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Quasinormal Ringdown: The Late Stage of Neutron Star Mergers

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Abstract

Perturbation techniques have been applied successfully to obtain the total radiated energy as well as wave forms for the collision of two black holes. Therefore, it appears promising to extend these methods to the case of colliding neutron stars as well. *Stationary* perturbations (quasinormal modes) both of black holes and of neutron stars have been investigated extensively. *Time-dependent* perturbations, yielding radiated energy and wave function for a given initial configuration, have mostly been studied for black holes. We review the basic techniques for treating perturbations of black holes and neutron stars, summarize the results, and discuss the problems that have to be solved if a time-dependent perturbation treatment is to be applied to the collision of neutron stars. We will limit this discussion to final objects which are spherically symmetric; however, the technique can be extended to other cases, such as final black holes or neutron stars carrying spin.

1 Introduction

The coalescence of massive bodies, in particular binary neutron stars or black holes, is one of the most fascinating topics in general relativity, and it carries considerable astrophysical significance. Such an event is expected to produce significant gravitational radiation, carrying a wealth of information about the process as well as about the objects involved in it. Several large interferometric gravitational wave detectors are currently under construction (Danzmann 1997). The availability of templates for the expected waveforms is vital for interpreting the data of these detectors, adding urgency to efforts to complete theoretical studies of such mergers.

The Grand Challenge alliance attempts to solve this problem for the coalescence of black holes by developing a fully relativistic, three-dimensional computer code (Finn 1997). More recently, projects are getting underway to do similar calculations for neutron stars.

Figure 1 shows a schematic representation of the coalescence of a binary system. We can distinguish three major phases:

- I: A semi-stable phase, where the two compact objects are in a close orbit. The orbit slowly decays due to energy lost by gravitational radiation. This phase has been treated in post-Newtonian approximations to various orders.

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- II:** After the orbit has decayed past a certain limit, it becomes unstable, and the actual merger of the two objects occurs. Newtonian calculations of the neutron star mergers have been carried out (Rasio and Shapiro (1992), Rasio (1997) and references therein, Ruffert et al. (1996), Zhuge et al. (1994,1996)). However, owing to the strength of the gravitational field, this phase will generally require a relativistic treatment. Calculations involving post-Newtonian or other approximations have been done, e.g., by Taniguchi and Nakamura (1996), Baumgarte et al. (1997), Wilson et al. (1996), and Shibata (1997). It remains to be seen if these results prove reliable, or if a fully relativistic numerical treatment is necessary to obtain correct results.
- III:** After the merger, the final object has essentially formed, but has not yet reached its final, stationary state. The deviations from this final state will be radiated away during the ringdown phase. This phase can be treated using perturbation techniques.

Figure 1 From (Vogt 1992): Schematic representation of the coalescence of a binary system.

The supercomputer calculations for binary black hole coalescence are extremely demanding, both in terms of techniques and algorithms to be developed, and in terms of the required computer resources. For neutron stars, the absence of a horizon eliminates some problems; on the other hand, the hydrodynamics of the stars' matter must now be included.

It turns out, though, that perturbation methods can actually be applied to a much broader range than what we have labelled as "phase III" above. Price and Pullin (1994)

first explored this for a head-on collision of initially static black holes. They assume that the two black holes are so close to each other at the beginning of the collision that they have already formed a common horizon, and treat this initial configuration as a perturbation of a single black hole (Fig. 2). Comparison with the results of supercomputer simulations shows that this approach yields realistic results for larger initial separations than one would expect a priori. It has further been shown (Abrahams and Cook 1994, Baker and Li 1997) that for head-on collisions, connecting Newtonian calculations for large separations directly to the perturbation treatment (in effect, eliminating phase II) is a viable and robust procedure.

The inclusion of second-order perturbation terms improves these results further, and also provides an independent error estimate for the first-order results (Gleiser et al. 1996).

Figure 2 $1 + 1 \rightarrow 1$: Treating a close collision of compact object as a perturbation of the final object.

It is not clear yet whether the perturbation approach will give similarly excellent results for more complicated initial configurations, such as including spinning black holes or non-axisymmetric configurations. A further complication arises if the initial configuration — and therefore the final black hole as well — contains net angular momentum, which is obviously the case for the inspiral coalescence of binary black holes.

In any case, it is clear that the perturbation approach provides a very interesting alternative to computing the radiated energy and the wave forms of black hole collisions. It can serve as an alternative route in its own right, and also as a benchmark for testing fully relativistic numerical codes.

The question thus arises naturally whether perturbation techniques can be used to study collisions of neutron stars as well. Not much work has been done in this direction. Stationary perturbations of neutron stars (quasinormal modes) have been studied extensively. However, if we want to specify an initial configuration, and then determine the total energy and the wave form of the gravitational radiation that is emitted in the ensuing collision, then we need to treat time-dependent perturbations. We will also have to find a way to derive initial data for the perturbations from the physical configuration that we wish to study.

2 Perturbations of compact objects

2.1 Perturbations of black holes

Spherically symmetric equilibrium state

The spherically symmetric equilibrium state of a black hole is, of course, the well-known Schwarzschild spacetime:

$$(ds^2)_0 = - \left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 (d\vartheta^2 + \sin^2 \vartheta d\phi^2). \quad (1)$$

Perturbation of the equilibrium

We will write a perturbed Schwarzschild metric in the usual way:

$$ds^2 = (ds^2)_0 + h_{\mu\nu} dx^\mu dx^\nu, \quad (2)$$

where h is assumed to be small in an appropriate sense.

At this point, we have not chosen a particular gauge for the perturbation. Using a complete set of tensor spherical harmonics $T_{\mu\nu}^{lmn}(\vartheta, \phi)$ (Zerilli 1970a, Moncrief 1974), we may represent the perturbation as the sum

$$h_{\mu\nu}(t, r, \vartheta, \phi) = \sum_{l=0}^{\infty} \sum_{m=-l}^l \sum_{n=1}^{10} C^{lmn}(t, r) T_{\mu\nu}^{lmn}(\vartheta, \phi). \quad (3)$$

Just as their scalar counterparts, the tensor spherical harmonics possess distinct values of l and m . In addition, they can be classified into even and odd parity. For each value of l and m , there are 10 tensors; we use the letter n as an additional label to distinguish them.

This kind of decomposition makes sense here: Due to the spherical background, perturbations belonging to different values of l and m or to different parities will not mix. Therefore, we can examine the contribution of each (l, m, P) mode separately, adding up the final result.

From the coefficients $C^{lmn}(t, r)$ in Eq. (3), we can construct a scalar function (usually called the Zerilli function) (Detweiler 1979), separately for each l and m and for even and odd parity contributions. This construction can be found in (Abrahams and Price 1996). Here we only note that it is linear, and write it symbolically as

$$Q^+(t, r) = \mathcal{L}^+(C^{n, \text{even}}) \quad (4)$$

$$Q^\times(t, r) = \mathcal{L}^\times(C^{n, \text{odd}}). \quad (5)$$

From now on, we are omitting the indices l, m .

After considerable analytical acrobatics (Regge and Wheeler 1957, Zerilli 1970b, Chandrasekhar 1975, Chandrasekhar 1983), the linearized field equations lead to a differential equation for the Zerilli function which has the form of a wave equation with a potential:

$$\left[\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial r_*^2} + V(r_*) \right] Q(t, r) = 0, \quad (6)$$

where M is the mass of the background black hole, and r_* is the tortoise coordinate $r_* = r + 2M \ln(r/2M - 1)$.

While the form of Eq. (6) is the same for odd and even parity, the potentials are different:

Even parity (Zerilli potential):

$$V(r) = \left(1 - \frac{2M}{r}\right) \left[\frac{72M^3}{r^5 \lambda^2} - \frac{12M}{r^3 \lambda^2} (l-1)(l+2) \left(1 - \frac{3M}{r}\right) + \frac{(l-1)l(l+2)(l+1)}{r^2 \lambda} \right], \quad (7)$$

$$\lambda = l(l+1) - 2 + \frac{6M}{r}.$$

Odd parity (Regge-Wheeler potential):

$$V(r) = \left(1 - \frac{2M}{r}\right) \left[\frac{l(l+1)}{r^2} - \frac{6M}{r^3} \right]. \quad (8)$$

Note that the potentials, and therefore the wave equation, depend on l , but not on m .

