Post-Newtonian Expansion of Gravitational Waves from a Particle in Circular Orbits around a Rotating Black Hole
– Effects of Black Hole Absorption –

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When a particle moves around a Kerr black hole, it radiates gravitational waves. Some of these waves are absorbed by the black hole. We calculate such absorption of gravitational waves induced by a particle of mass $\mu$ in a circular orbit on an equatorial plane around a Kerr black hole of mass $M$. We assume that the velocity of the particle $v$ is much smaller than the speed of light $c$ and calculate the energy absorption rate analytically. We adopt an analytic technique for the Teukolsky equation developed by Mano, Suzuki and Takasugi. We obtain the energy absorption rate to $O((v/c)^8)$ compared to the lowest order. We find that the black hole absorption occurs at $O((v/c)^5)$ beyond the Newtonian-quadrupole luminosity at infinity in the case when the black hole is rotating, which is $O((v/c)^3)$ lower than the non-rotating case. Using the energy absorption rate, we investigate its effects on the orbital evolution of coalescing compact binaries.

§1. Introduction

Among the possible sources of gravitational waves, coalescing compact binaries are the most promising candidates for detection by near-future, ground based laser interferometric detectors such as LIGO, VIRGO, GEO600 and TAMA. When a signal of gravitational waves is detected, we will attempt to extract parameters for binaries, such as masses and spins etc., from inspiral wave forms using the matched filtering technique. In this method, parameters for binaries are determined by cross-correlating the noisy signal from detectors with theoretical templates. If the signal and the templates lose phase with each other by one cycle over $\sim 10^3 - 10^4$ cycles as the waves sweep through the interferometer band, their cross correlation will be significantly reduced. This means that, in order to extract information optimally, we need to make theoretical templates which are accurate to better than one cycle during an entire sweep through the interferometer’s band.\(^1\)

The standard method to calculate inspiraling wave forms from coalescing binaries is the post-Newtonian expansion of the Einstein equations, in which the orbital velocity $v$ of the binaries are assumed to be small compared to the speed of light. Since, for coalescing binaries, the orbital velocity is not so small when the frequency of gravitational waves is in LIGO/VIRGO band, it is necessary to carry the post-Newtonian expansion up to extremely high order in $v$. A post-Newtonian wave generation formalism which can handle the high order calculation has been developed by Blanchet, Damour and Iyer.\(^2,3\) Based on this formalism, calculations have been carried out up to post-$5/2$-Newtonian order, or $O(v^5)$ beyond the leading order quadrupole formula.\(^4\) Another formalism has been developed up to $O(v^4)$ by Will and Wiseman.\(^5\) This formalism is based on that of Epstein and Wagoner.\(^6\)

Another post-Newtonian expansion techniques based on black hole perturbation formalism has also been developed. In this analysis, one considers gravitational waves
from a particle of mass \( \mu \) orbiting a black hole of mass \( M \) when \( \mu \ll M \). Although this method is restricted to the case \( \mu \ll M \), one can calculate very high order post-Newtonian corrections to gravitational waves using a relatively simple algorithm in contrast with the standard post-Newtonian analysis. The gravitational radiation at infinity was calculated using this method to \( O(v^8) \) by Tagoshi, Shibata, Tanaka and Sasaki\(^7\) in a Kerr black hole case, and to \( O(v^{11}) \) by Tanaka, Tagoshi and Sasaki\(^8\) in a Schwarzschild black hole case.

Despite these works, a study of gravitational waves which are absorbed by a black hole has been developed very little in the context of the post-Newtonian approximation. The post-Newtonian approximation of the absorption of gravitational waves into the black hole horizon was first calculated by Poisson and Sasaki in the case when a test particle is in a circular orbit around a Schwarzschild black hole.\(^9\) In this case, since the effect of the black hole absorption appears at \( O(v^8) \), this effect is negligible for the orbital evolution of coalescing compact binaries in the above laser interferometer’s band. However, as we shall see in this paper, black hole absorption appears at \( O(v^5) \) when a black hole is rotating. Thus it is important to investigate the effect of black hole absorption to the orbital evolution of coalescing compact binaries when at least one of the stars is a rotating black hole.

In order to calculate the post-Newtonian expansion of ingoing gravitational waves into a Schwarzschild black hole, Poisson and Sasaki\(^9\) used two types of representations of a solution of the homogeneous Teukolsky equation. One is expressed in terms of spherical Bessel functions which can be used at large radius, and the other is expressed in terms of a hypergeometric function which can be used near the horizon. Then the two types of expressions are matched at some region where both formulas can be applied. They obtained formulas for a solution of the Teukolsky equation which can be used to calculate ingoing gravitational waves to \( O(v^{13}) \), although they presented formulas for ingoing waves only to \( O(v^8) \).

On the other hand, another analytic technique for the Teukolsky equation was found by Mano, Suzuki and Takasugi.\(^10\) Since this method is very powerful for the calculation of the post-Newtonian expansion of the Teukolsky equation, we adopt this method in this paper.

This paper is organized as follows. In §2, we develop the Teukolsky equation. In §3, first we describe the analytic techniques for the Teukolsky equation and then we solve the Teukolsky equation analytically assuming the frequency of the wave is small. In §4, the energy absorption rate is calculated to \( O(v^8) \) compared to the lowest order. In §5, we consider the effect of black hole absorption on the orbital evolution of coalescing compact binaries. Section 6 is devoted to summary and discussion.

Throughout this paper we use units in which \( c = G = 1 \).
§2. The Teukolsky formalism

In the Teukolsky formalism, the gravitational perturbations of the Kerr black hole are described by a Newman-Penrose quantity $\psi_4 = -C_{\alpha\beta\gamma\delta} n^\alpha \bar{m}^\beta n^\gamma \bar{m}^\delta$. Here $C_{\alpha\beta\gamma\delta}$ is the Weyl tensor, $n^\alpha = ((r^2+a^2), -\Delta, 0, a)/(2\Sigma)$, $m^\alpha = (ia \sin \theta, 0, 1, i/\sin \theta)/(\sqrt{2}(r + ia \cos \theta))$. We use the Boyer-Lindquist coordinates $(t, r, \theta, \phi)$ and $\Sigma = r^2 + a^2 \cos^2 \theta$, $\Delta = r^2 - 2Mr + a^2$, where $a$ is the spin parameter of a Kerr black hole. We decompose $\psi_4$ into Fourier-harmonic components according to

\[
(r - ia \cos \theta)^4 \psi_4 = \sum_{\ell m} \int d\omega e^{-i\omega t + i\ell \phi} -2S_{\ell m}(\theta) R_{\ell m\omega}(r).
\]  

(2.1)

The radial function $R_{\ell m\omega}$ and the angular function $s_{S_{\ell m}}(\theta)$ satisfy the Teukolsky equations with $s = -2$ as

\[
\Delta^2 \frac{d}{dr} \left( \frac{1}{\Delta} \frac{dR_{\ell m\omega}}{dr} \right) - V(r) R_{\ell m\omega} = T_{\ell m\omega},
\]

(2.2)

\[
\left[ \frac{1}{\sin \theta} \frac{d}{d\theta} \left( \frac{d}{d\theta} \right) - a^2 \omega^2 \sin^2 \theta - \frac{(m - 2 \cos \theta)^2}{\sin^2 \theta} + 4a \omega \cos \theta - 2 + 2ma + \lambda \right] -2S_{\ell m} = 0.
\]

(2.3)

The potential $V(r)$ is given by

\[
V(r) = -\frac{K^2}{\Delta} + 4i(r - M)K + 8i\omega r + \lambda,
\]

(2.4)

where $K = (r^2 + a^2)\omega - ma$ and $\lambda$ is the eigenvalue of $-2S_{\ell m}$. The angular function $s_{S_{\ell m}}(\theta)$ is the spin-weighted spheroidal harmonic which can be normalized as

\[
\int_0^\pi |s_{S_{\ell m}}|^2 \sin \theta d\theta = 1.
\]

(2.5)

The source term $T_{\ell m\omega}$ is given in a paper TSTS.

We define two kinds of homogeneous solutions of the radial Teukolsky equation:

\[
R_{\ell m\omega}^{\text{in}} \to \begin{cases} C_{\ell m\omega} \Delta e^{-ikr^*} & \text{for } r \to r_+ \\ r^3 A_{\ell m\omega}^{\text{out}} e^{i\omega r^*} + r^{-1} A_{\ell m\omega}^{\text{in}} e^{-i\omega r^*} & \text{for } r \to +\infty, \end{cases}
\]

(2.6)

\[
R_{\ell m\omega}^{\text{up}} \to \begin{cases} B_{\ell m\omega}^{\text{out}} e^{ikr^*} + \Delta^2 B_{\ell m\omega}^{\text{in}} e^{-ikr^*} & \text{for } r \to r_+ \\ D_{\ell m\omega}^{\text{out}} e^{i\omega r^*} & \text{for } r \to +\infty, \end{cases}
\]

(2.7)

where $k = \omega - ma/2M r_+$, and $r^*$ is the tortoise coordinate defined by

\[
r^* = \int \frac{dr^*}{dr} dr = r + \frac{2Mr_+}{r_+ - r_-} \ln \frac{r - r_+}{2M} - \frac{2Mr_-}{r_+ - r_-} \ln \frac{r - r_-}{2M},
\]

(2.8)

where $r_\pm = M \pm \sqrt{M^2 - a^2}$. 

