauge singlet fields $M_1$ and $M_2$ (each one
transforming as adj. + 1 under the global flavour symmetry). The dual superpotential can be read off from the branes to be:

$$\tilde{W} = \tilde{s}_x x^3 + \tilde{s}_y y^3 + \tilde{q} y q + M_1 \tilde{q} x q + M_2 \tilde{q} q. \quad (7.217)$$

In the potential finite setup with \( N_f = N_c \) this is indeed self-dual, but this time we obtain unwanted quartic operators on the dual side. This seems rather annoying. Both sides have the right matter content to be finite, but the nonrenormalizable quartic terms in \( W \) prevent the dual theory from being finite.

The solution of this puzzle is deeply connected to the way these dualities arise in the brane picture. As shown by [59], the brane only realizes the configuration where the \( s_i \to \infty \). The appearance of the quartic terms on the dual side are going to be remnants of this limit, where new degrees of freedom become massless. This is exactly what happens in the case of just one adjoint, which we discussed in the previous section. Here [59] showed that in addition to the terms present in the duality found by usual field theory arguments by [60] we get an additional singlet field with a quartic superpotential coupling in the \( s \to \infty \) limit seen by the branes. To achieve a finite theory we have to adjust the \( s \). Since they are fixed to be infinity from the brane point of view, the brane construction will not help us to understand the dualities of the finite theories. We need a duality for arbitrary values of the couplings, as is usually obtained by field theoretic reasoning. Thus we should look for a generalization of the duality of [60] to the 2 adjoint case. Probably this has to be obtained by the usual ‘guess and check’ procedure. Perhaps one can deform one of the 2 adjoint dualities of [10] to derive this wanted duality. This is clearly very important, since this theory seems to be a very good candidate for a finite, S-dual \( N = 1 \) theory.

8. Conclusions

In the present paper we have been searching for the dual gauge theories of all-loop finite, \( N = 1 \) supersymmetric gauge theories, following the lines of [5]. Using established methods for such searching we have been able to construct the duals of almost all known \( N = 1 \) supersymmetric, chiral gauge theories (with the exception of \( E_6 \) models and \( SO(10) \) con-
taining antispinors) with vanishing one-loop $\beta$-function. These theories have first been promoted to all-loop finite ones, by adding appropriate superpotential and in turn by meeting the requirements of all loop-finiteness. In addition certain vector-like, all-loop finite, $N = 1$ gauge theories and their duals have been discussed, first in the standard field theory framework, and second also using the derivation of gauge theories from branes. However the brane picture still encounters several difficulties in the corresponding hunting for finite gauge theories, as we have seen in the discussion of specific examples.

From our search, one chiral, $N = 1$, all-loop finite gauge theory has been singled out as a candidate of having S-duality as the $N = 4$ gauge theories. It is based on the gauge group $SO(10)$ and has matter content consisting of eight vector and eight spinor superfields. The dual of this theory is based on the $SU(17) \times Sp(14)$ gauge group. We found that there is strong evidence that after spontaneous symmetry breaking of the first gauge factor to $SU(9)$ the infrared theory is finite. So, given that both the electric as well as the magnetic theories are finite, there exist marginal operators, common to both theories, which can be used to continuously interpolate between the strongly (weakly) coupled electric theory and the weakly (strongly) coupled magnetic theory, i.e. there should be $N = 1$ S-duality.

The dual of the realistic finite unified theory [2] based on the $SU(5)$ has been determined and discussed in some detail. However the resulting dual theory is rather complicated and it does not give, for far, any hint for a useful use of it. On the other hand, we should note that duals constructed using the deconfinement method, as it was the case for determining the dual of the finite unified $SU(5)$ model, are not unique. Therefore we cannot exclude the possibility that more interesting dual theories can be constructed, which also provide in the infrared the same physics as the original ones.

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Appendix

Here we present the matter content of the dual of the realistic $SU(5)$. For simplicity of notation we consider the case without tree-level superpotential in the original theory, so that we have the full non-abelian global symmetry group and can organize the fields in a little bit more transparent fashion.

The superpotential are the usual $Mq\tilde{q}$ with all the meson fields appearing with the corresponding quarks. It is no problem to translate the tree-level superpotential required for finiteness in this dual language. Some terms turn out to be mass terms, but many are mapped to baryon like excitations, which fields to the tenth power and hence are non-renormalizable. The global symmetry structure is broken to the same discrete subgroup already present in the original theory. For simplicity we will only present the matter content of the theory without superpotential.