While the functional forms of the Zerilli and the Regge-Wheeler potential appear to be quite different, Fig. 3 shows that their values differ only slightly from each other.

Figure 3 Potentials in the wave equation for time dependent perturbations of Schwarzschild black holes.

2.2 Perturbations of neutron stars

Spherically symmetric equilibrium state

Due to the extra freedom associated with the material of the star, the determination of the equilibrium state of a neutron star is somewhat more involved than for a black hole. The following is only a brief summary; extensive discussions can be found in just about any textbook on general relativity, e.g. Misner et al. (1973) and Weinberg (1972).

We start by writing the metric (inside as well as outside the star) in the form

$$ds^2 = (ds^2)_0 = -e^{\nu(r)} dt^2 + e^{\lambda(r)} dr^2 + r^2 (d\vartheta^2 + \sin^2 \vartheta d\phi^2). \quad (9)$$

Assuming that the material of the neutron star behaves like an ideal fluid, we have for the energy-momentum tensor

$$T_{\mu\nu} = pg_{\mu\nu} + (p + \rho)u_\mu u_\nu, \quad (10)$$

where $\rho(r)$ is the total energy density and $p(r)$ the total pressure.

Conservation of energy-momentum, i.e., the condition $T_{\nu;\mu}^\mu = 0$, yields

$$\frac{d\nu}{dr} = -2(\rho + p)^{-1} \frac{dp}{dr}. \quad (11)$$

Using the mass function $m(r)$, defined as

$$m(r) = \int_0^r 4\pi r'^2 \rho dr', \quad (12)$$

in connection with the field equations, we obtain

$$e^{-\lambda} = 1 - \frac{2m}{r}. \quad (13)$$

Again using the field equations, as well as Eq. (11), we obtain the relativistic generalization for the condition of hydrostatic equilibrium:

$$\frac{dp}{dr} = -\frac{(\rho + p)(m + 4\pi r^3 p)}{r(r - 2m)}. \quad (14)$$

Equations (14) and (11), together with Eq. (12), are usually referred to as the Oppenheimer-Volkov (OV) or Tolman-Oppenheimer-Volkov (TOV) equations (Weinberg 1972, Misner et al. 1973, Shapiro and Teukolsky 1983).

The equation of state for the star's material,

$$p = p(\rho), \quad (15)$$

completes the required equations for the determination of the unknown functions. Realistic equations of state are usually given numerically; either in the form of tables which must be interpolated, or in the form of subroutines which compute the required values "on the fly". Polytropic equations of state are popular as analytic approximations.

We can use the equation of state (15) to eliminate $p(r)$ from Eqs. (14) and (12), providing us with a system of two ordinary differential equations for the two unknown functions $\rho(r)$ and $m(r)$. One initial condition is given by $m(r = 0) = 0$. This leaves us with a choice for one additional parameter, i.e., the central energy density $\rho(0)$. Spherically symmetric stars consisting of an ideal fluid (with given equation of state) therefore form a one-parameter family.

Once $\rho(r)$ and $m(r)$ are determined, we can use the equations above to find $p(r)$ and $\lambda(r)$. For the other function in the metric, it turns out that

$$\nu(r) = -\lambda(r). \quad (16)$$

Using $m(r = 0) = 0$ and $\rho(0)$, we integrate outward until we reach a radius $r = R_{\text{star}}$ such that $p(r = R_{\text{star}}) = 0$, which defines the radius of the star. Outside the star, we obtain, of course, a Schwarzschild metric with the mass parameter given by the total mass of the star:

$$M = \int_0^{R_{\text{star}}} 4\pi r^2 \rho dr. \quad (17)$$

Perturbation of the star's equilibrium

We will again denote the perturbation of the metric with $h_{\mu\nu}(t, r, \vartheta, \phi)$. In addition to the metric perturbations, we now have to deal with the perturbations of the energy density, $\delta\epsilon$, and of the pressure, δp . The Lagrangian displacement of the fluid elements is called δu_μ .

Since $\delta\epsilon$ and δp are scalar quantities, the usual scalar spherical harmonics can be used for the separation of the angular variables ϑ and ϕ :

$$\delta\epsilon = \rho^{lm}(t, r) Y_{lm}(\vartheta, \phi), \quad (18)$$

where a sum over l and m is implied. The pressure perturbation δp is determined from $\delta\epsilon$ via the equation of state.

Using vector spherical harmonics for δu_μ , we have

$$\begin{aligned} \delta u_0 &= u_0^{lm}(t, r) Y_{lm}(\vartheta, \phi) = -S_1^{lm}(t, r) Y_{lm}(\vartheta, \phi) \\ \delta u_1 &= u_1^{lm}(t, r) Y_{lm}(\vartheta, \phi) \\ \delta u_2 &= u_2^{lm}(t, r) \frac{\partial}{\partial \vartheta} Y_{lm}(\vartheta, \phi) - u_3^{lm}(t, r) \frac{1}{\sin \vartheta} \frac{\partial}{\partial \phi} Y_{lm}(\vartheta, \phi) \\ \delta u_3 &= u_2^{lm}(t, r) \frac{\partial}{\partial \phi} Y_{lm}(\vartheta, \phi) + u_3^{lm}(t, r) \sin \vartheta \frac{\partial}{\partial \vartheta} Y_{lm}(\vartheta, \phi). \end{aligned} \quad (19)$$

The normalization of the four-velocity has been used to determine u_0 in terms of S_1 , which is related to the (tt) component of the metric perturbation. Owing to the form of the vector spherical harmonics, the expansion coefficients do not have a one-to-one relationship to the components of the displacement velocity.

Instead of writing out the complete expansion of the metric perturbation in terms of tensor spherical harmonics, we will just indicate which components of $h_{\mu\nu}$ give rise to which expansion coefficients:

$$h_{\mu\nu} \rightarrow \begin{pmatrix} S_1 & S_2 & V_1, V_2 \\ * & S_3 & V_3, V_4 \\ * & * & T_1, T_2, T_3 \end{pmatrix} \begin{matrix} t \\ r \\ \vartheta, \phi \end{matrix}. \quad (20)$$

The perturbation of the energy-momentum tensor can be expressed in terms of these quantities with the help of Eq. (10); we only need it to first order in the perturbations.

We will use the field equations in their first-order form [see App. 5, Eqs. (59) and (60)]. We therefore need an expansion for the extrinsic curvature as well:

$$k_{ij} \rightarrow \begin{pmatrix} K_1 & K_2, K_3 \\ * & K_4, K_5, K_6 \end{pmatrix} \begin{matrix} r \\ \vartheta, \phi \end{matrix}. \quad (21)$$

Due to the static nature of the background metric, the perturbation of the extrinsic curvature is identical to the total extrinsic curvature.

Linearizing the field equations around the spherically symmetric background yields a system of first order equations for the coefficients of these quantities. It can be written schematically as

$$\frac{\partial}{\partial t} \vec{Q}(t, r) = \underline{A}(r) \frac{\partial^2}{\partial r^2} \vec{P}(t, r) + \underline{B}(r) \frac{\partial}{\partial r} \vec{P}(t, r) + \underline{C}(r) \vec{P}(t, r). \quad (22)$$

These are only the time evolution equations; in addition, the field equations yield 4 constraint equations.