We solve the Teukolsky equation by using the Green function method. A solution
of the Teukolsky equation which has a purely outgoing property at infinity and a purely
ingoing property at the horizon is given by

$$R^\ell_m \omega = \frac{1}{W} \left\{ R^{up}_{\ell m} \int_{r_+}^r dr' R^{in}_{\ell m} T_{\ell m} \Delta^{-2} + R^{in}_{\ell m} \int_r^\infty dr' R^{up}_{\ell m} T_{\ell m} \Delta^{-2} \right\}, \quad (2.9)$$

where the Wronskian $W$ is given by

$$W = 2i \omega D_{\ell m} A_{\ell m}^in. \quad (2.10)$$

Then, $\psi_4$ has an asymptotic property at the horizon such that

$$R_{\ell m} \omega (r \to r_+) \to \tilde{Z}^H_{\ell m} \Delta e^{-i k r^*}, \quad (2.11)$$

where

$$\tilde{Z}^H_{\ell m} = \frac{C_{\ell m}}{2i \omega D_{\ell m} A_{\ell m}^in} \int_{r_+}^\infty dr' R^{up}_{\ell m} T_{\ell m} \Delta^{-2}. \quad (2.12)$$

The energy flux formula is given by Teukolsky and Press\cite{12} as

$$\left( \frac{dE_{\text{hole}}}{dt d\Omega} \right) = \sum_{\ell m} \int \frac{d\omega}{2 \pi} \frac{d^2 S_{\ell m}}{2 \ell m} \frac{128 \omega k (k^2 + 4 \tilde{\epsilon}^2) (k^2 + 16 \tilde{\epsilon}^2) (2 M r_+)^5}{|C|^2} |\tilde{Z}^H_{\ell m}|^2, \quad (2.13)$$

where $\tilde{\epsilon} = \kappa/(4 r_+)$ and

$$|C|^2 = (Q^2 + 4 a \omega m - 4 a^2 \omega^2)[(Q - 2)^2 + 36 a \omega m - 36 a^2 \omega^2]$$

$$+ (2 Q - 1)(96 a^2 \omega^2 - 48 a \omega m) + 144 a^2 (M^2 - a^2), \quad (2.14)$$

and $Q = \lambda + 2.$

The post-Newtonian expansion of these formulas can be obtained by solving the
Teukolsky equation by setting $\epsilon \equiv 2 M \omega \sim O(v^3), z = \omega r \sim O(v)$ and by assuming
$\epsilon \ll z \ll 1.$ Here, we define $v \equiv (M/r_0)^{1/2},$ where $r_0$ is the orbital radius of the test
particle.

The solution of the spin weighted spheroidal harmonics was given explicitly in a
previous paper\cite{7} up to $O(\epsilon^2)$ which is accurate enough for the present purpose. In the
next section, we explain the method to obtain the homogeneous solutions of the radial
equation.

§3. Analytic solutions of the homogeneous Teukolsky equation and the
post-Newtonian expansion

New analytic representations of the homogeneous Teukolsky equation were found
by Mano, Suzuki and Takasugi\cite{10,13}. In this paper, we adopt variables used by Mano
and Takasugi.\cite{13} Since all the details are given in that paper, we only describe the
outline of the method here. In this section, the value of the spin $s$ is assumed to be
$s = -2.$ In this method, the solutions of the Teukolsky equation are represented by
two kinds of expansions. One is a series of hypergeometric functions which is defined as

$$R^{\nu}_{0, s} = e^{-i \kappa \tilde{x}} (\tilde{x})^\nu + i e^+ (\tilde{x} - 1)^{-s - \nu^+}$$
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\[
\times \sum_{n=-\infty}^{\infty} \frac{\Gamma(1-s-2i\epsilon_+)}{\Gamma(n+\nu+1-i\tau)+\Gamma(n+\nu+1-s-i\epsilon)} a_{n}^{\nu}\]
\[
\times \bar{x}^n F \left( -n - \nu - i\tau, -n - \nu - s - i\epsilon, -2n - 2\nu; \frac{1}{\bar{x}} \right), \tag{3.15}
\]

where \( F(a, b, c, x) \) is the hypergeometric function, and
\[
\bar{x} = \frac{\omega}{\epsilon \kappa} (r-r_-), \quad \kappa = \sqrt{1-q^2}, \quad q = \frac{a}{M}, \quad \tau = \frac{\epsilon - mq}{\kappa}. \tag{3.16}
\]

The coefficients \( a_{n}^{\nu} \) obey the three terms recurrence relation
\[
a_{n}^{\nu} a_{n+1}^{\nu} + \beta_{n}^{\nu} a_{n}^{\nu} + \gamma_{n}^{\nu} a_{n-1}^{\nu} = 0, \tag{3.17}
\]
where
\[
\alpha_{n}^{\nu} = \frac{i\kappa (n+\nu + 1 + s + i\epsilon) (n+\nu + 1 + s - i\epsilon) (n+\nu + 1 + i\tau)}{(n+\nu + 1) (2n + 2\nu + 3)},
\]
\[
\beta_{n}^{\nu} = -\lambda - s (s+1) + (n+\nu) (n+\nu + 1) + \epsilon^2 + \epsilon (\epsilon - mq) + (n+\nu + 1),
\]
\[
\gamma_{n}^{\nu} = -\frac{i\kappa (n+\nu - s + i\epsilon) (n+\nu - s - i\epsilon) (n+\nu - i\tau)}{(n+\nu) (2n + 2\nu - 1)}. \tag{3.18}
\]

The series converges if \( \nu \) satisfies the equation
\[
R_n(\nu)L_{n-1}(\nu) = 1, \tag{3.19}
\]
where \( R_n(\nu) \) and \( L_n(\nu) \) are continued fractions defined by
\[
R_n(\nu) = \frac{a_{n}^{\nu}}{a_{n-1}^{\nu}}, \quad L_n(\nu) = \frac{a_{n+1}^{\nu}}{a_{n}^{\nu}} = -\frac{\gamma_{n}^{\nu} a_{n+1}^{\nu} + \alpha_{n}^{\nu} a_{n}^{\nu}}{\beta_{n}^{\nu} a_{n}^{\nu} + \alpha_{n}^{\nu} a_{n+1}^{\nu} + \gamma_{n}^{\nu} a_{n-1}^{\nu}}, \tag{3.20}
\]

Then the expansion equation (3.15) converges if \( r < \infty \).

The ingoing solution \( R_{\nu}^{in} \) is given by
\[
R_{\nu}^{in} = e^{i\kappa}(R_{0}^{\nu} + R_{0}^{-\nu-1}). \tag{3.21}
\]

The other solution is a series of Coulomb wave functions, defined as\(^{15}\)
\[
R_{C,s}^{\nu} = \bar{z}^{-1-s} \left( 1 - \frac{\epsilon \kappa}{\bar{z}} \right)^{s-i\epsilon_+} \sum_{n=-\infty}^{\infty} (-i)^n \frac{(n+1+s+i\epsilon)}{(n+1-s+i\epsilon)} a_{n}^{\nu} F_{n+\nu,s}(\bar{z}), \tag{3.22}
\]

where \( (a)_n = \Gamma(n+a)/\Gamma(a) \), and \( F_{n+\nu,s}(\bar{z}) \) is the Coulomb wave function defined by
\[
F_{n+\nu}(z) = e^{-iz} (2z)^{n+\nu} \frac{\Gamma(n+\nu + 1 - s + i\epsilon)}{\Gamma(2n + 2\nu + 2)} \times \Phi(n+\nu + 1 - s + i\epsilon, 2n + 2\nu + 2; 2iz),
\]
where $\Phi(a, b, z)$ is a confluent hypergeometric function. The expansion coefficients $a_n^\nu$ obey the same recurrence relation (3.17). The expansion equation (3.22) converges if $\nu$ is a solution of Eq. (3.19). In that case, the series converge for $\hat{z} > \epsilon\kappa$.