Under the symmetry group

$$\tilde{G} = [SU(10) \times SU(4) \times SU(2)_A \times SU(2)_B \times SU(2)_C]_{\text{local}} \times [SU(4) \times SU(7)]_{\text{global}}$$

the matter fields transform as

$$\alpha \sim (1,1,2,1;1,1), \quad \beta \sim (1,1,1,2;1,1),$$

$$\gamma \sim (1,1,1,1;2;1,1), \quad p_2 \sim (1,1,1,1;1,1),$$
\[ q \sim (10, 1, 1, 1; \tilde{4}, 1), \quad \tilde{q} \sim (\overline{10}, 1, 1, 1; 1, \overline{7}), \]
\[ a_d \sim (10, 1, 2, 1; 1, 1), \quad A_d \sim (\overline{10}, 1, 1, 1; 1, 1), \]
\[ b_d \sim (10, 1, 1, 1; 1, 1), \quad B_d \sim (\overline{10}, 1, 1, 1; 1, 1), \]
\[ c_d \sim (10, 1, 1, 2; 1, 1), \quad C_d \sim (\overline{10}, 1, 1, 1; 1, 1), \]
\[ x_d \sim (10, \tilde{4}, 1, 1, 1; 1, 1), \quad \tilde{x}_d \sim (\overline{10}, 1, 1, 1; 1, 1), \]
\[ p^d_5 \sim (10, 1, 1, 1; 1, 1), \quad \tilde{p}^d_5 \sim (\overline{10}, 1, 1, 1; 1, 1), \]
\[ M \sim (1, 1, 1, 1; 4, 7), \quad M_1 \sim (1, 1, 1, 1; 4, 1), \]
\[ M_2 \sim (1, 1, 1, 1; 4, 1), \quad M_3 \sim (1, 1, 1, 1; 4, 1), \]
\[ M_4 \sim (1, \tilde{4}, 1, 1, 1; 4, 1), \quad M_5 \sim (1, 1, 1, 1; 4, 1), \]
\[ M_6 \sim (1, 1, 2, 1; 1, 1), \quad M_7 \sim (1, 1, 2, 1; 1, 1), \]
\[ M_8 \sim (1, 1, 2, 1; 1, 1), \quad M_9 \sim (1, 1, 2, 1; 1, 1), \]
\[ M_{10} \sim (1, \tilde{4}, 2, 1; 1, 1) \quad M_{11} \sim (1, 1, 2, 1; 1, 1), \]
\[ M_{12} \sim (1, 1, 2, 1; 1, 1), \quad M_{13} \sim (1, 1, 2, 1; 1, 1), \]
\[ M_{14} \sim (1, 1, 2, 1; 1, 1), \quad M_{15} \sim (1, 1, 2, 1; 1, 1), \]
\[ M_{16} \sim (1, \tilde{4}, 2, 1; 1, 1), \quad M_{17} \sim (1, 1, 2, 1; 1, 1), \]
\[ M_{18} \sim (1, 1, 2, 1; 1, 1), \quad M_{19} \sim (1, 1, 2, 1; 1, 1), \]
\[ M_{20} \sim (1, 1, 1, 2; 1, 1), \quad M_{21} \sim (1, 1, 1, 2; 1, 1), \]
\[ M_{22} \sim (1, \tilde{4}, 1, 2; 1, 1), \quad M_{23} \sim (1, 1, 1, 2; 1, 1), \]
\[ M_{24} \sim (1, 4, 1, 1; 1, 7), \quad M_{25} \sim (1, 4, 1, 1; 1, 1), \]
\[ M_{26} \sim (1, 4, 1, 1; 1, 1), \quad M_{27} \sim (1, 4, 1, 1; 1, 1), \]
\[ M_{28} \sim (1, 1, 1, 1; 1, 1), \quad M_{29} \sim (1, 4, 1, 1; 1, 1), \]
\[ M_{30} \sim (1, 1, 1, 1; 1, 1), \quad M_{31} \sim (1, 1, 1, 1; 1, 1), \]
\[ M_{32} \sim (1, 1, 1, 1; 1, 1), \quad M_{33} \sim (1, 1, 1, 1; 1, 1), \]
\[ M_{34} \sim (1, \tilde{4}, 1, 1; 1, 1), \quad M_{35} \sim (1, 1, 1, 1; 1, 1), \]
\[ h \sim (1, 15, 1, 1; 1, 1), \]