Just as in the case of black holes, these equations depend on l , but not on m . Again, they separate into a set for even perturbations and one for odd perturbations. The “vectors” $\vec{Q}$ and $\vec{P}$ for these are

$$\vec{Q}^{\text{even}} = (S_3, V_3, T_1, T_2, K_1, K_2, K_4, K_5, \rho, u_1, u_2) \quad (23)$$

$$\vec{P}^{\text{even}} = (S_1, S_2, S_3, V_1, V_3, T_1, T_2, K_1, K_2, K_4, K_5, \rho, u_1, u_2)$$

$$\vec{Q}^{\text{odd}} = (V_4, T_3, K_3, K_6, u_3) \quad (24)$$

$$\vec{P}^{\text{odd}} = (V_2, V_4, T_3, K_3, K_6, u_3).$$

We obviously have four quantities, S_1, S_2, V_1, V_2 , which appear only on the right-hand side of the equation; we do not have any evolution equation for them. They are the expansion coefficients associated with the lapse and the shift vector.

We might think of using the constraint equations to determine these coefficients using the values of the others. However, if the constraint equations are satisfied on an initial slice, then the time evolution equations guarantee that they will remain satisfied for all time. Therefore, the constraint equations do not provide any information that is not already contained in the time evolution equations.

The answer to this problem is that we have to impose gauge conditions. This may seem inappropriate, since we have already fixed the gauge by using the specific form (9) for the background metric of the spherically symmetric neutron star. However, we are still free to specify conditions for the metric in terms of the first order perturbation quantities. This will only affect the total metric to first order in the perturbation quantities, and leave the background metric intact.

A gauge commonly used for neutron star perturbations is the Regge-Wheeler gauge. It is characterized by the conditions

$$\begin{aligned} \frac{\partial}{\partial \vartheta} (h_{02} \sin \vartheta) &= \frac{\partial}{\partial \varphi} \left(\frac{h_{03}}{\sin \vartheta} \right) \\ \frac{\partial}{\partial \vartheta} (h_{12} \sin \vartheta) &= \frac{\partial}{\partial \varphi} \left(\frac{h_{13}}{\sin \vartheta} \right) \\ h_{23} &= 0 \quad h_{33} = \sin^2 \vartheta h_{22}. \end{aligned} \quad (25)$$

There is no particularly intuitive justification for this choice; its main advantage is that it allows a considerable simplification of Eq. (22). As expected, the coefficients S_1, S_2, V_1, V_2 are fixed by these conditions. In addition, T_1, T_3, V_3, K_4 either have trivial solutions or are identical to other coefficients. We are now left with 8 equations for 8 unknowns for even perturbations, and 4 equations for 4 unknowns for odd perturbations.

Odd perturbations

In the case of odd perturbations, the equation for u_3 , the only matter coefficient involved, has a very simple solution

$$u_3(t, r) = f(r). \quad (26)$$

This solution represents stationary, differential rotations. Since we assume the star to consist of an ideal fluid, there are no restoring forces between adjacent fluid shells which could lead to oscillations or to braking the rotation. Furthermore, this motion does not give rise to gravitational radiation. Therefore, metric perturbations are decoupled from matter perturbations in the odd case.

The equations for the matter perturbation coefficients can be reduced to a wave equation for a single coefficient. This equation depends on the background metric, but not on other perturbation quantities. The situation is therefore much the same as for black holes: The neutron star simply provides the background for the evolution of the metric perturbations, but, at least in first order, it is not influenced by them.

Odd perturbations have been studied by Chandrasekhar and Ferrari (1991) and by Kokkotas (1994). Due to their similarity with black hole perturbations, we will not discuss them any further. In the following, whenever we mention neutron star perturbations, we will mean specifically even parity perturbations.

Even perturbations

Equations (22) and (24) represent 8 partial differential equations which are of first order in t , but their right-hand sides still contain second derivatives with respect to r . Introducing further auxiliary functions results in a hyperbolic system of 12 partial differential equations which are of first order in space and time.

This system has six eigenvalues which are different from zero, they come in three pairs having equal absolute value and opposite sign. They represent three dynamical degrees of freedom, two of which are associated with gravitational radiation, and one with matter oscillations.

The equations governing even-parity perturbations cannot be simplified further by analytical means. They must be integrated numerically, using conditions for regularity at the origin, for continuity at the star's surface, and for outgoing radiation at spatial infinity as boundary conditions.

3 Quasi-stationary perturbations

Since the coefficients in the differential equations for perturbations both of black holes and neutron stars are not explicitly time dependent, we can study single-frequency solutions with a harmonic time dependence:

$$Q(t, r) = \phi(\omega, r) e^{i\omega t}. \quad (27)$$

3.1 Quasinormal modes of black holes

Inserting Eq. (27) into the time-dependent wave equation (6), we obtain a time-independent, ordinary differential equation for $\phi(\omega, r_*)$. In the case of black holes, this equation becomes

$$\phi''(\omega, r_*) + (\omega^2 - V(r_*)) \phi(\omega, r_*) = 0. \quad (28)$$

This equation resembles a time-independent Schrödinger equation with potential $V(r_*)$.

When perturbations of black holes were first studied numerically, oscillations at characteristic frequencies were found to dominate the response at relatively late times (Press 1971, Cunningham et al. 1978). This suggested that there are oscillatory modes which are equivalent to normal modes of an oscillating system, which could be obtained by subjecting Eq. (6) to a normal mode analysis. However, since the potential $V(r_*)$ is positive everywhere, we cannot impose on Eq. (28) the usual boundary conditions for bound systems, i.e., vanishing of the wave function for large values of the radial coordinate r_* .

Instead, boundary conditions were derived ad hoc by requiring that there should be no radiation coming in from infinity, nor emerging from the black hole. This leads to the requirement

$$\phi(\omega, r_*) \sim e^{i\omega r_*} \quad \text{as } r_* \rightarrow -\infty \quad \text{and} \quad \phi(\omega, r_*) \sim e^{-i\omega r_*} \quad \text{as } r_* \rightarrow \infty. \quad (29)$$

More precisely, in order to obtain the harmonic solution and the time-independent version of Eq. (6), we have to subject Eq. (6) and its solution to a Laplace transform, rather than a Fourier transform (Nollert and Schmidt 1992). This leads to an *inhomogeneous* time-independent equation,

$$\phi''(\omega, r_*) + (\omega^2 - V(r_*)) \phi(\omega, r_*) = -\dot{Q}_0(r_*) - i\omega Q_0(r_*) \equiv J(\omega, r_*) \quad (30)$$

which explicitly contains the initial data for the time-dependent solution on its right-hand side. The complete time-dependent solution is then obtained via the inverse Laplace transform:

$$Q(t, r_*) = \frac{1}{2\pi} \int_{-\infty+i\epsilon}^{\infty+i\epsilon} \phi(\omega, r_*) e^{i\omega t} d\omega, \quad (31)$$

where ϵ is an arbitrarily small positive number.

The solution for the time-independent equation (30) is obtained via a Green function

$$\phi(\omega, r_*) = \int_{x_2}^{x_1} G(\omega, r_*, r'_*) J(\omega, r'_*) dr'_*, \quad (32)$$

with the integral running over a range $x_2 \leq r_* \leq x_1$ where the initial data is non-zero. The Green function $G(\omega, r_*, r'_*)$ is constructed in the usual way from solutions of the homogeneous version of Eq. (30) that satisfy a boundary condition at only *one* boundary each (Nollert and Schmidt 1992). It then turns out that this Green function has poles in the complex frequency plane which lead to the modes at the characteristic frequencies found in the numerical experiments. Analyzing the properties of the Green function further allows to identify other characteristic contributions to the time-dependent perturbation response, such as the late-time power-law tail (Leaver 1986, Andersson 1997).