This solution is decomposed into a pair of solutions, the incoming solution at infinity, $R_{\nu; s}^\nu$, and the outgoing solution at infinity, $R_{\nu; \hat{s}}^\nu$. Explicitly, we have

$$R_{\nu; s}^\nu = R_{\nu; \hat{s}}^\nu + R_{\nu; s}^{\nu*},$$

where

$$R_{\nu; s}^\nu = 2^{\nu} e^{-\pi \epsilon} e^{i\pi (\nu + 1 - s)} \frac{\Gamma(\nu + 1 - s + i\epsilon)}{\Gamma(\nu + 1 - s - i\epsilon)} e^{-i\hat{z} \nu + i\epsilon + (\hat{z} - \epsilon\kappa) - s - i\epsilon} \times \sum_{n = -\infty}^{\infty} i^n a_n^\nu (2\hat{z})^n \Psi(n + \nu + 1 + s + i\epsilon, 2n + 2\nu + 2; 2i\hat{z}),$$

$$R_{\nu; s}^{\nu*} = 2^{\nu} e^{-\pi \epsilon} e^{-i\pi(\nu + 1 + s)} e^{i\hat{z} \nu + i\epsilon + (\hat{z} - \epsilon\kappa) - s - i\epsilon} \times \sum_{n = -\infty}^{\infty} i^n$$

where $\Psi(a, b, z)$ is a confluent hypergeometric function. The upgoing solution $R_{\ell m\omega}^{\nu\up}$ is then given by

$$R_{\ell m\omega}^{\nu\up} = R_{\nu; \hat{s}}^\nu.$$

We see that the above two kinds of solutions have a very wide convergence region. We also see that the two solutions $R_{\nu; s}^\nu$ and $R_{\nu; \hat{s}}^\nu$ are not linearly independent of each other. Then $R_{\nu; s}^\nu$ must be proportional to $R_{\nu; \hat{s}}^\nu$. The relation between them is given by

$$R_{\nu; s}^\nu = K_\nu R_{\nu; \hat{s}}^\nu,$$

where

$$K_\nu(s) = \left(\frac{2\epsilon\kappa}{\epsilon\kappa}\right)^{\nu + s - \bar{r}} 2^{-s - \bar{r}} \frac{\Gamma(1 - s - 2i\epsilon_+)}{\Gamma(\nu + 1 + s - 2i\epsilon)} \frac{\Gamma(\nu + 2 + 1 + 2i\epsilon)}{\Gamma(\nu + 1 + s - 2i\epsilon)} \times \frac{\Gamma(\nu + \nu + 1 + s + i\epsilon)}{\Gamma(\nu + 1 + s + i\epsilon)} \frac{\Gamma(\nu + 1 + i\epsilon)}{\Gamma(\nu + 1 + s - 2i\epsilon)}$$

$$\times \left(\sum_{n = \bar{r}}^{\infty} \frac{(\nu + 1 - s - i\epsilon)_n a_n^\nu(s)}{(n - \bar{r})! (\nu + 1 - s - i\epsilon)_n}ight)^* \times \left(\sum_{n = -\infty}^{\bar{r}} \frac{(-1)^n (\nu + 1 + s - i\epsilon)_n a_n^\nu(s)}{(\nu + 1 - s + i\epsilon)_n (\nu + 1 - s + i\epsilon)_n}ight)^{-1},$$

where $\bar{r}$ can be any integer and $K_\nu(s)$ is independent of the choice of $\bar{r}$. This relation holds in a region where both expansion equations (3.15) and (3.22) converge.

Using this relation, it is possible to obtain the asymptotic amplitudes of $R_{\ell m\omega}^{\nu\up}$ at infinity. They are given by

$$A_{\ell m\omega}^{\nu\up} = \frac{e^{i\kappa\omega}}{\sqrt{\omega}} \left[ K_\nu(s) - ie^{-i\pi\nu} \sin \pi(\nu + s + i\epsilon) K_{\nu - 1}(s) \right] A_{\nu; \hat{s}}^\nu e^{-i\epsilon \ln \omega},$$

$$A_{\ell m\omega}^{\nu\out} = \frac{e^{i\kappa\omega}}{\sqrt{\omega}^{1/2s}} \left[ K_\nu(s) + ie^{i\pi\nu} K_{\nu - 1}(s) \right] A_{\nu; s}^\nu e^{i\epsilon \ln \omega},$$
where
\[ A_{\nu}^\nu = 2^{-1-s+i\epsilon} e^{i(\pi/2)(\nu+1-s)} e^{-i\pi/2} \frac{\Gamma(\nu+1-s+i\epsilon)}{\Gamma(\nu+1-s-i\epsilon)} \sum_{n=-\infty}^{\infty} a_n^\nu, \quad (3.30) \]
\[ A_{\nu}^\nu e^{-i(\pi/2)(\nu+1-s)} e^{-i\pi/2} \sum_{n=-\infty}^{\infty} (-1)^n \frac{(\nu+1-s+i\epsilon)n}{(\nu+1-s-i\epsilon)n} a_n^\nu. \quad (3.31) \]

The asymptotic amplitude at the outer horizon \( C_{\ell m \omega} \) is given by
\[ C_{\ell m \omega} = \left( \frac{\epsilon \kappa}{\omega} \right)^{2s} e^{i \epsilon \ln \kappa} \sum_{n=-\infty}^{\infty} a_n^\nu. \quad (3.32) \]

The asymptotic amplitude \( D_{\ell m \omega} \) of \( R_{\ell m \omega}^{up} \) at infinity is also given by
\[ D_{\ell m \omega} = 2^{-1-s+i\epsilon} e^{-i\pi/2(\nu+1-s)} e^{-\frac{\pi}{2}(\nu+1-s)} \sum_{n=-\infty}^{\infty} \frac{(\nu+1-s+i\epsilon)n}{(\nu+1-s-i\epsilon)n} a_n^\nu (-1)^n \frac{e^{i \epsilon \ln \epsilon}}{\omega^{1+2s}}. \quad (3.33) \]

Next, we explain the relation of the method above to the post-Newtonian expansion. It is easy to see that if we set \( a_0^\nu = 1 \) at \( n = 0 \), the order of \( a_n^\nu \) in \( \epsilon \) increases with \( |n| \). Basically, this is because \( \alpha_n^\nu \) and \( \gamma_n^\nu \) contain an overall factor \( \epsilon \). Further, from the behavior of each term of the expansion equation (3.22) as a function of \( z \), we see that the post-Newtonian order of the expansion equation (3.22) increases. This can be checked by setting \( \epsilon \sim v \) and \( z \sim \nu \), where \( v = (M/r_0)^{1/2} \), and \( r_0 \) is the orbital radius of the test particle. Then, the above expansion is very useful for the post-Newtonian expansion. Note that these increases of the post-Newtonian order are not monotonic when \( n < 0 \). Therefore we must be careful to treat the expansion for \( n < 0 \). This is not, however, a serious problem because we can obtain the desired post-Newtonian accuracy by summing up the terms up to the appropriately large \( |n| \).

Finally, we briefly explain the procedure of the analytic calculation. We first calculate the eigenvalue \( \lambda \) in a power series of \( \epsilon \) using a method due to Fackerell and Crossman. Next we determine \( \nu \) by solving Eq. (3.19). This equation can also be solved in a power series of \( \epsilon \). Then we evaluate the expansion coefficients \( a_n^\nu \). This can be done conveniently by using continued fractions Eqs. (3.20) and setting \( a_0^\nu = 1 \). Once we obtain the \( a_n^\nu \), it is straightforward to calculate asymptotic amplitudes \( A_{\ell m \omega}^{in} \), \( C_{\ell m \omega} \) and \( D_{\ell m \omega} \). It is also straightforward but tedious to calculate \( R_{\ell m \omega}^{up} \) from Eq. (3.25) by assuming \( \epsilon \ll z \ll 1 \). These results are given in Appendix B.

\section{The black hole absorption to \( O(\nu^8) \)}

In this section, we evaluate Eq. (2.13). The calculation is almost the same as those given in section III of a paper TSTS. First, we solve the geodesic equation for circular motion in the equatorial plane. Next, we calculate the source term \( T_{\ell m \omega} \). The result is given by
\[ T_{\ell m \omega} = \int_{-\infty}^{\infty} dt e^{i \omega t - i m \varphi(t)} \left[ (A_{n} n_{0} + A_{\bar{m}} n_{0} + A_{\bar{m} \bar{m}}) \delta(r - r_{0}) + \{(A_{\bar{m} n_{1}} + A_{\bar{m} \bar{m}}) \delta(r - r_{0})\}_{,rr} + \{A_{\bar{m} \bar{m}} 2 \delta(r - r_{0})\}_{,rr}, \delta_{= \pi/2} \right], \quad (4.34) \]
where $A_{n,0}$, etc., are given in Appendix A. Inserting Eq. (4.34) into Eq. (2.12), we obtain

\[
\tilde{Z}_{\ell m,\omega}^H = \frac{2\pi C_{\ell m,\omega} \delta(\omega - m\Omega)}{2i\omega A^m_{\ell m,\omega} D_{\ell m,\omega}} \left[ R_{\ell m,\omega}^{up} \{ A_{n,0} + A_{\bar{m},0} + A_{\bar{m},\bar{m}} \} ight.
\]

\[
- \frac{dR_{\ell m,\omega}^{up}}{dr} \{ A_{\bar{m},1} + A_{\bar{m},\bar{m}} \} \Big|_{r=r_0, \theta=\pi/2}
\]

\[
\equiv \delta(\omega - m\Omega) Z_{\ell m}^H, \quad (4.35)
\]

where $\Omega$ is the orbital angular frequency given by

\[
\Omega = \frac{M_{1/2}}{r_{1/0}^{3/2}} \left[ 1 - qv^3 + q^2v^6 + O(v^9) \right]. \quad (4.36)
\]