Numerical results

It turns out that there can be no quasinormal frequencies with negative or vanishing imaginary part. This makes physical sense: Based on the time dependence of Eq. (27), a positive imaginary part yields a damped oscillation, the damping being due to energy being carried off as gravitational radiation. On the other hand, the boundary conditions of Eq. (29) imply that the quasinormal mode solutions diverge, as functions of r_* , both at the horizon and at spatial infinity. This is not as unphysical as it may seem: A contribution

to the wave function very far from the source must have originated at the source a very long time ago, when the wave function had to be much larger than at a later time. This illustrates that the paradigm of a stationary situation — one that has existed for an arbitrarily long time and will continue without change for an arbitrarily long time — is not really appropriate here: An arbitrarily long time ago, the perturbations at the source would have had to be arbitrarily large. Instead, we have to take into account that the perturbations started at a definite time $t = 0$, and they will die exponentially with time. This is the reason why we talk here about a quasi-stationary, rather than a stationary, picture.

In fact, the Green function approach described above takes this into account explicitly; the Green function itself, and the solutions used to construct it, do not diverge along the integral path for the inverse Laplace transformation.

Also, note that there is no divergence along the characteristics of the wave equation, i.e., towards past or future null infinity.

Nevertheless, the presence of exponentially growing versus exponentially dying solutions has been a stumbling block for the numerical determination of quasinormal frequencies with large imaginary parts. For a long time, this restricted the known frequencies to the fundamental (least damped) one, and maybe one or two higher modes (Chandrasekhar and Detweiler 1975, Ferrari and Mashhoon 1984). Questions that remained unsolved included the total number of modes (finite or infinite), or whether the frequencies were unconstrained or confined to a finite part of the complex plane. Adapting a continued fraction technique used in molecular physics, Leaver (1986) successfully determined frequencies with imaginary parts as high as $\sim 30 \times 2M$. Nollert and Schmidt (1992) found a way to express the Wronskian appearing in the construction of the Green function (32) as a convergent series, confirming Leaver's numerical results. Figure 4 shows the first 11 (or 12) quasinormal frequencies for $l = 2$.

Since then, various other techniques have been introduced (e.g. (Andersson and Linnæus 1992)), providing essentially identical results. A somewhat lingering debate remains over the existence or non-existence of an imaginary frequency at $\omega = 4i$ (for $l = 2$). Bachelot and Motet-Bachelot (1993) have presented a proof that the number of quasinormal modes is indeed countably infinite. Extending the continued fraction method pioneered by Leaver (1986), Nollert (1993) found a way to compute quasinormal frequencies with arbitrarily large imaginary parts.

3.2 Quasinormal modes of neutron stars

Using the harmonic time dependence of Eq. (27) for all perturbation quantities, we arrive at a system of 5 ordinary differential equations of first order. Detweiler and Lindblom (1985) have shown that this can be reduced to a system of 4 first-order equations. We do not have a horizon now, and the domain of integration therefore ranges from $r = 0$ to $r \rightarrow \infty$. Boundary conditions have to be imposed at $r = 0$, at the surface of the star, $r = R_{\text{star}}$, and at $r \rightarrow \infty$. The conditions and the restrictions they impose are:

- At the center ($r = 0$): The solutions have to be regular. This restricts 3 degrees of freedom.
- At the surface of the star: The Lagrangian change of density must vanish. However, this is not a real restriction: If the surface is defined by the pressure being zero in the equilibrium state, and if the density/pressure variation vanishes in the initial

Figure 4 The first 11 (12) quasinormal frequencies of a Schwarzschild black hole for $l = 2$.

data, then the perturbation equations will guarantee that it continues to vanish at all times.

- At $r \rightarrow \infty$, the geometry of spacetime is identical to the black hole case, and we can impose the same boundary condition, resulting in purely outgoing radiation at spatial infinity. This restricts one degree of freedom.

Numerical results

Neutron stars, unlike black holes, have a non-relativistic limit. Such a non-relativistic star will have a set of modes, associated with oscillations of the star's fluid. As long as we assume the star to consist of an ideal fluid and ignore internal dissipation, these oscillations will continue undamped forever, since there is no gravitational radiation to carry away energy. A relativistic star loses energy through gravitational radiation, but we expect this loss to be slow compared to the period of oscillation. We can therefore expect a set of modes with oscillation periods close to their Newtonian counterparts, and a damping rate small compared to the inverse oscillation period.

Such modes are indeed found, they are commonly called p -modes (' p ' standing for pressure: the force driving the oscillation is the pressure of the fluid). Nobody has yet shown

explicitly, though, that these relativistic modes will indeed approach the undamped modes of a Newtonian star if gravity is 'turned off' in some way.

Figure 5 From (Andersson et al. 1995): Some p -modes (bottom) and the two branches of w -modes of a relativistic neutron star.

Figure 5 shows the frequencies of all known types of quasinormal modes of relativistic neutron stars. In addition to these slowly damped p -modes, Kokkotas and Schutz (1992) found another set of modes which are much more strongly damped; their damping time is on the same order as the oscillation period. Inspired by a model system with very similar characteristics, Kokkotas and Schutz (1986) offer the interpretation that these modes are associated with the spacetime metric, rather than the material of the star.

They call these modes w -modes; ' w ' standing for (gravitational) wave. Since the metric perturbations contain less energy and are radiated away faster than the p -modes, the w -modes have to be much more strongly damped. In this respect, they should be analogous to quasinormal modes of a black hole, in particular since the metric of a spherically symmetric star outside the star itself is identical to the Schwarzschild metric. The w -modes, however, seem to be unbounded in their real parts, while the quasinormal modes of a black hole are bounded in the real parts and unbounded in the imaginary parts.

Leins et al. (1993) found another branch of w -modes which appears to be more closely related to quasinormal modes of a black hole. For any given star, however, there is only a finite number of these w_{II} -modes; this number increases if the star is made more compact. It turns out that neither the frequencies of the w -modes nor those of the w_{II} -modes approach the quasinormal modes of a black hole if the (background) star is made as compact as general relativity allows (i.e., the star's Schwarzschild radius is $8/9$ of its actual radius) (Leins et al. 1993).

3.3 Are quasinormal modes physically significant?

We have seen that quasinormal modes of black holes and neutron stars can be formally defined and numerically computed. At this point, the question arises whether they are meaningful and significant for the study of gravitational waves originating from perturbations of black holes and neutron stars, and ultimately for the study of close collisions.

In fact, quasinormal modes were first studied because they were found in numerical studies of collapsing stars and black hole collisions (Press 1971, Vishveshwara 1970). Figure 6 shows gravitational waves being emitted from a perturbed Oppenheimer-Snyder dust collapse (Cunningham et al. 1979). At times between about $8 \times 4M$ and $30 \times 4M$, most of the radiation is made up of characteristic oscillations in the fundamental quasinormal mode. Subtracting the contribution of the fundamental mode from the total signal, we find that the remainder is two orders of magnitude smaller than the total signal!

However, at later time, the oscillations are swamped by a non-oscillatory power-law tail. This tail can obviously not be represented by a quasinormal mode, or any combination of them. Therefore, quasinormal modes do not contain the full information about the time evolution of perturbations. In addition to the tail, we notice in Fig. 4 that there are no quasinormal frequencies with large real parts, corresponding to fast oscillations. Therefore, perturbations containing fast oscillations cannot be represented by quasinormal modes.

The mathematical reason for this is that the differential equation (28), together with the boundary conditions (29), does not form a self-adjoint system. Linear bound systems of the kind that we usually study (such as a harmonic oscillator) form self-adjoint systems, guaranteeing that the set of normal modes will be orthogonal and complete. Therefore, studying the time evolution of normal modes will yield all the information one can obtain about the evolution of arbitrary initial data.

The situation for neutron stars is slightly different from black holes: The w -modes contain frequencies with large real parts. However, the time evolution of arbitrary initial data will still show a tail, which cannot be described by quasinormal ringing.

The 'missing contributions', such as the tail, high-frequency oscillations, and the response at early time, can be traced to the properties of the Green function (32) for the solution of the time-independent equation. Extensive studies have been performed by

Figure 6 From (Cunningham et al. 1979): Quasinormal ringing for quadrupole gravitational radiation, originating from a perturbed Oppenheimer-Snyder collapse.

Leaver (1986) and by Andersson (1997).

Figure 7 shows the extraction of waveforms from a fully non-linear computation of a head-on collision of two black holes. Again, quasinormal ringing is a prominent feature of the signal. Also shown is a fit with two modes $l = 2$. It turns out that they provide an excellent approximation to the original data, except for a short time after the onset of the signal. This result is rather surprising, since the quasinormal modes are based on a linearized perturbation treatment, while the data in Fig. 7 comes from a fully non-linear computation of an event which involves very strong curvature effects. Quasinormal ringing appears to be a very robust feature of processes which eventually form a stationary black hole, and probably of processes resulting in neutron stars as well.

4 Time-dependent perturbation analysis

Why would we want to use time-dependent perturbation theory, even though the quasi-stationary picture of quasinormal modes seems to describe the time evolution of the perturbation so nicely?

We have already seen that the set of quasinormal modes does not yield a complete description of the time dependent wave function. As far as the power-law tail is concerned, we may choose to ignore this fact, since it contains only a tiny fraction of the total radiated energy. However, the figures in the last section also show that there is a time period from the onset of the signal, lasting about one or two full oscillations, where the quasinormal mode fits are completely inadequate. This time period contains a considerable part of the total energy and cannot be ignored.

Furthermore, we need to be able to quantify the strength of excitation of a given quasinormal mode for a specific set of initial data. This can be achieved by fitting the

Figure 7 From (Anninos et al. 1993): Waveforms of gravitational radiation resulting from the head-on collision of two black holes, extracted at $R = 40M$ for $\ell = 2$. The cross marks the point where the quasinormal mode fit begins.

modes to the time-dependent signal, but only if we have already determined this signal. An alternative is to deform the contour of integration in the inverse Laplace transform (31) to include the singularities that form the quasinormal modes (Sun and Price 1988). This procedure works well for initial data with compact support. Realistic astrophysical data, however, falls off far from the source, but it does not vanish completely. Together with the exponential growth of the quasinormal mode solution in space, this creates a problem when trying to determine the excitation coefficient. These problems have been studied (Sun and Price 1988, 1990), but it may be easier to do the time evolution explicitly.

4.1 Initial data for black hole collisions

We have seen that the time evolution for perturbations of black holes is exceedingly simple. Before we can do the time evolution, though, we need to find appropriate initial data. Specifically, we need to find the internal geometry and the extrinsic curvature of an initial slice of spacetime such that they satisfy the constraint equations, and that they can be regarded as describing two black holes which are close together and about to collide and merge. (See the appendix for the definition of the extrinsic curvature, its relationship to the internal geometry of the spatial slices, and the constraint equations.)

Constraint equations, conformally flat version

These four constraint equations are (Bowen and York 1980)

$${}^3R - K_{ij}K^{ij} + K^2 = \rho \quad \text{Hamiltonian constraint} \quad (33)$$

$$\nabla^i(K_{ij} - g_{ij}K) = J_j \quad \text{Momentum constraint,} \quad (34)$$

where all indices i, j run from 1...3. g_{ij} is the interior geometry of the initial slice, 3R the corresponding curvature scalar, and K_{ij} the extrinsic curvature, $K = \text{Tr}(K_{ij})$. ρ is the energy density and J_j the current density on the initial slice. In vacuum, we have

$$\rho = 0 \quad J_j = 0. \quad (35)$$

Equations (33) are four non-linear, coupled partial differential equations. Considerable simplification can be achieved if we assume that the initial geometry is conformally flat:

$$g_{ij} = \phi^4 \eta_{ij}, \quad (36)$$

where η_{ij} is the metric of flat space. The extrinsic curvature then becomes

$$K_{ij} = \phi^{-2} \hat{K}_{ij}. \quad (37)$$

With this choice, the constraint equations are

$$\hat{\nabla}^2 \phi = -\frac{1}{8} \phi^{-7} \hat{K}_{ij} \hat{K}^{ij} \quad \text{Hamiltonian constraint} \quad (38)$$

$$\hat{\nabla}^i \hat{K}_{ij} = 0 \quad \text{Momentum constraint.} \quad (39)$$

All quantities and operators in the conformal space are distinguished from their “physical” counterparts by a caret.

Equations (38) are much simpler than Eqs. (33): The momentum constraint is linear in K , and it has now become independent of ϕ . It can be solved first, leaving only the Hamiltonian constraint as one non-linear equation for the scalar function ϕ .

Note that this is not an approximation: If ϕ and $\hat{K}_{ij}$ satisfy Eqs. (38), then g_{ij} and K_{ij} will satisfy Eqs. (33) *exactly*. It is, however, a specific choice; and it is possible that not all solutions of the original Eqs. (33) will satisfy this pattern. In fact, this will generally be the case, since we can easily choose a slicing of a given spacetime in such a way that some spatial slice is not conformally flat.

Initial deformation of spacetime

To simplify things further, we will assume for now that the two black holes are initially at rest. The time derivative of the interior geometry of the spatial slices will then vanish initially. Setting the shift vector β^i to zero, and using the slicing of the Schwarzschild background spacetime, $\alpha = \sqrt{1 - 2M/r}$, we conclude that

$$K_{ij} = 0 = \hat{K}_{ij} \quad (40)$$

in this case. The momentum constraint is then satisfied trivially, and we are left with only the Hamiltonian constraint, which takes the simple form:

$$\hat{\nabla}^2 \phi = 0. \quad (41)$$

A simple solution, of course, is $\phi = 1$. This is not a very interesting solution, since it simply represents flat space; there are no black holes at all. There are many different solutions, characterized by the boundary conditions they satisfy. Misner (1963) has given an algorithm that allows the construction of solutions with an arbitrary number of momentarily static wormholes. A wormhole is a spatial slice of the Kruskal spacetime which contains two isometric parts (Fig. 8). The isometry condition at the throat defines the boundary condition at $r = a = m/2$. This boundary condition singles out the solution $\phi = 1 - a/r$ for $r > a$, and $\phi' = 1 - a/r'$, with $r' = (m/2)^2 1/r$, for $r < a$.

Figure 8 Schematic representation of a “wormhole” spatial slice of a Schwarzschild spacetime.

A solution describing two wormholes is more complicated; it consists of an infinite series of contributions. We now have isometry conditions across both throats (Fig. 9) with their associated boundary conditions. In either the upper or lower sheet, these wormholes look like two momentarily static black holes, which will then proceed to collide head-on.

For several reasons, wormhole solutions are very popular in numerical relativity: They contain neither a singularity at the center, nor matter inside the horizon. They are given analytically and have well-known properties. They satisfy simple boundary conditions at the throats, which makes it possible to evolve only one sheet on the computer, saving memory and CPU time.

We should keep in mind, though, that wormhole solutions are simply one class of solutions of the constraint equations; there is no guarantee that they correspond to the physical situation of two black holes that are about to collide. It is clear that one wormhole will look from the outside like a Schwarzschild black hole. If we place two wormholes far enough apart that they are both in each other's asymptotic regime, it is safe to interpret them as two black holes far apart. However, it will generally be too expensive to follow the time evolution numerically from a point where the two black holes are asymptotically far apart. For the perturbation approach to be applicable, we also have to require them to be close initially, generally even closer than for numerical calculations.

It is not clear if the two-wormhole Misner solution is a good description of two close, initially static black holes. On the other hand, this configuration is not very realistic physically; there are probably no static black holes in the universe which are somehow held apart and are suddenly let go. We may therefore assume that the two-wormhole solution is as good an approximation for the initial data of an initially static black hole

Figure 9 Schematic representation of a spatial slice containing two wormholes of equal size.

collision as any other, and feel justified in using it.

Initial derivative of metric perturbation

If we want to allow the black holes to have momentum or spin initially, then we need to determine the extrinsic curvature, as a solution of the momentum constraint, for the initial spatial slice.