From Eqs. (2.13) and (4.35), the time averaged energy absorption rate becomes

\[
\left( \frac{dE}{dt} \right)_H = \sum_{\ell m} \left[ \frac{128\omega k(k^2 + 4\ell^2)(k^2 + 16\ell^2)(2Mr_+)^5}{|C|^2} \right] Z_{\ell m}^H \Big|_{\omega=m\Omega}
\]

\[
\equiv \sum_{\ell m} \left( \frac{dE}{dt} \right)_{\ell,m}. \quad (4.37)
\]

Using a property of $-2S_{\ell m}^\omega(\theta)$ at $\theta = \pi/2$, it is straightforward to show that $\tilde{T}_{\ell,-m,-\omega} = (-1)^\ell T_{\ell,m,\omega}$ where $T_{\ell,m,\omega}$ is the complex conjugate of $T_{\ell,m,\omega}$. Since the homogeneous Teukolsky equation is invariant under complex conjugation followed by $m \to -m$ and $\omega \to -\omega$, we have $\tilde{Z}_{\ell,-m,-\omega} = (-1)^\ell Z_{\ell,m,\omega}$. Then, from Eq. (4.37), we have $(dE/dt)_{\ell,-m} = (dE/dt)_{\ell,m}$.

In order to express the post-Newtonian corrections to the black hole absorption, we define $\eta_{\ell,m}$ as

\[
\left( \frac{dE}{dt} \right)_{\ell,m} = \frac{1}{2} \left( \frac{dE}{dt} \right)_N v^5 \eta_{\ell,m}^H, \quad (4.38)
\]

where $(dE/dt)_N$ is the Newtonian quadrupole luminosity at infinity,

\[
\left( \frac{dE}{dt} \right)_N = \frac{32\mu^2 M^3}{5r_0^5} = \frac{32}{5} \left( \frac{\mu}{M} \right)^2 v^{10}.
\]

In Appendix C, we give $\eta_{\ell,m}$ for $m > 0$. The results for $m < 0$ are given by $\eta_{\ell,m} = \eta_{\ell,-m}$.

The total absorption rate is given by

\[
\left( \frac{dE}{dt} \right)_H = \left( \frac{dE}{dt} \right)_N v^5 \left[ -\frac{3}{4} q^3 - \frac{1}{4} q + \left( -q - \frac{33}{16} q^3 \right) v^2 
\]

\[
+ \left( \frac{7}{2} q^4 + 2qB_2 + \frac{1}{2} + 6q^3B_2 + \frac{85}{12} q^2 + 3q^4\kappa + \frac{1}{2} + \frac{13}{2} \kappa q^2 \right) v^3 
\]

\[
+ \left( -\frac{4651}{336} q^3 - \frac{43}{7} q - \frac{17}{56} q^5 \right) v^4 
\]

\[
+ \left( \frac{569}{24} q^2 + \frac{371}{48} q^4 + 18q^3B_2 - \frac{3}{4} q^3B_1 + 2\kappa + 2 + \frac{33}{4} \kappa q^2 \right) v^5 \right].
\]
Post-Newtonian Expansion of Gravitational Waves

\[ A_n = \frac{1}{2} \left[ \psi^{(0)} \left( 3 + \frac{nq}{\sqrt{1 - q^2}} \right) + \psi^{(0)} \left( 3 - \frac{nq}{\sqrt{1 - q^2}} \right) \right], \]
\[ B_n = \frac{1}{2i} \left[ \psi^{(0)} \left( 3 + \frac{niq}{\sqrt{1 - q^2}} \right) - \psi^{(0)} \left( 3 - \frac{niq}{\sqrt{1 - q^2}} \right) \right], \]
\[ C_n = \frac{1}{2} \left[ \psi^{(1)} \left( 3 + \frac{niq}{\sqrt{1 - q^2}} \right) + \psi^{(1)} \left( 3 - \frac{niq}{\sqrt{1 - q^2}} \right) \right], \quad (4.40) \]

and \( \psi^{(n)}(z) \) is the polygamma function.

We see that the absorption effect starts at \( O(v^5) \) beyond the quadrupole formula in the case \( q \neq 0 \). If we set \( q = 0 \), the above formula reduces to
\[ \left( \frac{dE}{dt} \right)_H = \left( \frac{dE}{dt} \right)_N \left( v^8 + O(v^{10}) \right), \quad (4.41) \]

which was derived by Poisson and Sasaki. Then, the black hole absorption is more important in the case \( q \neq 0 \). We also notice that Eq. (4.39) can be negative when \( q > 0 \). This is an example of super radiance phenomenon.

It is not clear that Eq. (4.39) has a finite limit as \( |q| \to 1 \). However by using the formulas
\[ \lim_{q \to \pm 1} \psi^{(0)} \left( 3 + \frac{niq}{\sqrt{1 - q^2}} \right) = \ln n - \ln \kappa + i \frac{q \pi}{|q|^2}, \quad (4.42) \]
\[ \lim_{q \to \pm 1} \psi^{(k)} \left( 3 + \frac{niq}{\sqrt{1 - q^2}} \right) = 0, \quad (k \neq 0) \quad (4.43) \]

we can obtain the limit of \( (dE/dt)_H \) as
\[ \lim_{q \to \pm 1} \left( \frac{dE}{dt} \right)_H = \left( \frac{dE}{dt} \right)_N \left[ -\frac{q}{|q|} - \frac{49}{16} \frac{q}{|q|} v^2 + \left( 4 \pi + \frac{133}{12} \right) v^3 \right. \]
\[ \left. - \frac{6817}{336} \frac{q}{|q|} v^4 + \left( \frac{535}{16} + \frac{97}{8} \pi \right) v^5 \right. \]
\[ + \left( \frac{3424}{105} \ln(2) + \frac{1712}{105} \gamma \frac{q}{|q|} + \frac{3424}{105} \frac{q}{|q|} \ln(\nu) \right. \]
\[ \left. - \frac{3647533}{22050} \frac{q}{|q|} - \frac{289}{6} \frac{q}{|q|} \pi - 16/3 \pi^2 \frac{q}{|q|} \right) v^6 \]
\[ + \left( \frac{84955}{336} + \frac{55873}{672} \pi \right) v^7 \]
\[ + \left( \frac{14077}{60} \frac{q}{|q|} \ln(2) + \frac{16441}{140} \gamma \frac{q}{|q|} + \frac{34987}{210} \frac{q}{|q|} \ln(\nu) - \frac{193}{12} \pi^2 \frac{q}{|q|} \right. \]
\[ \left. - \frac{4057965601}{6350400} \frac{q}{|q|} - \frac{1289}{9} \frac{q}{|q|} \pi \right) v^8 \]. \quad (4.44) \]

In Appendix D, we give \( (dE/dt)_H \) written in terms of \( x \equiv (M\Omega)^{1/3} \).

§5. The orbital evolution of coalescing compact binaries

Using the above results, we estimate the effect of black hole absorption on the orbital evolution of the coalescing compact binaries. We ignore the finite mass effect in
the post-Newtonian formulas and interpret $M$ as the total mass and $\mu$ as the reduced mass of the system.

The total cycle $N$ of gravitational waves from an inspiraling binary is given by

$$N = \int_{v_i}^{v_f} \frac{\Omega}{\pi} \frac{dE}{dv} \, dv \Omega \frac{dE}{dv},$$

(5.45)

where $v_i = (M/r_i)^{1/2}$, $v_f = (M/r_f)^{1/2}$, and $r_i$ and $r_f$ are the initial and final orbital separation of the binary. The energy loss rate is given by $\frac{dE}{dt} = (\frac{dE}{dt})_0 + (\frac{dE}{dt})_\infty$, where $(\frac{dE}{dt})_\infty$ is the luminosity at infinity.