Bowen and York (1980) have found analytic solutions for the extrinsic curvature of one black hole carrying momentum or spin. These are simply

$$\hat{K}_{ij} = \frac{3}{2r^2} [2P_{(i}n_{j)} - (\eta_{ij} - n_i n_j)P^c n_c] \quad (42)$$

$$\hat{K}_{ij} = \frac{3}{r^3} (\epsilon_{kil} S^l n^k n_j + \epsilon_{kjl} S^l n^k n_i), \quad (43)$$

where P is the momentum, S the spin, n denotes a normal vector pointing from the center to the point where the extrinsic curvature is evaluated; and ϵ_{kjl} is the completely antisymmetric tensor.

Since the conformal form of the momentum constraint (38) is linear, we may simply add contributions from several black holes. In doing so, each contribution has to be computed with r and n relative to the center of the respective black hole. We generally use only the lowest order in L/r , which coincides with the lowest value of l , for the total extrinsic curvature.

While the Misner solution for the conformal factor is symmetrized to make the two sheets of the wormhole spacetime isometric, the expressions (38) for the extrinsic curvature satisfy no such symmetry condition. It is possible to symmetrize tensors as well as scalar functions (Bowen and York 1980, Kulkarni 1984). However, we do not make use of this possibility, since the terms added for symmetrization are of higher order in L/r .

We need to multiply the conformal extrinsic curvature by ϕ^{-2} in order to obtain the extrinsic curvature in the original, physical space [Eq. (37)]. In this step, we simply use the conformal factor derived from the isotropic form of the Schwarzschild metric, rather than the correct solution for ϕ :

$$\phi^4 = \left(1 + \frac{M}{2\bar{r}}\right)^4, \quad (44)$$

where $\bar{r}$ is the isotropic Schwarzschild radial coordinate. Again, the difference is of higher order in L/r and needs not be considered in lowest order.

As an example, the extrinsic curvature of two initially static, counterrotating black holes (Cosmic Screw), in standard Schwarzschild coordinates, is given by

$$K = \frac{6SL}{r^2} \phi^2 \sin^2 \vartheta \left[\sin \vartheta \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} + \frac{4}{r} \frac{2\bar{r} + M}{2\bar{r} - M} \cos \vartheta \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \right]. \quad (45)$$

Of course, the Hamiltonian constraint will not be as simple any more as it was in the initially static case. In general, we will not be able to find an analytic solution, and we will need to solve it numerically. However, if the momentum or spin is small, and if we are only interested in the lowest order, with respect to P or S , of the wave form or radiated energy, then we can use the fact that the right-hand side of the Hamiltonian constraint contains K quadratically:

$$\hat{\nabla}^2 \phi \sim \phi^{-7} \hat{K} \cdot \hat{K} \approx 0. \quad (46)$$

This essentially tells us that initial momentum or spin will have a linear influence on the derivative of the spatial geometry, but it will deform the initial geometry itself only to higher order.

To summarize, the following approximations are made in the construction of initial data for black hole collisions which do not start out with an initially static configuration:

- The source term in the Hamiltonian constraint is ignored; the Misner solution for two wormholes is used.
- Only the lowest order in L/r , or l , is used in the extrinsic curvature.
- The extrinsic curvature is not symmetrized on the two sheets of the initial spatial slice.
- The conformal factor of a Schwarzschild spacetime is used when converting the conformal into the physical extrinsic curvature.

Each of these approximations is technical and used only to simplify the calculations. They could be relaxed if the use of more accurate initial data made a significant difference for the results to the order where the close approximation is valid.

The strategy, then, is to use the Misner solution of the initially static case for the conformal factor ϕ , and the extrinsic curvature obtained from Eqs. (42) and (38) to determine the initial time derivative.

Initial data for the Zerilli equation

We now have to convert the initial spatial geometry and the extrinsic curvature into initial data for the Zerilli function. First, the spatial geometry has to be converted into a Schwarzschild background metric plus a perturbation:

$$g_{ij} = \phi \eta_{ij} = (g_0)_{ij} + h_{ij}, \quad (47)$$

where g_0 is a spatial slice of the Schwarzschild metric. This step has been worked out by Price and Pullin (1994).

The (spatial) perturbation h_{ij} is then decomposed into tensor spherical harmonics as before, using only those tensors which do not have any non-zero time components:

$$h_{ij}(r, \vartheta, \phi) = \sum_{l=0}^{\infty} \sum_{m=-l}^l \sum_{n=5}^{10} C^{lmn}(r) T_{ij}^{lmn}(\vartheta, \phi). \quad (48)$$

The value of the Zerilli function itself is once again obtained by applying a linear operator to the resulting coefficients (Abrahams and Price 1996):

$$\text{Even parity: } Q_0^+(r) = Q^+(t=0, r) = \bar{\mathcal{L}}^+(C^{n,\text{even}}) \quad (49)$$

$$\text{Odd parity: } Q_0^\times(r) = Q^\times(t=0, r) = \bar{\mathcal{L}}^\times(C^{n,\text{odd}}). \quad (50)$$

The initial values for the derivative of the Zerilli function follow from the extrinsic curvature. Since the shift vector vanishes in the Schwarzschild spacetime, if using standard Schwarzschild coordinates, we simply have to apply the same procedure as above to the initial derivative of the spatial geometry (Abrahams and Price 1996), where α is the lapse function:

$$\partial_t g_{ij} = -2\alpha(r) K_{ij}(r, \vartheta, \phi) = \sum_{l=0}^{\infty} \sum_{m=-l}^l \sum_{n=5}^{10} \tilde{C}^{lmn}(r) T_{ij}^{lmn}(\vartheta, \phi). \quad (51)$$

$$\text{Even parity: } \dot{Q}_0^+(r) = \partial_t Q^+(t=0, r) = \bar{\mathcal{L}}^+(\tilde{C}^{p,\text{even}}), \quad (52)$$

$$\text{Odd parity: } \dot{Q}_0^\times(r) = \partial_t Q^\times(t=0, r) = \bar{\mathcal{L}}^\times(\tilde{C}^{p,\text{odd}}). \quad (53)$$

4.2 Results for black hole collisions

We have actually evolved only one scalar function which was derived from the full metric perturbation. It is possible to reconstruct the complete perturbed metric from the Zerilli function. However, we do not really need to do this, since we are mainly interested in the total energy radiated away as gravitational radiation, and in the wave form seen by an asymptotic observer. Therefore, all we need are certain components of the perturbed metric far away from the source.

Wave forms and total radiated energy

Taking the limit for large values of r , the power radiated into a solid angle element in a fixed direction is given by

$$\frac{dP}{d\Omega}(r \rightarrow \infty) = \frac{1}{16\pi} \left[\left(\frac{\partial}{\partial t} \frac{h_{\varphi\vartheta}}{\sin\vartheta} \right)^2 + \frac{1}{4} \left(\frac{\partial}{\partial t} \frac{h_{\varphi\varphi}}{\sin^2\vartheta} + \frac{\partial}{\partial t} h_{\vartheta\vartheta} \right)^2 \right] \quad (54)$$

(Landau and Lifshitz 1975, Cunningham et al. 1978, Cunningham et al. 1979, Abrahams and Price 1996).

Integrated over all angles, the total power radiated away towards spatial infinity is

$$P(r \rightarrow \infty) = \frac{1}{32\pi} \sum_{l,m} \left(\left(\frac{\partial}{\partial t} Q_{lm}^+ \right)^2 + (Q_{lm}^\times)^2 \right). \quad (55)$$

Contributions from different values of l and m and from different parities can therefore simply be added for the total radiated energy. The wave form observed in a specific direction, however, will show interference between these different contributions; these interferences only cancel out upon integrating over all angles.

Different configurations

The first configuration studied in the close approximation with the perturbation approach was the initially static collision (Price and Pullin 1994). The results are shown in Fig. 10, together with results of fully non-linear supercomputer calculations, and estimates from the two-phase approximation (Abrahams and Cook 1994, Baker and Li 1997).

The most striking result is the excellent agreement between the close approximation and the full numerical calculations. The two black holes are surrounded by a common apparent horizon only for $\mu_0 < 1.36$. Even though the logarithmic scale of the energy de-emphasizes the errors to some extent, it is quite amazing that the result of the close approximation yields even anything reasonable for an initial configuration that barely forms a single black hole at all.

One possible explanation for this surprisingly large range of validity of the close approximation is that most non-linear effects in the metric occur close to the horizon, and most of the gravitational radiation generated close to the horizon disappears into the black hole, anyway. A distant observer therefore never sees the differences between the correct metric distortions and those resulting from the close approximation.

This spectacular result encouraged the extension of this method to more complicated initial configurations. Allowing initial momentum such that the black holes approach each other can significantly increase the total radiated energy, as shown in Fig. 11. This situation is also somewhat more realistic, since it can be interpreted as the final stage of a collision that starts with the two black holes far apart.

It is somewhat surprising to see that the radiated energy actually decreases at first if the black holes are given some initial momentum. Only after increasing the initial momentum does the energy rise, eventually becoming much larger than the original, unboosted contribution. This dip can actually be understood as an interference between the contribution from the collision itself and from the initial momentum (Baker et al. 1997). The perturbation approach provides additional insight in this case: Since it is

Figure 10 [From: Baker and Li (Baker and Li 1997)] Total radiated energy according to the close approximation, for an initially static collision of two black holes. The parameter μ_0 determines the initial separation.

Figure 11 Total radiated energy according for black hole collisions, starting with the black holes moving towards each other. Three different initial separations are shown.

linear, we can identify and follow the individual contributions through the whole process of generating initial data, computing the time evolution, and obtaining the radiated energy. This is not possible in the non-linear numerical calculation.

Table 1 lists the leading contributions to the energy emitted as gravitational radiation for the configurations we discuss. Both the contribution from the conformal factor itself,

		Contribution	Even	Odd
Initially Static	$l = 2$ $m = 0$	Misner	$Q = Q_M$ $\dot{Q} = 0$	
Boosted Head-On	$l = 2$ $m = 0$	Misner Momentum	$Q = Q_M$ $\dot{Q} = \dot{Q}_P$	
Boosted Head-On	$l = 2$ $m = 0$	Misner	$Q = Q_M$ $\dot{Q} = 0$	
		Momentum	$Q = 0$ $\dot{Q} = \dot{Q}_P$	
Cosmic Screw	$l = 2$ $m = 0$	Misner	$Q = Q_M$ $\dot{Q} = 0$	
		Spin		$Q = 0$ $\dot{Q} = \dot{Q}_S$

Table 1 Leading contribution to the total energy emitted as gravitational radiation for the configurations discussed in the text.

representing the initially static collision with $P = 0$, and the contribution from the extrinsic curvature due to the momentum, are characterized by $l = 2$, $m = 0$, and even parity. Therefore, they combine to generate one Zerilli function. However, we can choose to determine the initial data, as well as the time evolved Zerilli function, separately for these two contributions. We then have to add them before computing the energy according to Eq. (55), leading to an interference and the dip in Fig. 11.

It turns out that the agreement between the linearized perturbation calculation and the non-linear, numerical results extends not only to rather large separations, as we have seen above, but also to rather large values of the initial momentum. This is again surprising, since we had explicitly used the assumption, in generating the initial data, that the initial momentum is small [cf. Eq. (46)].

Another step towards astrophysically realistic collisions is to allow the black holes to carry spin. In order to be able to use the Schwarzschild metric as the background metric for the perturbation approach, we do not want overall spin in the initial (and therefore final) configuration. We will therefore assume that the spins of the two black holes are anti-parallel and of equal magnitude, such that their sum vanishes. If the spins are furthermore aligned along the direction of collision, then the whole process will still be axisymmetric. Such a configuration is commonly referred to as a “cosmic screw”.

Figure 12 shows that initial spin dramatically increases the total radiated energy, up to a factor of about 400 if $S = M^2$. Note that $S = 1M^2$ does not correspond to an extremal Kerr black hole; rather, S has to approach infinity for an extremal Kerr black hole [see Fig. 7 in (Choptuik and Unruh 1986)]. If the two black holes are assumed to have collapsed from neutron stars which have been rotating as fast as millisecond pulsars, their spin will be much smaller, roughly $S = 0.05M^2$. However, the black holes might be spun up by accretion after collapsing, and black holes in the centers of galaxies could

Figure 12 Total radiated energy for black hole collisions where the black holes carry spin, antiparallel and aligned with the direction of collision (cosmic screw). Two different initial separations are shown.

have spin parameters close to M^2 as well.

Contrary to the boosted case, there is no dip in the radiated energy as the spins are increased from zero. Table 1 shows that the conformal factor (Misner contribution) and the extrinsic curvature (spin contribution) generate different Zerilli functions. Therefore, we do not expect an interference when calculating the energy according to Eq. (55).

Further configurations have been studied, in particular non-axisymmetric collisions and pseudo-inspiral with small orbital angular momentum (Nollert 1997, Baker et al. 1997).

4.3 Initial data for neutron star perturbations

Unlike the black hole case, there are no solutions readily available as initial data for neutron star collisions. The constraint equations now have to be solved including the matter source terms. Before we can even do that, we have to find the matter distribution and currents, which in turn depend on the prior evolution of the star, or on the (quasi-)equilibrium configuration that we want to start with.

Possible approaches to solving this problem include:

- Educated guess, just as in the black hole case. We need to find a solution of the constraint equations which represents both the two original stars that merged, and the one star that they merge into, which provides the background for the perturbation calculation. In the case of the black holes, the wormhole solutions have these properties: The individual throats, which continue to exist separately

inside a potential common horizon, represent the two original black holes, while the common horizon characterizes the final black hole. No similar construction, as intuitively appealing as this, is available for the neutron star collision. It seems unlikely that analytic solutions can be found which retain this “double identity” for both the metric and the matter of the star.

- From fully relativistic numerical calculations of phase II. This may seem to defeat the purpose to some extent: If we have to start with a full numerical calculation, why not just keep running it into phase III, rather than switching to the perturbation technique? Numerical calculations are likely to be very expensive in terms of computer resources, it may therefore be desirable to stop the calculation as early as possible, and continue with a less demanding method. Even running numerical and perturbation calculations in parallel makes sense: It provides a check for the numerical code, and it may offer insights, such as in the black hole case, which would not be possible with a numerical calculation alone.
- From Newtonian or post-Newtonian calculations of phase I and II. Quasistable orbits of neutron star binaries can be computed to several post-Newtonian orders [e.g. Lincoln and Will (1990), (1996), and Shibata and Taniguchi (1997)]. It is also possible that hydrodynamic effects will dominate the actual merger phase (II) to such an extent that a treatment of the merger involving a post-Newtonian or other approximation (e.g. Taniguchi and Nakamura (1996), Baumgarte et al. (1997), Wilson et al. (1996), and Shibata (1996)), joined directly to a perturbation calculation, will actually provide a good approximation of the whole collision process, starting with a rather distant orbit and ending with the final equilibrium star. Comparison with fully relativistic numerical calculations will reveal just how important the effects of “full relativity” really are.
- For testing purposes, we may use initial data derived from the quasi-stationary results. If we chose a quasinormal mode of a particular frequency as initial data, then the time evolution will have to reproduce the known behaviour of this quasinormal mode (Ruoff 1996). This has no astrophysical relevance, of course, since quasinormal modes do not correspond directly to any physical state of the system.

4.4 Results for neutron star perturbations

No perturbation results are yet available for astrophysically interesting configurations. Codes exist for integrating the time-dependent perturbation equations (Ruoff 1996, Allen et al. 1997), but they have only been applied to test problems so far.