We set $r_f = 6M$ and define $r_i$ to be the radius at which the frequency of the wave is 10Hz. We evaluate $\Delta N$ which is the difference of the total cycle $N$ caused by the formulas for $\frac{dE}{dt}$ with and without the black hole absorption effect. The value of $\Delta N$ is calculated explicitly by

$$\Delta N = \left| \int_{v_i}^{v_f} \frac{\Omega}{\pi} (\frac{dE}{dt})_0 + (\frac{dE}{dt})_\infty \, dv \, \pi \frac{dE}{dv} - \int_{v_i}^{v_f} \frac{\Omega}{\pi} (\frac{dE}{dt})_\infty \, dv \, \pi \frac{dE}{dv} \right|.$$  

(5.46)

The results for the typical NS-BH systems are given in Table I. We find that the effect of black hole absorption is very small when $q = 0$. We also find that the black hole absorption is more important in cases for which $q > 0$ than in cases for which $q < 0$. This is because, when $q < 0$ the total cycle is much smaller than the case $q > 0$. We also notice that the black hole absorption is more important when the mass of the black hole is large. The reason is as follows.

Table I. The difference of the total cycle $N$ caused by the formulas for $\frac{dE}{dt}$ with and without the black hole absorption effect for typical neutron star(NS)-black hole(BH) binaries with mass $(M_{NS}, M_{BH})$.

<table>
<thead>
<tr>
<th>$q$</th>
<th>$(1.4M_\odot, 10M_\odot)$</th>
<th>$(1.4M_\odot, 40M_\odot)$</th>
<th>$(1.4M_\odot, 70M_\odot)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$&lt; 10^{-2}$</td>
<td>0.02</td>
<td>0.1</td>
</tr>
<tr>
<td>0.9</td>
<td>0.5</td>
<td>1.1</td>
<td>1.5</td>
</tr>
<tr>
<td>$-0.9$</td>
<td>0.3</td>
<td>0.5</td>
<td>0.4</td>
</tr>
</tbody>
</table>

When the black hole mass becomes larger, the initial non-dimensional orbital radius $r_i/M$ becomes smaller and thus $v_i$ becomes larger. When $q > 0$, this effect is dominant. Hence, although the total cycle $N$ becomes small when the black hole mass is large, the effect of the black hole absorption becomes larger.

§6. Summary

We have calculated expressions for the gravitational waves which are absorbed by a rotating black hole and obtained the energy absorption rate by the black hole to $O(v^{13})$ beyond the Newtonian quadrupole luminosity at infinity. We adopted a method developed by Mano et al.\(^{(10)}\) to calculate the post-Newtonian expansion of the Teukolsky equation analytically. We find that black hole absorption occurs at $O(v^3)$ beyond the quadrupole formula. This order is $O(v^3)$ lower than in the non-rotating case. The energy absorption rate obtained here displays a property of super radiance.

Using the formulas obtained above, we have estimated the effect of the black hole absorption on the orbital evolution of coalescing neutron star-black hole binaries. We
calculated its effect on the cycle of waves from the binaries in the laser interferometer’s band. We found that the effect of the black hole absorption is more important in the case when \( q > 0 \). (This is the case when a particle moves in the direction of the black hole rotation.) We also found that black hole absorption is important in the case when the mass of black hole is large.

Gal’tsov\(^{17}\) gave formulas for black hole absorption from the radiation reaction force on the particle. Gal’tsov’s formula is expressed as

\[
\frac{dE}{dt} = \left( \frac{dE}{dt} \right)_N \frac{v^5}{2} \left\{ v^3 \left( 1 + \sqrt{1 - q^2} \right) - \frac{q}{2} \right\} \left( 1 + 3q^2 \right). \tag{6.47}
\]

This formula agrees with our formula Eq. (4.39) to lowest order.

In this paper, we have calculated the post-Newtonian formulas only to \( O(v^8) \) beyond the lowest order. However, it is straightforward to calculate higher orders with our method.

The analysis in this paper has been restricted to the case in which a test particle moves in a circular orbit on the equatorial plane. However, it has been suggested\(^{7}\) that the inclination of the orbital plane from the equatorial plane will significantly affect the orbital phase evolution. Hence, the effect of the orbital inclination should be taken into account in future works.

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**Appendix A**

---
The Functions in \( Z_{lmw}^H \)---

In this appendix, we give functions which appear in \( Z_{lmw}^H \).

The functions \( A \) in Eq. (4.34) are given by

\[
A_{n0} = \frac{-2}{\sqrt{2\pi} \Delta^2} C_{n0} \rho^{-2} \bar{\rho}^{-1} L_1^+ \{ \rho^{-4} L_2^+ (\rho^3 S) \},
\]
\[
A_{\tilde{m}0} = \frac{2}{\sqrt{2\pi} \Delta} C_{\tilde{m}0} \rho^{-3} \left[ \left( L_2^+ S \right) \left( \frac{iK}{\Delta} + \rho + \bar{\rho} \right) - a \sin \theta S \frac{K}{\Delta} (\bar{\rho} - \rho) \right],
\]
\[
A_{\tilde{m}1} = \frac{-1}{\sqrt{2\pi}} \rho^{-3} \bar{\rho} C_{\tilde{m}1} \left[ -i \left( \frac{K}{\Delta} \right)_r - \frac{K^2}{\Delta^2} + 2i\rho \frac{K}{\Delta} \right],
\]
\[
A_{\tilde{m}2} = \frac{2}{\sqrt{2\pi} \Delta} \rho^{-3} \bar{\rho} C_{\tilde{m}2} \left[ i \frac{K}{\Delta} + \rho \right],
\]
\[
A_{\tilde{m}0} = \frac{-1}{\sqrt{2\pi}} \rho^{-3} \bar{\rho} C_{\tilde{m}0} \left[ i \frac{K}{\Delta} \right],
\]
\[
A_{\tilde{m}1} = \frac{2}{\sqrt{2\pi} \Delta} \rho^{-3} \bar{\rho} C_{\tilde{m}1} \left[ i \frac{K}{\Delta} + \rho \right],
\]
\[
A_{\tilde{m}2} = \frac{-1}{\sqrt{2\pi}} \rho^{-3} \bar{\rho} C_{\tilde{m}2} S,
\]

---
where $S$ denotes $\mathcal{S}_\ell^\omega_m$.

The functions $C$ in the above formulas are given by

\[ C_{nn} = \frac{\mu}{4\Sigma^3} \left[ E(r^2 + a^2) - al_z \right]^2, \]

\[ C_{\bar{m}n} = -\frac{\mu \rho}{2\sqrt{2} \Sigma^2} \left[ E(r^2 + a^2) - al_z \right] \left[ i \sin \theta \left( aE - \frac{l_z}{\sin^2 \theta} \right) \right], \]  \hspace{1cm} (A.1)

\[ C_{\bar{m}\bar{m}} = \frac{\mu \rho^2}{2\Sigma i} \left[ i \sin \theta \left( aE - \frac{l_z}{\sin^2 \theta} \right) \right]^2, \]

and $\dot{t}$ is given by the geodesic equations of the particle as

\[ r_0^2 \ddot{t} = -\left( aE - l_z \right)a + \frac{r^2 + a^2}{\Delta} \left( E(r^2 + a^2) - al_z \right). \]  \hspace{1cm} (A.2)

The specific energy $E$ and the angular momentum $l_z$ are given by

\[ E = \frac{1 - 2v^2 + qv^3}{(1 - 3v^2 + 2qv^3)^{1/2}}, \]

\[ l_z = \frac{r_0 v(1 - 2qv^3 + q^2 v^4)}{(1 - 3v^2 + 2qv^3)^{1/2}}. \]  \hspace{1cm} (A.3)
Appendix B

The Functions in § 3

(a) $\ell = 2$

$$A_{\ell m \omega}^{\text{in}} = \frac{1}{\omega \kappa^4} e^{\frac{i \pi}{2} (\nu+3) \kappa} e^{-i \ln \epsilon}$$

$$+ \left( \frac{15}{4} + \left( -\frac{15}{2} i \gamma - \frac{25}{12} m q - \frac{15}{8} \pi - \frac{15}{4} i \psi(0) \left( 3 + \frac{imq}{\kappa} \right) - \frac{15}{4} \ln(2) + \frac{125}{16} i \right) \epsilon \right)$$

$$+ \left( \frac{1089}{56} \gamma + \frac{725}{2352} \kappa^2 + \frac{1089}{112} \ln(2) - \frac{15}{2} \gamma^2 + \frac{12625}{21168} m^2 q^2 + \frac{35}{32} \pi^2 \right)$$

$$- \frac{535}{144} \left( \frac{15}{4} i \kappa \gamma + \frac{125}{32} i \pi + \frac{25}{6} i \gamma m q + \frac{25}{24} m \pi q - \frac{15}{2} \gamma \ln(2) \right)$$

$$- \frac{15}{2} \left( \frac{15}{4} i \psi(0) \left( 3 + \frac{imq}{\kappa} \right) - \frac{15}{8} \ln(2) \psi(0) \left( 3 + \frac{imq}{\kappa} \right) + \frac{107}{56} \ln(\epsilon) + \frac{107}{56} \ln(\kappa) \right)$$

$$- \frac{20573}{960} \left( \frac{15}{8} \psi(1) \left( 3 + \frac{imq}{\kappa} \right) + \frac{25}{12} i \pi m q \ln(2) + \frac{25}{12} i \pi \psi(0) \left( 3 + \frac{imq}{\kappa} \right) \right)$$