Ruoff (1996) has used results of stationary calculations performed by Leins et al. (1993). Instead of quasinormal frequencies, he used two arbitrary frequencies (one real, i.e., purely oscillatory without damping, and one complex, with damping). He has to omit the boundary condition at spatial infinity in order to obtain time-independent solutions for these frequencies. This implies that there is incoming radiation from infinity, but that does not present a problem for these test calculations. We simply have to make sure that the point of observation, the outer spatial boundary of the computational grid, and the time period of observation are chosen in such a way that the outer boundaries are not in the past light cone of the observation point at any moment during the observation. This also takes care of the problem that the time-independent solutions, whether they

are quasinormal modes or not, do not vanish, but rather grow, outside the computational grid.

Figure 13 From (Ruoff 1996): Initial data for the energy density perturbation (right) and one metric perturbation (left), using stationary solutions for the complex frequency $\omega = (1.3 + 0.1i)\text{km}^{-1}$.

The equation of state for the background star is based on a Machleidt potential A (MPA) (Wu et al. 1991). The central pressure was chosen to be $p_c = 9.08 \times 10^{34} \text{dyn/cm}^2$, corresponding to a central density of $\epsilon_c = 1.14 \times 10^{15} \text{g/cm}^3$. This results in a total mass of $M = 0.802M_\odot$ and a radius of $R = 10.09 \text{ km}$.

Figure 13 shows part of the initial data for the complex frequency $\omega = (1.3 + 0.1i)\text{km}^{-1}$. Shown are one of the metric perturbations, S_1 , and the perturbation of the energy density, ρ .

Figure 14 From (Ruoff 1996): The energy density perturbation ρ and one metric perturbation S_1 as functions of time and space, using the initial data shown in Fig. 13.

Figure 14 shows the evolution of the same perturbation quantities as a function of the radial coordinate r and of time. In Fig. 15, the time dependence of S_1 at a fixed point inside the star, $r = 6.18 \text{ km}$, is compared with the amplitude expected from the stationary calculation. The time-dependent integration is in excellent agreement with the expected behaviour.

Figure 15 From (Ruoff 1996): Comparison of the time dependence of the metric perturbation S_1 at a fixed point in space with the behaviour expected from the stationary calculation.

5 Discussion

The perturbation method for close collisions works surprisingly well for collisions of black holes. This could be due to the non-linear effects manifesting themselves mostly close to the horizon; most of the gravitational radiation carrying this information is therefore captured by the black hole and does not reach an asymptotic observer.

In fact, it turns out that a collision of two widely separated black holes can be treated using Newtonian calculations when they are far apart, directly joining them to a perturbation calculation when the black holes reach a certain separation. The results of this procedure are very robust with respect to the choice of the point where the transition from Newtonian to perturbation approach is made (Abrahams and Cook 1994, Baker and Li 1997), and agree well with fully non-linear numerical calculations (Baker et al. 1997).

It appears at first that the approach followed here should limit us not only to small separations, but also to small values of initial momentum or spin for the black holes. Again, it turns out that much larger values can be allowed than expected. The reason here is probably that the contributions of momentum or spin quickly become much larger than the one from the initial deformation of the spatial slice. This, in turn, depends on the term ϕ^{-7} on the right-hand side of the Hamiltonian constraint, which prevents the conformal factor from becoming large even if the extrinsic curvature is not small any more. The mistake we make in ignoring the influence of the extrinsic curvature K on the conformal factor via the Hamiltonian constraint can thus be ignored. This essentially

means that details of the initial deformation, which are difficult to calculate, may not be important for the gravitational radiation any more when initial momentum or spin become fairly large.

All of these favorable coincidences do not occur in the neutron star case. There is no horizon which could swallow part of the gravitational radiation, destroying information about strong non-linear effects. The constraint equations now have source terms depending on the fluid of the star, making estimates about the behaviour of their solutions for large separations and large momenta or spins more difficult.

Studying neutron star collisions in the perturbation approximation nevertheless makes sense: It will provide important checks for the much more complex numerical codes that are being developed. In addition, it is quite possible that the perturbation method will still work much better than expected, just as it did for black holes. In order to find out, we will have to follow this approach and examine the results we obtain.

The major problem in doing so lies in obtaining appropriate initial data. Other than for black holes, there are no analytical solutions available that we can use as an 'educated guess' for the correct initial data. We do not have the concept of two throats inside a common horizon any more, which allowed us to retain the signature of two individual black holes within the one black hole that formed in the collision. Therefore, we need to obtain initial data from numerical calculations specifically designed for this purpose, such as solving the constraint equations with source terms based on post-Newtonian results, or on results of a fully non-linear calculation of the merger phase.

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Appendix

The 3+1 decomposition of spacetime

In order to describe the "time" evolution of a relativistic system, we may choose to represent the four-dimensional spacetime as a collection of three-dimensional space-like slices which are then "stacked" in a timelike direction.

Figure 16 shows the relationship between a slice at time t and one at time $t + dt$. The lapse function $\alpha(x^i)$ gives the distance between two points on adjacent slices along a normal direction. Conversely, choosing a lapse function $\alpha(x^i)$ will result in a specific slicing of a four-dimensional spacetime. The shift vector $\beta^i(x^i)$ represents the freedom to choose coordinates independently on each slice.

Each of these slices has an interior, three-dimensional geometry, described by g_{ij} . The four-dimensional spacetime metric is directly related to this interior geometry, the lapse function, and the shift vector:

$$g_{\mu\nu} = \begin{pmatrix} \beta_k \beta^k - \alpha^2 & \beta_j \\ \beta_i & g_{ij} \end{pmatrix}. \quad (56)$$

Figure 16 3 + 1 decomposition of spacetime into spatial slices (one spatial dimension suppressed).

As an example, we have for the Schwarzschild metric:

$$\alpha = \sqrt{1 - \frac{2M}{r}} \quad \vec{\beta} = 0. \quad (57)$$

The three-dimensional extrinsic curvature K_{ij} describes how each slice is embedded in the four-dimensional spacetime:

$$K_{ij} = -n_{i;j}. \quad (58)$$

It is also related to the change of interior geometry from one slice to the next:

$$\partial_t g_{ij} = -2\alpha K_{ij} + g_{ij,k}\beta^k + g_{ki}\beta_{;j}^k + g_{kj}\beta_{;i}^k. \quad (59)$$

This dual role can be easily understood from Fig. 16: If we know the geometry in the neighborhood of a point x^i , we know how the normal vector “tilts” when we move from one point to another. This change of the normal vector defines the extrinsic curvature according to Eq. (58). On the other hand, if we know all normal vectors around x^i , as well as the lapse function $\alpha(x^i)$, then we can construct the surface at a later time $t + dt$ (assuming for a moment that $\beta^i = 0$), and then derive its internal geometry $g_{ij}(t + dt)$. Eventually, this leads to Eq. (59).

The time derivative of the extrinsic curvature is given by:

$$\begin{aligned} \partial_t K_{ij} = & \alpha \left[R_{ij} + K K_{ij} - 2 K_{ik} K^k_j - 8 \pi \left(T_{ij} - \frac{1}{2} T g_{ij} \right) \right] \\ & - \alpha_{;j;i} + K_{ij,k} \beta^k + K_{ik} \beta^k_j + K_{jk} \beta^k_i, \end{aligned} \quad (60)$$

where $K = \text{Tr}(K_{ij})$, $T = \text{Tr}(T_{ij})$, and R_{ij} is the three-dimensional curvature tensor of the spatial slice.

Choosing g_{ij} and K_{ij} as the unknown functions, we can regard Eqs. (59) and (60) as the first-order version of the gravitational field equations.

The interior geometry and the extrinsic curvature for any given spatial slice are not independent of each other. Four equations among the field equations are not related to the time evolution of g_{ij} and K_{ij} , but rather represent conditions which have to be satisfied in each spatial slice by itself. They are therefore called constraint equations. In the second order formulation of the field equations, the constraint equations contain at most first derivatives in time. In the first order version of Eqs. (59) and (60), they do not contain any time derivative at all. They do, however, always contain derivatives with respect to the spatial coordinates.

The explicit form of the constraint equations is given by Eqs. (33).

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