$$+ \frac{15}{8} i \pi \ln(2) + \frac{15}{8} i \pi \psi(0) \left( 3 + \frac{imq}{\kappa} \right) - \frac{15}{4} \psi(1) \left( 3 + \frac{imq}{\kappa} \right) \kappa^{-1} \right) \epsilon^2 \right) \right)$$

$$C_{\ell m \omega} = \left( \frac{\omega}{\epsilon \kappa} \right)^4 e^{i \epsilon + \ln \kappa} \left\{ 1 + \left( \frac{5}{6} i \kappa - \frac{5}{18} m q \right) \epsilon \right\}$$

$$+ \left( \frac{325}{7938} m^2 q^2 + \frac{5}{18} - \frac{15}{49} \kappa^2 - \frac{85}{378} i \kappa m q \right) \epsilon^2 \right\}$$

$$D_{\ell m \omega} = \omega^3 2 i e^{-\frac{i}{2} (\nu-1)} e^{i \epsilon \ln \epsilon} \left\{ 2 + \left( -\frac{1}{3} i \kappa - \pi + \frac{1}{9} m q \right) \epsilon \right\}$$

$$+ \left( \frac{1}{4} \pi^2 - \frac{1}{18} i \kappa m q + \frac{2}{49} q^2 - \frac{1}{3} i m q - \frac{11}{3969} m^2 q^2 \right)$$

$$- \frac{1}{18} m \pi q + \frac{1}{6} i \kappa \pi - \frac{67}{441} \right) \epsilon^2 \right\}$$

$$R_{\ell m \omega}^{\text{up}} = -\frac{3}{z} - z + \frac{3}{2} i z + \frac{1}{2} z^2 - \frac{1}{8} i z^3 + \frac{31}{40} z^4 + \frac{43}{80} i z^5 - \frac{117}{560} z^6 - \frac{769}{13440} i z^7$$

$$+ \epsilon \left( \frac{3}{2} i + m q \right) \frac{1}{z^2} + \left( -3 \gamma + \kappa + 3 i \pi - \frac{1}{6} i m q \right) \frac{1}{z} - \frac{9}{4} i + 3 i \gamma$$
\[-i\kappa + 3\pi + \frac{1}{3}mq + \left(-\frac{5}{2} + \frac{3}{2}i\gamma - \frac{1}{2}\kappa - \frac{3}{2}i\pi - \frac{1}{4}imq\right)z\]

\[\left(\frac{85}{48}i - \frac{1}{2}i\gamma + \frac{1}{6}i\kappa - \frac{1}{2}i\pi - \frac{1}{2}mq\right)z^2\]

\[\left(-\frac{13}{6} - \frac{1}{8}\gamma + \frac{1}{24}\kappa + \frac{1}{8}i\pi - \frac{269}{720}imq\right)z^3\]

\[\left(-\frac{1559}{2400}i + \frac{1}{40}i\gamma + \frac{31}{120}i\kappa - \frac{3}{8}\pi + \frac{49}{180}mq + \frac{4}{5}i\ln(2) + \frac{4}{5}i\ln(z)\right)z^4\]

\[+e^2\left(-\frac{3}{4}i - \frac{3}{28}i\kappa^2 + mq + \frac{11}{56}i\kappa^2q^2\right)z^{-3} + \left(-\frac{3}{2}\gamma + \frac{1}{2}\kappa + \frac{1}{14}\kappa^2\right)\]

\[+\frac{3}{2}i\pi - \frac{7}{12}imq - i\gamma mq + \frac{1}{3}i\kappa mq - m\pi q - \frac{31}{504}m^2q^2\right)z^{-2}\]

\[\left(\frac{183}{28}i - \frac{107}{70}i\gamma + \frac{3}{2}i\kappa^2 - \gamma + \frac{1}{2}i\gamma - \frac{4}{49}i\kappa^2 + 3\gamma\pi - \kappa\pi - \frac{7}{4}i\pi - \frac{1}{4}mq\right)\]

\[\left(-\frac{1}{6}\gamma mq + \frac{19}{252}\kappa mq + \frac{1}{6}im\pi q + \frac{781}{21168}im^2q^2 - \frac{107}{70}i\ln(2)\right)\]

\[\left(-\frac{179}{280} - \frac{129}{140}\gamma + \frac{3}{2}\gamma^2 + \frac{3}{4}\kappa - \gamma - \frac{11}{392}\kappa^2 + \frac{9}{4}i\pi\right)\]

\[\left(-3i\gamma\pi + i\kappa\pi - \frac{7}{4}\pi^2 + \frac{13}{24}imq - \frac{1}{3}i\gamma mq + \frac{23}{252}i\kappa mq - \frac{1}{3}m\pi q\right)\]

\[\left(-\frac{563}{21168}m^2q^2 - \frac{107}{70}\ln(2) - \frac{107}{70}\ln(z) + \left(-\frac{13751}{3360}i\right)\right)\]

\[\left(\frac{457}{140}i\gamma - \frac{3}{4}i\gamma^2 - \frac{5}{6}i\kappa + \frac{1}{2}i\gamma\kappa + \frac{17}{4704}i\kappa^2 + \frac{5}{2}\kappa\pi\right)\]

\[\left(-\frac{3}{2}\gamma\pi + \frac{1}{2}\kappa\pi + \frac{7}{8}i\pi^2 + \frac{2}{3}mq - \frac{1}{4}i\gamma mq\right)\]

\[\left(\frac{37}{504}\kappa mq + \frac{1}{4}im\pi q + \frac{751}{28224}im^2q^2 + \frac{107}{140}i\ln(2) + \frac{107}{140}i\ln(z)\right)z\]

\[+e^3\left(\left(-\frac{3}{8}i - \frac{9}{56}i\kappa^2 + \frac{3}{4}mq + \frac{15}{112}\kappa^2mq + \frac{33}{112}im^2q^2\right)\right)\]

\[-\frac{3}{112}m^3q^3\right)z^{-4} + \left(-\frac{1}{3}\gamma + \frac{3}{4}i\kappa - \frac{1}{7}\kappa^2 - \frac{3}{28}\gamma\kappa^2 + \frac{1}{28}\kappa^3\right)\]

\[+\frac{3}{4}i\pi + \frac{3}{28}i\kappa^2i\pi - \frac{157}{168}imq - i\gamma mq + \frac{1}{3}im\pi q + \frac{5}{84}i\kappa^2mq - m\pi q\]

\[+\left(\frac{41}{504}m^2q^2 + \frac{11}{56}\gamma m^2q^2 - \frac{11}{168}\kappa^2mq^2 - \frac{11}{56}im^2i\pi q^2\right)\]

\[+\left(-\frac{23}{1008}im^3q^3\right)z^{-3} + \left(-\frac{1}{3}i\kappa mq + \frac{31}{504}i\gamma m^2q^2\right)\]

\[\left(-\frac{1}{72}i\kappa mq + i\gamma mq - \frac{1}{2}\gamma^2mq + \frac{7}{12}m\pi^2q + \frac{31}{504}m^2\pi q^2 + \frac{107}{210}mq\ln(2)\right)\]

\[+\left(\frac{107}{210}mq\ln(z) + \frac{2335}{42336}im^2q^2 - \frac{1}{2}i\gamma\kappa + \frac{1}{3}\gamma m\pi q + \frac{7}{12}im\pi q - \frac{25}{168}i\kappa^2q^2\right)\]

\[\left(-\frac{1}{14}i\gamma^2 - \frac{239}{14112}\kappa^2mq + \pi + \frac{1269}{280}i + \frac{3}{2}\gamma\pi - \frac{1}{2}\kappa\pi - \frac{1}{14}\kappa^2\pi + \frac{25}{504}mq^3\right)\]
\[
\begin{align*}
&\quad \frac{69}{196} \kappa^2 + \frac{33}{140} i\gamma - \frac{31}{168} \kappa - \frac{107}{140} i\ln(2) - \frac{107}{140} i\ln(z) + \frac{3}{4} i\gamma^2 \\
&- \frac{7}{8} i\pi^2 - \frac{1}{8} \kappa^3 - \frac{31}{420} \gamma mq + \frac{103}{504} \kappa mq - \frac{421}{210} mq - \frac{239}{127008} m^3 q^3 \right) z^{-2} \\
&\quad + \epsilon^4 \left( -\frac{3}{16} i - \frac{9}{56} i\kappa^2 - \frac{1}{112} i\kappa^4 + \frac{1}{2} mq + \frac{15}{56} \kappa^2 mq + \frac{33}{112} \kappa^2 q^2 \\
&\quad + \frac{11}{252} \kappa^2 m^2 q^2 - \frac{3}{56} m^3 q^3 - \frac{23}{8064} \kappa^4 q^4 \right) z^{-5} \\
&\quad (b) \ \ell = 3
\end{align*}
\]

\[
\begin{align*}
A_{\ell m \omega}^{\text{in}} &= \frac{1}{\omega \kappa \varepsilon} e^{\frac{1}{2} \pi i (\nu+3) \kappa} e^{-i\epsilon \ln\epsilon} \left( \frac{945}{2} \frac{\kappa}{3 \kappa + i\kappa} \\
&\quad - 63 \frac{\kappa}{(3 \kappa + i\kappa)^2} \left( 90 \kappa \pi + 30 \kappa m q + 180 i \kappa \ln(2) - 60 \kappa m q \ln(2) \\
&\quad + 45 \kappa mq + 15 \kappa^2 q^2 + 360 i \kappa \gamma - 120 \gamma mq - 591 i \kappa + 197 \kappa m q \\
&\quad + 180 i \psi(0) \left( \frac{3 \kappa + i\kappa}{\kappa} \right) \kappa - 60 \psi(0) \left( \frac{3 \kappa + i\kappa}{\kappa} q \right) - 60 i \right) \right), \\
C_{\ell m \omega} &= \left( \frac{\omega}{\epsilon \kappa} \right)^4 e^{i\epsilon \ln\epsilon} \left( 1 + \left( \frac{11}{72} \kappa m q + \frac{2}{3} i\kappa \right) \epsilon \right), \\
D_{\ell m \omega} &= \omega^2 e^{-\frac{5}{2} i(\nu-1) \epsilon} e^{i\epsilon \ln\epsilon} \left( 2 + \left( -\pi + \frac{1}{36} \kappa m q - \frac{2}{3} i\kappa \right) \epsilon \right), \\
R_{\ell m \omega}^{\text{up}} &= -45 \frac{i}{z^2} - 30 z^{-1} + \frac{15}{2} i + \frac{5}{8} i z^2 \\
&\quad + \epsilon \left( \frac{45}{4} \kappa m q - 45 i \kappa + 90 i \left( -\frac{1}{2} + \frac{1}{2} \kappa \right) \right) \zeta^{-3} \\
&\quad + \left( 45 i \pi + 30 + \frac{15}{2} \kappa - 45 \gamma - \frac{15}{8} \kappa m q \right) \zeta^{-2} + \left( 30 \pi - 45 i \right) \\
&\quad + \frac{35}{24} \kappa m q + 30 i \gamma - 5 i \kappa \right) \zeta^{-1} \\
&\quad + \epsilon^2 \left( \frac{135}{8} \kappa m q - \frac{75}{2} i\kappa^2 \right) \\
&\quad + 135 i \kappa \left( -\frac{1}{2} + \frac{1}{2} \kappa \right)^2 - \frac{135}{4} \left( -\frac{1}{2} + \frac{1}{2} \kappa \right) \kappa m q + \frac{15}{8} \kappa m^2 q^2 \\
&\quad - 45 i \left( -\frac{1}{2} + \frac{1}{2} \kappa \right)^2 + 2 \left( -1 + \kappa \right) \left( -\frac{1}{2} + \frac{1}{2} \kappa \right) \right) \zeta^{-4}.
\end{align*}
\]
(c) $\ell = 4$

\[
A_{\ell m \omega}^{\text{in}} = \frac{1}{\omega} \frac{1}{\kappa^5 e^6} e^{\frac{2\pi i}{3}(\nu + 3)} e^{i \epsilon \kappa} e^{-i \epsilon \ln \epsilon} \left\{ \frac{\kappa^2}{(3 \kappa + i m q)(4 \kappa + i m q)} \right\},
\]

\[
C_{\ell m \omega} = \left( \frac{\omega}{\ell \kappa} \right)^4 e^{i \epsilon \kappa} (1 + O(\epsilon)),
\]

\[
D_{\ell m \omega} = \omega^3 e^{-\frac{2}{3}(\nu - 1)} e^{i \epsilon \ln \epsilon} (2 + O(\epsilon)),
\]

\[
R_{\ell m \omega}^{\text{up}} = -\frac{630 \epsilon}{\kappa^3}.
\]

\[\text{Appendix C}\]

The Energy Absorption Rate for Each Mode

The energy absorption rate for each mode to $O(v^8)$ are given by

\[
\eta_{2,2} = -\frac{1}{4} q - \frac{3}{4} q^3 + v^2 \left( -\frac{3}{4} q - \frac{9}{4} q^3 \right)
+ \left( \frac{27}{4} q^2 + 4 q B_2 + \frac{15}{4} q^4 + 6 q^3 B_2 + \frac{13}{2} \kappa q^2 + 3 q^4 \kappa + \frac{1}{2} \kappa + \frac{1}{2} \right) v^3
+ \left( -\frac{199}{12} q - \frac{593}{42} q^3 + \frac{2}{7} q^5 \right) v^4
+ \left( \frac{721}{36} q^2 + 6 q B_2 + \frac{127}{12} q^4 + 18 q^3 B_2 + \frac{39}{2} \kappa q^2 + 9 q^4 \kappa
+ \frac{3}{2} \kappa + \frac{3}{2} \right) v^5
+ \left( -\frac{607076}{11025} q - 4 B_2 + \frac{428}{105} \gamma q + \frac{2}{3} \pi^2 q + \frac{428}{105} q \ln 2
- 4 q C_2 - 12 q^3 C_2 - 36 q^4 B_2 - 6 q^5 B_2 + \frac{428}{35} q^3 \gamma + \frac{428}{35} q^3 \ln 2
+ 2 q^3 \pi^2 + \frac{428}{105} q \ln \kappa + \frac{428}{105} q \ln \kappa + \frac{428}{35} \kappa \ln \kappa + \frac{428}{35} \kappa \ln \kappa
+ \frac{6 \kappa^7}{2 \kappa} + \frac{428}{35} q^3 A_2
+ 8 q B^2 \kappa - 24 q^3 B_2 \kappa + \frac{856}{35} q \ln v + \frac{856}{35} q^3 \ln v - \frac{4 B_2}{\kappa} - 32 \frac{q^3}{\kappa}
- 31 \frac{q^5}{\kappa} + 57 \frac{q^5}{\kappa} - 48 \frac{q^2 B_2}{\kappa} + 28 \frac{q^4 B_2}{\kappa} - 24 \frac{q^3 C_2}{\kappa} - 8 \frac{q C_2}{\kappa}
+ 24 \frac{B_2}{\kappa}\frac{q^6}{\kappa} + \frac{11883052}{99225} q^3 + \frac{548}{27} q^5 \right) v^6
+ \left( -\frac{16747}{126} q^2 + \frac{15893}{189} q^4 - \frac{155}{126} q^6 + 123 \kappa q^2 + \frac{1142}{21} q^4 \kappa - \frac{8}{7} \kappa q^6
\]
\[ \eta_{2,1} = v^2 \left( \frac{3}{16} q^3 - \frac{1}{4} q \right) + \left( -\frac{1}{4} q^4 + \frac{1}{3} q^2 \right) v^3 \]
\[ + \frac{1}{12} q^5 + \frac{8}{9} q^3 - \frac{4}{3} q^2 \right) v^4 \]
\[ + \left( -\frac{137}{48} q^4 - \frac{3}{4} q^3 B_1 + \frac{265}{72} q^2 + q B_1 - \frac{3}{4} q^4 + \frac{1}{2} \kappa + \frac{7}{8} \kappa q^2 + \frac{1}{2} \right) v^5 \]
\[ + \left( -\frac{382}{63} q + \frac{865}{756} q^3 + \frac{2321}{1008} q^5 - \frac{2}{3} \kappa q - \frac{7}{6} \kappa q^3 + q^5 + q^4 B_1 - \frac{4}{3} q^2 B_2 \right) v^6 \]
\[ + \left( \frac{8}{3} q^2 - \frac{32}{9} q^3 B_1 - \frac{1405}{3024} q^6 - \frac{65}{18} q^4 - \frac{123269}{9072} q^4 \right) v^7 \]
\[ + \left( -\frac{6292400}{176400} q - 2 B_1 + \frac{107}{105} \gamma q + \frac{1}{6} \pi^2 q + \frac{107}{105} q \ln 2 - \frac{107}{140} q^3 \gamma \right) v^8. \]

\[ \eta_{3,3} = \left( -\frac{75}{112} q^5 - \frac{555}{896} q^3 - \frac{15}{224} q \right) v^4 + \left( -\frac{45}{112} q - \frac{1665}{448} q^3 - \frac{225}{56} q^5 \right) v^6 \]
\[ + \left( \frac{225}{28} q^5 + \frac{375}{112} q^6 + \frac{15}{112} + \frac{1905}{448} \kappa q^2 + \frac{2055}{224} q^4 \kappa + \frac{15}{112} \kappa \right) v^8. \]
\[ + \frac{1665}{224} q^3 B_3 + \frac{45}{56} q B_3 + \frac{10995}{896} q^4 + \frac{2055}{448} q^2 v^7 
+ \left( -\frac{17697}{896} q^3 - \frac{18315}{896} q^5 + \frac{125}{112} q^7 - \frac{481}{224} q \right) v^8. \]  

(C.3)

\[ \eta_{3.2} = \left( \frac{25}{189} q^5 - \frac{5}{63} q - \frac{110}{567} q^3 \right) v^6 
+ \left( \frac{5}{42} q^2 + \frac{55}{189} q^4 - \frac{25}{126} q^6 \right) v^7 
+ \left( \frac{25}{336} q^7 - \frac{40}{63} q - \frac{14485}{9072} q^3 + \frac{205}{216} q^5 \right) v^8. \]  

(C.4)

\[ \eta_{3.1} = \left( -\frac{1}{224} q + \frac{59}{8064} q^3 - \frac{1}{336} q^5 \right) v^4 
+ \left( \frac{767}{12096} q^3 - \frac{13}{504} q^5 - \frac{13}{336} q \right) v^6 
+ \left( \frac{1}{84} q^5 B_1 + \frac{1}{56} q B_1 + \frac{31}{4032} \kappa q^2 - \frac{55}{2016} q^4 \kappa + \frac{1}{84} \kappa q^6 + \frac{109}{3024} q^6 \right) v^7 
\]  
\[ - \frac{6395}{72576} q^4 - \frac{59}{2016} q^3 B_1 + \frac{1}{112} \frac{4032}{4032} q^2 + \frac{1}{112} \kappa \right) v^7 
+ \left( \frac{1}{336} q^7 - \frac{431}{2105} \frac{72576}{72576} q^3 - \frac{3271}{2192} q^5 \right) v^8. \]  

(C.5)

\[ \eta_{4.4} = -\frac{5}{2268} v^8 q \left( 9 + 7 q^2 \right) \left( 3 q^2 + 1 \right) \left( 15 q^2 + 1 \right). \]  

(C.6)

\[ \eta_{4.2} = -\frac{5}{63504} v^8 q \left( 5 q^2 - 9 \right) \left( 3 q^2 - 4 \right) \left( 3 q^2 + 1 \right). \]  

(C.7)

**Appendix D**

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**The Alternative Form of \((dE/dt)_H\)**

In this section, we describe the absorption rate \((dE/dt)_H\) by means of \(x \equiv (M \Omega)^{1/3}\). Here we define

\[ \left( \frac{dE}{dt} \right)_Q = \frac{32}{5} \left( \frac{\mu}{M} \right)^2 x^{10}. \]

Using the relation

\[ v = x (1 + \frac{1}{3} qx^3 + \frac{2}{9} q^2 x^6 + O(x^9)), \]

we have

\[ \left( \frac{dE}{dt} \right)_H = \left( \frac{dE}{dt} \right)_Q x^5 \left[ \frac{1}{4} q - \frac{3}{4} q^3 + (-q - \frac{33}{16} q^3) x^2 \right] \]
\[
(2 q B_2 + \frac{1}{2} + \frac{13}{2} \kappa q^2 + \frac{35}{6} q^2 - \frac{1}{4} q^4 + \frac{1}{2} \kappa \kappa)
+ 3 q^4 \kappa + 6 q^3 B_2 \big| x^3 \big)
\]
\[
+ \left( - \frac{43}{7} q - \frac{17}{56} q^5 - \frac{4651}{336} q^3 \big| x^4 \right)
\]
\[
+ \left( \frac{433}{24} q^2 - \frac{95}{24} q^4 + 2 - \frac{3}{4} q^2 B_2 + 2 \kappa + \frac{33}{4} q^4 \kappa + 6 q B_2 \right)
\]
\[
+ 18 q^3 B_2 + \frac{163}{8} \kappa q^2 + q B_1 \big| x^5 \right)
\]
\[
+ \left( - \frac{2586329}{44100} q - 4 B_2 - \frac{1640747}{19600} q^3 + 19 q^5 \kappa + \frac{428}{105} \gamma q + \frac{2}{3} \pi^2 q \right)
\]
\[
+ \left( \frac{428}{105} q \ln(2) - 4 q C_2 - 12 q^3 C_2 - 4 B_2 q^3 \right)
\]
\[
+ \left( 2 q^2 \pi^2 + \frac{428}{105} q \ln(\kappa) + \frac{428}{35} q A_2 + \frac{2}{3} q^2 \ln(\kappa) + 6 \frac{q^5}{\kappa} \right)
\]
\[
+ \left( \frac{428}{35} q^3 A_2 - 8 q^3 B_2^2 - 24 q^3 B_2 - 4 \frac{B_2}{\kappa} - 32 \frac{q^3}{\kappa} - 31 \frac{q}{\kappa} \right)
\]
\[
+ \left( 57 q^5 \kappa + 4 q^4 B_1 - \frac{4}{3} q^2 B_1 + \frac{7}{3} \kappa q + \frac{227}{6} \kappa q^3 + \frac{455}{16} q^5 \right)
\]
\[
- \frac{48}{\kappa} q^2 B_2 + 28 q^4 B_2 - \frac{24}{\kappa} q^2 C_2 - 8 \frac{q C_2}{\kappa} \right)
\]
\[
+ \left( 24 \frac{q^4 B_2}{\kappa} + \frac{856}{105} q \ln(x) + \frac{856}{35} q^3 \ln(x) \big| x^6 \right)
\]
\[
+ \left( \frac{19687}{168} q^2 - \frac{145}{336} q^6 - \frac{4729}{1008} q^4 + \frac{899}{168} q B_1 - \frac{41}{28} q^6 \kappa \right)
\]
\[
+ \left( \frac{45}{56} q B_3 - \frac{803}{224} q^3 B_1 + \frac{1665}{224} q^3 B_3 + \frac{86}{7} + \frac{719}{12} \kappa q^4 \kappa \right)
\]
\[
+ \left( \frac{796}{21} q B_2 + \frac{86}{7} q B_2 - \frac{16}{7} q^5 B_2 + \frac{225}{28} q^5 B_3 \right)
\]
\[
+ \left( \frac{9}{28} q^5 B_1 + \frac{22201}{168} \kappa q^2 + \frac{2372}{21} \kappa q^3 B_2 \right) \big| x^7 \right)
\]
\[
\left( - \frac{19366807}{88200} q - 12 B_2 - 2 B_1 - \frac{2062220497}{6350400} q^3 - q C_1 + 55 q^5 \kappa \right)
\]
\[
+ \left( \frac{1061}{35} \gamma q + \frac{13}{6} \pi^2 q + \frac{995}{21} q \ln(2) - 12 q C_2 - 36 q^3 C_2 \right)
\]
\[
+ \left( \frac{52}{3} q^4 B_2 - \frac{1136}{9} q^2 B_2^2 + \frac{12197}{140} q^3 \gamma + \frac{3873}{28} q^3 \ln(2) \right)
\]
\[
+ \left( \frac{47}{8} q^3 \pi^2 + \frac{1391}{105} q \ln(\kappa) + \frac{428}{35} q A_2 + \frac{5029}{140} q^3 \ln(\kappa) \right)
\]
\[
+ \left( \frac{37}{6} q^7 + \frac{1284}{35} q^3 A_2 - 24 q^3 B_2^2 - 72 q^3 B_2 - 12 \frac{B_2}{\kappa} \right)
\]
\[
- \frac{341}{4} q^3 \kappa - \frac{637}{6} \kappa \kappa + \frac{741}{4} \kappa \kappa \right)
\]
\[
+ \left( \frac{43}{6} q^4 B_1 - \frac{163}{18} q^2 B_1 + \frac{3}{2} q^2 B_1^2 + \frac{107}{105} \kappa A_1 \right)
$$\begin{align*}
&\left. -\frac{107}{140} q^3 A_1 - 2 q B_1^2 + \frac{3}{4} q^3 C_1 - 2 \frac{B_1}{\kappa} + \frac{40}{3} \kappa q + \frac{815}{6} \kappa q^3 \\
&+ \frac{1265}{18} q^5 + \frac{25}{252} q^7 - 144 \frac{q^2 B_2}{\kappa} + 84 \frac{q^4 B_2}{\kappa} - 72 \frac{q^3 C_2}{\kappa} + 24 \frac{q C_2}{\kappa} \\
&+ 72 \frac{q^6 B_2}{\kappa} - 3 \frac{q^6 B_1}{\kappa} + \frac{13}{2} \frac{q^4 B_1}{\kappa} \\
&- \frac{3}{2} \frac{q^2 B_1}{\kappa} - 2 \frac{q C_1}{\kappa} + \frac{3}{2} q^3 C_1 + \frac{4574}{105} q \ln(x) + \frac{8613}{70} q^3 \ln(x) x^8 \right].
\end{align*}$$

References